Abstract

We show that the reals in the minimal iterable inner model having $n$ Woodin cardinals are precisely those which are $\Delta^1_{n+2}$ definable from some countable ordinal. (One direction here is due to Hugh Woodin.) It follows that this model satisfies "There is a $\Delta^1_{n+2}$ well-order of the reals". We also describe some other connections between the descriptive set theory of projective sets and inner models with finitely many Woodin cardinals.

0. Introduction

Let $M_n$ be the canonical minimal inner model satisfying "There are $n$ Woodin cardinals", where $n < \omega$. We shall show in this paper that $M_n$ is $\Sigma^1_{n+1}$ correct and satisfies "$\mathbb{R}$ has a $\Delta^1_{n+2}$ well-order".

D.A. Martin and the author proved versions of these results in 1986, and announced them in [6]. Since then, the work of [7, 8] has produced inner models which are (superficially, anyway) somewhat different from those to which those 1986 results applied. The newer models carry a fine structure theory which makes it much easier to determine their properties, and in particular, it is much easier to prove the analogs of those 1986 results. This is what we shall do here. We shall let the sands of time drift over whatever it was Martin and the author proved concerning the older models. Nevertheless, we wish to emphasize that the main ideas of this paper are part of that earlier joint work.

Our results imply that every real in $M_n$ is $\Delta^1_{n+2}$ in some countable ordinal. In 1988, Woodin proved the converse (for the older models, but his proof goes over with no change to the newer ones). Thus $\mathbb{R} \cap M_n$ is precisely the set of reals $\Delta^1_{n+2}$ in a countable ordinal. This set is familiar to descriptive set theorists: for $n$ even, it is $C_n$, and for $n$ odd, it is $Q_n$ (cf. [3]). Since $\mathbb{R} \cap M_{2k} = C_{2k}$, $M_{2k}$ is $\Sigma^1_{2k+2}$ correct, something our original proof of correctness did not show.

Section 1 is devoted to preliminary definitions and lemmas concerning the comparison process. In Section 2 we use a comparison argument to show that every real in...
$M_n$ is $\Delta^1_{n+2}$ in a countable ordinal. Section 3 contains miscellaneous further applications of the techniques of Section 2. In Section 4 we sketch our original proof of correctness for $M_n$, and then present Woodin's results in this area. We also indicate some other ways the $M_n$'s appear naturally in the descriptive set theory of projective sets.

We shall assume the reader is familiar with [7, 8].

1. Preliminaries

We shall use the Levy hierarchy over $(HC, \in)$, the hereditarily countable sets, for our quantifier calculations. Recall that $\Sigma^m_n = \Sigma^m_{n+1}$ in the codes", for all $n \geq 1$. We use extensively the Spector-Gandy theorem [9, 6E.7]. The natural extension of this theorem to $HC$ is the following.

**Spector-Gandy Theorem.** Let $n \geq 2$ be even, and assume $\Delta^m_n$ determinacy. Let $R(a,b,c)$ be a $\Pi^m_n$ relation on $HC$, and let

$$Q(a,c) \iff \exists b \in \Delta^m_n(a) R(a,b,c).$$

Then $Q$ is $\Pi^m_n$.

One must be careful here. We take $\Sigma^m_n(a)$ to be the class of relations on $HC$ which are $\Sigma^m_n$ definable over $(HC, \in)$ from parameters in $TC(\{a\})$, the transitive closure of $\{a\}$. Similarly for $\Pi^m_n(a)$ and $\Delta^m_n(a)$. This meaning for "$\Delta^m_n(a)$" is necessary to insure Lemma C below. Notice that $\{b\} \in \Sigma^m_n(a)$ implies $b \in \Delta^m_n(a)$. We doubt that the converse is true, but have no counterexample.

The Spector-Gandy theorem of [9] is just the theorem above with $a, b$ and $c$ restricted to range over reals. It takes some additional work to prove the full theorem, so we shall give a proof here. (There is a general theorem on definable equivalence relations, due to Kechris, of which this is a special case.) We could have avoided most of this by using the ordinary analytical hierarchy for our quantifier calculations, and coding countable mice and iteration trees and the like by reals.

The main step toward the full Spector–Gandy theorem is due to A.S. Kechris. Let $WO$ be the set of (reals coding) well-orders of $\omega$, and $|x| = \text{order type of } x$, for $x \in WO$.

**Lemma A** (Kechris). Let $n \geq 2$ be even, and assume $\Sigma^1_n$ determinacy. Let $S \subseteq WO$ be $\Sigma^1_n$, and suppose $\{|x| : x \in S\}$ is bounded in $\omega_1$. Then $\{|x| : x \in S\}$ has a $\Delta^1_{n+1}$ bound; that is, $\exists \alpha < \delta^1_{n+1} \forall x \in S(|x| < \alpha)$. 

**Proof (sketch).** Let $S(x) \iff \exists y R(x,y)$, where $R$ is $\Pi^1_n$. Consider the following "Solovay game": I plays $z$, and II plays $x, y$. Player I loses unless $z \in WO$. If $z \in WO$, then II wins iff $R(x,y)$ and $|z| \leq |x|$. Since $S$ is bounded, I has a winning strategy. But the game is $\Sigma^1_n$ for him, so by 3rd periodicity [9, 6E.1], I has a $\Delta^1_{n+1}$ winning strategy $\sigma$. 

Now $Y = \{z \mid z$ is a play for $I$ according to $\sigma\}$ is a $\Sigma^1_1(\sigma)$ subset of $WO$, so $\forall z \in Y \ (|z| < \delta^1_1(\sigma))$. Since $\sigma$ is $A^1_{n+1}$, $\delta^1_1(\sigma) < \delta^1_{n+1}$, and we are done. □

Let $WF$ be the set of (reals coding) well-founded trees on $\omega$. Recall the standard coding of hereditarily countable sets by elements of $WF$: we define the set $c(T)$ coded by $T$ by induction on $rk(T)$. Let

$$T_{\langle n \rangle} = \{s \in \omega^{< \omega} \mid \langle n \rangle \supseteq s \in T\},$$

and set

$$c(T) = \{c(T_{\langle n \rangle}) \mid \langle n \rangle \in T\}.$$

Then $c$ is a $\Sigma^1_{HC}$ map of $WF$ onto $HC$. Let $R$ be a relation on $HC$, and

$$R^* = \{(x_1, \ldots, x_k) \mid \forall i \leq k \ (x_i \in WF) \land R(c(x_1), \ldots, c(x_k))\}$$

be its coded version. Then for any $n \geq 1$, $R$ is $\Sigma^1_n$ iff $R^*$ is $\Sigma^1_{n+1}$.

**Lemma B.** Let $n + 2$ be even, and assume $\Delta^1_{n+1}$-determinacy. Suppose $b \in \Delta^1_{n+1}(a) \cap HC$, and $c(x) = a$. Then $\exists y \in \Delta^1_{n+1}(x)(c(y) = b)$.

**Proof.** We take the case $n = 2$; the proof is the same in the general case.

Note first that the set of all $z \in WO$ such that $\exists y \in WF (c(y) \in b \land rk(c(y)) \geq |z|)$ is $\Sigma^1_1(x)$. It follows from Lemma A that $rk(b) < \delta^1_2(x)$.

Now fix a good coding of the $\Delta^1_2(x)$ reals. That is, we have a $\Pi^1_2(x)$ set $D \subseteq \omega$, and for each $e \in D$ a real $[e]^x$ so that $\Delta^1_2(x) \cap \omega = \{[e]^x \mid e \in D\}$. Moreover, there is a $\Pi^1_2(x)$ relation $R$ and a $\Sigma^1_1(x)$ relation $S$ such that $\forall e \in D ([e]^x(n) = m \iff R(e, n, m) \land S(e, n, m))$. Fix also a $\Pi^1_2(x)$ norm $\varphi$ on a complete $\Pi^1_2(x)$ set $P \subseteq \omega$; our determinacy hypothesis implies that there is such a norm. We assume $\varphi$ is "regular", that is, ran $\varphi$ is transitive.

Let $TC(b)$ be the transitive closure of $b$. We use effective transfinite induction to produce a partial $\Delta^1_2(x)$ function $\pi: \omega \times \omega \rightarrow \omega$ such that $P \times \omega \subseteq \text{dom } \pi$, and for all $e \in P$

\begin{equation}
(*) \quad (d \in TC(b) \land rk(d) \leq \varphi(e) \Rightarrow \exists n(d = c([\pi(e, n)]^x))).
\end{equation}

We shall present the inductive definition of $\pi$ informally. It can be formalized using the recursion theorem as usual.

Suppose then $e \in P$ and $\pi(e', n) \downarrow$ whenever $\varphi(e') < \varphi(e)$ and $n \in \omega$. Suppose also $\pi(e', n) \in D$ and $[\pi(e', n)]^x \in WF$ whenever $\varphi(e') < \varphi(e)$ and $n \in \omega$. Suppose finally $(*)$ holds at $e'$ such that $\varphi(e') < \varphi(e)$. Now for any $d \in HC$, let

$$H_d = \{(e', n) \mid \varphi(e') < \varphi(e) \land c([\pi(e', n)]^x) \in d\}.$$  

$H_d$ is essentially a real, and letting

$$S = \{H_d \mid d \in TC(b)\},$$
we see that $S$ is $\Sigma^1_3(x)$ (uniformly in $e$ and an index for $\pi$ as a partial $\Delta^1_3(x)$ function). Since $S$ is countable, we can (uniformly in $e$ and an index for $\pi$) find a $\Delta^1_3(x)$ real $z$ so that $S \subseteq \{z_i | i \in \omega\}$.

Notice that if $d \in TC(b)$ and $rk(d) \leq \varphi(e)$, then

$$d = \{c([\pi(e', n)]^x) | (e', n) \in H_d\}.$$

We can now define $\pi(e, i)$, for $i \in \omega$, to be a $\Delta^1_3(x)$ index for a tree $[\pi(e, i)]^x \in WF$ such that

$$\langle a \rangle \cap s \in [\pi(e, i)]^x \quad \text{iff} \quad (a = (e', n) \text{ for some } (e', n) \in (z)_i \text{ such that } \varphi(e') < \varphi(e) \text{ and } s \in [\pi(e', n)]^x).$$

We then have that if $(z)_i = H_d$, where $d \in TC(b)$ and $rk(d) \leq \varphi(e)$, then

$$c([\pi(e, i)]^x) = \{c([\pi(e', n)]^x) | (e', n) \in H_d\} = d.$$

Thus $(\ast)$ holds at $e$. Moreover, $\pi(e, n) \in D$ and $[\pi(e, n)]^x \in WF$ for all $n < \omega$. So our induction hypotheses on $\pi$ continue to hold at $e$.

Now let $e \in P$ be such that $rk(b) \leq \varphi(e)$. Set

$$H = \{n | c([\pi(e, n)]^x) \in b\}.$$

Since $b$ is $\Delta^1_2HC(a)$, $H$ is $\Delta^1_2HC(a)$, and therefore $H$ is $\Delta^1_3(x)$. Put

$$T = \{ \langle n \rangle \cap s | n \in H \wedge s \in [\pi(e, n)]^x \}.$$

Then $T$ is $\Delta^1_3(x)$ and $c(T) = b$. $\square$

Lemma B is not true for $n$ odd, since $C_n$ is a countable $\Delta^1_nHC$ set having no $\Delta^1_n+1$ code. Finally, we have the converse to Lemma B.

Lemma C. Let $n \geq 2$ be even, and assume $\Delta^1_{n-1}HC$-determinacy. Let $a, b \in HC$, and suppose $\forall x (c(x) = a \Rightarrow \exists y \in \Delta^1_{n+1}(x) (c(y) = b))$. Then $b \in \Delta^1_nHC(a)$.

Proof (sketch). Let $\{[\pi]^x | e \in D^x\}$ be a good parametrization of the $\Delta^1_3(x)$ reals, uniformly in $x$. (Again, we take $n = 2$ for no good reason.) We may assume $a$ is transitive. Consider the space $\omega a$, which is of course homeomorphic to $\omega \omega$. Comeager many $\pi \in \omega a$ map $\omega$ onto $a$ and to such $\pi$ we associate canonically an $x_\pi$ such that $c(x_\pi) = a$. Since $\Sigma^1_1$ sets are Baire, we can fix $e \in \omega$ and a nbd $p$ so that for comeager many $\pi \supseteq p$ in $\omega a$, $c([\pi]^x^\ast) = b$. (See [2], for such arguments.) Because "forcing for $\Delta^1_3$ formulae is $\Delta^1_n$" (see [2]), this means that $\{y | c(y) \in b\}$ is $\Delta^1_n(x, z)$ whenever $c(x) = a$ and $c(z) = p$, uniformly in such $x$ and $z$. Thus $b \in \Delta^1_nHC(a)$. $\square$

We can now easily prove the Spector-Gandy theorem. Let

$$Q(a, c) \iff \exists b \in \Delta^1_nHC(a) R(a, b, c),$$
where \( R \) is \( \Pi_n^{HC} \) and \( n \) is even. Let \( R^*(x, y, z) \iff R(c(x), c(y), c(z)) \) and \( Q^*(x, z) \iff Q(c(x), c(z)) \). So \( R^* \) is \( \Pi_{n+1}^1 \), and we must see \( Q^* \) is \( \Pi_{n+1}^1 \). But
\[
Q^*(x, z) \iff \exists y \in A_{n+1}^1(x) \exists y' \in A_{n+1}^1(x') \exists (c(y') = (c(y)))
\]
by Lemmas B and C. By the Spector–Gandy theorem on \( \mathbb{R} \), \( Q^* \) is \( \Pi_{n+1}^1 \), so we are done.

In what follows, we shall often omit the superscript “HC” from “\( \Sigma_n^{HC} \)”, etc. Thus \( \Sigma_{n+1}^1 = \Sigma_n \) in what follows; we hope this causes no confusion.

We now turn to the inner model theory.

Definition 1.1. A premouse \( \mathcal{M} \) is \( n \)-small above \( \delta \) iff whenever \( \kappa \) is the critical point of an extender on the \( \mathcal{M} \)-sequence, and \( \delta < \kappa \), then
\[
\mathcal{J}_n^\kappa \not\vdash \text{there are } n \text{ Woodin cardinals } > \delta.
\]
We say \( \mathcal{M} \) is \( n \)-small iff \( \mathcal{M} \) is \( n \)-small above \( 0 \).

Let \( \mathcal{C}' = \langle \mathcal{N}_\zeta | \mathcal{N}_\zeta \text{ is defined} \rangle \) be the construction of tame mice using full background extenders from \( V \), as described in [8, Section 1]. Fix \( n < \omega \). The model \( M_n \) is produced by an initial segment of \( \mathcal{C}' \).

Suppose first that all \( \mathcal{N}_\zeta \) are \( n \)-small. Then all \( \mathcal{N}_\zeta \) are meek, and so \( \mathcal{N}_\infty \) is defined (as the limit of the \( \mathcal{N}_\zeta \)'s as \( \zeta \to \infty \)). We set \( M_n = \mathcal{N}_\infty \). On the other hand, suppose \( \zeta \) is least such that \( \mathcal{N}_\zeta \) is not \( n \)-small. Notice \( \mathcal{N}_\zeta \) is active. Then we define \( M_n^\# = \mathcal{C}_\infty(\mathcal{N}_\zeta) \), and letting \( \mathcal{P} \) be the \( OR \)th iterate of \( M_n^\# \) via its last extender, we set \( M_n = \mathcal{P}^\mathcal{M}_R \).

If \( M_n^\# \) exists, then it is essentially the type of a club class of indiscernibles for \( M_n \), and \( M_n \) is the hull of those indiscernibles. In any event, \( M_n \) is a premouse of proper class size, and \( M_n \) and all its levels \( \mathcal{J}_n^\mathcal{M} \) are \( n \)-small and \( \omega \)-sound. \( M_0 \) is just \( L \). From [8] and [7, Section 11] we have at once the following theorem.

Theorem 1.2. Let \( n < \omega \), and suppose there are \( n \) Woodin cardinals; then \( M_n \) satisfies “There are \( n \) Woodin cardinals”.

The proof of Theorem 1.2 also shows that if there are \( n \) Woodin cardinals and a \( P_\infty(\kappa) \)-measurable above, then \( M_n^\# \) exists. If we were more careful about what we meant by \( M_n^\# \), we could reduce the hypothesis here to \( n \) Woodins plus indiscernibles for \( L(V_\delta) \), where \( \delta \) is the \( n \)th Woodin.

The definability of \( \mathbb{R} \cap M_n \) and its members comes (as usual) from the comparison lemma for \( n \)-small mice: a real belongs to \( M_n \) just in case it belongs to some sufficiently iterable \( n \)-small premouse. We shall describe an iterability condition which is \( \Pi_n^{HC} \), and which suffices to guarantee comparability with realizable premise (i.e. those which are embeddable in a model of \( \mathcal{C}' \); cf. [8]). We call this condition \( \Pi_n \)-iterability.
Roughly speaking, $\mathcal{M}$ is $\Pi_n$-iterable just in case player II wins a certain variant $W_\mathcal{G}_\alpha(\mathcal{M}, n)$, the weak iteration game of length $n$ on $\mathcal{M}$. The assertion that II wins $W_\mathcal{G}_\alpha(\mathcal{M}, n)$ itself is $\Pi_{2n}$. However, the unique branches results from [6, Section 2] enable us to piece definability restrictions on the iteration trees and branches played by I and II in $W_\mathcal{G}_\alpha(\mathcal{M}, n)$. We arrive thereby at a variant game $\mathcal{I}(\mathcal{M}, n)$ such that

(a) if $\mathcal{M}$ is realizable, then II has a winning strategy in $\mathcal{I}(\mathcal{M}, n)$,
(b) if II has a winning strategy in $\mathcal{I}(\mathcal{M}, n)$, then $\mathcal{M}$ can be compared with any realizable $\mathcal{N}$, and
(c) $\{\mathcal{M} \mid \text{II has a winning strategy in } \mathcal{I}(\mathcal{M}, n)\}$ is $\Pi_n$.

The Spector–Gandy theorem figures heavily in the proof of (c).

In order to define $\mathcal{I}(\mathcal{M}, n)$, we must introduce some terminology.

**Definition 1.3.** Let $\mathcal{M}$ be a premouse, and $\delta < OR^\mathcal{M}$.

(a) $k(\mathcal{M}, \delta)$ is the unique $k < \omega$ such that $\mathcal{M}$ is $k$-sound, $k + 1$ solid, and $\rho_{k+1}(\mathcal{M}) \leq \delta < \rho_k(\mathcal{M})$, if such a $k$ exists, and $k(\mathcal{M}, \delta) \uparrow$ otherwise.

(b) $\mathcal{M}$ is a $\delta$-mouse iff $k(\mathcal{M}, \delta) \downarrow$, and letting $k = k(\mathcal{M}, \delta)$, $\mathcal{M} = H_{k+1}(\delta \cup \rho_{k+1}(\mathcal{M}))$.

(c) A putative iteration tree $T$ on $\mathcal{M}$ is above $\delta$ iff $\text{crit}(E_{\mathcal{M}}^T) \geq \delta$ for all $\alpha + 1 < lh T$.

(d) $\delta$ is a cutpoint of $\mathcal{M}$ iff for no $E$ on the $\mathcal{M}$-sequence do we have $\text{crit}(E) < \delta \leq lh E$.

The definability restrictions leading from $W_\mathcal{G}_\alpha(\mathcal{M}, n)$ to $\mathcal{I}(\mathcal{M}, n)$ are somewhat different in the cases $n$ odd and $n$ even. (This reflects the "periodicity of order two" in the projective hierarchy.) We begin with the case $n$ is even, and $n \geq 2$. Let $\mathcal{M}$ be a countable $\delta$-mouse. $\mathcal{I}(\mathcal{M}, \delta, n)$ is the following variant of the weak iteration game. There are $n$ rounds, Before beginning round $k$, where $1 < k \leq n$, we have a $\delta$-mouse $\mathcal{M}_k$. We begin with $\delta_1 = \delta$ and $\mathcal{M}_1 = \mathcal{M}$. Round $k$ is played as follows:

I must play a countable, $\omega$-maximal, putative iteration tree $T$ on $\mathcal{M}_k$ such that $T$ is above $\delta_k$. Player II can then either accept $T$ or play a maximal well-founded branch $b$ of $T$ such that $b \in A_n(\langle \mathcal{M}_k, T \rangle)$, with the proviso that he cannot accept $T$ if it has a last, ill-founded model. If II accepts $T$, then we set $\mathcal{M}_{k+1} = \text{last model of } T$, and $\delta_{k+1} = \sup \{\text{v}(E_{\mathcal{M}}^T) \mid \alpha + 1 < lh T\}$. If II plays $b$, then we set $\mathcal{M}_{k+1} = \mathcal{M}_k^b$ and $\delta_{k+1} = \sup \{\text{v}(E_{\mathcal{M}}^b) \mid \alpha \in b\}$. We now go on to round $k + 1$, unless $k = n$, in which case the game is over. The first player to violate a rule of $\mathcal{I}(\mathcal{M}, \delta, n)$ loses the game, and if no one violates a rule, then II wins the game.

**Definition 1.4.** $\mathcal{M}$ is $\Pi_n$-iterable above $\delta$ iff II has a winning strategy in $\mathcal{I}(\mathcal{M}, \delta, n)$. $\mathcal{M}$ is $\Pi_n$-iterable iff $\mathcal{M}$ is $\Pi_n$-iterable above $0$.

The important respect in which $\mathcal{I}(\mathcal{M}, \delta, n)$ differs from the weak iteration game of length $n$ is that the branches played by II must be $\Delta_n$ in the trees played by I. This is what makes $\Pi_n$-iterability a $\Pi_n$ condition.

**Lemma 1.5.** Let $n \geq 2$ be even, and assume $\Delta_n^{HC}$ determinacy. Then $\{\langle \mathcal{M}, \delta \rangle \in HC \mid \mathcal{M}$ is $\Pi_n$-iterable above $\delta\}$ is $\Pi_n^{HC}$. 
Proof. By inspecting the rules of $\mathcal{I}(\mathcal{M}, \delta, n)$, one sees that there is a $\Pi_1$ relation $R$ on $HC$ such that $\mathcal{M}$ is $\Pi_n$-iterable above $S$ if and only if

$$\forall \mathcal{T} \exists b_1 \in A_n(\mathcal{T}_1) \ldots \forall \mathcal{T}_n \exists b_n \in A_n(\mathcal{T}_n) \ R(\mathcal{T}_1, b_1, \ldots, \mathcal{T}_n, b_n, \mathcal{M}).$$

The lemma now follows from the Spector–Gandy theorem. □

In order to define $\Pi_n$-iterability for $n$ odd, we introduce the following notion. Let $\mathcal{T}$ be an iteration tree on $\mathcal{M}$, $b$ a branch of $\mathcal{T}$, and $a \in OR$. We say $b$ is $a$-good just in case whenever $\mathcal{N} = \mathcal{M}_{\mathcal{T}}^a$ or $\mathcal{N}$ is the $\alpha$th iterate of some initial segment $\mathcal{P}$ of $\mathcal{M}_{\mathcal{T}}^a$ via a single extender $E$ on the $\mathcal{P}$-sequence (and the images of $E$), then either $\mathcal{N}$ is well-founded or $a \in wfp(\mathcal{N})$. Clearly, if $b$ is realizable then it is $a$-good for all $a$; in fact, it is enough that $\mathcal{M}_{\mathcal{T}}^a$ be well-founded and iterable with respect to linear trees. On the other hand, $a$-goodness is simply definable: there is a $\Sigma_1^1$ relation $S(x, y)$ such that if $x$ is a real coding a triple $(\mathcal{A}, \mathcal{S}, a) \in HC$ such that $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ and $a \in OR$, then $S(x, y)$ iff $y$ codes an $a$-good branch $b$ of $\mathcal{T}$.

Now let $n \geq 1$ be odd, and $\mathcal{M}$ a countable $\delta$-mouse. Again, $\mathcal{I}(\mathcal{M}, \delta, n)$ is played in $n$ rounds. Before beginning round $k$ we have $(\mathcal{M}_k, \delta_k)$, where $\mathcal{M}_k$ is a $\delta_k$-mouse. For $k = 1$, we set $(\mathcal{M}_1, \delta_1) = (\mathcal{M}, \delta)$. If $k$ is odd and $1 \leq k \leq n$, then round $k$ is played as follows: I must play a countable, $\omega$-maximal, putative iteration tree $\mathcal{T}$ on $\mathcal{M}_k$ such that $\mathcal{T}$ is above $\delta_k$. In addition, I plays $x_k \in HC$. II can now either accept $\mathcal{T}$, provided it has a last, well-founded model, or play a maximal branch $b$ of $\mathcal{T}$. If $k < n$, then we demand that $b$ be well-founded. If $k = n$, then we only demand that $x_k \in OR \Rightarrow b$ is $x_k$-good. If II accepts $\mathcal{T}$, we set $\mathcal{M}_{k+1} = \text{last model of } \mathcal{T}$, and $\delta_{k+1} = \sup\{v(E^x_\mathcal{T}) | x + 1 < lh \mathcal{T}\}$. If II plays the branch $b$, then we set $\mathcal{M}_{k+1} = \mathcal{M}_{\mathcal{T}}^b$ and $\delta_{k+1} = \sup\{v(E^x_\mathcal{T}) | x \in b\}$.

If $k < n$ is even, then the rules for round $k$ are just as in the case $k$ is odd, except that I must play a tree $\mathcal{T}$ such that $\mathcal{T}$ is $A_{(n+1)-k}(x_{k-1}, \mathcal{T}_{k-1}, b_{k-1})$.

The first player to violate a rule of $\mathcal{I}(\mathcal{M}, \delta, n)$ loses the game, and if no one violates any rules, then II wins.

**Definition 1.6.** Let $n \geq 1$ be odd. Then $\mathcal{M}$ is $\Pi_n$-iterable above $\delta$ iff II has a winning strategy in $\mathcal{I}(\mathcal{M}, \delta, n)$. $\mathcal{M}$ is $\Pi_n$-iterable iff $\mathcal{M}$ is $\Pi_n$-iterable above $0$.

**Lemma 1.7.** Let $n \geq 1$ be odd. Then $(\mathcal{M}, \delta) \in HC \mid \mathcal{M}$ is $\Pi_n$-iterable above $\delta$} is $\Pi_n^{HC}$.

**Proof (sketch).** By induction on $n$. For $n = 1$, $\Pi_n$-iterability is just a slight strengthening of the $\Pi_1^1$ mouse condition of [6, Section 6], adapted to fine structural mice and trees. One easily checks that it remains "$\Pi_1^1$ in the codes", hence $\Pi_1^{HC}$. For $n > 1$, $\Pi_n$-iterability is in the form

$$\mathcal{M} \text{ is } \Pi_n\text{-iterable above } \delta \iff \forall \mathcal{T} \forall x \exists b \forall U \in A_{n-1}(\langle x, \mathcal{T}, b \rangle) \exists c(R(\langle \mathcal{T}, x, b, U, c \rangle, \mathcal{M}) \lor f((x, b, U, c)) \text{ is } \Pi_{n-2}\text{-iterable above } g(U, c)).$$
where $R, f, \text{ and } g$ are $\Delta_1^{HC}$. By the generalized Spector–Gandy theorem, and our induction hypothesis, $\Pi_n$-iterability is $\Pi_n^{HC}$.

We remark in passing that although $\Pi_1$-iterability is stronger than the $\Pi_2$-mouse condition of [6, Section 6], for 1-small mice the two conditions are equivalent. In what follows, the extra clause in $\Pi_1$-iterability (concerning iterates of $\mathcal{M}_n^T$) plays no role, except in Lemma 3.1.

If $n$ is even, then $\Pi_n$-iterability suffices for comparison (see [8]), but it is not clear that the countable initial segments of $M_n$ are $\Pi_n$-iterable. If $n$ is odd, we have the dual problem: the countable initial segments of $M_n$ are $\Pi_n$-iterable, but it is not clear that this suffices for comparison. Lemma 2.2 will solve both problems.

**Definition 1.8**. Let $\mathcal{M}$ and $\mathcal{N}$ be premice; then $\mathcal{M} \preceq \mathcal{N}$ iff $\exists \alpha (\mathcal{M} = \mathcal{F}_\infty^\alpha)$.

That is, $\mathcal{M} \preceq \mathcal{N}$ iff $\mathcal{M}$ is a (perhaps improper) initial segment of $\mathcal{N}$.

**Definition 1.9**. Let $\mathcal{M}$ and $\mathcal{N}$ be premice. A coiteration of $\mathcal{M}$ and $\mathcal{N}$ is a sequence $\langle (\mathcal{F}_\alpha, \mathcal{U}_\alpha) \mid \alpha < \theta \rangle$ such that

1. $\mathcal{F}_\alpha$ and $\mathcal{U}_\alpha$ are $\omega$-maximal iteration trees on $\mathcal{M}$ and $\mathcal{N}$, respectively,
2. $\alpha < \beta \Rightarrow \mathcal{F}_\beta$ extends $\mathcal{F}_\alpha$ and $\mathcal{U}_\beta$ extends $\mathcal{U}_\alpha$,
3. $\alpha$ limit $\Rightarrow (\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$ and $\mathcal{U}_\alpha = \bigcup_{\beta < \alpha} \mathcal{U}_\beta$),
4. $lh \mathcal{F}_\alpha$ a limit $\Rightarrow lh \mathcal{F}_{\alpha+1} = lh \mathcal{F}_\alpha + 1$, and $lh \mathcal{U}_\alpha$ a limit $\Rightarrow lh \mathcal{U}_{\alpha+1} = lh \mathcal{U}_\alpha + 1$,
5. $(lh \mathcal{F}_\alpha$ a limit $\land lh \mathcal{U}_\alpha$ a successor) $\Rightarrow lh \mathcal{U}_{\alpha+1} = lh \mathcal{U}_\alpha$, and $(lh \mathcal{U}_\alpha$ a limit $\land lh \mathcal{F}_\alpha$ a successor) $\Rightarrow lh \mathcal{F}_{\alpha+1} = lh \mathcal{F}_\alpha$,
6. if $lh \mathcal{F}_\alpha$ and $lh \mathcal{U}_\alpha$ are both successor ordinals, and $\alpha + 1 < \theta$, then $\mathcal{F}_{\alpha+1}$ and $\mathcal{U}_{\alpha+1}$ are determined by “iterating the least disagreement” between the last models in $\mathcal{F}_\alpha$ and $\mathcal{U}_\alpha$, and by the rules for $\omega$-maximal iteration trees (Section 7 of [7]).

A coiteration of $\mathcal{M}$ and $\mathcal{N}$ is determined by the cofinal well-founded branches of $\mathcal{F}_\alpha$ or $\mathcal{U}_\alpha$ used to produce $\mathcal{F}_{\alpha+1}$ or $\mathcal{U}_{\alpha+1}$ in the case $lh \mathcal{F}_\alpha$ or $lh \mathcal{U}_\alpha$ is a limit. This means that $\bigcup_{\alpha < \theta} \mathcal{F}_\alpha$ and $\bigcup_{\alpha < \theta} \mathcal{U}_\alpha$ determine $\langle (\mathcal{F}_\alpha, \mathcal{U}_\alpha) \mid \alpha < \theta \rangle$, so that we can identify the two, and speak of an appropriate pair $(\mathcal{F}, \mathcal{U})$ of iteration trees on $\mathcal{M}$ and $\mathcal{N}$, respectively as a coiteration.

**Definition 1.10**. Let $(\mathcal{F}, \mathcal{U})$ be a coiteration of $\mathcal{M}$ and $\mathcal{N}$. We say $(\mathcal{F}, \mathcal{U})$ is terminal iff either

1. $lh \mathcal{F}$ is a limit and $\mathcal{F}$ has no cofinal well-founded branch, or $lh \mathcal{U}$ is a limit and $\mathcal{U}$ has no cofinal well-founded branch, or
2. $\mathcal{F}$ and $\mathcal{U}$ have last models $\mathcal{P}$ and $\mathcal{Q}$, $\mathcal{P} \not\subseteq \mathcal{Q}$ and $\mathcal{Q} \not\subseteq \mathcal{P}$, and one of the ultrapowers determined by iterating the least disagreement between $\mathcal{P}$ and $\mathcal{Q}$, according to the rules for $\omega$-maximal trees, is ill-founded, or
3. $\mathcal{F}$ and $\mathcal{U}$ have last models $\mathcal{P}$ and $\mathcal{Q}$, and $\mathcal{P} \not\subseteq \mathcal{Q}$ or $\mathcal{Q} \not\subseteq \mathcal{P}$.
Clearly, a coiteration is terminal just in case it has no proper extension to a longer coiteration. Definition 1.10 simply enumerates the ways this can happen. We shall call a terminal coiteration successful just in case (3) of Definition 1.10 holds.

We conclude this section with three lemmas which are implicit in [7, 8].

Lemma 1.11. Let $\mathcal{M}$ and $\mathcal{N}$ be $\delta$-mice, where $\delta$ is a cutpoint of $\mathcal{M}$ and $\mathcal{N}$. Suppose $\mathcal{J}_\delta^\mathcal{M} = \mathcal{J}_\delta^\mathcal{N}$. Let $(\mathcal{F}, \mathcal{U})$ be a coiteration of $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{P}$ the last model of $\mathcal{F}$, and $2$ the last model of $\mathcal{U}$, and suppose $\mathcal{P} \models 2$. Then $\mathcal{M} \models \mathcal{N}$.

Proof. Let $\mathcal{P} = \mathcal{M}_\delta^\mathcal{F}$ and $2 = \mathcal{N}_\delta^\mathcal{P}$. Let $k = k(\mathcal{M}, \delta)$. Since $\delta$ is a cutpoint and $\mathcal{J}_\delta^\mathcal{M} = \mathcal{J}_\delta^\mathcal{N}$, $\mathcal{F}$ and $\mathcal{U}$ are above $\delta$.

Claim 1. There is no dropping on $[0, \alpha]_\mathcal{F}$; that is, $D^\mathcal{F} \cap [0, \alpha]_\mathcal{F} = \emptyset$ and $\deg^\mathcal{F}(\alpha) = k$.

Proof. If not, then $\mathcal{P}$ is not $\omega$-sound, so $\mathcal{P}$ is not a proper initial segment of $2$, so $\mathcal{P} = 2$. Let $\gamma + 1$ be the site of the last drop in $[0, \alpha]_\mathcal{F}$, so that for $n = \deg^\mathcal{F}(\gamma + 1)$ we have that $i_{\gamma+1, \alpha} \circ i_{\gamma+1}^\mathcal{F}$ is an $n$-embedding from $\mathcal{M}_{\gamma+1}^\mathcal{F}$ into $\mathcal{P}$, and that $\mathcal{P}$ is $n$-sound but not $n + 1$-sound. Let $\kappa = \text{crit}(i_{\gamma+1, \alpha} \circ i_{\gamma+1}^\mathcal{F})$, and notice $\rho_{n+1}(\mathcal{P}) \leq \kappa$ and $\kappa$ is least not in $H_{n+1}^\mathcal{F}(\rho_{n+1}(\mathcal{P})) \cup \{\rho_{n+1}(\mathcal{P})\}$. (This latter hull is just the image of $\mathcal{M}_{\gamma+1}^\mathcal{F}$ under $i_{\gamma+1, \alpha} \circ i_{\gamma+1}^\mathcal{F}$.) Since $\kappa > \delta$ and $\mathcal{P} = 2$, there must be a drop of some kind in $[0, \beta]_\mathcal{F}$. Let $\eta + 1$ be the site of the last drop in $[0, \beta]_\mathcal{U}$. We then have $n = \deg^\mathcal{U}(\eta + 1)$, and $\mathcal{N}_{\eta+1} = \mathcal{C}_{\eta+1}(2) = \mathcal{C}_{\eta+1}(\mathcal{P}) = \mathcal{M}_{\eta+1}^\mathcal{F}$. Also, $i_{\eta+1, \beta} \circ i_{\eta+1}^\mathcal{U} = i_{\eta+1, \beta} \circ i_{\eta+1}^\mathcal{P}$, since each is just the natural embedding from $\mathcal{C}_{\eta+1}(\mathcal{P})$ into $\mathcal{P}$. It follows that $E^\mathcal{F}_{\eta+1}$ is compatible with $E^\mathcal{U}_{\eta+1}$, which cannot happen in a coiteration. □

Claim 2. $\mathcal{P} = \mathcal{M}$.

Proof. Otherwise, $\alpha > 0$. By Claim 1, $i_{0\alpha}^\mathcal{F}$ is defined and is a $k$-embedding. Also, $\mathcal{P}$ is not $k + 1$-sound, so $\mathcal{P} = 2$. Let $\gamma = \text{crit}(i_{0\alpha}^\mathcal{F})$, and note $\gamma \geq \delta$. Then $\rho_{k+1}(\mathcal{P}) \leq \gamma \leq \kappa$, and $\kappa$ is least not in $H_{k+1}^\mathcal{F}(\rho_{k+1}(\mathcal{P}))$. If there is a drop on $[0, \beta]_\mathcal{U}$, then we can argue to a contradiction as in the previous claim. Otherwise, $\mathcal{N}$ is defined. Since $\mathcal{N}$ is a $\delta$-mouse, $H_{j+1}(\delta \cup \rho_{j+1}(\mathcal{N})) = \mathcal{N}$ for some $j$ such that $\rho_{j+1}(\mathcal{N}) \leq \delta < \rho_j(\mathcal{N})$. Since $\mathcal{P} = 2$, $j = k$ and $i_{0\beta}^\mathcal{F} = i_{0\beta}^\mathcal{F}$. This means the first extenders used on $[0, \alpha]_\mathcal{F}$ and $[0, \beta]_\mathcal{U}$ are compatible, a contradiction. □

Claim 3. $\mathcal{N} = 2$.

Proof. Otherwise $\beta > 0$. If $D^\mathcal{U} \cap [0, \beta]_\mathcal{U} \neq \emptyset$, then $\mathcal{P}$ is a proper initial segment of $2$. (See Section 7 of [7], or the proof of Claim 1.) Since $\mathcal{P} = \mathcal{M}$ is a $\delta$-mouse, $|\mathcal{M}| = \delta$ in $2$. But $lh E^\mathcal{U}_\delta$ is a cardinal of $2$ above $\delta$, so $OR^\mathcal{U} < lh E^\mathcal{U}_\delta$. This means that $E^\mathcal{U}_\delta$ was not part of a disagreement, a contradiction. So we may assume $D^\mathcal{U} \cap [0, \beta]_\mathcal{U} = \emptyset$. The argument also shows $\mathcal{P} = 2$. But then $crit i_{0\beta}^\mathcal{F} \in H_{k+1}(\delta \cup \rho_{k+1}(2))$, and $i_{0\beta}^\mathcal{F}$ is a $k$-embedding with critical point above $\delta$, a contradiction. □
Lemma 1.12. Let $\mathcal{M}$ be a $\delta$-mouse, and $\mathcal{T}$ an $\omega$-maximal iteration tree on $\mathcal{M}$ above $\delta$, with $\mathcal{T}$ of limit length. Let $b$ and $c$ be distinct cofinal well-founded branches of $\mathcal{T}$. Then $\mathcal{M}_b \equiv \mathcal{M}_c$.

Proof (sketch; see Claim 4 in 6.2 of [7] for more detail). Suppose $\mathcal{M}_b \preceq \mathcal{M}_c$. Since $\mathcal{M}_b$ is not $\omega$-sound, $\mathcal{M}_b = \mathcal{M}_c$. If there is any dropping in $(b - c) \cup (c - b)$, either in height of model or in degree, then as in the proof of Claim 1 of Lemma 1.11, we get $\alpha \neq \beta$ such that $E_{\alpha}^\mathcal{T}$ is compatible with $E_{\beta}^\mathcal{T}$. This cannot happen in an iteration tree. (See the remark following 5.1 of [7].) Now let $\eta$ be largest in $b \cap c$. We have just shown that $i_{\eta b}$ and $i_{\eta c}$ exist and are $\deg(\eta)$ embeddings. Let $n = \deg(\eta)$, and

$$\rho = \sup \{lh E_{\xi} | \xi + 1 \in b \cap c\}.$$ 

Since $\mathcal{M}$ is a $\delta$-mouse, we have $\mathcal{M}_n = H_{\rho + 1}(\rho \cup P_{\rho + 1}(\mathcal{M}_n))$. Since $\mathcal{M}_b = \mathcal{M}_c$, this implies $i_{\eta b} = i_{\eta c}$. But then, letting $\gamma$ and $\xi$ be the next elements in $b$ and $c$ after $\eta$, we have $E_{\gamma - 1}^\mathcal{T}$ compatible with $E_{\xi - 1}^\mathcal{T}$, a contradiction. $\square$

Our final preliminary lemma states that definable coiterations of countable premice cannot last $\omega_1$ steps. It is a straightforward generalization of an argument in [6, Section 6].

Lemma 1.13. Let $0 \leq n < \omega$, and suppose there are $n$ Woodin cardinals with a measurable above them all. Let $\mathcal{M}$ and $\mathcal{N}$ be countable premice, and $\langle (\mathcal{T}_\alpha, \mathcal{M}_\alpha) | \alpha < \theta \rangle$ a coiteration of $\mathcal{M}$ and $\mathcal{N}$ which is $\Sigma^1_{n+1}$. Then $\theta < \omega_1$.

Proof. Fix a $\Sigma^1_{n+1}$ formula $\sigma$ and parameter $t \in HC$ such that $\forall x, \sigma$

$$x = (\mathcal{T}_\alpha, \mathcal{M}_\alpha) \text{ iff } HC \models \sigma[\alpha, x, t].$$

Let $\varphi(v_0, v_1, v_2)$ be the following $\Pi^1_{n+2}$ formula in the language of set theory:

$$\forall x \in OR[\exists x \sigma(\alpha, x, v_0) \land \forall y \sigma(\alpha, y, v_0)] \land \forall x, \beta \in OR \land \forall y \sigma(\alpha, x, v_0) \land \sigma(\alpha, y, v_0) \land \alpha < \beta \Rightarrow ((x)_0 \text{ and } (y)_0 \text{ are iteration trees on } v_1 \text{ such that } (y)_0 \text{ extends } (x)_0) \land \forall x \in OR(\alpha(x + 1, x, v_0) \Rightarrow (x)_0 \text{ and } (x)_1 \text{ have successor length}).$$

It is clear that $HC \models \varphi[t, \mathcal{M}, \mathcal{N}]$.

Now let $G$ be $V$-generic for $Col(\omega, \omega_1)$. We claim that

$$V[G] \models (HC \models \varphi[t, \mathcal{M}, \mathcal{N}]).$$

This follows from our large cardinal hypothesis. Since $\varphi$ is $\Pi^1_{n+2}$, it can be translated into a $\Pi^1_{n+3}$ formula. Thus it is enough to see that any $\Pi^1_{n+3}$ formula true in $V$ of a real in $V$ is also true in $V[G]$ of that real. For $n = 0$, this is just the Martin–Solovay absoluteness theorem (see [4]). For $n > 0$, we can use [5] to generalize the Martin–Solovay proof. For by [5], all $\Pi^1_{n+1}$ sets of reals are homogeneously Suslin, and hence all $\Sigma^1_{n+2}$ sets are weakly homogeneously Suslin. But then the construction of [4]
produces a tree $T$ in $V$ such that $p[T]$ is the universal $\Sigma^4_{1+3}$ set in every generic extension of $V$ by a poset of size less than the least measurable cardinal. Since $Col(\omega, \omega_1)$ is such a poset, we have proved our claim.

Inspecting $\phi$, we see that in $V[G]$ there are unique iteration trees $T_*$ and $U_*$ of length at least $\omega_1^V + 1$ extending $T$ and $U$ and such that $HC^{V[G]} = \sigma[\omega_1^V + 1, (T_*, U_*)]$. By the homogeneity of $Col(\omega, \omega_1)$, $T_*$ and $U_*$ are in $V$. This means that $T$ and $U$ have cofinal branches of order type $\omega_1$ in $V$. The standard $\diamondsuit$-like proof that the comparison process terminates now yields a contradiction. $\Box$

2. A complexity bound on the reals of $M_n$

In this section we use $\Pi_n$-iterability to provide definitions, from countable ordinal parameters, of the reals in $M_n$. The main result is 2.2, which solves the problems involving $\Pi_n$-iterability which we described in Section 1.

Recall that, for any iteration tree $T$ of limit length, $\delta(T)$ is the supremum of the lengths of the extenders used in $T$. If $b$ is any cofinal branch of $T$, then $M_b^T \models \delta(T)$ is a limit cardinal.

**Definition 2.1.** Let $T$ be an iteration tree on $M$ of limit length, and $b$ a cofinal well-founded branch of $T$. Then

$$Q(b, T) = J_x^\#T,$$

where $x$ is the largest $\beta$ such that $\beta = \delta(T)$ or

$$\delta(T) < \beta \leq OR^\#T$$

and $J_x^\#T \models \delta(T)$ is Woodin.

It is easy to check that if $M$ is a $\delta$-mouse and $T$ is an $\omega$-maximal tree on $M$ above $\delta$, and $\delta(T) < OR^{Q(b, T)}$, then $Q(b, T)$ is a $\delta(T)$ mouse. If $Q(b, T) = J_\beta^\#T$ where $\beta < OR^\#T$, then $(J_{\beta + 1}^\#T - J_\beta^\#T) \cap P(\delta(T)) \neq \emptyset$ because $\delta(T)$ is not Woodin in $J_{\beta + 1}^\#T$. Since $J_\beta^\#T$ is $\omega$-sound, $Q(b, T)$ is a $\delta(T)$-mouse. If $Q(b, T) = M_b^T$, and there is no dropping in model or degree along $b$, then $Q(b, T)$ is a $\delta(T)$ mouse with $k(Q(b, T), \delta(T)) = k(M_b^T, \delta)$. Otherwise, letting $n$ be the degree at the last drop along $b$, $Q(b, T)$ is a $\delta(T)$-mouse with $k(Q(b, T), \delta(T)) = n$, and $\rho_{n+1}(Q(b, T))$ less than or equal to the critical point at that stage.

Notice that if $M$ is $n + 1$ small above $\delta$, then $\delta(T)$ is a cutpoint of $Q(b, T)$, and $Q(b, T)$ is $n$-small above $\delta(T)$. Nevertheless, $Q(b, T)$ is large enough that its iterability would characterize $b$ as the “good” branch of $T$ (see (1) of Lemma 2.2 for a precise statement).

We now show that $\Pi_n$-iterability behaves properly.

**Lemma 2.2.** Let $1 \leq n < \omega$, and suppose there are $n - 1$ Woodin cardinals with a measurable above them all.

1. Let $M$ be a countable $\delta$-mouse which is $n$ small above $\delta$, where $\delta$ is a cutpoint of $M$. Suppose $M$ is $k(M, \delta)$ realizable via $\pi$. Let $T$ be an $\omega$ maximal iteration tree on $M$ of
countable limit length, such that $\mathcal{T}$ is above $\delta$, and suppose $\mathcal{M}_x^\mathcal{T}$ is $(\pi, \mathcal{T})$ realizable whenever $x < \text{lh} \mathcal{T}$. Let $b$ be the unique cofinal $(\pi, \mathcal{T})$ realizable branch of $\mathcal{T}$. Then $b$ is the unique cofinal branch $c$ of $\mathcal{T}$ such that $\mathcal{M}_x^\mathcal{T}$ is well-founded and, if $n \geq 2$, $Q(c, \mathcal{T})$ is $\Pi_{n-1}$-iterateable above $\delta(\mathcal{T})$.

(2) Let $\mathcal{M}$ be a countable $\delta$-mouse which is $n$ small above $\delta$, where $\delta$ is a cutpoint of $\mathcal{M}$. Suppose $\mathcal{M}$ is $k(\mathcal{M}, \delta)$ realizable. Then $\mathcal{M}$ is $\Pi_n$-iterateable above $\delta$.

(3) Suppose $\mathcal{M}$ and $\mathcal{N}$ are countable $\delta$-mice which are $n$-small above $\delta$, $\delta$ is a cutpoint of $\mathcal{M}$ and $\mathcal{N}$, and $\mathcal{I}_S^\mathcal{M} = \mathcal{I}_S^\mathcal{N}$. Suppose $\mathcal{M}$ is $k(\mathcal{M}, \delta)$ realizable and $\mathcal{N}$ is $\Pi_n$-iterateable above $\delta$. Then $\mathcal{M} \preceq \mathcal{N}$ or $\mathcal{N} \preceq \mathcal{M}$.

Proof. By induction on $n$. We begin with (1) for $n$.

Let $\mathcal{M}$, $\delta$, $\pi$, and $\mathcal{T}$ be as in (1). Let $b$ be a cofinal $(\pi, \mathcal{T})$ realizable branch of $\mathcal{T}$; it is shown in [8] that there is a unique such branch. So $\mathcal{M}_x^\mathcal{T}$ is well-founded. Also, if $n \geq 2$ then $Q(b, \mathcal{T})$ is $\Pi_{n-1}$-iterateable above $\delta(\mathcal{T})$ by (2) for $n - 1$. Suppose toward a contradiction that there is a second cofinal branch $c$ of $\mathcal{T}$ with these properties of $b$.

Since $\delta(\mathcal{T})$ is a limit cardinal in $\mathcal{M}_x^\mathcal{T}$ and $\mathcal{M}_c^\mathcal{T}$, it does not index an extender on either sequence. Thus $\mathcal{I}_{Q(b, \mathcal{T})}^\mathcal{T} = \mathcal{I}_{Q(c, \mathcal{T})}^\mathcal{T}$.

We claim $Q(b, \mathcal{T}) \subseteq Q(c, \mathcal{T})$ or $Q(c, \mathcal{T}) \subseteq Q(b, \mathcal{T})$. For this, suppose first $n = 1$. Say $\text{OR} \cap Q(b, \mathcal{T}) \subseteq Q(c, \mathcal{T})$; the other case is similar. If $Q(b, \mathcal{T}) \notin Q(c, \mathcal{T})$ in this case, there must be an extender $E$ on one of the sequences with $\delta(\mathcal{T}) < \text{lh} E \leq \text{OR} \cap Q(b, \mathcal{T})$. Since $\delta(\mathcal{T})$ is a cutpoint, $\text{crit} E > \delta(\mathcal{T})$. But then one of $Q(b, \mathcal{T})$ and $Q(c, \mathcal{T})$ is not $1$-small, which contradicts the $1$-smallness of $\mathcal{M}$. Next, suppose $n \geq 2$. Then if neither of $Q(b, \mathcal{T})$ and $Q(c, \mathcal{T})$ is an initial segment of the other, then $\delta(\mathcal{T}) \in \text{OR} \cap Q(b, \mathcal{T}) \cap Q(c, \mathcal{T})$. This implies that $Q(b, \mathcal{T})$ and $Q(c, \mathcal{T})$ are $\delta(\mathcal{T})$ mice which are $n - 1$ small above $\delta(\mathcal{T})$. $Q(b, \mathcal{T})$ is appropriately realizable since $\mathcal{M}_x^\mathcal{T}$ is, and $Q(c, \mathcal{T})$ is $\Pi_{n-1}$-iterateable above $\delta(\mathcal{T})$. The desired conclusion follows from (3) for $n - 1$.

Suppose $Q(b, \mathcal{T})$ is a proper initial segment of $Q(c, \mathcal{T})$. If $Q(b, \mathcal{T}) = \mathcal{M}_x^\mathcal{T}$, this implies $\mathcal{M}_x^\mathcal{T} \preceq \mathcal{M}_c^\mathcal{T}$, contrary to Lemma 1.12. If $Q(b, \mathcal{T})$ is a proper initial segment of $\mathcal{M}_x^\mathcal{T}$, then definable over $Q(b, \mathcal{T})$ is a function $f : \delta(\mathcal{T}) \to \delta(\mathcal{T})$ witnessing $\delta(\mathcal{T})$ is not Woodin via the extenders in $\mathcal{I}_{Q(b, \mathcal{T})}^\mathcal{T}$. But then $f \in Q(c, \mathcal{T})$, which contradicts the fact that $\delta(\mathcal{T})$ is Woodin in $Q(c, \mathcal{T})$. Similarly, $Q(c, \mathcal{T})$ cannot be a proper initial segment of $Q(b, \mathcal{T})$, so $Q(b, \mathcal{T}) = Q(c, \mathcal{T})$.

If $Q(b, \mathcal{T}) = \mathcal{M}_x^\mathcal{T}$ or $Q(c, \mathcal{T}) = \mathcal{M}_c^\mathcal{T}$ this contradicts Lemma 1.12. The alternative is that $Q(b, \mathcal{T}) \in \mathcal{M}_x^\mathcal{T} \cap \mathcal{M}_c^\mathcal{T}$, but by [7, Section 6], the extenders of $\mathcal{I}_{Q(b, \mathcal{T})}^\mathcal{T}$ witness Woodinness for $\delta(\mathcal{T})$ with respect to functions $f$ in $\mathcal{M}_x^\mathcal{T} \cap \mathcal{M}_c^\mathcal{T}$. This contradiction completes the proof of (1) for $n$.

Next we prove (2) for $n$. Let $\mathcal{M}$ and $\delta$ be as in (2). We claim that the following is a winning strategy for II in $\mathcal{I}(\mathcal{M}, \delta, n)$: play a (in fact, the unique) maximal realizable branch of the tree just played by I, unless there is no such branch, in which case accept I's tree. Theorem 1.7 and Corollary 1.9 of [8] imply that this strategy wins for II, provided the definability restrictions on the branches played by II in $\mathcal{I}(\mathcal{M}, \delta, n)$ are met. If $n$ is odd, there are no such restrictions. If $n$ is even, and II plays $b$ at round $k$,
then we must have $b \in A_n(\langle \mathcal{M}_k, \mathcal{T}_k \rangle)$, where $\mathcal{T}_k$ is the tree on $\mathcal{M}_k$ above $\delta_k$ just played by $I$. But $\mathcal{M}_k$ is $n$-small above $\delta$, so by (1) for $n$, $\{b\}$ is $\Pi_{n-1}(\langle \mathcal{M}_k, \mathcal{T}_k \rangle)$, and thus $b \in A_n(\langle \mathcal{M}_k, \mathcal{T}_k \rangle)$.

Finally, we prove (3) for $n$. Let $\mathcal{M}$ and $\mathcal{N}$ be as in the hypotheses.

Suppose first $n = 1$. In this case, the argument is essentially given in [6, Section 6]. Notice first that if $\mathcal{T}$ is any $\omega$-maximal iteration tree on $\mathcal{M}$ or $\mathcal{N}$ which is above $\delta$, then $\mathcal{T}$ has at most one cofinal well-founded branch. For suppose $b$ and $c$ were distinct such branches, and suppose, without loss of generality, $\mathcal{O}^\mathcal{T}_b \leq \mathcal{O}^\mathcal{T}_c$. By Lemma 1.12, $\mathcal{M}_b^\mathcal{T} \not\equiv \mathcal{M}_c^\mathcal{T}$, so there is an extender $E$ on one of the $\mathcal{M}_b^\mathcal{T}$ and $\mathcal{M}_c^\mathcal{T}$ sequences such that $\delta(\mathcal{T}) < \lambda \mathcal{E} \leq \mathcal{O}^\mathcal{T}_b^\mathcal{E}$. Fix such an $E$ with $\lambda \mathcal{E}$ minimal, and let $\kappa = \operatorname{crit} E$. Since $\delta(\mathcal{T})$ is Woodin in $\mathcal{M}_b^\mathcal{T}$ with respect to functions in $\mathcal{M}_b^\mathcal{T}$, $\kappa > \delta(\mathcal{T})$ and $\mathcal{J}_{\mathcal{T}}(\mathcal{J}_{\mathcal{T}}(\mathcal{J}_{\mathcal{T}}) = \mathcal{J}_{\mathcal{T}}^{\mathcal{T}})$ satisfies that $\delta$ is Woodin. This means that one of $\mathcal{M}_b^\mathcal{T}$ and $\mathcal{M}_c^\mathcal{T}$, the one whose sequence has $E$, is not 1-small above $\delta$. This contradicts our hypotheses on $\mathcal{M}$ and $\mathcal{N}$ and the assumption that $\mathcal{T}$ is above $\delta$.

It follows that for any $\theta < \omega$, there is at most one coiteration of $\mathcal{M}$ and $\mathcal{N}$ of $\lambda$-length $\theta$, and if $\theta = \omega$, this coiteration is $\Sigma^\mathcal{H}_1(\langle \mathcal{M}, \mathcal{N} \rangle)$. By Lemma 1.13 there is a terminal coiteration $(\mathcal{T}, \mathcal{U})$ of $\mathcal{M}$ and $\mathcal{N}$ of length $\theta < \omega$. We are done if this coiteration is successful, since then Lemma 1.11 implies $\mathcal{M} \equiv \mathcal{N}$ or $\mathcal{N} \equiv \mathcal{M}$. So assume that $(\mathcal{T}, \mathcal{U})$ is not successful.

If $\lambda$ is a limit $< \lambda \mathcal{T}$, then $[0, \lambda]_\mathcal{T}$ is the unique cofinal well-founded branch of $\mathcal{T}$, and similarly for $\mathcal{U}$. Since $\mathcal{M}$ is realizable, the failure of iterability represented by (1) or (2) of Definition 1.10 cannot happen in $\mathcal{T}$, so it must happen in $\mathcal{U}$. Let $\mathcal{U}^+ = \mathcal{U}$ if (1) of Definition 1.10 applies to $\mathcal{U}$, and $\mathcal{U}^+$ be the putative tree extending $\mathcal{U}$ given by (2), with its last model ill-founded, if (2) of Definition 1.10 applies to $\mathcal{U}$. Since $\mathcal{N}$ is $\Pi_1$-iterable above $\delta$, we have a winning strategy $\mathcal{S}$ for $\Pi$ in $\mathcal{S}(\mathcal{N}, \delta, 1)$. If $I$ plays $(\mathcal{U}^+, \alpha)$, where $\alpha < \omega_1$, then $\mathcal{S}$ cannot accept, so $\mathcal{S}$ must respond with a maximal branch $b_\lambda$ of $\mathcal{U}^+$ such that $\mathcal{N}_{\mathcal{U}^+}^{\mathcal{U}^+}$ is $\alpha$-well-founded. Now $N_{\mathcal{U}^+}$ cannot be fully well-founded. [If $\sup b_\lambda < \lambda \mathcal{U}^+$ this contradicts the uniqueness of $[0, \sup b_\lambda]$; if $\sup b_\lambda = \lambda \mathcal{U}^+$ this contradicts the fact that $(\mathcal{T}, \mathcal{U})$ is terminal.] Thus we can find $\lambda < \lambda \mathcal{U}^+$ such that for $\omega_1$ many distinct $b_\lambda$, $\sup b_\lambda = \lambda$.

Let $\langle (\mathcal{T}_\alpha, \mathcal{U}_\alpha) \mid \alpha < \theta \rangle$ be the coiteration identified with $(\mathcal{T}, \mathcal{U})$, so that $\mathcal{T} = \mathcal{T}_\alpha$ and $\mathcal{U} = \mathcal{U}_\alpha$. (The length of the coiteration is a successor since it is terminal.) Let $\alpha < \theta$ be least so that $\lambda \mathcal{U}_\alpha \geq \lambda$. Since $\alpha$ is a limit, $\alpha$ is a limit and $\lambda \mathcal{U}_\alpha = \lambda$. If $\mathcal{T}_\alpha$ has a last model, then call it $\mathcal{P}$; otherwise let $\mathcal{P} = \mathcal{M}_b^\mathcal{T}_\alpha$ for $b$ the unique cofinal well-founded branch of $\mathcal{T}_\alpha$. In either case, whenever $\sup b_\lambda = \lambda$, $\mathcal{P}$ agrees with $\mathcal{N}_{\mathcal{U}_\alpha}^{\mathcal{U}_\alpha}$ below $\delta(\mathcal{U} \alpha)$. We can find $\gamma \neq \eta$ such that the well-founded parts of $\mathcal{N}_{\mathcal{U}_\alpha}^{\mathcal{U}_\alpha}$ and $\mathcal{N}_{\mathcal{U}_\alpha}^{\mathcal{U}_\alpha}$ are longer than $\mathcal{O}^\mathcal{P}$, and $\lambda = \sup b_\lambda = \sup b_\eta$. This implies that $\mathcal{P} \not\equiv \delta(\mathcal{U} \alpha)$ is Woodin. Since $\mathcal{M}$ and $\mathcal{N}$ are 1-small above $\delta$, $\delta < \delta(\mathcal{U} \alpha)$, and $\mathcal{T}$ and $\mathcal{U}$ above $\delta$, we have that $\mathcal{P}$ is an initial segment of $\mathcal{N}_{\mathcal{U}_\alpha}^{\mathcal{U}_\alpha}$. As in the proof of Lemma 1.11, this implies $\mathcal{M} \equiv \mathcal{N}$ (The ill-foundedness of $\mathcal{N}_{\mathcal{U}_\alpha}^{\mathcal{U}_\alpha}$ above $\mathcal{O}^\mathcal{P}$ does not affect the proof of Lemma 1.11.)

We now prove (3) in the case $n$ is even. We define a coiteration $\langle (\mathcal{T}_\alpha, \mathcal{U}_\alpha) \mid \alpha < \theta \rangle$ of $\mathcal{M}$ and $\mathcal{N}$ by induction. For this, it suffices to define $\mathcal{T}_{\alpha+1}$ (resp., $\mathcal{U}_{\alpha+1}$) in the case $\lambda \mathcal{T}_\alpha$ (resp., $\lambda \mathcal{U}_\alpha$) is a limit.
Fix a \( k(M, \delta) \) realization \( \pi \) of \( M \). If \( lh \mathcal{F}_z \) is a limit, then let \( b \) be a (the unique) cofinal \((\pi, \mathcal{F})\) realizable branch of \( \mathcal{F}_z \) if there is one. Let \( \mathcal{F}_{x+1} \) be the iteration tree extending \( \mathcal{F}_x \) with length \( lh \mathcal{F}_x + 1 \) and such that \( b = [0, lh \mathcal{F}_x]_{7_{x+1}} \). If there is no such \( b \), then stop the construction. If \( lh \mathcal{U}_x \) is a limit, then let \( c \) be the unique cofinal well-founded branch of \( \mathcal{U}_x \) such that \( \mathcal{N}_c^{\mathcal{U}_x} \) is \( \Pi_{n-1} \)-iterable above \( \delta(\mathcal{U}_x) \). Let \( lh \mathcal{U}_{x+1} = lh \mathcal{U}_x + 1 \) and \( c = [0, lh \mathcal{U}_x]_{7_{x+1}} \). If it is not the case that there is a unique such \( c \), then stop the construction.

Suppose this construction produces a coiteration \( \langle (\mathcal{F}_z, \mathcal{U}_z) | z < \omega_1 \rangle \) of length \( \omega_1 \). By (1) for \( n \), the branch \( b \) of \( \mathcal{F}_z \) chosen when \( lh \mathcal{F}_z \) is a limit is \( \Delta_n(\langle \mathcal{F}, \mathcal{F}_z \rangle) \), uniformly in \( \mathcal{F}_z \). Clearly, the branch \( c \) of \( \mathcal{U}_z \) chosen when \( lh \mathcal{U}_z \) is a limit is uniformly \( \Delta_n(\langle N, \mathcal{U}_z \rangle) \). It follows that \( \langle (\mathcal{F}_z, \mathcal{U}_z) | z < \omega_1 \rangle \) is \( \Sigma_n(\langle \mathcal{F}, N \rangle) \), which contradicts Lemma 1.13. Therefore, the construction must either stop for one of the reasons described above, or produce a terminal coiteration.

Suppose the construction produces a terminal coiteration \( \langle (\mathcal{F}_z, \mathcal{U}_z) | z \leq \theta \rangle \). (Terminal coiterations have successor length.) If this coiteration is successful, then by Lemma 1.11 we are done, so assume otherwise. Now the failure-of-iterability clauses (1) and (2) of Definition 1.10 cannot apply to \( \mathcal{F}_0 \), by 1.7 of [8] and the fact (coming from (1) for \( n \)) that we always chose the unique realizable \( b \) in extending \( \mathcal{F}_z \) when \( lh \mathcal{F}_z \) was a limit. So one of (1) and (2) of Definition 1.10 applies to \( \mathcal{U}_0 \). If (2) applies, let \( \mathcal{U} \) be the putative tree extending \( \mathcal{U}_0 \) whose last model is ill-founded which is given by (2); if (1) applies, let \( \mathcal{U} = \mathcal{U}_0 \). Since \( N \) is \( \Pi_n \)-iterable, we have \( c \) such that \( (\mathcal{U}, c) \) is a winning position in \( \mathcal{I}(N, \delta, n) \) for \( II \). We cannot have \( c = \text{accept} \), since then \( \mathcal{U} \) has successor length. and since it comes from (2) of Definition 1.10 it is not acceptable. So \( c \) is a maximal branch of \( \mathcal{U} \). Let \( \delta_1 = \sup \{|v(E_u^u)| u \in c\} = \delta(\mathcal{U} | \sup c) \). Since \( II \) wins \( \mathcal{I}(N, \delta, n) \) from \( (\mathcal{U}, c) \), and since \( n \) is even, \( \mathcal{N}_c^{\mathcal{U}} \) is \( \Pi_{n-1} \)-iterable above \( \delta_1 \). (This is where we use \( n \) even.) If \( c \) is cofinal in \( \mathcal{U} \), this contradicts the fact that (1) or (2) of Definition 1.10 applied to \( \mathcal{U}_0 \). If \( c \) is not cofinal, it means we stopped the construction without extending \( \mathcal{U} / (\sup c) \). This contradiction implies our coiteration was indeed successful, provided it was terminal.

It remains to show that the construction does not stop for one of the reasons described. Now if \( lh \mathcal{F}_z \) is a limit, then \( \mathcal{F}_z \) has a maximal realizable branch \( b \) by [8, 1.7]. Since we did not stop the construction before \( z \), \( b \) is cofinal. So if the construction stops at \( z \), it is because of \( \mathcal{U}_z \). The argument of the preceding paragraph shows that \( \mathcal{U}_z \) has a cofinal branch \( c \) such that \( \mathcal{N}_c^{\mathcal{U}_z} \) is \( \Pi_{n-1} \)-iterable above \( \delta(\mathcal{U}_z) \). (Again, here we use that \( n \) is even.) So it must be that there is a second cofinal branch \( d \) of \( \mathcal{U}_z \) with \( \mathcal{N}_d^{\mathcal{U}_z} \) \( \Pi_{n-1} \)-iterable above \( \delta(\mathcal{U}_z) \).

Let \( \mathcal{P} = \mathcal{M}^{\mathcal{F}_z}_b \), where \( b \) is as above, if \( lh \mathcal{F}_z \) is a limit, and let \( \mathcal{P} \) be the last model of \( \mathcal{F}_z \) otherwise. Let \( \mathcal{P} = Q(c, \mathcal{U}_z) \) and \( \mathcal{R} = Q(d, \mathcal{U}_z) \). If \( \mathcal{P} \subseteq \mathcal{R} \), then \( \mathcal{P} \subseteq \mathcal{N}_c^{\mathcal{U}_z} \), so by Lemma 1.11 \( \mathcal{M} \subseteq \mathcal{N} \) and our coiteration succeeded at step 0, a contradiction. Thus \( \mathcal{P} \not\subseteq \mathcal{R} \), and similarly \( \mathcal{P} \not\subseteq \mathcal{R} \). It follows that \( \delta(\mathcal{U}_z) \in OR \cap \mathcal{P} \). Let \( \mathcal{I} = \mathcal{I}_b^\mathcal{P} \), where \( \beta \) is largest such that \( \beta = \delta(\mathcal{U}_z) \) or \( \mathcal{I}^\mathcal{P} \vdash \delta(\mathcal{U}_z) \) is Woodin.
$\mathcal{P}$ and $\mathcal{Q}$ are $n-1$ small $\delta(\mathcal{Q})$-mice agreeing through $\delta(\mathcal{Q})$, $\mathcal{P}$ is realizable, and $\mathcal{Q}$ is $\Pi_{n-1}$-iterable above $\delta(\mathcal{Q})$. By (3) for $n-1$, $\mathcal{P} \equiv \mathcal{Q}$ or $\mathcal{Q} \equiv \mathcal{P}$. Similarly, $\mathcal{P} \equiv \mathcal{R}$ or $\mathcal{R} \equiv \mathcal{P}$.

Suppose $\mathcal{P} \neq \mathcal{P}$. Then definable over $\mathcal{P}$ is a function witnessing $\delta(\mathcal{Q})$ is not Woodin in $\mathcal{P}$. This implies $\mathcal{P} = \mathcal{Q}$ and $\mathcal{P} = \mathcal{Q}$. If $\mathcal{Q} = \mathcal{N}^{\mathcal{Q}}_c$ or $\mathcal{R} = \mathcal{N}^{\mathcal{R}}_d$, then $\mathcal{N}^{\mathcal{R}}_c \mathcal{N}^{\mathcal{Q}}_d$ or $\mathcal{N}^{\mathcal{R}}_d \mathcal{N}^{\mathcal{Q}}_c$, which finishes the proof by Lemma 1.11. It follows that $\mathcal{Q} = \mathcal{R} \in \mathcal{N}^{\mathcal{R}}_d \cap \mathcal{N}^{\mathcal{Q}}_c$. But then there is a function definable over $\mathcal{Q}$ witnessing that $\delta(\mathcal{Q})$ is not Woodin in $\mathcal{N}^{\mathcal{Q}}_c$, while there can be no such function in $\mathcal{N}^{\mathcal{R}}_c \cap \mathcal{N}^{\mathcal{Q}}_d$. This contradiction implies $\mathcal{P} = \mathcal{P}$.

But if $\mathcal{P} = \mathcal{P}$ then $\mathcal{P} \neq \mathcal{Q}$, so $\mathcal{Q}$ is a proper initial segment of $\mathcal{P}$. This means $\delta(\mathcal{Q})$ is Woodin with respect to functions definable over $\mathcal{Q}$, so $\mathcal{Q} = \mathcal{N}^{\mathcal{Q}}_c$. But then $\mathcal{N}^{\mathcal{Q}}_c \mathcal{N}^{\mathcal{Q}}_d$, which by Lemma 1.11 finishes the proof.

Finally, we prove (3) in the case $n > 1$ and $n$ is odd. Once more we define a coiteration of $\mathcal{M}$ and $\mathcal{N}$. Fix a $k.(\mathcal{M}, \delta)$ realization $\pi$ of $\mathcal{M}$. If $lhKx$ is a limit, then we extend $Kx$ to $Kx_{+1}$ by choosing the unique cofinal $(\pi, Kx)$ realizable branch of $Kx$, just as in the $n$ even case. Using (1) for $n$, we see that there will always be a unique such branch. Now suppose $lhKx$ is a limit. If $lhKx$ is a limit, let $\mathcal{P} = \mathcal{M}_{b}^{Kx}$, where $b$ is the unique cofinal $(\pi, Kx)$ realizable branch of $Kx$. Let $\mathcal{P}$ be the last model of $Kx$. Suppose that there is a unique cofinal branch $c$ of $\mathcal{Q}$ such that $\langle(\mathcal{U}, \langle\mathcal{P}, Kx\rangle), c\rangle$ is a winning position for $I$ in $\mathcal{I}(\mathcal{N}, \delta, n)$. In this case we let $\mathcal{K}_{x+1}$ be the tree extending $\mathcal{K}$ such that $lh\mathcal{K}_{x+1} = lh\mathcal{K}_{x} + 1$ and $c = [0, lh\mathcal{K}_{x}].$ If it is not the case that there is a unique such $c$, then we stop the construction.

The construction above cannot yield a coiteration $\langle(\mathcal{K}, \mathcal{Q})| \alpha < \omega_1 \rangle$ of length $\omega_1$. For if so, then by (1) for $n$ we see that $\mathcal{K}_{x+1}$ is $A_n(\langle\mathcal{K}, \mathcal{Q}\rangle)$, uniformly, while $\mathcal{K}_{x+1}$ is uniformly $A_{n-1}(\langle\mathcal{K}_{x+1}, \mathcal{Q}\rangle)$ because the property of being a winning position of length $1$ in $\mathcal{I}(\mathcal{M}, \delta, n)$ is $\Sigma_{n-1}$ when $n$ is odd. (This uses the Spector–Gandy theorem.) It follows that $\langle(\mathcal{K}, \mathcal{Q})| \alpha < \omega_1 \rangle$ is $\Sigma_n(\langle\mathcal{M}, \mathcal{N}\rangle)$, which contradicts Lemma 1.13.

As in the case that $n$ is even, if the construction produces a terminal coiteration, then this coiteration must be successful, and by Lemma 1.11 we are done.

It remains to see that the construction cannot stop for the reason we gave while describing it. So suppose $lh\mathcal{K}_{x}$ is a limit and it is not the case that there is a unique cofinal branch $c$ of $\mathcal{Q}$ such that $\langle(\mathcal{Q}, \langle\mathcal{P}, Kx\rangle), c\rangle$ is winning for $I$ in $\mathcal{I}(\mathcal{N}, \delta, n)$, where $\mathcal{P}$ comes from $Kx$ as described earlier. Since $\mathcal{N}$ is $\Pi_{n-1}$-iterable above $\delta$, there is a maximal branch $c$ of $\mathcal{Q}$ such that $\langle(\mathcal{Q}, \langle\mathcal{P}, Kx\rangle), c\rangle$ is winning for $I$ in $\mathcal{I}(\mathcal{M}, \delta, n)$, and since the construction did not stop before $x$, this branch is cofinal. Thus there must be a second branch $d \neq c$ with these properties of $c$.

Let $\mathcal{K} = \mathcal{K}(c, \mathcal{Q})$ and $\mathcal{R} = Q(d, \mathcal{Q})$. Let

$$\mathcal{P} = \mathcal{P}^\beta, \quad \text{where} \quad \beta = \delta(\mathcal{Q}) \quad \text{or} \quad \beta^\mathcal{P} = \delta(\mathcal{Q}) \quad \text{is Woodin.}$$

Arguing as we did in the case $n$ is even, we can show that $\mathcal{P}$ is incomparable with one of $\mathcal{K}$ and $\mathcal{R}$, and by symmetry we may as well assume $\mathcal{P}$ is incomparable with $\mathcal{Q}$. That is, $\mathcal{P} \not\equiv \mathcal{Q}$ and $\mathcal{Q} \not\equiv \mathcal{P}$. 


We now define a coiteration \( ((S_i, \gamma_i)_{i < \theta}) \) of \( \mathcal{S} \) and \( \mathcal{L} \). Notice \( \mathcal{S} \) and \( \mathcal{L} \) are \( \delta(\mathcal{U}_\alpha) \) mice which agree below \( \delta(\mathcal{U}_\alpha) \), and \( \delta(\mathcal{U}_\alpha) \) is a cutpoint of both. Also, \( \mathcal{S} = k(\mathcal{S}, \delta(\mathcal{U}_\alpha)) \) realizable. Thus \( \mathcal{S}' \) and \( \mathcal{L}' \) will stay above \( \delta(\mathcal{U}_\alpha) \), and when \( lh \mathcal{S}' \) is a limit we can choose its unique cofinal realizable branch to form \( \mathcal{S}'_{\gamma+1} \). When \( lh \mathcal{L}' \) is a limit, we choose its unique cofinal branch \( a \) such that \( Q^\mathcal{L}_{a} \) is \( \Pi_{n-2} \)-iterable above \( \delta(\mathcal{U}_\alpha) \), and use \( a \) to form \( \mathcal{L}'_{\gamma+1} \). If it is not the case that there is a unique such \( a \), we stop the construction.

Again, the coiteration cannot last \( \omega_1 \) steps, and if it terminates it must do so successfully. This cannot happen, however, because \( \mathcal{L} \) is incomparable with \( \mathcal{S} \). So the construction stops at some stage \( \gamma < \omega_1 \).

We claim that \( \mathcal{U}_\gamma \) has a cofinal branch \( a \) such that \( \mathcal{L}_a^{\mathcal{U}_\gamma} \) is \( \Pi_{n-2} \)-iterable above \( \delta(\mathcal{U}_\gamma) \). For suppose otherwise. Define

\[
S(\eta) \iff \text{there is a coiteration } ((\mathcal{U}_\alpha, \mathcal{L}_\alpha))_{i < \eta} \text{ of } (\mathcal{S}, \mathcal{L}) \text{ such that } \forall \alpha < \eta
\]

(a) \( lh \mathcal{U}_\alpha \) a limit \( \Rightarrow \mathcal{M}_{lh \mathcal{U}_\alpha}^{\mathcal{U}_\alpha} \) is \( \Pi_{n-2} \)-iterable above \( \delta(\mathcal{U}_\alpha) \), and

(b) \( lh \mathcal{L}_\alpha \) a limit \( \Rightarrow \mathcal{M}_{lh \mathcal{L}_\alpha}^{\mathcal{L}_\alpha} \) is \( \Pi_{n-2} \)-iterable above \( \delta(\mathcal{L}_\alpha) \).

By our uniqueness hypothesis on \( ((\mathcal{S}_i, \mathcal{L}_i))_{i < \eta} \), any coiteration of \( (\mathcal{S}, \mathcal{L}) \) witnessing the truth of \( S(\eta) \) must satisfy \( \mathcal{U}_\alpha = \mathcal{S}_\alpha \) and \( \mathcal{L}_\alpha = \mathcal{L}_\alpha \) for all \( \alpha < \min(\eta, \gamma) \).

Since \( \mathcal{U}_\gamma \) has no cofinal appropriately iterable branch, \( S(\eta) \) is true precisely when \( \eta \leq \gamma \). Now clearly \( S \) is \( \Sigma_{n-1}^{HC}(\langle \mathcal{S}, \mathcal{L} \rangle) \), and so by Kechris' theorem (Lemma A), \( \gamma \in \Delta_{n-1}^{HC}(\langle \mathcal{S}, \mathcal{L} \rangle) \). It follows that \( \mathcal{U}_\gamma \in \Delta_{n-1}^{HC}(\langle \mathcal{S}, \mathcal{L} \rangle) \). This means that I can play \( \mathcal{U}_\gamma \) as his second move in \( \mathcal{I}(\mathcal{N}, \delta, n) \) without losing immediately. Letting \( a \) be such that \( \langle (\mathcal{U}_\alpha, (\mathcal{S}, \mathcal{L}_\alpha)), c, \mathcal{U}_\gamma, a \rangle \) is a winning position of length 2 for II in \( \mathcal{I}(\mathcal{N}, \delta, n) \), this means that \( a \) is a maximal branch of \( \mathcal{U}_\gamma \) which is appropriately iterable. Since our construction did not stop before \( \gamma \), \( a \) is cofinal in \( \mathcal{U}_\gamma \). This proves the claim at the beginning of this paragraph.

Thus our construction stopped because \( \mathcal{U}_\gamma \) has a second cofinal branch \( e \) with these properties of \( a \).

Let \( \mathcal{S}' \) be the last model of \( \mathcal{S}_\gamma \), or the direct limit along its unique cofinal realizable branch if \( lh \mathcal{S}_\gamma \) is a limit. Let \( \mathcal{L}' = Q(a, \mathcal{U}_\gamma) \) and \( \mathcal{R}' = Q(e, \mathcal{U}_\gamma) \). Let

\[
\mathcal{S}' = \mathcal{S}_\beta, \quad \text{where } \beta \text{ is largest such that } \beta = \delta(\mathcal{U}_\gamma) \text{ or } \mathcal{S}_\beta \vdash \delta(\mathcal{U}_\gamma) \text{ is Woodin.}
\]

As above, \( \mathcal{S}' \) is incomparable with one of \( \mathcal{L}' \) and \( \mathcal{R}' \). This contradicts our induction hypothesis (3) for \( n-2 \). This contradiction finishes the proof. \( \square \)

By applying Lemma 2.2 to \( \omega \)-mice we obtain immediately the main result of this section.

**Theorem 2.3.** Let \( 1 \leq n < \omega \), and suppose there are \( n-1 \) Woodin cardinals with a measurable above them all. Then \( \forall \alpha \in (\mathbb{R} \cap M_\alpha) \exists \omega_1(x \text{ is } \Delta_{n+1}^{HC}(\alpha)) \).
Proof. Suppose $x$ is the $\alpha$th real in $M_n$ in its natural order of construction. Then

$$z = x \iff \exists \mathcal{P} (\mathcal{P} \mbox{ is an } n\mbox{-small } \omega\mbox{-mouse } \wedge \mathcal{P} \mbox{ is } \Pi_n\mbox{-iterable } \wedge z \mbox{ is the } \alpha\mbox{th real in } \mathcal{P}).$$

This follows at once from Lemma 2.2 for $\Rightarrow$, we use part (2), and for $\Leftarrow$, we use part (3). The formula displayed shows that $\{x\}$ is $\Sigma_{n+1}(a)$, so $x$ is $\Delta_{n+1}(a)$. $\square$

3. Further applications of comparison

We shall show in the next section that $\mathbb{R} \cap M_n$ is precisely the set of reals $\Delta_{n+1}$ in a countable ordinal. The set has been studied extensively by purely descriptive set theoretic means in [1, 3], and elsewhere. Various facts proved in these papers can also be proved using only the methods of this section. As an example, we shall show that $\mathbb{R} \cap M_n$ is $\Sigma_{n+1}$, if $n$ is even, and $\Pi_{n+1}$, if $n$ is odd.

In the $n$ odd case, we need the following slight extension of 2.2.

Lemma 3.1. Suppose $1 \leq n < \omega$ and $n$ is odd, and suppose there are $n - 1$ Woodin cardinals with a measurable above them all. Suppose $\delta$ is a cutpoint of $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{J}_n^\mathcal{M} = \mathcal{J}_n^\mathcal{N}$, $\mathcal{M}$ is a $\delta$-mouse which is $k(\mathcal{M}, \delta)$ realizable, and $\mathcal{N}$ is $\Pi_1\delta$-iterable above $\delta$. Suppose $\mathcal{M}$ is $n$-small above $\delta$ and $\mathcal{N}$ is tame but not $n$-small. Then $\mathcal{M} \subseteq \mathcal{N}$.

Proof (sketch). As in (3) of Lemma 2.2. By induction on $n$, we see that there is an iteration tree $\mathcal{T}$ on $\mathcal{M}$ with last model $\mathcal{P}$ and a putative iteration tree $\mathcal{U}$ on $\mathcal{N}$ with last model 2 so that $\mathcal{P} \leq 2$ or $2 \leq \mathcal{P}$ (2 may be ill-founded; then "$\mathcal{P} \leq 2$" means $\exists \beta \in \omega fp(2)$ ($\mathcal{P} = \mathcal{J}_\beta^\mathcal{P}$)). If $2 \leq \mathcal{P}$, then as usual, there is no dropping in $\mathcal{U}$ on the branch below 2. But then 2 is not $n$-small above $\delta$, while $\mathcal{P}$ is, contradiction. So $\mathcal{P} \leq 2$, and by the proof of Lemma 1.11, $\mathcal{M} \subseteq \mathcal{N}$. [In the $n = 1$ case of the induction we need the clause in $\Pi_1\delta$-iterability which goes beyond the $\Pi_2^1$ mouse condition of [6]. As in the $n = 1$ case of the proof of (3) of Lemma 2.2, we are done unless the coiteration of $\mathcal{M}$ and $\mathcal{N}$ produces a putative iteration tree $\mathcal{U}$ on $\mathcal{N}$ of limit length such that $\mathcal{U}$ has no cofinal well-founded branch, but $\forall x < \omega_1$ ($\mathcal{U}$ has a cofinal $x$-good branch). Let $\mathcal{P}$ be the last model of $\mathcal{P}_n$ or the direct limit along its unique cofinal well-founded branch. Let $b$ be a cofinal $x$-good branch of $\mathcal{U}$, where $x > OR^\mathcal{P}$. In the proof of Lemma 2.2, we argued that $\mathcal{P} \leq \mathcal{M}_b^{\mathcal{U}}$, but this used the 1-smallness of $\mathcal{N}$ and need not be true here. However, let $\delta = \delta(\mathcal{U})$; then since $\mathcal{M}$ is 1-small the $\mathcal{P}$-sequence has no extenders $E$ such that $lh E \geq \delta$. Suppose $E$ is the first extender on the $\mathcal{M}_b^{\mathcal{U}}$-sequence with length $\geq \delta$; if there is no such extender then we are done. Since $\mathcal{N}$ is tame, $\text{crit } E > \delta$. Since $b$ is $x$-good, $x \in \omega fp(Q)$, where $Q$ is the $x$th iterate of $\mathcal{M}_b^{\mathcal{U}}$ by $E$. But then $\mathcal{P} \leq Q$, as desired.] $\square$

We wish to thank Mitch Rudominer for finding the flaw in our original attempt to prove Lemma 3.1 with the weaker notion of $\Pi_1$-iterability of [6].
Theorem 3.2. Let $1 \leq n < \omega$, and $n$ odd. Suppose there are $n$ Woodin cardinals with a $P_\delta(\kappa)$-measurable above. Then $\mathbb{R} \cap M_n$ is $\Pi^{HC}_{n+1}$.

Proof. Our hypothesis implies $M_n^\#$ exists. We claim that for $x \in \mathbb{R}$

$$x \in M_n \iff \forall \mathcal{P} [ (\mathcal{P} \text{ is countable } \land \mathcal{P} \text{ is } \Pi_n \text{-iterable} \land \mathcal{P} \text{ is not } n\text{-small} ) \Rightarrow x \in \mathcal{P} ].$$

For $\Rightarrow$, we use Lemma 3.1. For the other direction, taking $P = M_n^\#$, we see that $x \in M_n^\#$, and hence $x \in M_n$. □

With more care one can reduce the large cardinal hypothesis of Theorem 3.2 to $n$ Woodins plus indiscernibles for $L(V_\delta)$, where $\delta$ is the $n$th Woodin. The existence of $n$ Woodin cardinals does not suffice for Theorem 3.2 by itself. [Take $n = 1$. Suppose there is a Woodin cardinal in $V$, and $\mathbb{R} \cap V = \mathbb{R} \cap M_1$. This supposition is consistent relative to the existence of one Woodin by Jensen's trick (cf. [6, Section 6]). Let $x$ be Cohen generic over $V$. One can easily see that $M_1^V = M_1^{[x]}$. Suppose $\varphi(v)$ is a $\Pi_3^1$ formula defining $\mathbb{R} \cap M_1$ in $V[x]$. Then $V[x] \models \exists v \to \varphi(v)$, so $V \models \exists v \to \varphi(v)$ by Martin–Solovay absoluteness. But if $y \in V$ and $V \models \neg \varphi[y]$, then $y \in M_1^{[x]}$ and $V[x] \models \neg \varphi[y]$, a contradiction.]

Let $M_n(x)$ be the result of relativising the construction of $M_n$ to $x$, where $x \in \mathbb{R}$. A slight extension of the proof of Theorem 3.2 shows that if $B(x, y, z)$ is a $\Pi_{n+1}^1$ relation, and

$$A(x, z) \iff \exists y \in M_n(x) B(x, y, z),$$

then $A$ is $\Pi_{n+2}^1$. (Here $n$ is odd.)

For the $n$ even case, we need the following lemma.

Lemma 3.3. Suppose $n$ is even, $1 < n < \omega$, and there are $n$ Woodin cardinals with a $P_\delta(\kappa)$ measurable above. Let $\mathcal{M}$ be a $\Pi_n$-iterable, $n$-small $\omega$-mouse. Let $\mathcal{N}$ be realizable and not $n$-small. Then $\mathcal{M} \preceq \mathcal{N}$.

Proof (sketch). We may as well assume $\mathcal{N} = M_n^\#$, since $M_n^\# \preceq \mathcal{J}$ whenever $\mathcal{J}$ is realizable and not $n$-small. So $\mathcal{N}$ is $n+1$-small. We follow closely the proof of (3) of Lemma 2.2 for $n+1$. In that case $\mathcal{M}$ was realizable; here it is $\Pi_n$-iterable. However, because $n$ is even, this is enough. Where we used (1) of Lemma 2.2 to build $\mathcal{J}$ on $\mathcal{M}$ at limit stages, we use here the following: for any $\mathcal{F}$ on $\mathcal{M}$ of limit length, $\mathcal{F}$ has at most one cofinal branch $b$ such that $\langle \mathcal{F}, b \rangle$ is a winning position for II in $\mathcal{J}(\mathcal{M}, \omega, n)$. This is proved just as was (1) of Lemma 2.2. □

Theorem 3.4. Suppose there are $n$ Woodins and a $P_\delta(\kappa)$ measurable above, where $1 < n < \omega$ is even. Then $\mathbb{R} \cap M_n$ is $\Sigma_{n+1}^{HC}$. 
Proof. For \( x \in \mathbb{R} \), we have
\[
    x \in M_n \iff \exists \mathcal{P} (\mathcal{P} \text{ is countable} \land \mathcal{P} \text{ is } n\text{-small} \land \mathcal{P} \text{ is } \Pi_n\text{-iterable} \land x \in \mathcal{P}).
\]

We do not know whether Theorem 3.4 can be proved under the hypothesis that there are \( n \) Woodin cardinals. Of course, for \( n = 0 \) it can. In any event, \( n \) Woodins plus indiscernibles suffices.

If there \( n \) Woodins plus indiscernibles, then \( \mathbb{R} \cap M_n \) is not \( \Delta_{n+1}^{HC} \). It follows that Lemma 3.3 fails when \( n \) is odd; there are \( \Pi_n\)-iterable \( \omega \) mice which are \( n \)-small but are not in \( M_n \). If \( \mathcal{P} \) is one of these "nonstandard" mice, then \( \mathbb{R} \cap M_n \subseteq \mathcal{P} \), by (3) of Lemma 2.2. For \( n \) even, Lemma 3.1 is true, but vacuously: every \( \Pi_n\)-iterable \( \omega \)-mouse is \( n \)-small.

Finally, let us consider the well-order of \( \mathbb{R} \cap M_n \) given by order of construction. Let us define
\[
    x S_n y \iff \exists \mathcal{P} (\mathcal{P} \text{ is countable and } \Pi_n\text{-iterable} \land \mathcal{P} \text{ is an } \omega\text{-mouse} \land \mathcal{P} \models x \text{ is constructed before } y).
\]

So \( S_n \) is \( \Sigma_n^{HC} \). It is easy to see, using Lemma 2.2, that \( S_n \cap M_n \) is the order of construction of \( M_n \).

Since \( M_1 \) is \( \Pi_1 \) correct by Shoenfield's theorem, \( S_1^{M_1} = S_1 \cap M_1 \). It follows that \( M_1 \) satisfies "\( \mathbb{R} \) has a \( \Delta_2^{HC} \) well-order". In order to show that \( S_n^{M_n} = S_n \cap M_n \) for \( n > 1 \), we need the correctness results of the next section. However, Jensen's trick \( \{6, 6.17\} \) shows that it is consistent, relative to the existence of \( n \) Woodins, that there are \( n \) Woodins and \( \mathbb{R} = \mathbb{R} \cap M_n \) (so \( S_n \cap M_n = S_n^{M_n} \)). This gives the following theorem.

Theorem 3.5. Let \( n < \omega \), and suppose there are \( n \) Woodin cardinals. Then there is a proper class models of ZFC + "There are \( n \) Woodins" + "\( \mathbb{R} \) has a \( \Delta_{n+1}^{HC} \) wellorder".

4. Correctness

If \( M \) and \( N \) are transitive and \( M \subseteq N \), then we say \( M \) is \( \Sigma_n \) correct in \( N \) iff whenever \( a \in HC^M \) and \( \varphi \) is a \( \Sigma_n \) formula of the language of set theory, then \( HC^M \models \varphi[a] \) iff \( HC^N \models \varphi[a] \). We say \( M \) is \( \Sigma_n \) correct iff \( M \) is \( \Sigma_n \) correct in \( V \).

We shall begin by sketching our original argument that \( M_n \) is \( \Sigma_n \) correct. This result is best possible when \( n \) is odd, but not when \( n \) is even, and in neither case does it give the proper lower bound on \( \mathbb{R} \cap M_n \). We then describe work of Hugh Woodin, using different methods, which gives the optimal correctness and proper lower bound on \( \mathbb{R} \cap M_n \) in all cases. We have included our original argument because we find it interesting, and because it has one small consequence Woodin's methods do not seem to give.
Our proof is based on the construction of homogeneity systems in [5], with which we assume familiarity. Woodin’s work does not require [5], and the reader who would like to go directly to it should skip Theorem 4.1 and Corollaries 4.2–4.4.

First, some terminology. Let \( T \) be a tree on \( \omega \times \kappa \); then a homogeneity system for \( T \) is a sequence \( \vec{\mu} = \langle \mu_s \mid s \in \omega^{< \omega} \rangle \) of measures witnessing that \( T \) is homogeneous. If \( M \) is an inner model, then \( \vec{\mu} \models M = \langle \mu_s \cap M \mid s \in \omega^{< \omega} \rangle \). Notice that if \( T \) and \( \vec{\mu} \models M \) belong to \( M \) then

\[
M \models \vec{\mu} \models M \text{ is a homogeneity system for } T.
\]

We use similar terminology in the case \( T \) is a tree on \( (\omega \times \omega) \times \kappa \), or in general, on \( \omega^n \times \kappa \) for some \( n < \omega \). If \( \vec{\mu} \) is a homogeneity system for a tree \( T \) on \( \omega^{n+1} \times \kappa \), and \( \gamma \in OR \), then \( ms(T, \vec{\mu}, \gamma) \) is the Martin–Solovay tree on \( \omega^n \times \gamma \) constructed from \( T \) and \( \vec{\mu} \). [Say \( n = 1 \). Let \( \langle r_i \mid i < \omega \rangle \) be a one one recursive enumeration of \( \omega^{< \omega} \) such that \( \forall i (t \leq r_i \Rightarrow \exists j < i (t = r_j)) \). Actually, \( ms(T, \vec{\mu}, \gamma) \) depends on this enumeration, but we assume one such enumeration has been fixed throughout. Let \( (s, t), (u, v) \in (\omega \times \omega)^{< \omega} \), with \( s \leq u \) and \( t \leq v \); then \( i_{s, t}(u, v) \) is the canonical embedding from \( Ult(V, s, t) \) into \( Ult(V, t) \).

We shall show that if \( M \) is any transitive model of ZFC satisfying “there are \( n \) Woodin cardinals and a measurable above them all”, where \( 0 \leq n < \omega \), and if there are \( M \)-extenders witnessing these large cardinal hypotheses in \( M \) which have \( V \)-extensions, then \( M \) is \( \Sigma^1_{n+2} \) correct.

**Theorem 4.1.** Let \( M \) be a transitive model of ZFC, and suppose

\[
M \models \delta \text{ is Woodin via } \mathcal{E}.
\]

Suppose also that every \( E \in \mathcal{E} \) has a \( V \)-extension. Suppose \( \vec{\mu} \) is a \( \delta^+ \)-additive homogeneity system for a tree \( T \) on \( \omega^{n+1} \times \kappa \), and that \( T, \vec{\mu} \models M \in M \). Then for all sufficiently large \( \gamma \in OR^M \), letting \( U \) be such that

\[
M \models U = ms(T, \vec{\mu} \models M, \gamma),
\]

we have

(a) \( (\forall \vec{x} \in V)(\vec{x} \in p[U] \leftrightarrow \forall y((\vec{x}, y) \notin p[T])) \).

and

(b) for all \( \kappa < \delta \), there is a \( \kappa \)-additive homogeneity system \( \vec{\nu} \) for \( U \) such that \( \vec{\nu} \models M \in M \).
Proof (sketch). We assume $n = 1$ for notational simplicity.

Working inside $M$, let $\gamma$ be large enough that there are strong limit cardinals $c_2 > c_1 > c_0 > \delta$, with $T \in V_{c_0}$, such that structure $(V_{c_2}, \in, c_0, \delta, T, a)_{a \in V_\gamma}$ is elementarily equivalent to the structure $(V_{c_1}, \in, c_1, \delta, T, a)_{a \in V_\gamma}$. Set $U = ms(T, \mu \upharpoonright M, \gamma)$. Still working inside $M$, we have from [5] that $U$ is $\kappa$-homogeneous for all $\kappa < \delta$, and $\forall x(x \in p[U] \iff \forall y((x,y) \not\in p[T]))$. We shall show that the construction of [5] guarantees that these properties of $U$ go over to $V$.

First, notice that $U$ is isomorphic to a subtree of the full $ms(T, \mu, \gamma)$ (computed in $V$). The embedding from $U$ into $ms(T, \mu, \gamma)$ comes from the natural maps

$$\pi_s: \{ f \in M | f: T_\delta \rightarrow \gamma \} / \mu_s \cap M \rightarrow \{ f \upharpoonright f: T_\delta \rightarrow \gamma \} / \mu_s,$$

defined for $s \in \omega^{<\omega}$. Thus for any $x$ in $V$, $x \in p[U] \Rightarrow x \in p[ms(T, \mu, \gamma)] \Rightarrow \forall y((x,y) \not\in p[T])$.

For the converse to this, we need the construction of [5]. We work inside $M$ again. Let $W$ be the tree order on $\omega$ given by: $0Wn$ for all $n > 0, 2nW2k$ iff $n < k$, and $(2n + 1)W(2k + 1)$ iff $r_{n+1} \not\equiv r_{k+1}$. So $W$ has branches $b = \{2n|n \in \omega\}$ and, for each $y \in \omega^\omega, c_y = \{0\} \cup \{2n + 1|n < k \in y\}$. Using the homogeneity system $\mu_M$ and the method of [5], we associate to each $s \in \omega^{<\omega}$ an iteration tree $W_s$ of length $2 \dim(s) + 1$ on $V$ (which, for the moment, is $M$) whose tree order is $W \upharpoonright 2 \dim(s) + 1$. We have $s \subseteq t \Rightarrow W_s$ extends $W_t$. Since $\delta$ is Woodin via extenders in $\mathcal{E}$, we can arrange that each $W_s$ uses only extenders from $\mathcal{E}$ and its images.

Now we go back to $V$. Fix a map $h$ assigning $V$-extensions to the extenders in $\mathcal{E}$. For $s \in \omega^{<\omega}$ we define an iteration tree $W_s^\ast$ and embeddings

$$\pi_s^n: M_n^W \rightarrow i_{\omega, \omega}^W(M)$$

for $n < lh W_s$ such that $\pi_s^n \upharpoonright lh E_{n-1}^W = \text{identity}$. We have

$$E_n^W = i_{\omega, \omega}^W(h)(\pi_n^W(E_n^W)).$$

This determines the model in $W_s^\ast$ with index $n + 1$, and we get $\pi_s^{n+1}$ in the obvious way. ($W_s$ and $W_s^\ast$ have the same tree order.) The reader can check through the details of this simple copying construction.

For $x \in \omega^\omega$, let $W_x = \bigcup_{n \in \omega} W_x \upharpoonright n$ and $W_x^\ast = \bigcup_{n \in \omega} W_x \upharpoonright n$. Let

$$k_x: M \rightarrow M_{W_x}$$

be the canonical embedding along the branch $b$ of $W_x$, and for $t \in \omega^\omega$ let

$$i_{xy}: M \rightarrow M_{W_x}$$

be the embedding along the branch $c_y$ of $W_x$. Similarly, we have $k_x^\ast$ and $i_x^\ast$ coming from the iteration tree $W_s^\ast$. The maps $\pi_s^n$ from the copying construction commute with the iteration tree embeddings, and so give us, for $x, y \in \omega^\omega$

$$\pi_x: M_{W_x} \rightarrow i_{xy}^\ast(M)$$
and
\[ \pi_{xy} : \mathcal{M}_{c_y} \rightarrow i_{0, c_y}^* (M). \]

Now let \( x \in \omega \) be such that \( \forall y ((x, y) \notin p[T]) \), or equivalently, \( \forall y (T_{x,y} \text{ is well-founded}) \). The construction of [5] guarantees that for all \( y \in \omega \), \( \bigcup_{n < \omega} i_{xy}(T_{x \upharpoonright n, y \upharpoonright n}) \) has an infinite branch. (Of course, \( x \) and \( y \), and so \( T_{x,y} \), may not be in \( M \); however, the construction is "continuous" in \( x \) and \( y \).) Since
\[ i_{xy}^* (T_{x,y}) = \bigcup_{n < \omega} \pi_{xy} (i_{xy}(T_{x \upharpoonright n, y \upharpoonright n})), \]
we see that \( i_{xy}^* (T_{x,y}) \) is ill-founded for all \( y \). Since \( T_{x,y} \) is well-founded, \( \mathcal{M}_{c_y} \vDash "i_{xy}^*(T_{x,y}) \text{ is well-founded}" \), and thus \( \mathcal{M}_{c_y} \vDash "i_{xy}^*(T_{x,y}) \text{ is ill-founded}" \), for all \( y \). In fact, the ill-foundedness is continuous in \( y \), and this guarantees that \( \mathcal{M}_b \vDash "i_{xy}^*(T_{x,y}) \text{ is ill-founded}" \), and thus \( \mathcal{M}_b \vDash "i_{xy}^*(T_{x,y}) \text{ is well-founded}" \). Since \( \forall y (T_{x,y} \text{ is well-founded}) \), the construction of [5] guarantees that \( \bigcup_{n < \omega} k_s(U_{x \upharpoonright n}) \) has an infinite branch. For \( s \in \omega^{<\omega} \), letting \( k = \text{dom}(s) \), we have \( X_s \in \mu_{\{k \mid \text{dom}(r,k) \in M \}} \cap M \) and ordinals \( e(s, t) \in \mathcal{M}_{2 \text{dom}(s)} \) defined for all \( t \in X_s \). We have that whenever \( s \leq s' \), \( \text{dom}(s) = k \), \( \text{dom}(s') = k' \), \( r_k \subseteq r_{k'} \), \( t \in X_s \), \( t' \in X_{s'} \), and \( t \subseteq t' \), then
\[ e(s', t') < i_{2k, 2k}^*(e(s, t)). \]

For \( s \in \omega^{<\omega} \), let
\[ e_s = [ \lambda t . e(s, t)]^M_{\mu_{\{k \mid \text{dom}(r,k) \in M \}} \cap M}, \]
where \( k = \text{dom}(s) \), and where the superscript \( M \) indicates that the ultrapower is to be computed using functions from \( M \). Let
\[ e_k = i_{2k, 2k}^*(e_s \upharpoonright k). \]
One can show that \( \langle e_k \mid k \in \omega \rangle \) is an infinite branch of \( \bigcup_{n < \omega} k_s(U_{x \upharpoonright n}) \). (The key is that the embeddings \( i_{2k, 2k}^* \) along \( b \) in \( W_x \) and the embeddings in \( M \) from \( \text{Ult}(M, \mu_{\{k \mid \text{dom}(r,k) \in M \}} \cap M) \) into \( \text{Ult}(M, \mu_{\{k' \mid \text{dom}(r,k') \} \cap M} \) leave each other fixed.)

But then \( \langle \pi_k(e_k) \mid k \in \omega \rangle \) is an infinite branch of \( \bigcup_{n < \omega} k_s^*(U_{x \upharpoonright n}) \) which is just \( k_s^*(U_x) \). Since \( \mathcal{M}_b \vDash "k_s^*(U_x) \text{ is ill-founded}" \), \( k_s^* \) is elementary, \( U_x \) is ill-founded. This completes the proof of (a) of Theorem 4.1.

For (b), let us define measures over \( V \) witnessing homogeneity for \( U \) as follows. For \( s \in \omega^{<\omega} \), let
\[ \tau_s = \langle i_{2n, 2 \text{dom}(s)}(e_s \upharpoonright n) \mid n < \text{dom}(s) \rangle. \]
Then for \( A \supseteq U_s \), we put
\[ A \in v_s \iff \tau_s \subseteq i_{0, 2 \text{dom}(s)}^*(A). \]
One then sees that \( \text{Ult}(V, \langle v_x \mid k \in \omega \rangle) \) embeds into \( \mathcal{M}_b \vDash "i_{xy}^*(T_{x,y}) \text{ is well-founded}" \), so that as shown above \( \mathcal{M}_b \vDash "i_{xy}^*(T_{x,y}) \text{ is ill-founded}" \), then \( \text{Ult}(V, \langle v_x \mid k \in \omega \rangle) \) is well-founded. So \( v \) is indeed a homogeneity system for \( U \). Also, if \( A \subseteq U_s \) and \( A \in M \), then
since \( \pi_2^{\text{dom}(s)}(i_{0,2}^{\text{dom}(s)}(A)) = i_{0,2}^{\text{dom}(s)}(A) \), we have

\[
A \in (\nu_s \cap M) \iff \tau_s \in i_{0,2}^{\text{dom}(s)}(A).
\]

But \( \langle \tau_s | s \in \omega^{<\omega} \rangle \in M \), so \( \forall \nu M \in M \), as desired. 

**Corollary 4.2.** Let \( M \) be a transitive model of \( \text{ZFC} \), and \( 0 \leq n < \omega \). Suppose \( \delta_0, \ldots, \delta_{n-1}, \in M \) and \( M \models \forall i < n (\delta_i \text{ is Woodin via } \mathcal{E}) \), and every \( E \in \mathcal{E} \) has a \( V \)-extension. Suppose \( \kappa > \delta_{n-1} \), and

\[
M \models \mathcal{U} \text{ is a } (\kappa, \kappa + 1) \text{ extender on } \kappa,
\]

and \( \mathcal{U} \) has a \( V \)-extension. Then there is a tree \( T \in M \) and, for all \( \gamma < \delta_0 \), a \( \gamma \)-additive homogeneity system \( \mu_\gamma \) for \( T \) such that

1. \( p[T] \) is the universal \( \Pi^1_{n+1} \) subset of \( \omega^\omega \), both in \( V \) and \( M \), and
2. \( \mu_\gamma \models T \in M \).

**Proof.** In the case \( n = 0 \), \( T \) is the Shoenfield tree for \( \Pi^1_1 \) on \( \omega \times \kappa \). If \( n > 0 \), then we use induction and Theorem 4.1.

**Corollary 4.3.** Under the hypotheses of Corollary 4.2, \( M \) is \( \Sigma_n^{n+1} \) correct.

**Corollary 4.4.** Suppose there are \( n \) Woodin cardinals, where \( n < \omega \). Then \( M_n \) is \( \Sigma_n \) correct.

**Proof.** We may assume \( n > 0 \). If \( M_n = \mathcal{N}_n^{\omega_1} \), then \( M_n \) has \( n - 1 \) Woodin cardinals and a measurable above, and sufficiently many of its active extenders admit \( V \)-extensions. Thus Corollary 4.3 applies.

Otherwise, \( M_n \) is an iterate of \( \mathcal{C}_\omega(\mathcal{N}_n) \), truncated at \( \text{OR} \). In this case, \( \mathcal{N}_n \) has \( n - 1 \) Woodins and a measurable above, all via extenders admitting \( V \)-extensions. So \( \mathcal{N}_n \) is \( \Sigma_n \) correct, and thus \( M_n \) is \( \Sigma_n \) correct.

The proof just given produces \( L^{[\widetilde{E}]} \)-type models \( M \), and trees \( T \) for \( \Pi^1_1 \) admitting homogeneity systems \( \mu_\gamma \), such that \( T, \mu \models T \in M \). Woodin's method does not seem to give this.

**Theorem 4.5.** Suppose there are \( n \) Woodin cardinals; then \( M_n \) satisfies "\( \mathbb{R} \) has a \( \Delta^4_{n+2} \) well-order".

**Proof.** Let \( <^* \) be the usual order of construction on \( \mathbb{R} \cap M_n \). We claim that for \( x, y \in M_n \)

\[
x <^* y \iff M_n \models \exists \mathcal{P}(\mathcal{P} \text{ is } n \text{-small } \land \mathcal{P} \text{ is } \Pi^1_n \text{-iterable} \land x, y \in \mathcal{P} \land x \text{ is constructed before } y \text{ in } \mathcal{P}.
\]
The main point here is that \( \Pi_n \)-iterability is absolute between \( M_n \) and \( V \) by Corollary 4.4. Thus if \( x <^* y \), then we can take \( P = \mathcal{F}_y^{M_x} \) where \( x, y \in \mathcal{F}_y^{M_x} \). Since \( P \) is \( \Pi_n \)-iterable in \( V \), it is \( \Pi_n \)-iterable in \( M_n \), so witnesses the right-hand side. Conversely, let \( P \) have the properties stated on the right hand side in \( M_n \). Then \( P \) has these properties in \( V \). But if \( y \leq_* x \), we can find a realizable \( \mathcal{Z} = \mathcal{F}_y^{M_x} \) such that \( x, y \in \mathcal{Z} \) and \( y \leq_* x \) in \( \mathcal{Z} \). We may as well assume \( P \) and \( \mathcal{Z} \) are \( \omega \)-mice. By Lemma 2.2(3) then \( P \leq \mathcal{Z} \) or \( \mathcal{Z} \leq P \). This is a contradiction. 

Lemma 4.6 through Corollary 4.11 are due to Hugh Woodin, and are included with his permission.

The main result is the following lemma.

**Lemma 4.6 (Woodin).** Let \( \mathcal{M} \) be a countable, active, realizable premouse. Let \( n < \omega \) be even, and suppose \( P \) is a poset in \( \mathcal{M} \) such that \( \mathcal{M} \) there are \( n \) Woodin cardinals strictly greater than \( \text{card}(P) \). Then if \( G \) is \( P \)-generic over \( \mathcal{M} \), \( \mathcal{M}[G] \) is \( \Sigma_{n+1} \) correct.

**Proof.** By induction on \( n \). For \( n = 0 \), this is essentially Shoenfield's absoluteness theorem. Notice that since \( \mathcal{M} \) is active and realizable, we can iterate its last extender \( \omega_1 \) times to get \( \mathcal{M}^* \) with \( \omega_1 \in \mathcal{M}^* \). It is easy to see that \( G \) is \( P \)-generic over \( \mathcal{M}^* \), and \( HC \cdot A[G] = HC \cdot A^*[G] \). But \( \mathcal{M}^*[G] \) is \( \Sigma_1 \) correct by Shoenfield.

Now let \( n > 0 \). Let \( a \in HC \cdot A[G] \) and \( \varphi = \exists \psi \psi \) where \( \psi \) is \( \Pi_{n-1} \) and suppose \( HC \models \varphi[a] \). We must show \( HC \cdot A[G] \models \varphi[a] \). Pick \( b \) so that \( HC \models \psi[b, a] \), and let \( x \) be a real coding \( b \). We may assume that \( P \in V^\mathcal{M}_x \), where \( \mathcal{M} \) has \( n \) Woodin cardinals \( \geq \kappa \). Let \( \delta < \mu \) be the first two of these Woodin cardinals of \( \mathcal{M} \).

We now apply Woodin's theorem on genericity over \( L[\tilde{E}] \) models, stated as [8, 4.3]. This gives us a poset \( Q \in V^\mathcal{M}_{\delta+1} \) and an iteration tree \( \mathcal{S} \) on \( \mathcal{M} \) of countable length \( \theta + 1 \) such that

(a) \( \mathcal{M}_{\theta}^\mathcal{S} \) is realizable,
(b) \( D^\mathcal{S} = \emptyset \) (so \( i_{0\alpha}^\mathcal{S} \) is defined), and
(c) \( \text{crit} \ E_{\delta} > \kappa \) for all \( \alpha < \text{lh} \mathcal{S} \) (so that \( G \) is \( P \)-generic/\( \mathcal{M}_{\theta}^\mathcal{S} \), and \( x \) is \( i_{0\delta} \) generic/\( \mathcal{M}_{\theta}^\mathcal{S} [G] \).

Next, we can absorb \( x \) into a generic object for the stationary tower forcing over \( \mathcal{M}_{\theta}^\mathcal{S} [G] \) up to \( i_{0\delta}^\mathcal{S} (\mu) \). (It does not matter whether we use the \( P_{<i_{0\delta}^\mathcal{S} (\mu)} \) or the \( Q_{<i_{0\delta}^\mathcal{S} (\mu)} \) tower.) Let \( H \) be this generic. Let \( N \) be the generic ultrapower, and \( j: \mathcal{M}_{\theta}^\mathcal{S} [G] \rightarrow N \) the generic elementary embedding. Notice that the models we are discussing are countable, so generic objects really exist.

Now \( \mathcal{M}_{\theta}^\mathcal{S} [G][H] \) is a generic extension of \( \mathcal{M}_{\theta}^\mathcal{S} \) via a poset \( R \) of size \( i_{0\delta}^\mathcal{S} (\mu) \) in \( \mathcal{M}_{\theta}^\mathcal{S} \). \( \mathcal{M}_{\theta}^\mathcal{S} \) is realizable and there are \( n-2 \) Woodin cardinals of \( \mathcal{M}_{\theta}^\mathcal{S} \) above \( i_{0\delta}^\mathcal{S} (\mu) \). By induction, then, \( \mathcal{M}_{\theta}^\mathcal{S} [G][H] \models \psi[a, b] \) \( HC \). Since \( N \) and \( \mathcal{M}_{\theta}^\mathcal{S} [G][H] \) have the same reals, \( HC \models \psi[a, b] \). But \( j(a) = a \), so \( HC \cdot A[G] \models \varphi[a] \). Since \( \text{crit} E_{\delta} > \kappa \), this implies \( HC \cdot A[G] \models \varphi[a] \), as desired. 

\( \square \)
Lemma 4.6 is false for $n$ odd. For example, suppose $M_n^#$ exists. Then the sentence "$\exists \mathcal{P} (\mathcal{P}$ is $\Pi_n^*$-iterable $\land \mathcal{P}$ is not $n$-small)" is true in $HC^V$, but not in $HC^{M_*}$ (by Section 3). Thus $M_n^#$ is not $\Sigma_{n+1}$ correct when $n$ is odd. Equivalently, $M_n$ is not $\Sigma_{n+1}$ correct when $n$ is odd.

**Corollary 4.7** (Woodin). Let $n < \omega$ be even, and suppose $M_n^#$ exists. Then $M_n$ is $\Sigma_{n+1}$ correct.

We do not know how to prove Corollary 4.7 under the assumption that there are $n$ Woodin cardinals. Of course, for $n = 0$ this can be done. Woodin's method also shows that $M_n$ is $\Sigma_n$ correct when $n$ is odd, provided that $M_n$ satisfies "There are $n$ Woodin cardinals".

(Since there are $n$ Woodin cardinals in $M_n$, there are arbitrarily large $\beta < \omega_1^{M_*}$ such that $\mathcal{I}_{\beta}^{M_*}$ is active and satisfies "there are $n - 1$ Woodin cardinals". By Lemma 4.6, such $\mathcal{I}_{\beta}^{M_*}$ are $\Sigma_{n}$ correct. This implies $M_n$ is $\Sigma_{n}$ correct.)

We can now complete our characterization of $\mathbb{R} \cap M_n$.

**Theorem 4.8** (Woodin). Let $n < \omega$ and suppose $M_n^#$ exists. Let $x$ be a real which is $\Lambda_{n+1}(x)$, for some $\alpha < \omega_1$. Then $x \in M_n$.

**Proof.** We may assume $n > 0$. Let $\mathcal{P}$ be the result of iterating the bottom measurable cardinal of $M_n^# \times 1$ times. So $\mathcal{P}$ is countable, realizable, and has $n$ Woodin cardinals above $\alpha$. It will be enough to see $x \in \mathcal{P}$, and for this it will be enough to see $x \in \mathcal{P}[G]$ whenever $G$ is generic over $\mathcal{P}$ for $Col(\omega, \alpha)$. So fix such a $G$.

If $n$ is even, then $\mathcal{P}[G]$ is $\Sigma_{n+1}$ correct by Lemma 4.6. Since $x \in HC^{\mathcal{P}[G]}$, $x \in \mathcal{P}[G]$. Thus we may assume $n$ is odd.

Let $\delta$ be the image in $\mathcal{P}$ of the bottom Woodin cardinal of $M_n^#$. Let $Q \in V_{\delta+1}^{\mathcal{P}}$ be the poset given by Woodin's genericity theorem [8, 4.3]. Let $\varphi = \exists \forall \psi$ be a $\Sigma_{n+1}$ formula such that

$$n \in x \iff (HC^{V} \models \varphi[n, \bar{x}]).$$

We claim that

$$n \in x \iff (\mathcal{P}[G] \models \exists q \in Q (q \models \varphi[n, \bar{x}]^HC)).$$

Clearly this implies $x \in \mathcal{P}[G]$, and completes the proof.

Suppose first $n \in x$, and let $b$ be such that $HC \models \psi[b, n, \bar{x}]$. Woodin's genericity theorem gives a realizable countable iterate $\mathcal{R}_b$ of $\mathcal{P}$, where $i: \mathcal{P} \to \mathcal{R}_b$ has critical point $> \alpha$, and $H$ which is $i(Q)$ generic over $\mathcal{R}_b[G]$ such that $b \in HC^{\mathcal{R}_b[G,H]}$. There are $n - 1$ Woodin cardinals of $\mathcal{R}_b$ above $i(\delta)$, and $n - 1$ is even. The pair $(G, H)$ is $\mathcal{R}_b$ generic for a poset of size $i(\delta)$. Thus by Lemma 4.6, $\mathcal{R}_b[G, H]$ is $\Sigma_n$ correct. So $\mathcal{R}_b[G, H] \models (HC \models \psi[b, n, \bar{x}])$, and there is $q \in i(Q)$ and $p \in G$ such that $\mathcal{R}_b \models [p \models (q \models HC \models \varphi[n, \bar{x}])].$
Since \( i(p) \), \( \mathcal{P}[G] \) satisfies \( \exists q \in Q(q) \models \varphi[n,x]^H \). Suppose next that, in \( \mathcal{P}[G] \), \( q \) forces \( \varphi[n,x]^H \). Since \( \mathcal{P}[G] \) is countable, we have an \( H \) which is \( Q \) generic over \( \mathcal{P}[G] \) and such that \( q \in H \). In \( \mathcal{P}[G,H] \), \( HC \models \varphi[n,x] \). But \( \mathcal{P}[G,H] \) is \( \Sigma_n \) correct by Lemma 4.6, so \( n \in x \). □

**Corollary 4.9** (Woodin). Let \( n < \omega \) and suppose \( M_n^# \) exists. Then \( \mathbb{R} \cap M_n = C_{n+2} \) if \( n \) is even, and \( \mathbb{R} \cap M_n = Q_{n+2} \) if \( n \) is odd.

From well-known results concerning \( C_{n+2} \) and \( Q_{n+2} \), we see that if \( M_n^# \) exists, then \( M_n \) satisfies "\( \mathbb{R} \) has a \( \Delta_{n+2}^1 \)-good well-order" (see [3]).

The following is a counterpart to Lemma 4.6 in the case \( n \) is odd.

**Lemma 4.10** (Woodin). Let \( \mathcal{M} \) be a countable, active, realizable premouse. Let \( n < \omega \), and suppose \( \mathcal{M} \models \text{``there are } n \text{ Woodin cardinals}. \) Let \( x \in \mathcal{M} \cap \omega \omega \), and let \( y \in \omega \omega \) code \( \mathcal{M} \). Then every nonempty \( \Sigma_{n+2}^1(x) \) set of reals has a member recursive in \( y \).

**Proof.** By Corollary 4.7, we may assume \( n \) is odd. Let \( A(z) \iff \exists w B(z,w,x) \), where \( B \) is \( \Pi_{n+1}^1 \). Let \( \delta \) be the smallest Woodin cardinal of \( \mathcal{M} \), and \( Q \in V_{\delta+1}^\mathcal{M} \) be Woodin's every-real-generic poset. Arguing as in the \( n \) odd case of Theorem 4.8, we get a \( q \in Q \) so that

\[ \mathcal{M} \models (q \Vdash Q \exists w \exists z B(z,w,x)). \]

(The main point is that if \( i: \mathcal{M} \rightarrow \mathcal{P}, \mathcal{P} \) is realizable, and \( G \) is \( i(Q) \) generic/\( \mathcal{P}, \mathcal{P}[G] \) is \( \Pi_{n+1}^1 = \Pi_{n+1}^H \) correct by Theorem 4.5 for \( n - 1 \).) But now we can find \( G \) which is \( \mathcal{M} \)-generic over \( Q \) such that \( q \in G \) and every real in \( \mathcal{M}[G] \) is recursive in \( y \). Since \( \mathcal{M}[G] \) is \( \Pi_{n+1}^1 \) correct, there is a \( z \in \mathcal{M}[G] \) such that \( A(z) \). Since \( z \le_T y \), we are done. □

The smallest mouse as in Lemma 4.10 is just \( M_n^# \). Since \( M_n^# \) is projectible to \( \omega \), we can identify it with a real in a canonical way.

**Corollary 4.11** (Woodin). Every nonempty \( \Sigma_{n+2}^1 \) set of reals has a member recursive in \( M_n^# \).

Observe also that \( M_n^# \) is a \( \Pi_{n+2}^1 \) singleton, since it is determined by some conditions on its first order theory and the fact that it is \( \Pi_{n+1}^{HC} \)-iterable.

One can show that if \( n \) is odd, then \( M_n^# \) has the same \( \Delta_{n+2}^1 \) degree as \( y_{n+2} \), the least nontrivial \( \Pi_{n+2}^1 \) singleton.

Recall that for \( A \preceq \omega \), \( A \) is \( \Pi_1^1 \) iff \( A \) is \( \Sigma_1 \) over \( J_\delta \), where \( \delta = \delta_1^1 = \text{least admissible} \), and \( A \) is \( \Sigma_2^1 \) iff \( A \) is \( \Sigma_1 \) over \( J_\sigma \), where \( \sigma = \delta_2^1 = \text{least stable} \). We can compute analogous "Spector companions" for \( \Pi_{2n+1}^1 \) and \( \Sigma_{2n+2}^1 \), where \( n \geq 1 \), using the results of this paper.
As usual, $\delta_n^1$ is the supremum of the ranks of the $\Delta_n^1$ well-orders of $\omega$. The next theorem is a slight extension of some work of Woodin.

**Theorem 4.12.** Let $n < \omega$, and suppose $M_n^I$ exists. Let $\delta = \delta_{n+2}^1$. Then for any $A \subseteq \omega$, $A$ is $\Delta_{n+2}^1$ iff $A \in \mathcal{g}_\delta^{M_n^I}$. Moreover,

(a) if $n$ is odd, then $A$ is $\Pi_{n+2}^1$ iff $A$ is $\Sigma_1$ over $\mathcal{g}_\delta^{M_n^I}$, and

(b) if $n$ is even, then $A$ is $\Sigma_{n+2}^1$ iff $A$ is $\Sigma_1$ over $\mathcal{g}_\delta^{M_n^I}$.

**Proof.** We first prove (b). Let $n$ be even. Let $\psi(v)$ be a $\Sigma_{n+1}$ formula defining $A$ over $HC$. We claim that for $k \in \omega$,

$$k \in A \iff \exists x (\mathcal{g}_\delta^{M_n^I} \vdash \text{There are } n \text{ Woodin cardinals} \land \psi^{HC}[k]).$$

For if $k \in A$, then $M_n \models \psi^{HC}[k]$ by Lemma 4.6, so we get $\alpha$ as on the right-hand side. But if $\alpha$ is as on the right, then $\mathcal{g}_\delta^{M_n^I}$ is $\Sigma_1$ correct, so $\psi^{HC}(k)$ is true and $k \in A$. The $\Sigma_1$ correctness of $\mathcal{g}_\delta^{M_n^I}$ comes from Lemma 4.10: if $x \in ^\omega \omega \cap \mathcal{g}_\delta^{M_n^I}$, then there is a $y \in ^\omega \omega \cap \mathcal{g}_\delta^{M_n^I}$ which codes an active initial segment of $\mathcal{g}_\delta^{M_n^I}$ to which $x$ belongs and which satisfies "there are $n-1$ Woodin cardinals". Thus, by Lemma 4.10, $\mathcal{g}_\delta^{M_n^I}$ is correct for $\Sigma_{n-1+2}$ formulae, and hence for $\Sigma_n$ formulae. (One can also use Corollary 4.3 at this point.)

We claim that

$$k \in A \iff \exists x < \delta_{n+2}^1 (\mathcal{g}_\delta^{M_n^I} \vdash \text{There are } n \text{ Woodin cardinals} \land \psi^{HC}[k]).$$

For let $k \in A$. From the above we see that there is a countable, $n$-small, $\Pi_n$-iterable $\mathcal{P}$ which satisfies ""There are $n$ Woodin cardinals and $\psi^{HC}(k)$"". The set of reals coding such a $\mathcal{P}$ is $\Pi_n^{HC}$, or $\Pi_{n+1}^1$, and so has a $\Delta_{n+2}^1$ member. Thus, there is such a $\mathcal{P}$ with $OR^\mathcal{P} < \delta_{n+2}^1$. By taking $\mathcal{P}$ with $OR^\mathcal{P}$ minimal, we can arrange that $\mathcal{P}$ is an $\omega$-mouse. It follows then from Lemma 3.3 that $\mathcal{P} = \mathcal{g}_\delta^{M_n^I}$ for some $\alpha < \delta_{n+2}^1$.

This shows one direction of (b). For the other, let $A \subseteq \omega$ be definable over $\mathcal{g}_\delta^{M_n^I}$ by the $\Sigma_1$ formula $\varphi(v)$. The argument above shows $\mathcal{g}_\delta^{M_n^I} <_1 (M_n^I, \in, E^{M_n^I})$, so $A$ is definable over $(M_n^I, \in, E^{M_n^I})$ by $\varphi$. But then

$$k \in A \iff \exists \mathcal{P} (\mathcal{P} \text{ is } n \text{-small} \land \mathcal{P} \text{ is } \Pi_n \text{-iterable} \land \mathcal{P} \models \varphi[k]),$$

so $A$ is $\Sigma_{n+2}^1$.

We now prove (a). So let $n$ be odd, and let $A \subseteq \omega$ be $\Pi_{n+2}^1$. Let $k \in A$ iff $\forall x \in ^\omega \omega B(k, x)$, where $B$ is $\Sigma_{n+1}^1$, and let $\psi$ be a $\Sigma_n$ formula defining $B$ over $HC$.

For $\kappa$ Woodin, let $\mathcal{A}_\kappa$ be Woodin's every-real-generic poset, and for $G$ a $\mathcal{A}_\kappa$ generic object, let $x_G$ be the associated real. As in the proof of Theorem 4.8, we have

$$k \in A \iff \exists \kappa (\mathcal{g}_\delta^{M_n^I} \vdash \text{There are } n \text{ Woodin cardinals, and if }\kappa \text{ is the smallest, then } 0 \models \mathcal{g}_\delta^{M_n^I} \vdash \psi(k, x_G)).$$

(Note here that by Lemma 4.6, if $\mathcal{g}_\delta^{M_n^I} \vdash "\kappa \text{ is the smallest of } n \text{ Woodin cardinals}"$, then all generic extensions of $\mathcal{g}_\delta^{M_n^I}$ via $\mathcal{A}_\kappa$ are $\Sigma_1$ correct.)
We claim that

\[ k \in A \iff \exists \alpha < \delta_{n+2}^1 (\mathcal{F}^*_\alpha \models \text{There are } n \text{ Woodin cardinals, and if } \kappa \text{ is the smallest, then } \emptyset \models \exists \psi(k, x_G)). \]

For let \( k \in A \). Say \( \alpha \) is good if \( \mathcal{F}^*_\alpha \) is an \( \omega \)-mouse which satisfies the condition on the right-hand side. So we know there are good \( \alpha \), and we want a good \( \alpha < \delta_{n+2}^1 \). But for any \( \alpha \),

\[
\alpha \text{ is good } \iff \exists x \in \mathbb{R} \cap M_n (x \text{ codes an } \omega \text{-mouse } \mathcal{P} \text{ such that } OR^{\mathcal{P}} = \omega \alpha, \\
\mathcal{P} \text{ is } \Pi_n \text{-iterable, and } \mathcal{P} \text{ satisfies there are } n \text{ Woodin cardinals, and if } \kappa \text{ is the smallest, then } \emptyset \models \exists \psi(k, x_G)).
\]

(For the \( \iff \) direction, note that for any \( \Pi_n \)-iterable \( \omega \)-mouse \( \mathcal{P} \), either \( \mathcal{P} \equiv M_n \), or \( \mathcal{F}^*_\beta \leq \mathcal{P} \) for \( \beta = \omega^1 \mathcal{M} \). This is implicit in the proof of Lemma 3.1.) The right-hand side of the equivalence above is \( \Pi^1_{n+2} \) in the codes; note here that the quantification \( \exists x \in \mathbb{R} \cap M_n \) is bounded in a way that preserves \( \Pi^1_{n+2} \). So the set of reals coding good \( \alpha \) is \( \Pi^1_{n+2} \). By Kechris' lemma (Lemma A), the set of \( \beta \) such that there is no good \( \alpha < \beta \), being \( \Sigma^1_{n+2} \) and bounded, has a bound \( < \delta_{n+2}^1 \). That is, there is a good \( \alpha < \delta_{n+2}^1 \), as desired.

We have the other direction of (a), as and the proof that the \( \mathcal{A}^1_{n+2} \) reals are just the reals in \( \mathcal{F}^*_\alpha \), to the reader. \( \square \)

The proof of Theorem 4.12 also shows that for \( n \) even, \( \delta_{n+2}^1 \) is the least "\( M_n \) stable" ordinal, that is, the least \( \delta \) such that \( \mathcal{F}^*_\delta \prec_1 (M_n, \in, E^M) \).

Theorem 4.12 is a refinement of Woodin's proof that \( \Pi^1_{2n+1} \) and \( \Sigma^1_{2n+2} \) have the prewell-ordering property, in the case that the space is \( \omega \). Woodin has proved that \( \Pi^1_{2n+1} \) (resp., \( \Sigma^1_{2n+2} \)) subsets of \( \omega \omega \) admit \( \Pi^1_{2n+1} \) (resp., \( \Sigma^1_{2n+2} \)) norms. It is not known how to obtain the scale property for \( \Pi^1_{2n+1} \) and \( \Sigma^1_{2n+2} \) by the methods of inner model theory.

References