An efficient algorithm to find optimal double loop networks

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Abstract

The problem of finding optimal diameter double loop networks with a fixed number of vertices has been widely studied. In this work, we give an algorithmic solution of the problem by using a geometrical approach.

Given a fixed number of vertices \( n \), the general problem is to find "steps" \( s_1, s_2 \in \mathbb{Z} \), such that the digraph \( G(n; s_1, s_2) \) with set of vertices \( V = \mathbb{Z} \) and adjacencies given by \( i \to i + s_1 \) (mod \( n \)) and \( i \to i + s_2 \) (mod \( n \)) has minimum diameter \( d(n) \). A lower bound of this diameter is known to be \( lb(n) = \lceil \sqrt{3n} \rceil - 2 \). So, given \( n \), the algorithm has as outputs \( s_1, s_2 \) and the minimum integer \( \kappa = \kappa(n) \) such that

\[ d(n; s_1, s_2) = d(n) = lb(n) + \kappa. \]

The running time complexity of the algorithm is \( O(\kappa^3 \log n) \). \( O(\kappa) \) is unknown but it is upper-bounded by \( O(\sqrt{n}) \).

Moreover, in most of the cases the algorithm also gives (as a by-product) an infinite family of digraphs with increasing order and diameter as above, to which the obtained digraph \( G(n; s_1, s_2) \) belongs.

1. Introduction

Double loop networks have been widely studied in the last years because of their relevance to the design of some interconnection or communication computer networks. For a survey about these networks see, for instance, the papers of Bermond et al. [2] or Hwang [8]. The digraphs that model such networks are usually called double fixed step or circulant digraphs. A double fixed-step digraph with \( n \) vertices,
denoted by \( G(n; s_1, s_2) \), is a digraph with set of vertices \( V = \mathbb{Z}_n \), the set of integers modulo \( n \), and adjacencies given by the rules

\[
\begin{align*}
  i &\rightarrow i + s_1 \pmod{n}, \\
  i &\rightarrow i + s_2 \pmod{n},
\end{align*}
\]

with \( i, s_1, s_2 \in V \). The integers \( s_1 \) and \( s_2 \) are called the steps of the digraph. The condition \( \gcd(n, s_1, s_2) = 1 \) must hold for the digraph to be strongly connected.

One of the parameters most studied in these digraphs is the diameter, because its minimization corresponds to the problem to minimize the message transmission delay of the corresponding networks. Other interesting parameters, not considered in this paper, are the average distance, the connectivity, the existence of simple routings algorithms, etc. This paper focuses on the minimization of the diameter, a problem that has been studied by many authors since the first paper of Wong and Coppersmith [12]. See Cheng and Hwang [3], Erdős and Hsu [4], Esquè et al. [5], Fiol et al. [7], Hwang and Xu [9], etc.

More precisely, we propose a new algorithm to find the minimum diameter for (any) given number of vertices. We think that the algorithm has a double interest: Its low running time complexity, and the fact that it also gives infinite families of digraphs with minimum or quasi-minimum diameters.

2. Geometrical study

In this section we give the basic ideas and results on which most of the studies of double fixed-step digraphs (and in particular the algorithm of Section 4) are based. For further details and proofs we refer the reader to [5–7, 11, 12].

We are mainly interested in showing, through an example, that each double fixed-step digraph has an associated L-shaped planar region and conversely. This planar region makes easier the study of the distance-related properties (e.g. diameter) of its corresponding digraph.

2.1. From a digraph to an L-shaped tile

Consider a (strongly connected) digraph \( G(n; s_1, s_2) \) and take the squared integral plane. Fix a zero onto a unit square and, from it, add \( s_1 \pmod{n} \) when we move horizontally to the next square and \( s_2 \pmod{n} \) when we move vertically. Then, the plane is covered with the elements of \( \mathbb{Z}_n \) as shown in Fig. 1 with the example \( G(8; 1, 3) \).

Now, choose a zero (in a cycle in Fig. 1), mark it and mark also all the numbers from 1 to \( n - 1 \) which are at the minimum possible distance from it in the digraph. This can be done by using a simple algorithm which considers the successive diagonals as shown in Fig. 1. Then the marked squares form an L-shaped tile which periodically tessellates the plane. The tile obtained in our example is also shown in the figure.
A generic L-shaped tile is characterized by its dimensions $(l, h, w, y)$, $l, h \geq 1$, $0 \leq w \leq l$, $1 \leq y \leq h$, see Fig. 2. Cheng and Hwang [3] proposed an $O(\log n)$ algorithm to compute $(l, h, w, y)$ for a given $G(n; s_1, s_2)$.

**2.2. From an L-shaped tile to a digraph**

From an L-shaped tile with dimensions $(l, h, w, y)$, $\gcd(l, h, w, y) = 1$, and area $n = lh - wy$, we can obtain the steps of the corresponding digraph with $n$ vertices as follows (see [5] for a proof).

Let us consider the integral matrix

$$M = \begin{pmatrix} l & -w \\ -y & h \end{pmatrix}$$
and compute the Smith normal form of \( M \), \( S(M) = \text{diag}(1,n) \). Then we have \( S(M) = LMR \), where \( L, R \) are two nonsingular unimodular integral matrices. If we denote \( L \) by

\[
L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},
\]

then the steps of the corresponding digraph are given by \( s_1 = \gamma \text{ (mod } n\text{)} \) and \( s_2 = \delta \text{ (mod } n\text{)} \). In fact, this pair of possible steps is not unique since the resulting matrix \( L \) in the factorization of \( S \) is not canonical. However, all the possible pairs are given by \( s_1 = \lambda s_1 \text{ (mod } n\text{)} \) and \( s_2 = \lambda s_2 \text{ (mod } n\text{)} \) with \( \lambda \in \mathbb{Z}^*_n \), see [7].

We can apply this method to the previous example. Take the L-shaped tile given in Fig. 1, which has dimensions \( l = h = 3, w = y = 1 \), then

\[
M = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}
\]

and we can compute \( S(M), L, R: \)

\[
S(M) = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},
\]

so the steps are \( s_1 = 1, s_2 = 3 \) or, more generally, \( s_1 = \lambda s_1 \text{ (mod } n\text{)} \) and \( s_2 = \lambda s_2 \text{ (mod } n\text{)} \) with \( \lambda \in \mathbb{Z}^*_n \). An alternative method of computing \( s_1 \) and \( s_2 \) from the dimensions \( (l, h, w, y) \) can be found in [7].

2.3. Optimal tiles vs. optimal digraphs

Let \( d(n; s_1, s_2) \) denote the diameter of the digraph \( G(n; s_1, s_2) \). Since this digraph is clearly vertex symmetric, we can find its diameter by computing the maximum distance from one fixed vertex, say 0, to the others. Then, in terms of the dimensions of its corresponding tile \( L: (l, h, w, u) \), we have

\[
d(n; s_1, s_2) = d_L = \max \{ l + h - w, l + h - y \} - 2,
\]

since, as shown in Fig. 2, the square placed in one of the two marked corners corresponds to a farthest vertex from vertex 0. Besides, by symmetry, we can always assume that \( w \leq y \), so that the diameter is

\[
d(n; s_1, s_2) = d_L = l + h - w - 2.
\]

Let us denote by \( d(n) \) the best-possible diameter of a double fixed-step digraph with \( n \) vertices, i.e., \( d(n) = \min_{s_1, s_2 \in \mathbb{Z}_n} d(n; s_1, s_2) \).

The exact value of \( d(n) \) for general \( n \) is unknown. Using the above geometrical approach, Wong and Coppersmith [12] showed that

\[
d(n) \geq \text{lb}(n) = \lceil \sqrt{3n} \rceil - 2.
\]
Let $\kappa = \kappa(n) = d(n) - lb(n)$. From (1), Cheng and Hwang's algorithm [3] to compute $(l, h, w, y)$ gives an algorithm to find $d(n)$ for any $n$, which has order at most $O(n^2 \log n)$. Besides, Hwang and Xu [9] managed to prove, using a heuristic method, that $d(n) \leq \sqrt{3n + 2(3n)^{1/4}} + 5$, the best-known result for general $n$.

From (3) and the above comments, it seems to be convenient to introduce the following terminology.

**Definition 1.** The double loop digraph $G(n; s_1, s_2)$ (resp. the tile $L: (l, h, w, y)$ with area $n = lh - wy$) will be called

- **tight** if $d(n; s_1, s_2) = lb(n)$ (resp. $d_L = lb(n)$);
- **$k$-tight** if $d(n; s_1, s_2) = lb(n) + k$ (resp. $d_L = lb(n) + k$);
- **optimal** if $d(n; s_1, s_2) = d(n) = lb(n) + \kappa_c$ (resp. $d_L = d(n)$).

3. **Construction and classification of tiles. Procreation**

In this section we study how to construct an L-shaped tile with given area and diameter. This study gives rise to a complete classification of the tiles into different families, which basically differ in their asymptotic behaviour. The consideration of such (infinite) families is interesting because it allows us to obtain, in most of the cases, infinite families of optimal fixed-step digraphs.

The obtained classification tables, on which our algorithm is based, give the diophantine equations and conditions for a searched tile (family) to be possible.

Let $L: (l, h, w, y)$ be a $k$-tight L-shaped tile with area $n = lh - wy$. Then, $d_L = l + h - w - 2 = lb(n) + k$ and so

$$w = l + h - \alpha,$$

where $\alpha = lb(n) + k = \lceil \sqrt{3n} \rceil + k$. Moreover, because of the assumption $w \leq y$, we can write

$$y = w + \beta = l + h - \alpha + \beta$$

for some integer $\beta \geq 0$. Then, using (4) and (5) we must have

$$n = lh - (l + h - \alpha)(l + h - \alpha + \beta).$$

For given values of $n, \alpha$ and $\beta$, Eq. (6) represents an ellipse on the plane $lh$. All the results reported in this section can be derived from this expression. In particular, the algorithm of Section 4 basically works by looking for the integral solutions of (6) for the possible values of $\alpha$ (i.e. $k$) and $\beta$. In the study of such values we discover that the value of $\beta = y - w$ is crucial in order to classify the different tiles. In particular, we come across the concept of procreation, defined below, which allows us to obtain infinite families of $k$-tight digraphs. First, let us show a result about the order of $\beta$, which is used later to compute the running time of the algorithm.
Lemma 1. For given positive integers \(n\) and \(k \leq \kappa(n)\), let \(L : (l, h, w, y)\), \(\beta = y - w \geq 0\), be any \(k\)-tight \(L\)-shaped tile with area \(n = lh - wy\). Then, for large \(n\), the order of \(\beta\) is upper-bounded by \(O(k)\).

Proof. By the above-mentioned results of Hwang and Xu [9], i.e. \(O(k) \leq O(\kappa) \leq \sqrt[4]{n}\), we can assume \(w \neq 0\). Indeed, if \(w = 0\) then \(n = lh\) and \(d_L \geq 2\sqrt{n} - 2\), but if \(L\) is \(k\)-tight, \(d_L = \sqrt[3]{3n} + k + u_n - 2\) with \(0 \leq u_n < 1\), and so \(O(k) \geq \sqrt{n}\), a contradiction.

If \(w \neq 0\), we can isolate \(\beta\) from (6)

\[
\beta = \frac{lh - n}{l + h - x} - l - h + x.
\]

By fixing \(n\) and \(x\) as parameters, we can consider \(\beta = \beta(l, h)\) as a function of the variables \(l\) and \(h\). This function has a maximum, \(\beta_{\text{max}}\), at

\[
l = h = \frac{3x + \sqrt{12n - 3x^2}}{6},
\]

with

\[
\beta_{\text{max}} = \frac{x - \sqrt{12n - 3x^2}}{2}.
\]

But, since \(x\) grows as \(\sqrt[3]{n} + k\), \(\beta_{\text{max}}\) grows as

\[
\beta_{\text{max}} \approx \frac{\sqrt[3]{3n} + k - \sqrt[3]{3n} - 6k \sqrt[3]{3n} - 3k^2}{2} = \frac{8k + 4 \frac{k^2}{\sqrt[3]{3n}}}{2 \left(1 + \frac{k}{\sqrt[3]{3n}} + \sqrt{1 - \frac{6k}{\sqrt[3]{3n}} - \frac{k^2}{n}}\right)}.
\]

where the right-hand side is obtained by multiplying both terms in the quotient of the left-hand side by the (nonzero) conjugate expression \(\sqrt[3]{3n} + k + \sqrt[3]{3n} - 6k \sqrt[3]{3n} - 3k^2\), and dividing both resulting terms by \(\sqrt[3]{3n}\). Considering that \(O(k) \leq \sqrt[4]{n}\), the above right-hand side expression gives

\[
\beta_{\text{max}} \approx \frac{8k + 4}{\sqrt[3]{3}} \frac{1}{2 \left(1 + \frac{1}{\sqrt[3]{9n}} + \sqrt{1 - \frac{6}{\sqrt[3]{9n}} - \frac{1}{n}}\right)}.
\]

Hence, for large \(n\), \(\beta_{\text{max}} \approx 2k\) and the result is proved. \(\square\)
To carry out our study, note that the expression of $lb(n)$ suggests to parametrize $N$ as

$$N = \bigcup_{x=0}^{x} \left[ 3x^2 + 1, 3(x+1)^2 \right]$$

since, for any $n \in I(x) = \left[ 3x^2 + 1, 3(x+1)^2 \right]$ we have

- $n \in I_1(x) = \left[ 3x^2 + 1, 3x^2 + 2x \right] \iff lb(n) = 3x - 1$;
- $n \in I_2(x) = \left[ 3x^2 + 2x + 1, 3x^2 + 4x + 1 \right] \iff lb(n) = 3x$;
- $n \in I_3(x) = \left[ 3x^2 + 4x + 2, 3x^2 + 6x + 3 \right] \iff lb(n) = 3x + 1$.

Then, for a generic $k$-tight tile $L(x)$ with dimensions $l = 2x + a$, $h = 2x + b$, and area $n = 3x^2 + Ax + B \in I_i(x)$, $1 \leq i \leq 3$, we have that $lb(n) = 3x + i - 2$, $x = 3x + k + i$ (and hence $w = x + a + b - k - i$, $y = x + a + b - k - i + \beta$). Thus, entering such values of $n, l, h$ and $x$ into (6), we get

- $n \in I_1(x): Ax + B = (2 + 2k - \beta)x + ab - (a + b - 1 - k)$
  $$\times (a + b - 1 - k + \beta); \quad \text{(7.1a)}$$
  $$1 \leq Ax + B \leq 2x. \quad \text{(7.1b)}$$
- $n \in I_2(x): Ax + B = (4 + 2k - \beta)x + ab - (a + b - 2 - k)$
  $$\times (a + b - 2 - k + \beta); \quad \text{(7.2a)}$$
  $$2x + 1 \leq Ax + B \leq 4x + 1. \quad \text{(7.2b)}$$
- $n \in I_3(x): Ax + B = (6 + 2k - \beta)x + ab - (a + b - 3 - k)$
  $$\times (a + b - 3 - k + \beta); \quad \text{(7.3a)}$$
  $$4x + 2 \leq Ax + B \leq 6x + 3. \quad \text{(7.3b)}$$

Hence, if the above equalities and inequalities hold for any $x \geq 0$ we must have

- $A = 2i + 2k - \beta$. \quad \text{(8a)}$
- $B = ab - (a + b - i - k)(a + b - i - k + \beta)$. \quad \text{(8b)}$

and

$$2i - 2 \leq A \leq 2i, \quad \text{(9)}$$

respectively.

At this point, it is useful to introduce the following definition
Definition 2. Let $L(0)$ be a $k$-tight L-shaped tile with dimensions $(l, h, w, y)$, $w \leq y$. We say that $L(0)$ procreates if all the L-shaped tiles

$$L(x) \rightarrow (l + 2x, h + 2x, w + x, y + x), \quad x = 1, 2, \ldots,$$

are also $k$-tight, i.e. they have diameter $d_{L(x)} = \lfloor b(n(x)) \rfloor + k$, where $n(x) = (l + 2x)(h + 2x) - (w + x)(y + x)$ is the area of $L(x)$.

The geometrical idea of this concept is depicted in Fig. 3.

Theorem 1. Let $L(0): (l, h, w, y) = (a, b, c, d)$, $\beta = d - c \geq 0$, be a $k$-tight L-shaped tile. Then $L(0)$ procreates iff

$$2k \leq \beta \leq 2(k + 1). \quad (10)$$

Proof. Let $n \in I_i(x), 1 \leq i \leq 3$, be the area of $L(0)$ and let $n(x) = 3x^2 + Ax + B$ be the area of $L(x): (a + 2x, b + 2x, c + x, d + x)$. From the above results, $L(0)$ procreates iff $A x + B$ satisfy (7.ia) and (7.ib) for any $x \geq 0$. Hence, from (9) and (8a), we have

$$2i - 2 \leq 2i + 2k - \beta \leq 2i,$$

which yields (10). $\Box$

The $k$-tight tiles that do not satisfy (10) will be called exceptions. There are two kinds of exceptions: $E_1(k)$ for $k \geq 0$ and $E_2(k)$ for $k \geq 1$.

The general situation is shown in Fig. 4. So we must study essentially three different cases:

- $E_1(k)$ exceptions: $\beta > 2(k + 1)$, i.e. $0 \leq w \leq y - 2k - 3$;
- Procreating tiles: $2k \leq \beta \leq 2(k + 1)$, i.e. $y - 2k - 2 \leq w \leq y - 2k$;

![Fig. 3. A generic procreation.](image-url)
**Table 1**

<table>
<thead>
<tr>
<th>n(x)</th>
<th>d_l</th>
<th>w, y</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x^2 - tx + FB_1_1$</td>
<td>$3x + k - 1$</td>
<td>$x + a + b - k - 1$</td>
</tr>
<tr>
<td>$FB_1 = ab - (a + b - k - 1)(a + b + k + t + 1)$</td>
<td>$3x + k - 1$</td>
<td>$x + a + b + k + t + 1$</td>
</tr>
<tr>
<td>$3x^2 + (2 - t)x + FB_2_1$</td>
<td>$3x + k - 1$</td>
<td>$x + a + b - k - 2$</td>
</tr>
<tr>
<td>$FB_2 = ab - (a + b - k - 2)(a + b + k + t)$</td>
<td>$3x + k$</td>
<td>$x + a + b + k + 1$</td>
</tr>
<tr>
<td>$3x^2 + (4 - t)x + FB_3_1$</td>
<td>$3x + k + 1$</td>
<td>$x + a + b + k + t - 1$</td>
</tr>
<tr>
<td>$FB_3 = ab - (a + b - k - 1)(a + b + k + t - 1)$</td>
<td>$3x + k + 1$</td>
<td>$x + a + b + k + t - 1$</td>
</tr>
<tr>
<td>$x = 2k + 2$ (procreating case)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- $E_2(k)$ exceptions: $\beta < 2k (k \geq 1)$, i.e. $y - 2k + 1 \leq w \leq y$.

From now on, the dimensions $l$ and $h$ of an L-shaped tile belonging to one of the Tables 1–5 will be given by using the parameter $x$ as follows: $l = 2x + a$, $h = 2x + b$.

**Theorem 2.** Let $L: (l, h, w, y)$ be an L-shaped tile with area $n = lh - wy$. Then,

(i) If $L \in E_1(k)$ then the dimensions of $L$ are covered by Table 1 with $\beta = 2(k + 1) + t$, $t \geq 1$.

(ii) If $L$ is a $k$-tight procreating L-shaped tile, then it belongs to one of the Tables 2–4.

(iii) If $L \in E_2(k)$, then it belongs to Table 5 with $\beta = 2k - s$, $s \geq 1$.

**Proof.** Tables 1–5 arise from the remarks in Theorem 1 and (7.ia), (7.ib), $i = 1, 2, 3$. We will only prove one case, the others being similar.

Let us show, for instance, the first row of Table 2, which corresponds to $n \in I_1(x) = [3x^2 + 1, 3x^2 + 2x] \ (lb(n) = 3x - 1)$ and $\beta = 2k + 2$ (procreating case). Then our tile has dimensions $l = 2x + a$, $h = 2x + b$, $w = x + a + b - k - 1$ and $y = x + b$. 

![Diagram of L-tiles and conditions](image-url)
Table 2
Procreating tiles in $I_1(x)$

<table>
<thead>
<tr>
<th>$n(x)$</th>
<th>$w$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x^2 + B_{1,1}$</td>
<td>$x + a + b - k - 1$</td>
<td>$x + a + b + k + 1$</td>
</tr>
<tr>
<td>$B_{1,1} = ab - (a + b - k - 1)(a + b + k + 1)$</td>
<td>$x + a + b - k - 1$</td>
<td>$x + a + b + k + 1$</td>
</tr>
<tr>
<td>$B_{1,1} \geq 1, x \geq \lceil B_{1,1}/2 \rceil$</td>
<td>$x + a + b - k - 1$</td>
<td>$x + a + b + k$</td>
</tr>
<tr>
<td>$3x^2 + x + B_{1,2}$</td>
<td>$x + a + b - k - 1$</td>
<td>$x + a + b + k$</td>
</tr>
<tr>
<td>$B_{1,2} = ab - (a + b - k - 1)(a + b + k)$</td>
<td>$x + a + b - k - 1$</td>
<td>$x + a + b + k$</td>
</tr>
<tr>
<td>$x \geq \max{1 - B_{1,2}, B_{1,2}}$</td>
<td>$x + a + b - k - 1$</td>
<td>$x + a + b + k$</td>
</tr>
<tr>
<td>$3x^2 + 2x + B_{1,3}$</td>
<td>$x + a + b - k - 1$</td>
<td>$x + a + b + k + 1$</td>
</tr>
<tr>
<td>$B_{1,3} = ab - (a + b - k - 1)(a + b + k - 1)$</td>
<td>$x + a + b - k - 1$</td>
<td>$x + a + b + k + 1$</td>
</tr>
<tr>
<td>$B_{1,3} \leq 0, x \geq \lceil (1 - B_{1,3})/2 \rceil$</td>
<td>$x + a + b - k - 1$</td>
<td>$x + a + b + k + 1$</td>
</tr>
</tbody>
</table>

Table 3
Procreating tiles in $I_2(x)$

<table>
<thead>
<tr>
<th>$n(x)$</th>
<th>$w$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x^2 + 2x + B_{2,1}$</td>
<td>$x + a + b - k - 2$</td>
<td>$x + a + b$</td>
</tr>
<tr>
<td>$B_{2,1} = ab - (a + b - k - 2)(a + b + k)$</td>
<td>$x + a + b - k - 2$</td>
<td>$x + a + b + k - 1$</td>
</tr>
<tr>
<td>$B_{2,1} \geq 1, x \geq \lceil B_{2,1}/2 \rceil$</td>
<td>$x + a + b - k - 2$</td>
<td>$x + a + b + k - 1$</td>
</tr>
<tr>
<td>$3x^2 + 3x + B_{2,2}$</td>
<td>$x + a + b - k - 2$</td>
<td>$x + a + b + k - 1$</td>
</tr>
<tr>
<td>$B_{2,2} = ab - (a + b - k - 2)(a + b + k - 1)$</td>
<td>$x + a + b - k - 2$</td>
<td>$x + a + b + k - 1$</td>
</tr>
<tr>
<td>$x \geq \max{1 - B_{2,2}, B_{2,2}}$</td>
<td>$x + a + b - k - 2$</td>
<td>$x + a + b + k - 1$</td>
</tr>
<tr>
<td>$3x^2 + 4x + B_{2,3}$</td>
<td>$x + a + b - k - 2$</td>
<td>$x + a + b + k - 2$</td>
</tr>
<tr>
<td>$B_{2,3} = ab - (a + b - k - 2)(a + b + k - 2)$</td>
<td>$x + a + b - k - 2$</td>
<td>$x + a + b + k - 2$</td>
</tr>
<tr>
<td>$B_{2,3} \leq 1, x \geq \lceil (1 - B_{2,3})/2 \rceil$</td>
<td>$x + a + b - k - 2$</td>
<td>$x + a + b + k - 2$</td>
</tr>
</tbody>
</table>

Table 4
Procreating tiles in $I_3(x)$

<table>
<thead>
<tr>
<th>$n(x)$</th>
<th>$w$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x^2 + 4x + B_{3,1}$</td>
<td>$x + a + b - k - 3$</td>
<td>$x + a + b + k - 1$</td>
</tr>
<tr>
<td>$B_{3,1} = ab - (a + b - k - 3)(a + b + k - 1)$</td>
<td>$x + a + b - k - 3$</td>
<td>$x + a + b + k - 1$</td>
</tr>
<tr>
<td>$B_{3,1} \geq 2, x \geq \lceil (B_{3,1} - 3)/2 \rceil$</td>
<td>$x + a + b - k - 3$</td>
<td>$x + a + b + k - 1$</td>
</tr>
<tr>
<td>$3x^2 + 5x + B_{3,2}$</td>
<td>$x + a + b - k - 3$</td>
<td>$x + a + b + k - 2$</td>
</tr>
<tr>
<td>$B_{3,2} = ab - (a + b - k - 3)(a + b + k - 2)$</td>
<td>$x + a + b - k - 3$</td>
<td>$x + a + b + k - 2$</td>
</tr>
<tr>
<td>$x \geq \max{2 - B_{3,2}, B_{3,2} - 3}$</td>
<td>$x + a + b - k - 3$</td>
<td>$x + a + b + k - 2$</td>
</tr>
<tr>
<td>$3x^2 + 6x + B_{3,3}$</td>
<td>$x + a + b - k - 3$</td>
<td>$x + a + b + k - 3$</td>
</tr>
<tr>
<td>$B_{3,3} = ab - (a + b - k - 3)(a + b + k - 3)$</td>
<td>$x + a + b - k - 3$</td>
<td>$x + a + b + k - 3$</td>
</tr>
<tr>
<td>$B_{3,3} \leq 3, x \geq \lceil (2B_{3,3})/2 \rceil$</td>
<td>$x + a + b - k - 3$</td>
<td>$x + a + b + k - 3$</td>
</tr>
</tbody>
</table>

$y = w + \beta = x + a + b + k + 1$. Moreover, with $n = 3x^2 + Ax + B$, (8.a) and (8.b) yield $A = 0$ and $B = B_{1,1} = ab - (a + b - 1 - k)(a + b + k + 1)$.

Hence, since $n \in I_1(x)$,

$$3x^2 + 1 \leq 3x^2 + B_{1,1} \leq 3x^2 + 2x,$$
Table 5
Tiles in $E_2(k)$

<table>
<thead>
<tr>
<th>$n(x)$</th>
<th>$d_L$</th>
<th>$w,y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3x^2 + (s + 2)x + EB_1$</td>
<td>$3x + k - 1$</td>
<td>$x + a + h - k - 1$</td>
</tr>
<tr>
<td>$EB_1 = ab - (a + b - k - 1)(a + b + k - s - 1)$</td>
<td>$[ (1 - EB_1)/(s + 2) ] \leq x \leq \lfloor EB_1/s \rfloor$</td>
<td>$x + a + h + k - s - 1$</td>
</tr>
<tr>
<td>$3x^2 + (s + 4)x + EB_2$</td>
<td>$3x + k$</td>
<td>$x + a + h - k - 2$</td>
</tr>
<tr>
<td>$EB_2 = ab - (a + b - k - 2)(a + b + k - s - 2)$</td>
<td>$[ (1 - EB_2)/(s + 2) ] \leq x \leq \lfloor (1 - EB_2)/s \rfloor$</td>
<td>$x + a + h + k - s - 2$</td>
</tr>
<tr>
<td>$3x^2 + (s + 6)x + EB_3$</td>
<td>$3x + k + 1$</td>
<td>$x + a + h - k - 3$</td>
</tr>
<tr>
<td>$EB_3 = ab - (a + b - k - 3)(a + b + k - s - 3)$</td>
<td>$[ (2 - EB_3)/(s + 2) ] \leq x \leq \lfloor (3 - EB_3)/s \rfloor$</td>
<td>$x + a + h + k - s - 3$</td>
</tr>
</tbody>
</table>

so that the next conditions must hold:

$$B_1 \geq 1, \ x \geq \left\lceil \frac{B_1}{2} \right\rceil.$$ 

4. Algorithm

We can now describe the algorithm based on the previous tables. Let us first consider a procreating example. Take $n = 37520$ and $k = 0$ (we want to know if there exists some tight L-shaped tile with area 37520). We know that $lb(37520) = 334$ and $334 = 1 \mod 3$, so the diameter is $3x + 1$ with $x = 111$ (we must look at Table 4) and $37520 = 3x^2 + 5x + 2$, $B_{3,2} = 2$. Now we must solve for integral values $a$ and $b$ the equation

$$ab - (a + b - 3)(a + b - 2) = 2,$$

which have three integral solutions (see Fig. 5). Take, for instance, $a = b = 2$.

The corresponding L-shaped tile is $L: (224, 224, 112, 113)$. Now, since gcd$(224, 112, 113) = 1$, we can find steps $s_1, s_2$ such that the digraph $G(37520; s_1, s_2)$ is tight, i.e., its diameter attains the value $lb(37520)$. Take the matrix given by the dimensions of $L$:

$$M = \begin{pmatrix} l & -w \\ -y & h \end{pmatrix} = \begin{pmatrix} 224 & -112 \\ -113 & 224 \end{pmatrix},$$

and its corresponding Smith normal form

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 37520 \end{pmatrix} = \begin{pmatrix} 56 & 111 \\ 113 & 224 \end{pmatrix} M \begin{pmatrix} 1 & -18592 \\ 0 & 1 \end{pmatrix},$$

so the steps are $s_1 = 113$ and $s_2 = 224$. 
2.5
2
1.5
1
0.5
0
1.5
2
2.5
3
Fig. 5. The ellipse $B_{3,2} = 2$.

We must remark that all the expressions of $B_{i,j}$, $i,j = 1,2,3$ are ellipsis. Hence, there will be always a finite number of solutions. The analogous expressions in Table 5 of the $E_2(k)$ exceptions, $EB_{i}$, $i = 1,2,3$, are also ellipsis in the variables $a$ and $b$ with $k$ and $s$ as parameters. The same comments apply for Table 1 of the $E_1(k)$ exceptions with $k$ and $t$ as parameters.

Moreover, we can take advantage of the procreating fact by finding an infinite family of tight digraphs based on the above example. We only need to consider generic dimensions depending on the parameter $x$:

\[
n(x) = 3x^2 + 5x + 2,
\]

\[
L: (2x + 2, 2x + 2, x + 1, x + 2),
\]

\[
d_L = d(n(x)),
\]

\[
x \geqslant \max\{2 - B_{3,2}, B_{3,2} - 3\} = 0.
\]

Since $\gcd(2x + 2, 2x + 2, x + 1, x + 2) = 1$, $x \geqslant 0$, we can find the corresponding steps. Consider the ring $R = \mathbb{Z}[x]/(3x^2 + 5x + 2)$ and proceed by analogy with $\mathbb{Z}_{37520}$. Consider the matrix over $R^{2 \times 2}$

\[
M = \begin{pmatrix} 2x + 2 & -x - 1 \\ -x - 2 & 2x + 2 \end{pmatrix}
\]

and find its corresponding Smith normal form over $R$:

\[
S = \begin{pmatrix} 1 & 0 \\ 0 & 3x^2 + 5x + 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3x + 4 & 3x + 3 \end{pmatrix}M \begin{pmatrix} -1 & x + 1 \\ 1 & x \end{pmatrix}.
\]

At this point, we can find the steps:

\[
s_1' = 3x + 4 \pmod{3x^2 + 5x + 2},
\]

\[
s_2' = 3x + 3 \pmod{3x^2 + 5x + 2}.
\]
If we particularize for \( x = 111 \) we get \( s'_1 = 337 \) and \( s'_2 = 336 \). Hence the possible pairs of steps are \((337\lambda, 336\lambda)\) with \( \lambda \in \mathbb{Z}^*_57520 \). The pair \((113, 224)\) obtained above corresponds to \( \lambda = 37\,409 \).

In general, the algorithm for searching a \( k \)-tight double loop digraph works as follows:

1. Given \( n \), find \( x, \ell_b(n) \) and \( i \in \{1,2,3\} \) such that \( n \in I_i(x) \). Assign \( k = 0 \).
2. For a fixed \( k \) and from \( \beta = 0 \) to \( \beta = \lfloor \beta_{\text{max}} \rfloor \), look for a \( k \)-tile \( L: (l, h, w, y) \) with \( y = w + \beta \) and \( \gcd(l, h, w, y) = 1 \). Computer explorations showed us that the following order of searching is more efficient than the natural one:
   - **Procreating case** (Tables 2–4): Use table \((i + 1)\) and the values of \( \beta \) are \( 2k, 2k + 1 \) and \( 2k + 2 \).
   - \( E_2(k) \) case (Table 5): Use row \( i \) of Table 5 and the variation of \( \beta \) is recommended to be from \( 2k - 1 \) to \( 0 \). This variation is expressed by the parameter \( t \) in Table 5.
   - \( E_1(k) \) case (Table 1): Use row \( i \) of Table 1. The variation of \( \beta \) is recommended to be from \( 2k + 3 \) to \( \lfloor \beta_{\text{max}} \rfloor \). This variation is given by the parameter \( t \) in Table 1.
3. If an \( L \)-shaped tile has been found in step 2, then there exists a \( k \)-tight (optimal) digraph with \( n \) vertices and steps \( s_1, s_2 \) (computed with constant cost) such that

\[
d(n; s_1, s_2) = \ell_b(n) + k = d(n),
\]

and the algorithm ends.

Otherwise there is no such digraph with such a \( k \). Then, assign \( k \leftarrow k + 1 \) and go to step 2.

Step 2 of the algorithm is the most interesting one. Essentially we must search for integral points in the plane which are on the ellipses

\[
FB_i = ab - (a + b - k - i)(a + b + k + t - i + 2)
\]

for \( i = 1, 2, 3 \) and \( 1 \leq t \leq \lfloor \beta_{\text{max}} \rfloor - 2(k + 1) \) in the \( E_1(k) \) case;

\[
B_{i,j} = ab - (a + b - k - i)(a + b + k - i - j + 3)
\]

for \( i, j = 1, 2, 3 \) in the procreating case and

\[
EB_i = ab - (a + b - k - i)(a + b + k - i - s)
\]

for \( i = 1, 2, 3 \) and \( 1 \leq s \leq 2k \) in the \( E_2(k) \) case.

One integer pair \((a, b)\) will be valid if \( b \in \left[ b_i^-, \left[ b_i^+ \right]\right] \) in the cases \( E_1(k) \) or \( E_2(k) \). or \( b \in \left[ [b_i^-], \left[ b_i^+ \right]\right] \) in the procreating case and \( a = a_{i,j}^+ \) (or \( a = a_{i,j}^- \)) in the procreating case or \( a = a_{i}^+ \) (or \( a = a_{i}^- \)) in the \( E_1(k) \) or \( E_2(k) \) cases, where these values must be integral numbers. These values can be obtained as follows.

**\( E_1(k) \) case:**

\[
b_i^+ = \frac{2i - t - 2 \pm 2\sqrt{m_i}}{3},
\]

\[
m_i = (2i - t - 2)^2 + 3[(k + i)(k + t - i + 2) - FB_i] \geq 0,
\]
\[ a_i^\pm = \frac{2i - t - 2 - b \pm \sqrt{q_i}}{2}, \quad b \in [\lceil b_i^-, \lceil b_i^+ \rceil], \]

\[ q_i = -3b^2 + 2(2i - t - 2)b + (2i - t - 2)^2 
+ 4[(k + i)(k + t - i + 2) - FB_i] \geq 0. \]

Procreating case:

\[ b_{i,j}^\pm = \frac{2i + j - 3 \pm 2\sqrt{m_{i,j}}}{3}, \]

\[ m_{i,j} = (2i + j - 3)^2 + 3[(k + i)(k - i - j + 3) - B_{i,j}] \geq 0, \]

\[ a_{i,j}^\pm = \frac{2i + j - b - 3 \pm \sqrt{q_{i,j}}}{2}, \quad b \in [\lceil b_{i,j}^-, \lceil b_{i,j}^+ \rceil], \]

\[ q_{i,j} = -3b^2 + 2(2i + j - 3)b + (2i + j - 3)^2 
+ 4[(k + i)(k - i - j + 3) - B_{i,j}] \geq 0. \]

\[ E_2(k) \text{ case:} \]

\[ b_i^+ = \frac{s + 2i \pm 2\sqrt{m_i}}{3}, \]

\[ m_i = (s + 2i)^2 + 3[(k + i)(k - s - i) - EB_i] \geq 0, \]

\[ a_i^+ = \frac{s + 2i - b \pm \sqrt{q_i}}{2}, \quad b \in [\lceil b_i^-, \lceil b_i^+ \rceil], \]

\[ q_i = -3b^2 + 2(s + 2i)b + (s + 2i)^2 + 4[(k + i)(k - s - i) - EB_i] \geq 0. \]

We can now compute the order of the algorithm. Let us first assume that \( k \) is fixed:

- Integral solution of the ellipsis (find the parameters \( l, h, w, y \)). The computing cost for searching integral points on the above-mentioned ellipsis is \( O(b^+ - b^-) \) which, from the above formulas, is \( O(k) \). From Lemma 1, in the worst case we must repeat this search \( O(\beta_{\max}) = O(k) \) times. So the total cost of this step is \( O(k^2) \).

- Check that gcd \((l, h, w, y) = 1 \). The order of the Euclidean algorithm to compute such a gcd is \( O(\log n) \) [10].

- Compute the Smith normal form (find the steps \( s_1, s_2 \)). The simple algorithm described in [11] to compute the Smith normal form gives also the needed unimodular matrix \( L \). In our case (i.e., \( 2 \times 2 \) matrices) its cost is \( O(\log^2 n) \). However, note that this step is only executed when a solution has been found. Therefore, its cost has no relevance to the total order of the algorithm computed below.
As a conclusion, the proposed algorithm has order \(O(k^2)\) if \(k = \kappa\) is given and \(O(\kappa^3)\) otherwise. According to the above-mentioned results of [9], \(O(\kappa)\) is at most \(O(n^{1/4})\). Hence, the cost of our algorithm is upper-bounded by \(O(n^{3/4}\log n)\). However, the numerical computations and the results of Coppersmith reported in [4] suggest that the order of \(\kappa\) might be as low as \(O(\log^{1/4} n)\).

References