Planar Graphs of Maximum Degree Seven are Class I

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Received December 1, 1998; published online June 25, 2001

In 1965, Vizing proved that planar graphs of maximum degree at least eight have the edge chromatic number equal to their maximum degree. He conjectured the same if the maximum degree is either six or seven. This article proves the maximum degree seven case.

1. INTRODUCTION

Given a (simple) graph $G$, let $\Delta(G)$ denote the maximum (vertex) degree of $G$. If the graph is clear from the context, then $\Delta$ is frequently used. For instance, this article is about planar graphs with $\Delta = 7$. The other parameter important for this article is the edge chromatic number of $G$, denoted $\chi_e(G)$. In 1964, Vizing [5] showed that every graph either has edge chromatic number $\Delta$ (known as a Class I graph) or $\Delta + 1$ (a Class II graph).

For planar graphs, more is known. As noted by Vizing [6], if $C_4$, $K_4$, the octahedron, and the icosahedron have one edge subdivided each, Class II planar graphs are produced for $\Delta \in \{2, 3, 4, 5\}$. He also showed that if $\Delta \geq 8$, then a planar graph is always Class I. His Planar Graph Conjecture is that every planar graph with $\Delta \geq 6$ is Class I. This article proves this conjecture for the $\Delta = 7$ case. The $\Delta = 6$ case remains open.

Combining the result of this paper, the Four Color Theorem (e.g., [2]), and a trick of Yap (see [1]), gives new proofs of two results of the authors:
that every planar graph with \( \Delta = 7 \) has a vertex-edge (total) 9-coloring [3], as well as an edge-face 9-coloring [4].

2. STRUCTURE OF CRITICAL GRAPHS

Let a connected graph be \( \Delta \)-critical if it has maximum degree \( \Delta \), is Class II, and each of its subgraphs on one less edge (throwing away isolated vertices) is Class I. A well-known result is that every Class II graph of maximum degree \( \Delta \) has a \( \Delta \)-critical subgraph. Thus, it suffices to show that no 7-critical graph is planar.

This section contains some useful results on the structure of \( \Delta \)-critical graphs. Although this paper is chiefly concerned with 7-critical planar graphs, the authors hope that the lemmas of this section may also prove useful in other contexts. To this end, the graphs considered in this section are not necessarily planar.

All proofs in this section start by deleting an edge \( xy \) of a \( \Delta \)-critical graph \( G \), and obtaining an edge \( \Delta \)-coloring of \( C - xy \) by means of the definition. It is useful to discuss some properties of this coloring. Some notation is useful. First, given a vertex \( x \) and a color \( c \), if \( x \) is incident with an edge which is colored \( c \), then \( x \) is said to see \( c \). Next, given a vertex \( x \) which sees a color \( c \), let \( xc \) mean the edge incident with \( x \) colored \( c \).

Also, given two colors \( j \) and \( k \), the subgraph of \( G \) induced by the edges colored either \( j \) or \( k \), call it \( G(j,k) \), has maximum degree two, and is thus the disjoint union of paths and cycles. Let a component of \( G(j,k) \) be a Kempe \( (j,k) \)-chain. Given an edge \( x \) colored \( j \) and a color \( k \) distinct from \( j \), Kempeing \( x \) to \( k \) means reversing the colors on the Kempe \( (j,k) \)-chain \( C \) containing \( x \), so that edges of \( C \) previously colored \( j \) are recolored with \( k \), and edges of \( C \) previously colored \( k \) are recolored with \( j \).

The following is the key lemma when dealing with colorings of \( G - xy \):

**Lemma 2.1.** Given a \( \Delta \)-critical graph \( G \), an edge \( xy \) of \( G \), and an edge \( \Delta \)-coloring of \( G - xy \), if \( x \) does not see \( j \) and \( y \) does not see \( k \), then \( x \) sees \( k \), \( y \) sees \( j \), and the Kempe \( (j,k) \)-chain containing \( xK \) also contains \( yj \).

This is easy to see, for otherwise, an edge \( \Delta \)-coloring of \( G \) is easily obtained. As this is such a basic tool which will be used very frequently, it will be used without reference.

The next lemma, due to Vizing [6], is the only structural result which he needed to prove his planar graph theorem, that a planar graph with maximum degree at least eight is Class I. This lemma has been used in many places, and has thus received a name, Vizing’s Adjacency Lemma, which this article will abbreviate with VAL. It is convenient to refer to a
vertex by its degree. Thus, a $j$-vertex is a vertex of degree $j$, an $(\leq j)$-vertex is a vertex of degree at most $j$, and so forth.

**Lemma 2.2 (Vizing’s Adjacency Lemma).** If $G$ is a $\Delta$-critical graph and $xy$ is an edge of $G$, then $x$ is adjacent to at least $(\Delta - \deg(y) + 1)$ $\Delta$-vertices other than $y$.

VAL thus gives some information about the vertices which are distance one from a given vertex. It is useful to have some information about the vertices which are distance two from a given vertex. The following lemma helps in this regard.

**Lemma 2.3.** Let $x$ be a $j$-vertex of a $\Delta$-critical graph which is adjacent to a $k$-vertex $y$. If $j < \Delta$, $k < \Delta$, then $x$ is adjacent to at least $2\Delta - j - k + 1$ vertices other than $y$.

**Proof.** Suppose that $G$ is a $\Delta$-critical graph with such $x$ and $y$. Since $G$ is critical, $G - xy$ has an edge $\Delta$-coloring. Each color appears at either $x$ or $y$, or $G$ has an edge $\Delta$-coloring. Thus, without loss of generality, the edges incident with $x$ in $G - xy$ are colored $1, \ldots, j - 1$, while those incident with $y$ are colored $\Delta - k + 2, \ldots, \Delta$.

Consider a neighbor $z$ of $x$ such that $xz$ is colored $b \in \{1, \ldots, \Delta - k + 1\}$. First, $z$ sees each color in $j, \ldots, \Delta$, or else coloring $xz$ with a color from $j, \ldots, \Delta$, and then coloring $xy$ with $b$ gives an edge $\Delta$-coloring of $G$. Next, $z$ sees each color in $1, \ldots, \Delta - k + 1$, or else $zj$ may be Kemped to one of $1, \ldots, \Delta - k + 1$, such that when $xz$ is colored $j$, and $xy$ is colored $b$, it gives an edge $\Delta$-coloring of $G$. (Here is the first implicit use of Lemma 2.1, that Kemping $zj$ does not affect the colors on $x$ so that $xz$ may be colored $j$; further such uses will not be noted.)

Consider a neighbor $w \neq y$ of $z$ such that $wz$ is colored $c \in \{j, \ldots, \Delta\}$. First, $w$ sees $b$, or else Kemping $xz$ to $c$, and coloring $xy$ with $b$ gives an edge $\Delta$-coloring of $G$. Next, $w$ sees each color in $j, \ldots, \Delta$, or else $wb$ may be Kemped to a color in $j, \ldots, \Delta$, such that Kemping $xz$ to $c$, and coloring $xy$ with $b$ gives an edge $\Delta$-coloring of $G$. Also, $w$ sees each color in $1, \ldots, \Delta - k + 1$, or else $wi$, where $i \in \{j, \ldots, \Delta\} - \{c\}$, may be Kemped to a color in $1, \ldots, \Delta - k + 1$, such that Kemping $wb$ to $i$, Kemping $xz$ to $c$, and coloring $xy$ with $b$ gives an edge $\Delta$-coloring of $G$. Thus, $\deg(w) \geq 2\Delta - j - k + 2$.

Consider a neighbor $v$ of $z$, distinct from $x$ and $y$, such that $vz$ is colored $d \in \{1, \ldots, \Delta - k + 1\}$. The final argument shows that $\deg(v) \geq 2\Delta - j - k + 2$ as well. First, $v$ sees each color in $j, \ldots, \Delta$, or else $vz$ may be Kemped to a color in $j, \ldots, \Delta$ to give the case of the previous paragraph. Next, $v$ sees each
color in 1, ..., \(A-k+1\), or else \(v_j\) may be Kemped to a color in 1, ..., \(A-k+1\), such that Kemping \(v_z\) to \(j\) gives the case of the previous paragraph. It follows that \(\deg(v) \geq 2A-j-k+2\).

The remaining lemmas in this section deal with vertices which happen to be in triangles. This is useful when dealing with planar graphs. While the previous lemma is in some sense a natural analogue of VAL, the following lemmas were designed to handle specific situations which arise in the planar graph conjecture. Besides Kemping, described above, it is useful in the proof of the following lemma, to swap the colors of two edges \(x\) and \(\beta\), meaning to assign \(x\) the color that \(\beta\) had, to assign \(\beta\) the color that \(x\) had, and to leave the colors of all other edges unchanged. Of course, the general swapping of the colors of two edges of a properly colored graph may not yield a proper coloring of that graph.

**Lemma 2.4.** No \(A\)-critical graph has distinct vertices \(x, y, z\) such that \(x\) is adjacent to \(y\) and \(z\), \(\deg(z) < 2A - \deg(x) - \deg(y) + 2\), and \(xz\) is in at least \(\deg(x) + \deg(y) - A - 2\) triangles not containing \(y\).

**Proof.** Suppose that \(G\) is a \(A\)-critical graph with such \(x, y, z\). First we prove that \(\deg(x) + \deg(y) \geq A + 3\). Since \(G\) is critical, from \(\deg(z) < 2A - \deg(x) - \deg(y) + 2\), one can conclude that \(\deg(z) < A\). If \(\deg(x) + \deg(y) = A + 2\), by VAL, \(x\) would be adjacent to at least \(A - \deg(y) + 1 = \deg(x) - 1\) \(A\)-vertices other than \(y\). Since \(\deg(z) < A\), one can conclude that \(\deg(x) + \deg(y) \geq A + 3\).

Since \(G\) is critical, \(G-xy\) has an edge \(A\)-coloring. Thus, \(xy\) sees all \(A\) colors, or it may be colored to give an edge \(A\)-coloring of \(G\). Without loss of generality, then, \(xz\) is colored 1, \(x\) sees 2, ..., \(\deg(x)-1\), and \(y\) sees \(\deg(x), ..., A\).

Assume \(y\) does not see 1. Without loss of generality, \(y\) sees \(A - \deg(y) + 2, ..., \deg(x) - 1\). First, \(z\) sees \(\deg(x), ..., A\), or else coloring \(xz\) with one of \(\deg(x), ..., A\) and \(xy\) with 1 gives an edge \(A\)-coloring of \(G\). It follows that \(\deg(x) \geq A - \deg(x) + 2\). From the upper bound on \(\deg(z)\), it follows that \(2 \leq A - \deg(y) + 1\), hence there is a \(c \in \{2, ..., A - \deg(y) + 1\}\) such that \(z\) does not see \(c\). Thus, Kemping \(z\) to \(c\), then coloring \(xz\) with \(A\) and \(xy\) with 1 gives an edge \(A\)-coloring of \(G\).

Thus, \(y\) sees 1. Hence if \(\deg(x) + \deg(y) = A + 3\), then \(y\) sees 1, \(\deg(x), ..., A\) and if \(\deg(x) + \deg(y) > A + 3\), without loss of generality, in addition to 1, \(\deg(x), ..., A, y\) also sees \(A - \deg(y) + 3, ..., \deg(x) - 1\).

From the bound on the triangles containing \(xz\), there is a \(w \neq y\) adjacent to \(x\) and \(z\) such that \(wx\) is colored a color in 2, ..., \(A - \deg(y) + 2\). Without loss of generality, \(wx\) is colored 2.

Assume \(wz\) is colored one of \(\deg(x), ..., A\). Without loss of generality, \(wz\) is colored \(A\). First, \(z\) sees 2, or else Kemping \(wx\) to \(A\) and coloring \(xy\) with
2 gives an edge $A$-coloring of $G$. If $\deg(x) = A$, it is clear that $\deg(z) \geq 3 = A - A + 3 = A - \deg(x) + 3$. If $\deg(x) \neq A$, $z$ also sees $\deg(x), \ldots, A - 1$, or else $z$ may be Kempe to a color in $\deg(x), \ldots, A - 1$ so that Kempeing $wx$ to $A$ and coloring $xy$ with 2 gives an edge $A$-coloring of $G$. Thus $z$ sees 1, 2, $\deg(x)$, $\ldots, A$ and we have $\deg(z) \geq A - \deg(x) + 3$. Since $\deg(z) \geq A - \deg(x) + 3$ in either case, from the upper bound on $\deg(z)$, it follows that $4 \leq A - \deg(y) + 2$. Thus, by the bound on $\deg(z)$, there are two colors of 3, ..., $A - \deg(y) + 2$ not seen by $z$, without loss of generality, $z$ sees neither 3 nor 4. In this case, Kempeing $wz$ to 3, $z2$ to $A$ and then to 4, $wz$ back to $A$, $wx$ to $A$, and coloring $xy$ with 2 gives an edge $A$-coloring of $G$.

Assume $wz$ is colored one of 3, ..., $A - \deg(y) + 2$. Without loss of generality, $wz$ is colored 3. First, $z$ sees $\deg(x), \ldots, A$, or else $wz$ may be Kempe to one of $\deg(x), \ldots, A$, yielding the previous case. As before, $4 \leq A - \deg(y) + 2$. In this case, $zA$ may be Kempe to one of 4, ..., $A - \deg(y) + 2$ so that Kempeing $wz$ to $A$ yields the previous case.

Thus, $wz$ is colored one of $A - \deg(y) + 3$, ..., $\deg(x) - 1$, which leads to $(\deg(x) - 1) - (A - \deg(y) + 3) \geq 0$ and thus $\deg(x) + \deg(y) \geq A + 4$. By symmetry, for each triangle $uxz$ with $u \neq y$, if $ux$ is colored a color in 2, ..., $A - \deg(y) + 2$, then $uz$ is colored a color in $A - \deg(y) + 3$, ..., $\deg(x) - 1$. Partition the triangles containing $xz$ into $T_1$, and $T_2$, such that $T_1$ is the set of triangles $txz$, such that each of $tx$ and $tz$ is colored a color in $A - \deg(y) + 3$, ..., $\deg(x) - 1$. From the hypothesis, $|T_2| \geq \deg(x) + \deg(y) = A - 2 - |T_1|$. Let $S$ be the set of colors of such $tz$ described above (so that $|S| = |T_1|$), and let $R := \{A - \deg(y) + 3, \ldots, \deg(x) - 1\} \setminus S$. Then $|R| = \deg(x) + \deg(y) - A - 3 - |S|$. It follows then that $|T_2| > |R|$. Thus, there is a color $r \in R$ and two triangles $sxz$ and $exz$ in $T_2$, such that one of $sx$, $sz$ is colored $r$, and one of $ex$, $ez$ is colored $r$. As the coloring is proper, it may be assumed that $sz$ and $ex$ are colored $r$, and that $r = \deg(x) - 1$. From the definition of $T_2$, by relabeling if necessary, it may be further assumed that $s = w$. Finally, again from the definition of $T_2$, it may be assumed that $wz$ is colored one of 2, ..., $A - \deg(y) + 2$, $\deg(x), \ldots, A$.

Assume $rz$ is colored one of $\deg(x), \ldots, A$. Without loss of generality, $rz$ is colored $A$. First, $z$ sees 2, or else swapping the colors on $ex$ and $ez$, and swapping the colors on $wx$ and $wz$, and coloring $xy$ with 2 gives an edge $A$-coloring of $G$. Next, $r$ sees $\deg(x), \ldots, A$, or else $rz$ may be Kempe to a color in $\deg(x), \ldots, A$ so that swapping the colors on $ex$ and $ez$, and on $wx$ and $wz$, and coloring $xy$ with 2 gives an edge $A$-coloring of $G$. Again, $4 \leq A - \deg(y) + 2$. Thus, without loss of generality, $z$ sees neither 3 nor 4. In this case, however, Kempeing $rz$ to 3, Kempeing $z2$ to $A$ and then to 4, Kempeing $rz$ back to $A$, swapping the colors on $ex$ and $ez$, and on $wx$ and $wz$, and coloring $xy$ with 2 gives an edge $A$-coloring of $G$.

Assume $rz$ is colored one of 2, ..., $A - \deg(y) + 2$. First, $z$ sees $\deg(x), \ldots, A$, or else $rz$ may be Kempe to a color in $\deg(x), \ldots, A$ to give the previous
case. Thus, without loss of generality, \( z \) does not see 3. In this case, however, Kemping \( z \) to 3, and \( vz \) to \( A \) gives the previous case. 

**Lemma 2.5.** No \( A \)-critical graph has distinct vertices \( v, w, x, y, z \) such that \( w \) is a \((A - 2)\)-vertex, \( \deg(x) + \deg(y) \leq A + 3 \), \( \deg(x) \geq 5 \), \( \deg(y) \geq 5 \), and \( vwz \) and \( xyz \) are triangles.

**Proof.** Suppose that \( G \) is a critical graph with such vertices \( v, w, x, y, z \). Since \( G \) is critical, \( G - xy \) has an edge \( A \)-coloring. Thus, \( xy \) sees all \( A \) colors, or it may be colored to give an edge \( A \)-coloring of \( G \). Let \( k := \deg(x) \). Without loss of generality, then, \( xz \) is colored 1, \( x \) sees 2, \( \ldots, k - 1 \), \( yz \) is colored \( k \), and \( y \) sees \( k + 1, \ldots, A \).

Assume \( y \) does not see 1. Without loss of generality, \( y \) does not see 2, \( \ldots, k - 2 \).

Assume \( wz \) is colored one of 2, \( \ldots, k - 2, k + 1, \ldots, A \). Without loss of generality, by symmetry of \( x \) and \( y \), \( wz \) is colored \( k - 2 \). First, \( w \) sees \( k \), or Kemping \( yz \) to \( k - 2 \) and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \). Next, \( w \) sees 1, \( \ldots, k - 3 \), or else \( wk \) may be Kemped to one of 1, \( \ldots, k - 3 \) so that Kemping \( yz \) to \( k - 2 \), and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \). In this case, \( w1 \) may be Kemped to one of \( k + 1, \ldots, A \), so that Kemping \( wk \) to 1, \( yz \) to \( k - 2 \), and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \).

Thus, \( wz \) is colored \( k - 1 \). By the previous paragraph, when interchanging the roles of the colors \( k - 1 \) and \( k - 2 \), \( y \) sees \( k - 1 \). Without loss of generality by the symmetry of \( x \) and \( y \), \( vz \) is colored \( k - 2 \).

Assume \( vw \) is colored \( k \). Here, \( w \) sees \( k - 2 \), or else Kemping \( yz \) to \( k - 2 \), and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \). Next, \( w \) sees \( k + 1, \ldots, A \), or else \( wk \) may be Kemped to one of \( k + 1, \ldots, A \) so that Kemping \( yz \) to \( k - 2 \), and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \). In this case, \( w1 \) may be Kemped to one of \( 1, \ldots, k - 3 \) so that Kemping \( wk \) to \( A \), \( yz \) to \( k - 2 \), and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \).

Assume \( vw \) is colored \( c \in \{1, \ldots, k - 3\} \). Then, \( w \) sees \( k \), or else Kemping \( vw \) to \( k \) gives the previous case. Also, \( w \) sees 1, \( \ldots, k - 3 \), or else \( wk \) may be Kemped to one of \( 1, \ldots, k - 3 \) so that Kemping \( vw \) to \( k \) gives the previous case. Since \( k \geq 5 \), there is \( d \in \{1, \ldots, k - 3\} \setminus \{c\} \) such that \( wd \) may be Kemped to one of \( k + 1, \ldots, A \) so that Kemping \( wk \) to \( d \), and \( vw \) to \( k \) gives the previous case.

Assume \( vw \) is colored one of \( k + 1, \ldots, A \). Without loss of generality, \( vw \) is colored \( A \). Here, \( w \) sees 1, \( \ldots, k - 3 \), or else \( vw \) may be Kemped to one of 1, \( \ldots, k - 3 \) to give the previous case. In this case, \( w1 \) may be Kemped to one of \( k, \ldots, A - 1 \) so that Kemping \( vw \) to 1 gives the previous case.

Thus, \( y \) sees 1.
Assume \( wz \) is colored one of 2, ..., \( k-1 \). Without loss of generality, \( wz \) is colored 2. First, \( w \) sees \( k \), or else Kempeing \( yz \) to 2 and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \). Also, \( w \) sees 3, ..., \( k-1 \), or else \( wk \) may be Kempe to one of 3, ..., \( k-1 \) so that Kempeing \( yz \) to 2 and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \). In this case, \( w3 \) may be Kempe to one of \( k+1, \ldots, A \) so that Kempeing \( wk \) to 3, \( yz \) to 2, and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \).

Thus, \( wz \) is colored one of \( k+1, \ldots, A \). Without loss of generality, \( wz \) is colored \( A \). First, \( w \) sees 2, ..., \( k-1 \), or else \( wz \) may be Kempe to one of 2, ..., \( k-1 \) to give the previous case. In this final case, \( w2 \) may be Kempe to one of \( k, \ldots, A-1 \) so that Kempeing \( wz \) to 2 gives the previous case.

**Lemma 2.6.** No \( A \)-critical graph has distinct vertices \( v, w, x, y, z \) such that \( v \) and \( w \) are \( (\leq A-1) \)-vertices, \( \deg(x) + \deg(y) \leq A + 3 \), \( \deg(x) \geq 4 \), \( \deg(y) \geq 4 \), \( xy \) is a triangle, and \( z \) is adjacent to \( v \) and \( w \).

**Proof.** Suppose that \( G \) is a critical graph with such vertices \( v, w, x, y, z \). Since \( G \) is critical, \( G - xy \) has an edge \( A \)-coloring. Let \( k := \deg(x) \). Without loss of generality, \( xz \) is colored \( k-1 \), \( x \) sees 1, ..., \( k-1 \), and \( y \) sees \( k, \ldots, A \), or else \( xy \) may be colored to give an edge \( A \)-coloring of \( G \). Since \( \deg(x) + \deg(y) \leq A + 3 \), without loss of generality, \( y \) does not see 2, ..., \( k-1 \).

Assume \( yz \) is not colored 1. Without loss of generality, \( yz \) is colored \( k \).

Without loss of generality via symmetry of \( v \) and \( w \), \( wz \) is not colored 1. Without loss of generality via symmetry of \( x \) and \( y \), \( wz \) is colored 2. First, \( w \) sees \( k \), or Kempeing \( yz \) to 2 and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \). Next, \( w \) sees 3, ..., \( k-1 \), or \( wk \) can be Kempe to a color in 3, ..., \( k-1 \) so that Kempeing \( yz \) to 2, and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \). Also, \( w \) sees \( k+1, \ldots, A \), or \( w3 \) can be Kempe to a color in \( k+1, \ldots, A \) so that Kempeing \( wk \) to 3, Kempeing \( yz \) to 2, and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \). Since \( \deg(w) \leq A-1 \), \( w \) does not see 1. Finally, \( y \) sees 1, or Kempeing \( wk \) to 1, Kempeing \( yz \) to 2, and coloring \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \).

Suppose \( vz \) is not colored 1. By symmetry of \( v \) and \( w \), \( v \) does not see 1, and \( v \) sees 2, ..., \( A \). Without loss of generality, \( vz \) is not colored \( A \). By symmetry of \( v \) and \( w \), Kempeing \( wA \) to 1 does not affect \( x \). But after Kempeing \( wA \) to 1, \( w \) sees 1, and nothing else changes; this was handled in the previous paragraph.

Thus, \( vz \) is colored 1. Here, \( v \) sees \( k \), or else one can recolor \( vz \) with \( k \), \( wz \) with 1, \( yz \) with 2, and \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \). Also, \( v \) sees 2, ..., \( k-1 \), or else \( vk \) may be Kempe to one of 2, ..., \( k-1 \) so that coloring \( vz \) with \( k \), \( wz \) with 1, \( yz \) with 2, and \( xy \) with \( k \) gives an edge \( A \)-coloring of \( G \). But in the final case, since \( \deg(v) \leq A-1 \), \( v3 \) may be
Kemped to one of \( k + 1, \ldots, \Delta \) so that Kemping \( w_k \) to 3, coloring \( v_z \) with \( k \), \( w_z \) with 1, \( y_z \) with 2, and \( x_y \) with \( k \) gives an edge \( \Delta \)-coloring of \( G \).

Thus, \( y_z \) is colored 1.

Suppose \( w_z \) is colored with a color in \( k, \ldots, \Delta \). Without loss of generality, \( w_z \) is colored \( k \). First, \( w \) sees \( k - 1 \), or else Kemping \( x_z \) to \( k \) and coloring \( x_y \) with \( k - 1 \) gives an edge \( \Delta \)-coloring of \( G \). Next, \( w \) sees \( k + 1, \ldots, \Delta \), or else \( w \) may be Kemped to one of \( k + 1, \ldots, \Delta \) so that Kemping \( x_z \) to \( k \), and coloring \( x_y \) with \( k - 1 \) gives an edge \( \Delta \)-coloring of \( G \). Also, \( w \) sees 2, \( k - 2 \), or else \( w \Delta \) may be Kemped to one of 2, \( k - 2 \) so that Kemping \( w(k - 1) \) to \( A \), Kemping \( x_z \) to \( k \), and coloring \( x_y \) with \( k - 1 \) gives an edge \( \Delta \)-coloring of \( G \). Since \( \deg(w) \leq \Delta - 1 \), \( w \) does not see 1. If Kemping \( w \Delta \) to 1 affects either \( x \) or \( y \), it yields a previous case. Thus, Kemping \( w \Delta \) to 1, \( w(k - 1) \) to \( A \), \( x_z \) to \( k \), and coloring \( x_y \) with \( k - 1 \) gives an edge \( \Delta \)-coloring of \( G \).

Thus, \( w_z \) is colored with a color in \( 2, \ldots, k - 1 \). Without loss of generality, \( w_z \) is colored 2. First, \( w \) sees \( k, \ldots, \Delta \), or else Kemping \( w_z \) to one of \( k, \ldots, \Delta \) yields the previous paragraph. Also, \( w \) sees \( 3, \ldots, k - 1 \), or else \( w_k \) may be Kemped to one of \( 3, \ldots, k - 1 \) so that Kemping \( w_z \) to \( k \) yields the previous paragraph. Finally, Kemping \( w_k \) to 1 does not affect \( x \) or \( y \), or else a previous case is obtained. Thus, Kemping \( w_k \) to 1 and then \( w_z \) to \( k \) yields the previous paragraph.

3. STRUCTURE OF PLANAR GRAPHS OF MAXIMUM DEGREE SEVEN

This section gives a proof of the main result. The technique used to prove the theorem is the Discharging Method, the same technique used to prove the Four Color Theorem [2]. As a starting point, an initial charge function \( ch \) is defined on \( V \cup F \) as follows: For each vertex \( x \), let \( ch(x) := 6 - \deg(x) \). For each face \( y \), let \( ch(y) := 2(3 - \deg(y)) \). The key, well-known observation is the following, which easily follows from Euler’s formula:

**Lemma 3.1.** For a connected plane graph,

\[
\sum_{x \in V \cup F} ch(x) = 12.
\]

Next, a modified charge function \( ch' \) is defined as a modification of \( ch \) by moving some charge locally among vertices and faces according to the following discharging rules. Each rule sends charge from a vertex of degree at most 6 to either a face of degree at least 4 or to a vertex of degree at
least 6 (possibly via another vertex). Let \( j \)-neighbor, \( j \)-face, etc., be defined analogous to \( j \)-vertex, etc. Let an \((i, j, k)\)-face be a 3-face incident with distinct vertices \( x, y, z \) such that \( \deg(x) = i \), \( \deg(y) = j \), and \( \deg(z) = k \).

1. For each 2-vertex \( x \), and for each 7-vertex \( y \) adjacent to \( x \), send 1 from \( x \) to \( y \).

2. For each 2-vertex \( x \), for each 7-vertex \( y \) adjacent to \( x \), and for each \((\geq 4)\)-face \( F \) incident with \( xy \), send \( \frac{1}{2} \) from \( x \) to \( F \), and send \( \frac{3}{2} \) from \( x \) via \( y \) to \( F \). (Note: Since each 2-vertex \( x \) is adjacent two 7-vertices \( y, z \) such that \( xy, xz \) are incident with \( F \), \( x \) actually sends \( \frac{3}{2} \) directly to \( F \).)

3. For each 3-vertex \( x \), and for each 7-vertex \( y \) adjacent to \( x \), if \( xy \) is incident with two 3-faces, then send 1 from \( x \) to \( y \), else send \( \frac{1}{2} \) from \( x \) to \( y \).

4. For each vertex \( x \) such that \( 3 \leq \deg(x) \leq 6 \), for each \((\geq 4)\)-face \( F \) incident with \( x \), and for each vertex \( y \) such that \( xy \) is incident with \( F \), send \( \frac{1}{2} \) from \( x \) to \( F \), and if \( \deg(y) = 7 \), send an additional \( \frac{1}{2} \) from \( x \) via \( y \) to \( F \). (Note: Since each vertex \( x \) with \( 3 \leq \deg(x) \leq 6 \), is adjacent two vertices \( y, z \) such that \( xy, xz \) are incident with \( F \), \( x \) actually sends \( \frac{3}{2} \) directly to \( F \).)

5. For each 3-vertex \( x \), for each 6-vertex \( y \) adjacent to \( x \), and for each 7-vertex \( z \) adjacent to \( y \), but not to \( x \), send 1 from \( x \) to \( z \).

6. For each 4-vertex \( x \) adjacent to a 5-vertex, and for each 7-vertex \( y \) adjacent to \( x \), send \( \frac{1}{2} \) from \( x \) to \( y \).

7. For each 4-vertex \( x \) not adjacent to a 5-vertex, and for each 7-vertex \( y \) adjacent to \( x \), if \( xy \) is incident with two 3-faces, then send \( \frac{1}{2} \) from \( x \) to \( y \), else send \( \frac{1}{2} \) from \( x \) to \( y \).

8. For each 4-vertex \( x \), and for each 6-vertex \( y \) adjacent to \( x \), send \( \frac{1}{2} \) from \( x \) to \( y \).

9. For each 5-vertex \( x \) adjacent to a 4-vertex, and for each 7-vertex \( y \) adjacent to \( x \), send \( \frac{1}{2} \) from \( x \) to \( y \).

10. For each 5-vertex \( x \) not adjacent to a 4-vertex, and for each \((\geq 6)\)-vertex \( y \) adjacent to \( x \) such that \( xy \) is incident with two 3-faces, if \( xy \) is incident with exactly one \((5, 5, 7)\)-face, then send \( \frac{3}{2} \) from \( x \) to \( y \), else send \( \frac{1}{2} \) from \( x \) to \( y \).

11. For each 6-vertex \( x \) not adjacent to a 3-vertex, and for each 7-vertex \( y \) adjacent to \( x \), if \( xy \) is incident with two \((4, 6, 7)\)-faces, then send \( \frac{2}{3} \) from \( x \) to \( y \), else if \( xy \) is not incident with two \((6, 7, 7)\)-faces, then send \( \frac{1}{2} \) from \( x \) to \( y \).

Now the proof of the main result may be given. The proof proceeds as follows. It is supposed that a 7-critical planar graph exists. Each face or vertex is examined according to its degree. The results of Section 2 are used to show that each such element has non-positive modified charge. This contradicts Lemma 3.1 to prove the theorem.
THEOREM 3.1. No 7-critical graph is planar.

Proof. Suppose that $G$ is a 7-critical planar graph. By Lemma 3.1, $\sum_{v \in V(x)} ch(x) = 12$. The rules only move charge around, and do not affect the sum, and so we have $\sum_{v \in V(x)} ch'(x) = 12$, as well. A contradiction follows by showing that every face and every vertex has non-positive modified charge.

Let $F_3$ be a 3-face. Thus, $ch(F_3) = 0$, and nothing sends charge into $F_3$, so $ch'(F_3) = 0$ as well.

Let $F_4$ be a 4-face. Let $x, y, z, w$ be the vertices incident with $F_4$ cyclically ordered according to the embedding of $G$. If $x$, say, is a 2-vertex, then $y, z, w$ are 7-vertices by VAL, and $z$ sends no charge into $F_4$, while each of $x, y, w$ sends $\frac{1}{2}$ by Rule 2, so that $ch'(F_4) = 0$. Otherwise, each vertex incident with $F_4$ sends at most $\frac{1}{2}$ into $F_4$ by Rule 4, and $ch'(F_4) \leq 0$.

Let $F_5$ be a $k$-face for $k \geq 5$. By definition, $ch(F_5) = 2(3 - k)$. Let $xyz$ be a sequence of vertices in the facial walk of $F_5$. If $x$ is a 2-vertex, then by VAL, $y$ and $z$ are 7-vertices, and the only charge $y$ sends into $F_5$ is $\frac{1}{2}$ from Rule 2. If $y$ is a 2-vertex, then $y$ sends $\frac{1}{2}$ into $F_5$ by Rule 2. If none of $x, y$, or $z$ is a 2-vertex, then $y$ sends $\frac{1}{2}$ at most twice into $F_5$ by applications of Rule 4. Since $F_5$ receives at most $\frac{1}{2}$ from each vertex incident with it, $ch(F_5) \leq 2(3 - k) + 2k/3 = (9 - 2k)/3 \leq 0$.

Let $v_2$ be a 2-vertex. Thus, $ch(v_2) = 4$. No rule sends charge into $v_2$. By VAL, $v_2$ is adjacent to two 7-vertices, and $v_2$ sends out 2 by Rule 1. Since $G$ is simple, and clearly not $K_4$, $v_2$ is incident with at least one $(\geq 4)$-face, and $v_2$ sends out at least 2 by Rule 2. Thus, $ch'(v_2) \leq 0$.

Let $v_3$ be a 3-vertex. Thus, $ch(v_3) = 3$. No rule sends charge into $v_3$. By VAL, $v_3$ is adjacent to three $(\geq 6)$-vertices, at least two of which are 7-vertices. By Rules 3 and 4, $v_3$ sends out at least 1 for each of its 7-neighbors. Thus, if $v_3$ is adjacent to three 7-vertices, $ch'(v_3) \leq 0$. Otherwise, $v_3$ sends out at least 1 by Rule 5, and $ch'(v_3) \leq 0$.

Let $v_4$ be a 4-vertex. Thus, $ch(v_4) = 2$. No rule sends charge into $v_4$. By VAL, $v_4$ is adjacent to four $(\geq 5)$-vertices, at least two of which are 7-vertices. If $v_4$ is adjacent to a 5-vertex, by VAL, $v_4$ is adjacent to three 7-vertices, $v_4$ sends out 2 by Rule 6, and $ch'(v_4) \leq 0$. Otherwise, by Rules 4 and 7, $v_4$ sends out at least $\frac{1}{2}$ for each of its 7-neighbors. By Rule 8, $v_4$ sends out at least $\frac{1}{2}$ for each of its 6-neighbors. Since $v_4$ has at least two 7-neighbors, $ch'(v_4) \leq 0$ here as well.

Let $v_5$ be a 5-vertex. Thus, $ch(v_5) = 1$. No rule sends charge into $v_5$. By VAL, $v_5$ is adjacent to five $(\geq 4)$-vertices, at least two of which are 7-vertices. If $v_5$ is adjacent to a 4-vertex, by VAL, $v_5$ is adjacent to four 7-vertices, $v_5$ sends out $\frac{1}{2}$ by Rule 9, and $ch'(v_5) \leq 0$. If $v_5$ is adjacent to five $(\geq 6)$-vertices, by Rules 4 and 10, it sends out at least $\frac{1}{2}$ to each of its neighbors, and $ch'(v_5) \leq 0$ again.
Thus $v_5$ is adjacent to a 5-vertex, and by VAL, $v_5$ is adjacent to at least three 7-vertices; by Rules 4 and 10, it sends at least \( \frac{1}{2} \) to each of them. If $v_5$ is also incident with an ($\geq 4$)-face, it sends \( \frac{1}{2} \) to it by Rule 4, and $ch'(v_5) \leq 0$. Otherwise, $v_5$ is incident with five 3-faces. In this case, if $v_5$ is adjacent to two 5-vertices, by Rule 10, it sends out \( \frac{2}{7} \) to each of two of its 7-neighbors, and \( \frac{1}{7} \) to the other 7-neighbor, and otherwise, by Rule 10, it sends out \( \frac{1}{2} \) to one of its 7-neighbors, and \( \frac{1}{7} \) to the other three ($\geq 6$)-neighbors. In either of these cases, $ch'(v_5) = 0$.

Let $v_6$ be a 6-vertex. Thus, $ch(v_6) = 0$. By VAL, $v_6$ is adjacent to six ($\geq 3$)-vertices. If $v_6$ is adjacent to a 3-vertex, then by VAL, it is adjacent to five 7-vertices, nothing sends charge into $v_6$, and $ch'(v_6) = 0$. If $v_6$ is adjacent to a 4-vertex, then by VAL, it is adjacent to four 7-vertices; $v_6$ receives at most \( \frac{1}{2} \) from each of its (\( \leq 5 \))-neighbors by Rules 8 and 10, but it sends out by Rules 4 and 11 at least what it receives, and thus $ch'(v_6) \leq 0$ here as well. If $v_6$ is adjacent to a 5-vertex, then by VAL, it is adjacent to three 7-vertices; $v_6$ receives \( \frac{1}{7} \) from each of its 5-neighbors by Rule 10, and it sends out by Rules 4 and 11 at least what it receives, since VAL shows that no edge incident with $v_6$ is incident with two (5, 5, 6)-faces, and thus, $ch'(v_6) \leq 0$. Otherwise, no rule sends charge into $v_6$, and $ch'(v_6) = 0$.

Let $v_7$ be a 7-vertex. Thus, $ch(v_7) = -1$. If $v_7$ is adjacent to a 2-vertex, then by VAL, $v_7$ is adjacent to six 7-vertices, and the only charge $v_7$ receives is 1 from its 2-neighbor by Rule 1. If $v_7$ is adjacent to a 6-vertex which is adjacent to a 3-vertex $y$, but $v_7$ is not adjacent to $y$, then by Lemma 2.3, $v_7$ is adjacent to six 7-vertices, and the only charge $v_7$ receives is 1 from $y$ by Rule 5. If there is a $j \in \{4, 5\}$ such that $v_7$ is adjacent to a $j$-vertex $x$ which is adjacent to a $(9 - j)$-vertex $y$, then by Lemma 2.3, every neighbor of $v_7$ besides $x$ and $y$ is a 7-vertex, and the only charge $v_7$ receives is at most 1 from $x$ and $y$ by Rules 6 and 9, and $ch'(v_7) = 0$.

If $v_7$ is adjacent to a 3-vertex, then by VAL, $v_7$ is adjacent to five 7-vertices. If $v_7$ is adjacent to two 3-vertices, then by Lemma 2.4, $v_7$ receives only \( \frac{1}{2} \) from each of its 3-neighbors by Rule 3, and $ch'(v_7) \leq 0$. Thus, assume that $v_7$ is adjacent to only one 3-vertex. If $v_7$ receives 1 from it from Rule 3, Lemma 2.4 says that $v_7$ receives no other charge. Otherwise each of the two neighbors of $v_7$ of degree at most six sends at most \( \frac{1}{2} \) into $v_7$. In either case, $ch'(v_7) \leq 0$.

Suppose $v_7$ is adjacent to a 4-vertex. By VAL, $v_7$ is adjacent to four 7-vertices.

If a 4-vertex $x$ sends \( \frac{1}{2} \) into $v_7$, then since VAL says that $x$ is not adjacent to a 4-vertex, Lemma 2.4 says that $v_7$ is adjacent to only one 4-vertex. By Lemma 2.5, no vertex sends \( \frac{1}{2} \) into $v_7$ by Rule 10. It follows that each of the at most two other (\( \leq 6 \))-neighbors of $v_7$ sends at most \( \frac{1}{2} \) into $v_7$, and $ch'(v_7) \leq 0$.

Thus, assume no 4-vertex $x$ sends \( \frac{1}{2} \) into $v_7$. Then, each 4-neighbor of $v_7$ sends \( \frac{1}{2} \) into $v_7$, and each (\( \leq 6 \))-neighbor of $v_7$ sends at most \( \frac{1}{2} \) into $v_7$. In this case, $ch'(v_7) \leq 0$ as well.
Thus, assume $v_7$ is not adjacent to a 4-vertex. If a 5-vertex sends $\frac{2}{5}$ into $v_7$ by Rule 10, then by Lemma 2.6, $v_7$ is adjacent to at most three ($\leq 6$)-vertices, and by examining the Rules, one of those three sends at most $\frac{1}{5}$ into $v_7$, while the other two send at most $\frac{2}{5}$, and $ch'(v_7) \leq 0$. By VAL, $v_7$ is adjacent to at most five ($\leq 6$)-vertices. In the only case which remains, each ($\leq 6$)-neighbor of $v_7$ sends at most $\frac{1}{5}$ into $v_7$, and $ch'(v_7) \leq 0$ in this final case as well.

4. PROJECTIVE PLANAR GRAPHS

In closing, it is appropriate to mention the similar problem for graphs which embed in the projective plane, even though Vizing never considered it. The results in Section 2 certainly apply to projective graphs. Also, Lemma 3.1 has an analogue for projective graphs with the 12 simply replaced with a 6. As the proof of Theorem 3.1 only used that this sum is positive, this article also gives a proof that projective graphs of maximum degree seven are Class 1. The same is easily seen to be true for maximum degree at least eight.

ACKNOWLEDGMENT

The authors thank the referees for their valuable suggestions and for carefully reading manuscript.

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