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A connectedness property of algebraic moment maps ☆

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Dedicated to Claudio Procesi on the occasion of his 60th birthday

1. Introduction

Let K be a connected compact Lie group and M a Hamiltonian K-manifold, i.e., a symplectic K-manifold equipped with a moment map $\mu: M \to \mathfrak{k}^* := (\operatorname{Lie} K)^*$. A theorem of Kirwan (implicitly in [Ki]) asserts: *if M is connected and compact then the level sets of μ are connected*. The purpose of this note is to prove such a statement in the category of algebraic varieties.

First, we reformulate Kirwan's theorem: consider the map $\psi: M \to \mathfrak{k}^*/K$ which is the composition of μ with the quotient map. For a point $x \in \mathfrak{k}^*$ let $H = K_x$ be its isotropy group and $y = Kx \in \mathfrak{k}^*/K$ its orbit. Then the fiber $\psi^{-1}(y)$ is isomorphic to the fiber product $K \times^H \mu^{-1}(x)$. Thus, since both K/H and H are connected, the connectedness of the fibers of μ is equivalent to the connectedness of the fibers of ψ. This formulation is more suitable for the algebraic category.

Let G be a connected reductive group (everything over $\mathbb{C}$) and Z a Hamiltonian G-variety with moment map $\mu: Z \to \mathfrak{g}^* = (\operatorname{Lie} G)^*$. Let $\mathfrak{g}^*/\!\!/G := \operatorname{Spec} \mathbb{C}[\mathfrak{g}^*]^G$ be the categorical quotient and let $\widetilde{\psi}: Z \to \mathfrak{g}^*/\!\!/G$ be the composition of μ with the quotient map. Since the latter is not an orbit map, the connection between fibers of μ and $\widetilde{\psi}$ is considerably more loose than in the differential category and we concentrate on $\widetilde{\psi}$ from now on.

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The morphism $\widetilde{\psi}$ is still not the right map, since sometimes not even its generic fibers are connected. An example is the action of $G = SL_2(\mathbb{C})$ on $Z = \mathbb{C}^2 \times (\mathbb{C}^2)^*$ (this is the cotangent bundle of $\mathbb{C}^2$). Then $\mathfrak{g}^* /\!\!/ G = \mathbf{A}^1$ and $\widetilde{\psi}(u, \alpha) = \alpha(u)^2$. In particular, the generic fiber of $\widetilde{\psi}$ has two connected components. This is remedied by looking at the map $\psi(u, \alpha) := \alpha(u)$ instead. Then $\widetilde{\psi}$ is the composition of ψ with the finite map $\mathbf{A}^1 \to \mathbf{A}^1,\ z \mapsto z^2$, the latter being responsible for the disconnected fibers.

This construction can be generalized as follows: the morphism $\widetilde{\psi}$ induces an homomorphism of algebras

$$\widetilde{\psi}^* : \mathbb{C}[\mathfrak{g}^* /\!\!/ G] \to \mathbb{C}[Z]. \tag{1.1}$$

Let $R \subseteq \mathbb{C}[Z]$ be the integral closure of the image and $L := \operatorname{Spec} R$. Then $\widetilde{\psi}$ factors through $\psi : Z \to L$. By construction, the generic fibers of ψ are now connected, even irreducible. Furthermore, we expect that the behavior of ψ is dramatically better than that. For example, when Z is connected and affine, then it is hoped that L is smooth and ψ is faithfully flat with reduced, connected fibers. If that were true then most of bad behavior of $\widetilde{\psi}$ were on account of the finite morphism $L \to \mathfrak{g}^* /\!\!/ G$.

So far, the theory of algebraic Hamiltonian varieties is not developed enough to convert this hope into a proof, but for one class we know a lot. That is, when Z is the cotangent bundle of a smooth G-variety X. Then one can prove [Kn1] that L is not only smooth but even an affine space. More precisely, $L = \mathfrak{a}^* / W_X$ where $\mathfrak{a}^*$ is a finite dimensional vector space and W_X a finite reflection group. In this setting, the main result of this paper is:

Theorem 1.1. *Let X be a connected, smooth G-variety and $Z := T_X^*$ its cotangent bundle. Then all fibers of $\psi : Z \to \mathfrak{a}^* / W_X$ are connected.*

Apart from its intrinsic interest this theorem is useful for further investigation of the fibers of ψ. For example, in [Kn4] it is used to prove that most fibers of ψ are reduced, provided X is affine. In turn, that latter result is crucial in the investigation of so-called collective functions on Hamiltonian manifolds in the differentiable category (also in [Kn4]).

The strategy of the proof of Theorem 1.1 is as follows: the core argument is an application of Zariski's connectedness theorem, a theorem which works only for proper morphisms. For that reason we construct a partial compactification of Z. Since this works only when X is homogeneous, the proof splits into two parts: (1) the proof of the connectedness theorem for homogeneous X and (2) the reduction of the general case to the homogeneous case.

In both parts certain irreducible subvarieties of the fibers of ψ are used to "tie" its different parts together, thereby proving connectivity (cf. Lemma 5.3 and the construction of E in Section 6).

Notation. All varieties are defined over an algebraically closed field of characteristic zero, which is denoted by $\mathbb{C}$. Throughout the paper, G is a connected reductive group. The Lie algebras of the algebraic groups $G, B, U, H, \ldots$ are denoted by the corresponding lower case Fraktur letters $\mathfrak{g}, \mathfrak{b}, \mathfrak{u}, \mathfrak{h}, \ldots$. For an affine G-variety X let $X/\!\!/G := \operatorname{Spec} \mathbb{C}[X]^G$ be the categorical quotient. For a subspace $U \subseteq V$ let $U^\perp \subseteq V^*$ denote its annihilator.

2. The moment map on cotangent bundles

In this section we review some results of [Kn1] about the geometry of the moment map on a cotangent bundle.

Let X be a smooth connected G-manifold. Let $B \subseteq G$ be a Borel subgroup with maximal unipotent subgroup U and maximal torus T. Then one can define the following important numerical invariants of X:

$$n := \dim X, \qquad n_u := \max_{x \in X} \dim Ux, \qquad n_b := \max_{x \in X} \dim Bx. \tag{2.1}$$

The difference $c := n - n_b$ is called the *complexity* of X while $r := n_b - n_u$ is its *rank*. Consider

$$X_0 := \{x \in X \mid \dim Ux = n_u,\ \dim Bx = n_b\}. \tag{2.2}$$

This is a dense open B-stable subset of X. We define the following subbundles of the cotangent bundle $T^*_{X_0}$:

$$\mathcal{B} := \dot{\bigcup_{x \in X_0}} (\mathfrak{b}x)^\perp \subseteq \mathcal{U} := \dot{\bigcup_{x \in X_0}} (\mathfrak{u}x)^\perp. \tag{2.3}$$

The fibers of the quotient are:

$$\begin{aligned}(\mathcal{U}/\mathcal{B})_x &= \mathcal{U}_x/\mathcal{B}_x = (\mathfrak{u}x)^\perp/(\mathfrak{b}x)^\perp = (\mathfrak{b}x/\mathfrak{u}x)^* \\ &= \big(\mathfrak{b}/(\mathfrak{u} + \mathfrak{b}_x)\big)^* \subseteq (\mathfrak{b}/\mathfrak{u})^*.\end{aligned} \tag{2.4}$$

Therefore, we can identify this fiber with a subspace $\mathfrak{a}^*_x$ of the "abstract" dual Cartan subalgebra $\mathfrak{t}^* = (\mathfrak{b}/\mathfrak{u})^*$. By construction, the ranks of the vector bundles $\mathcal{B}$ and $\mathcal{U}$ are c and $c + r$, respectively. Hence we have $\dim \mathfrak{a}^*_x = r$.

Lemma 2.1 [Kn1, 6.3]. *The space $\mathfrak{a}^* = \mathfrak{a}^*_x$ depends neither on the choice of $x \in X_0$ nor on the choice of a Borel subgroup B. In particular, we obtain a morphism $\mathcal{U} \twoheadrightarrow \mathfrak{a}^*$ which induces a trivialization $\mathcal{U}/\mathcal{B} \xrightarrow{\sim} \mathfrak{a}^* \times X_0$.*

The cotangent bundle T^*_X is a Hamiltonian G-variety, i.e., it carries a natural G-invariant symplectic structure and is equipped with a moment map. More precisely, let $\pi : T^*_X \to X$ be the natural projection. Then the moment map is

$$\mu : T^*_X \to \mathfrak{g}^*, \quad \alpha \mapsto \big[\xi \mapsto \alpha\big(\xi\pi(\alpha)\big)\big]. \tag{2.5}$$

Actually, we are more interested in the composition $\widetilde{\psi}: T_X^* \xrightarrow{\mu} \mathfrak{g}^* \to \mathfrak{g}^*/\!\!/G$. It induces a homomorphisms of algebras

$$\widetilde{\psi}^*: \mathbb{C}[\mathfrak{g}^*/\!\!/G] \to \mathbb{C}\big[T_X^*\big]. \tag{2.6}$$

Let L_X be the spectrum of the integral closure of the $\widetilde{\psi}^*\mathbb{C}[\mathfrak{g}^*]$ in $\mathbb{C}[T_X^*]$. Then we obtain a factorization of $\widetilde{\psi}$

$$T_X^* \xrightarrow{\psi} L_X \xrightarrow{\eta} \mathfrak{g}^*/\!\!/G, \tag{2.7}$$

where η is a finite morphism and ψ is dominant. Moreover, ψ is universal with these properties. An important property of ψ is that its generic fibers are irreducible.

Theorem 2.2 [Kn1, 6.2, 6.6a,b]. *The restriction of ψ to $\mathcal{U}$ factors through $\mathfrak{a}^*$:*

$$\mathcal{U} \to \mathfrak{a}^* \xrightarrow{\pi} L_X. \tag{2.8}$$

The morphism π is finite and surjective. More precisely, there is a finite reflection group $W_X \subseteq GL(\mathfrak{a}^*)$ (the little Weyl group of X) such that π induces an isomorphism $\mathfrak{a}^*/W_X \xrightarrow{\sim} L_X$. In particular, L_X is an affine space.

From now on, we identify L_X with $\mathfrak{a}^*/W_X$. It will also be convenient to use a less canonical way of stating (2.8). Let $x \in X_0$ and let $\mathfrak{a}'$ be a complement of $(\mathfrak{b}x)^\perp \subseteq T_{X,x}^*$ in $(\mathfrak{u}x)^\perp \subseteq T_{X,x}^*$. Then, by (2.4) we can identify $\mathfrak{a}'$ with $\mathfrak{a}^*$ and we have a commutative diagram

$$\begin{tikzcd} \mathfrak{a}' \arrow[r, hook] \arrow[d, "\sim"] & T_X^* \arrow[d, "\psi"] \\ \mathfrak{a}^* \arrow[r, two heads] & \mathfrak{a}^*/W_X. \end{tikzcd} \tag{2.9}$$

In other words, $\mathfrak{a}'$ is "almost" a section of ψ.

Theorem 2.3 [Kn1, 6.6c]. *The morphism $\psi: T_X^* \to \mathfrak{a}^*/W_X$ is faithfully flat. In particular, all fibers of ψ are of pure codimension r.*

Now we specialize everything to the case where $X = G/H$ is a homogeneous space. Then we can write $T_X^* = G \times^H \mathfrak{h}^\perp$ and all information about ψ is contained in the restriction

$$\psi_{\mathfrak{h}}: \mathfrak{h}^\perp \to \mathfrak{a}^*/W_X. \tag{2.10}$$

Theorem 2.4. *Let $X = G/H$ be a homogeneous variety.*

(i) *The rank r, the complexity c, and the little Weyl group $W_{G/H}$ depend only on the Lie algebra $\mathfrak{h}$.*

(ii) *The morphism* $\psi_{\mathfrak{h}}$ *is faithfully flat. All of its fibers are purely of codimension* r. *The generic fibers are irreducible.*

(iii) *Let* B *be a Borel subgroup of* G *and* $g := \dim G$. *Then*

$$\dim(\mathfrak{b}+\mathfrak{h}) \leqslant g - c \quad \textit{and} \quad \dim(\mathfrak{u}+\mathfrak{h}) \leqslant g - c - r. \tag{2.11}$$

Moreover, equality holds for an open set of Borel subgroups.

(iv) *Assume that equality holds in* (2.11). *Let* $\mathfrak{a}'$ *be any complement of* $\mathfrak{h}^\perp \cap \mathfrak{b}^\perp$ *in* $\mathfrak{h}^\perp \cap \mathfrak{u}^\perp$. *Then one can identify* $\mathfrak{a}'$ *with* $\mathfrak{a}^*$ *and the restriction of* $\psi_{\mathfrak{h}}$ *to* $\mathfrak{a}'$ *is the quotient map* $\mathfrak{a}^* \to \mathfrak{a}^*/W_X$.

Proof. (i) The connected component H^0 of H depends only on $\mathfrak{h}$. Moreover, the morphism $X^0 := G/H^0 \to X = G/H$ is étale. This implies that dimension, complexity, and rank of X^0 and X are the same (a direct consequence of the definitions). The same holds for the little Weyl group (see [Kn1, 6.5.3]).

(ii) By Theorem 2.3, the morphism $\psi_0 : T^*_{X^0} \to \mathfrak{a}^*/W_{X^0} = \mathfrak{a}^*/W_X$ is surjective and has fibers of codimension r. Moreover, the generic fibers are irreducible. Since, for $u \in \mathfrak{a}^*/W_X$ we have $\psi_0^{-1}(u) = G \times^{H^0} \psi_{\mathfrak{h}}^{-1}(u)$, the same holds for $\psi_{\mathfrak{h}}$. It follows that $\psi_{\mathfrak{h}}$ is faithfully flat [EGA, Section 15.4.2].

(iii) Let $x_0 = eH \in G/H$ be the base point. Then $\dim(\mathfrak{b}+\mathfrak{h}) = \dim Bx_0 + \dim \mathfrak{h}$ and $\dim(\mathfrak{u}+\mathfrak{h}) = \dim Ux_0 + \dim \mathfrak{h}$. Thus, (2.11) follows from the definition of c and r. Moreover, there is a point $gx_0 \in G/H$ such that Bgx_0 and Ugx_0 have maximal dimension. Thus, if we replace B by $g^{-1}Bg$ we obtain equalities in (2.11).

(iv) We have

$$\begin{aligned} \mathfrak{a}' \xrightarrow{\sim} \big(\mathfrak{h}^\perp \cap \mathfrak{u}^\perp\big)\big/\big(\mathfrak{h}^\perp \cap \mathfrak{b}^\perp\big) &= (\mathfrak{h}+\mathfrak{u})^\perp\big/(\mathfrak{h}+\mathfrak{b})^\perp \\ &= \big[(\mathfrak{h}+\mathfrak{b})/(\mathfrak{h}+\mathfrak{u})\big]^* = [\mathfrak{b}x_0/\mathfrak{u}x_0]^*. \end{aligned} \tag{2.12}$$

Now the assertion follows from Theorem 2.2. □

3. A partial compactification: construction

Ultimately, we would like to invoke Zariski's connectedness theorem which allows one to deduce the connectedness of all fibers from that of the generic fibers. The essential prerequisite for this theorem is that the morphism is proper. Therefore, we construct first a partial compactification of the cotangent bundle which renders the moment map proper. For this we need that $X = G/H$ is homogeneous. So we assume this until further notice.

Consider the Lie algebra $\mathfrak{h}$ as a point of the Grassmannian $\mathrm{Gr}(\mathfrak{g})$ of all subspaces of $\mathfrak{g}$. Let $Y \subseteq \mathrm{Gr}(\mathfrak{g})$ be the closure of the orbit $G \cdot \mathfrak{h}$. Recall that a subalgebra of $\mathfrak{g}$ is called *algebraic* if it is the Lie algebra of a closed subgroup of G. By Theorem 2.4(i) we may speak of the rank, the complexity, and the little Weyl group of an algebraic Lie subalgebra of $\mathfrak{g}$.

Lemma 3.1. *Every point $\mathfrak{m} \in Y$ represents an algebraic subalgebra of $\mathfrak{g}$ with the same dimension, complexity, and rank as $\mathfrak{h}$.*

Proof. First, we claim that there is a smooth affine curve C, a point $c_0 \in C$, and morphisms $\alpha' : C' := C \setminus \{c_0\} \to G$, $\alpha : C \to Y$ such that

$$\alpha(c) = \begin{cases} \alpha'(c) \cdot \mathfrak{h} & \text{for } c \neq c_0, \\ \mathfrak{m} & \text{for } c = c_0. \end{cases} \tag{3.1}$$

In fact, let $G \hookrightarrow G'$ be any completion and $\overline{G} \subseteq G' \times Y$ the closure of the set $\{(g, g \cdot \mathfrak{h}) \mid g \in G\}$. Then

$$G \hookrightarrow \overline{G}, \quad g \mapsto (g, g \cdot \mathfrak{h}), \tag{3.2}$$

is again a completion of G. Let $\gamma : \overline{G} \to Y$ be the second projection. This morphism is dominant and proper, hence surjective. Choose $x \in \overline{G}$ with $\gamma(x) = \mathfrak{m}$. Since G is open and dense in $\overline{G}$ there is an affine curve $C_1 \subseteq \overline{G}$ with $x \in C_1$ and $C_1 \setminus \{x\} \subseteq G$. Let $\widetilde{C} \to C_1$ be the normalization of C_1 and S the preimage of x. This is a non-empty finite subset of $\widetilde{C}$. The desired curve C is obtained by removing from $\widetilde{C}$ all points but one of S. The remaining point of S is called c_0. The morphism α is just the composition $C \to \overline{G} \to Y$. Moreover, the image of C' in $\overline{G}$ is contained in G giving $\alpha' : C' \to G$. Finally, (3.1) follows from (3.2) which proves the claim.

Using this curve C, consider the trivial group schemes $\mathcal{G} := G \times C/C$ and $\mathcal{G}' := G \times C'/C'$. Then $\mathcal{G}'$ contains the subgroup scheme

$$\begin{aligned} \mathcal{H}' &:= \dot{\bigcup_{c \in C'}} \alpha'(c) H \alpha'(c)^{-1} = \left\{ (\alpha'(c) h \alpha'(c)^{-1}, c) \,\middle|\, h \in H,\ c \in C' \right\} \\ &\cong H \times C'. \end{aligned} \tag{3.3}$$

Let $\mathcal{H}$ be the closure of $\mathcal{H}'$ in $\mathcal{G}$. Then every irreducible component of $\mathcal{H}$ maps dominantly to C which implies that $\mathcal{H}$ is flat over C; see, e.g., [Ha2, 9.7]. Therefore, the same holds true for the fiber product $\mathcal{H}^2 := \mathcal{H} \times_C \mathcal{H}$ which implies that $\mathcal{H}' \times_C \mathcal{H}'$ is dense in $\mathcal{H}^2$. In turn, the multiplication map $\mathcal{G} \times_C \mathcal{G} \to \mathcal{G}$ maps $\mathcal{H}^2$ into $\mathcal{H}$. In other words, $\mathcal{H}$ is a flat subgroup scheme of $\mathcal{G}$. In particular, the fiber $H_c := \mathcal{H} \times_C \{c\}$ is a subgroup of G for every $c \in C$. Any flat group scheme is smooth (Cartier, see [DG]). Therefore, the Lie algebras $\mathfrak{h}_c := \operatorname{Lie} H_c$ form a vector bundle over C. By construction, we have $\mathfrak{h}_c = \alpha'(c) \cdot \mathfrak{h}$ for all $c \neq c_0$. Thus, by (3.1), we have $\mathfrak{h}_c = \alpha(c)$ for all c. In particular, $\mathfrak{h}_{c_0} = \mathfrak{m}$. This shows that $\mathfrak{m} = \operatorname{Lie} H_{c_0}$ is algebraic.

Finally, it is known that the quotient $\mathcal{G}/\mathcal{H}$ exists ([An, 4.C]). This is a flat deformation of homogeneous G-varieties with fibers G/H_c. Thus dimension, complexity and rank do not change [Kn1, 2.5] which proves the second assertion. □

Now we construct a partial compactification of T_X^*. First, choose an equivariant embedding of $X = G/H$ into a projective space: $X = G/H \hookrightarrow \mathbf{P}^N$. Using the orbit map

$$G/H \to \mathrm{Gr}(\mathfrak{g}), \quad gH \mapsto g \cdot \mathfrak{h}, \tag{3.4}$$

we get a diagonal embedding of G/H in $\mathbf{P}^N \times \mathrm{Gr}(\mathfrak{g})$. Let X' be its closure. This is a projective, possibly singular G-variety. Let $\overline{X} \to X'$ be an equivariant desingularization (see [AHV]). Then we have a smooth completion $G/H \hookrightarrow \overline{X}$ such that the orbit map (3.4) extends to a morphism $\overline{X} \to \mathrm{Gr}(\mathfrak{g})$.

Let $\mathcal{M} \to \overline{X}$ be the pull-back of the tautological vector bundle over $\mathrm{Gr}(\mathfrak{g})$. For $x \in \overline{X}$ let $\mathfrak{m}_x := \mathcal{M}_x$ be its fiber over x. Then, by construction, we have $\mathfrak{m}_x = g \cdot \mathfrak{h}$ whenever $x = gH \in G/H$. Thus, by Lemma 3.1, every $\mathfrak{m}_x$ is an algebraic Lie subalgebra of $\mathfrak{g}$ with the same dimension, complexity, and rank as $\mathfrak{h}$.

By construction, the vector bundle $\mathcal{M}$ is a subbundle of the trivial vector bundle $\mathfrak{g} \times \overline{X}$. Let $\overline{Z} \to \overline{X}$ be its annihilator in the dual bundle $\mathfrak{g}^* \times \overline{X}$. This means that each fiber $\overline{Z}_x$ equals $\mathfrak{m}_x^\perp \subseteq \mathfrak{g}^*$. We claim that $\overline{Z}$ is a partial compactification of $Z := T_X^*$. In fact, the restriction of $\overline{Z}$ to the open orbit G/H is $G \times^H \mathfrak{h}^\perp$ which equals the cotangent bundle over X. Note that $\overline{Z}$ is equipped with a natural G-equivariant morphism

$$\overline{\mu} : \overline{Z} \hookrightarrow \mathfrak{g}^* \times \overline{X} \twoheadrightarrow \mathfrak{g}^*, \tag{3.5}$$

which, on each fiber, is simply the natural embedding $\mathfrak{m}_x^\perp \hookrightarrow \mathfrak{g}^*$. This means that $\overline{\mu}$ is an extension of the moment map $\mu : Z \to \mathfrak{g}^*$ to $\overline{Z}$. The point is now that $\overline{\mu}$ is a *proper* morphism as one sees from the factorization (3.5) and the fact that $\overline{X}$ is complete.

4. A partial compactification: properties

The first step to our goal is the following lemma.

Lemma 4.1. *The morphism $\psi : Z \to \mathfrak{a}^*/W_X$ extends to a morphism $\overline{\psi} : \overline{Z} \to \mathfrak{a}^*/W_X$. Moreover, all fibers of $\overline{\psi}$ are connected.*

Proof. Let $\Gamma_\psi \subseteq Z \times \mathfrak{a}^*/W_X$ be the graph of ψ and let $\overline{\Gamma}_\psi$ be its closure in $\overline{Z} \times \mathfrak{a}^*/W_X$. Then the projection $\overline{\Gamma}_\psi \to \overline{Z}$ is an isomorphism over X, hence birational. Now observe that the composition $Z \to \mathfrak{a}^*/W_X \to \mathfrak{g}^*/\!\!/G$ has an extension to $\overline{Z}$, namely, the composition $\overline{Z} \to \mathfrak{g}^* \to \mathfrak{g}^*/\!\!/G$. This implies that Γ_ψ, hence $\overline{\Gamma}_\psi$, is contained in the fiber product $\Gamma' := \overline{Z} \times_{\mathfrak{g}^*/\!\!/G} \mathfrak{a}^*/W_X$. Since $\mathfrak{a}^*/W_X \to \mathfrak{g}^*/\!\!/G$ is a finite morphism, so are $\Gamma' \to \overline{Z}$ and $\overline{\Gamma}_\psi \to \overline{Z}$. Thus, the latter is an isomorphism since $\overline{Z}$ is normal (even smooth). This shows that $\overline{\psi}$ exists.

To show connectedness of fibers, we need another construction of [Kn1]. Consider the moment map $\mu : Z \to \mathfrak{g}^*$ and let M be the spectrum of the integral closure of $\mu^*\mathbb{C}[\mathfrak{g}^*]$ in $\mathbb{C}[Z]$. Then μ factors as $Z \xrightarrow{\varphi} M \to \mathfrak{g}^*$. As above, one shows that φ extends to a morphism $\overline{\varphi} : \overline{Z} \to M$ which factors $\overline{\mu}$. This implies that also $\overline{\varphi}$ is proper. By construction of M, the generic fibers of φ are irreducible. Hence the same holds for $\overline{\varphi}$. Since M is normal we can apply Zariski's connectedness theorem [Ha2, 11.3] and conclude that all fibers of $\overline{\varphi}$ are connected. On the other hand, we have a commutative diagram:

$$\begin{tikzcd} \overline{Z} \arrow[r, "\overline{\varphi}"] \arrow[dr, "\overline{\psi}"'] & M \arrow[d, "\pi"] \\ & \mathfrak{a}^*/W_X. \end{tikzcd} \tag{4.1}$$

If f is a G-invariant regular function on Z which is integral over $\mathbb{C}[\mathfrak{g}^*]$ then it is also integral over $\mathbb{C}[\mathfrak{g}^*]^G$ (apply the Reynolds operator to an integral equation of f). This means that π is just the categorical quotient by G. Thus, every fiber of π contains a unique closed (connected) orbit which implies that π has connected fibers, as well. Choose $u \in \mathfrak{a}^*/W_X$. By applying the following lemma to the morphism $\overline{\psi}^{-1}(u) \to \pi^{-1}(u)$ we conclude that $\overline{\psi}^{-1}(u)$ is connected. □

Lemma 4.2. *Let $\varphi : X \to Y$ be a surjective morphism whose fibers are all connected. Assume φ is either closed* (*e.g., proper*) *or open* (*e.g., flat*). *Assume, moreover, that Y is connected. Then X is connected as well.*

Proof. Assume that φ is closed/open. Suppose that X is not connected. Then $X = X_1 \mathbin{\dot{\cup}} X_2$ is a disjoint union of non-empty closed/open subsets. Thus, $Y = \varphi(X_1) \cup \varphi(X_2)$ is a union of non-empty closed/open subsets. Since Y is connected, this union cannot be disjoint. Choose $y \in \varphi(X_1) \cap \varphi(X_2)$. Then the fiber $F = \varphi^{-1}(y)$ is the union of two disjoint non-empty closed/open subsets, namely, $F \cap X_1$ and $F \cap X_2$, which contradicts the connectedness of F. □

Next, we investigate the restriction of $\overline{\psi}$ to the fibers $\overline{Z}_x = \mathfrak{m}_x^\perp$. Let c and r be the complexity and rank of G/H. For a fixed Borel subgroup B let $\overline{X}_0$ be the set of $x \in \overline{X}$ such that

$$\dim(\mathfrak{b} + \mathfrak{m}_x) = g - c \quad \text{and} \quad \dim(\mathfrak{u} + \mathfrak{m}_x) = g - c - r. \tag{4.2}$$

Since $\mathfrak{m}_x$ has also complexity c and rank r (Lemma 3.1), this is an open subset of $\overline{X}$ which intersects each G-orbit non-trivially. Let

$$\overline{\mathcal{B}} = \dot{\bigcup_{x \in \overline{X}_0}} (\mathfrak{b} + \mathfrak{m}_x)^\perp \quad \text{and} \quad \overline{\mathcal{U}} = \dot{\bigcup_{x \in \overline{X}_0}} (\mathfrak{u} + \mathfrak{m}_x)^\perp. \tag{4.3}$$

Then $\overline{\mathcal{B}} \subseteq \overline{\mathcal{U}} \subseteq \overline{Z}$ are sub-vector bundles. Consider the fiber of the quotient $\mathfrak{a}_x^* = \overline{\mathcal{U}}_x/\overline{\mathcal{B}}_x = (\mathfrak{b}/\mathfrak{u} + \mathfrak{b} \cap \mathfrak{m}_x)^*$ of $(\mathfrak{b}/\mathfrak{u})^* = \mathfrak{t}^*$. For $x \in X \cap \overline{X}_0$ it is always the same space $\mathfrak{a}^*$ (Lemma 2.1). Thus, by continuity, $\mathfrak{a}_x^* = \mathfrak{a}^*$ for all $x \in \overline{X}_0$ and we obtain a projection $\overline{\mathcal{U}} \twoheadrightarrow \mathfrak{a}^*$.

Lemma 4.3. *For $x \in \overline{X}$, let $W(x)$ be the little Weyl group of $\mathfrak{m}_x$. Then $W(x) \subseteq W_X$. Moreover, let $\overline{\psi}_x$ be the restriction of $\overline{\psi}$ to $\overline{Z}_x = \mathfrak{m}_x^\perp$. Then $\overline{\psi}_x$ is the composition*

$$\mathfrak{m}_x^\perp \xrightarrow{\psi_{\mathfrak{m}_x}} \mathfrak{a}^*/W(x) \twoheadrightarrow \mathfrak{a}^*/W_X. \tag{4.4}$$

Proof. The compactification $\overline{Z}$ is constructed in such a way that the restriction of $\overline{\mu}: \overline{Z} \to \mathfrak{g}^*$ to a fiber $\overline{Z}_x$ is the natural embedding $\mathfrak{m}_x^\perp \hookrightarrow \mathfrak{g}^*$. Thus, we obtain a commutative diagram (without the dotted arrow):

$$\begin{tikzcd}
\mathfrak{m}_x^\perp \arrow[r, hook] \arrow[d] & \overline{Z} \arrow[d] \\
\mathfrak{a}^*/W(x) \arrow[r, dotted] \arrow[dr] & \mathfrak{a}^*/W_X \arrow[d] \\
 & \mathfrak{g}^*/\!\!/G.
\end{tikzcd} \tag{4.5}$$

The universal property (as integral closure) implies the existence of the dotted arrow. In particular, $W(x) \subseteq W_X$. $\square$

Lemma 4.4. *Let C be an irreducible component of a fiber $\overline{\psi}^{-1}(u)$ and let $x \in \overline{X}$. Then $C \cap \overline{Z}_x$ is of pure codimension r in $\overline{Z}_x$. Moreover, C is of pure codimension r in $\overline{Z}$, and the projection $C \to \overline{X}$ is dominant.*

Proof. Clearly, the codimension of C in $\overline{Z}$ is at most r. Therefore, the codimension of every irreducible component of $C \cap \overline{Z}_x$ in $\overline{Z}_x$ is also at most r. On the other hand, $C \cap \overline{Z}_x$ is contained in a fiber of $\overline{\psi}_x$ which is of pure codimension r by Theorem 2.4(ii) and (4.4). Hence $C \cap \overline{Z}_x$ is of pure codimension r in $\overline{Z}_x$.

Now consider the projection $p: C \to \overline{X}$. Let Y be the closure of its image. We have seen that the fibers of p have dimension $n - r$. Hence

$$2n - r \leqslant \dim C = (n - r) + \dim Y \leqslant (n - r) + n = 2n - r, \tag{4.6}$$

which implies $\dim C = 2n - r$ and $\dim Y = n$, hence $Y = X$. $\square$

Corollary 4.5. *For $u \in \mathfrak{a}^*/W_X$ consider the fibers $F = \psi^{-1}(u) \subseteq Z$ and $\overline{F} = \overline{\psi}^{-1}(u) \subseteq \overline{Z}$. Then $\overline{F}$ is the closure of F in $\overline{Z}$.*

Proof. By Lemma 4.4, every irreducible component of $\overline{\psi}^{-1}(u)$ meets the open subset $\overline{\psi}^{-1}(u) \cap Z = \psi^{-1}(u)$. Hence, $\overline{\psi}^{-1}(u)$ is the closure of F. $\square$

5. The proof of the connectedness theorem in the homogeneous case

First, we construct very special points in the fiber $\overline{F}$ by applying the following result.

Proposition 5.1. *Let Y be an affine $\mathbf{G}_m$-variety, V a finite dimensional vector space, and $\varphi: V \to Y$ a $\mathbf{G}_m$-equivariant morphism. Assume that V has a closed irreducible $\mathbf{G}_m$-stable subvariety S such that the restriction $\varphi|_S : S \to Y$ is finite and surjective. Let C be an irreducible component of a fiber of φ. Then $C \cap S \neq \emptyset$.*

Proof. Let C be an irreducible component of $F := \varphi^{-1}(y)$, $y \in Y$. Assume first $y = y_0 := \varphi(0)$. Then C is a closed $\mathbf{G}_m$-stable subset of V. Hence $0 \in C \cap S$ and we are done. Thus, we may assume $y \neq y_0$.

The origin 0 is the only closed orbit of V, hence of S. Since $\varphi|_S$ is finite and surjective, the fixed point y_0 is the only closed orbit of Y. This implies, in particular, that $\mathbf{G}_m y$ is closed in $Y' := Y \setminus \{y_0\}$. Let $V_0 := \varphi^{-1}(y_0)$ and $V' := V \setminus V_0 = \varphi^{-1}(Y')$. Then $\mathbf{G}_m F = \varphi^{-1}(\mathbf{G}_m y)$ is closed in V'. Let $E \subset \mathbf{G}_m$ be the isotropy group of y. From $\mathbf{G}_m F \twoheadrightarrow \mathbf{G}_m y = \mathbf{G}_m/E$ we obtain $\mathbf{G}_m F = \mathbf{G}_m \times^E F$. The group E is necessarily a finite (cyclic) group. Hence the map $\mathbf{G}_m \times F \to \mathbf{G}_m F$ is proper, which implies that $Z := \mathbf{G}_m C$ is closed in $\mathbf{G}_m F$. We conclude that Z is also closed in V' or, equivalently,

$$\overline{Z} \setminus Z \subseteq V_0. \tag{5.1}$$

Let $r := \dim S$. Using again that $\varphi|_S$ is finite and surjective we have that Y is irreducible of dimension r. Since C is an irreducible component of a fiber of $\varphi: V \to Y$, we have $\operatorname{codim}_V C \leqslant \dim Y = r$. Thus, $\operatorname{codim}_V \overline{Z} < r$. Since $0 \in \overline{Z}$ and $\dim S = r$, the intersection $\overline{Z} \cap S$ is non-empty of positive dimension (here we use the smoothness of V). On the other hand, the finiteness of $\varphi|_S$ implies that $V_0 \cap S$ is a finite set. Thus we obtain from (5.1) that $Z \cap S \neq \emptyset$, which is equivalent to $C \cap S \neq \emptyset$. □

Corollary 5.2. *Let $\mathfrak{a}' \subseteq \mathfrak{h}^\perp$ as in Theorem* 2.4(iv) *and let C be an irreducible component of a fiber of $\psi_{\mathfrak{h}} : \mathfrak{h}^\perp \to \mathfrak{a}^*/W_X$. Then $C \cap \mathfrak{a}' \neq \emptyset$.*

Next we study the intersection $C \cap \mathfrak{a}'$ for the Lie algebras $\mathfrak{m}_x$ simultaneously for an open set of $x \in X_0$. For this, let $\mathcal{A} \subseteq \overline{\mathcal{U}}$ be a complementary vector bundle to $\overline{\mathcal{B}}$ over an open subset $\overline{X}_1 \subseteq \overline{X}_0$. That exists in the neighborhood of every point of $\overline{X}_0$. Since the fiber is $\mathcal{A}_x = \overline{\mathcal{U}}_x/\overline{\mathcal{B}}_x = \mathfrak{a}^*$, there is a canonical trivialization $\tau : \mathfrak{a}^* \times \overline{X}_1 \xrightarrow{\sim} \mathcal{A}$.

Lemma 5.3. *Let C be an irreducible component of the fiber $F := \overline{\psi}^{-1}(u)$ and assume there is $\alpha \in \mathfrak{a}^*$, $x_0 \in \overline{X}_1$ with $\tau(\alpha, x_0) \in C$. Then $\tau(\alpha, x) \in C$ for every $x \in \overline{X}_1$.*

Proof. The restriction of $\overline{\psi}$ to $\mathcal{A}$ is the composition $\mathcal{A} \twoheadrightarrow \mathfrak{a}^* \xrightarrow{\pi} \mathfrak{a}^*/W_X$. Therefore, $F \cap \mathcal{A}$ is of pure dimension $n = \dim X$. More precisely, $F \cap \mathcal{A} = \tau(S \times \overline{X}_1)$ where S is the W_X-orbit $\pi^{-1}(u)$. On the other hand, C is of pure codimension r in $\overline{Z}$ and $\mathcal{A}$ is irreducible of dimension $n + r$. This implies that the dimension of every irreducible component of $C \cap \mathcal{A}$ is at least n. Since $C \cap \mathcal{A} \subseteq F \cap \mathcal{A}$, we conclude that $C \cap \mathcal{A}$ is the union of irreducible components of $F \cap \mathcal{A} \cong S \times \overline{X}_1$. Hence there is a subset S' of S with $C \cap \mathcal{A} = \tau(S' \times \overline{X}_1)$. By assumption we have $(\alpha, x_0) \in S' \times \overline{X}_1$. Hence $\alpha \in S'$ and $\tau(\alpha, x) \in C \cap \mathcal{A} \subseteq C$ for all $x \in \overline{X}_1$. $\square$

Now we are able to prove Theorem 1.1 for $X = G/H$. Consider the fiber $F = \psi^{-1}(u)$ and suppose that F is disconnected. Then $F = F_1 \dot{\cup} F_2$ is the disjoint union of non-empty closed subsets. By Corollary 4.5, the closure of F in $\overline{Z}$ is the fiber $\overline{F} = \overline{\psi}^{-1}(u)$ which is connected by Lemma 4.1. This implies $\overline{F_1} \cap \overline{F_2} \neq \emptyset$ but we can say even more. As a fiber of an equidimensional map between smooth varieties, F is locally a complete intersection. A theorem of Hartshorne [Ha1, 3.4] then asserts that F is even connected in codimension one, i.e., it stays connected upon removal of any subset of codimension two or higher. Applied to our situation, we conclude that $\overline{F_1} \cap \overline{F_2}$ has an irreducible component C_0 which is of codimension 1 in $\overline{F}$, hence of codimension $r + 1$ in $\overline{Z}$.

Since $C_0 \cap Z \subseteq \overline{F_1} \cap \overline{F_2} \cap Z = F_1 \cap F_2 = \emptyset$ we have $C_0 \cap Z = \emptyset$. Let X' be the closure of the image of C_0 in $\overline{X}$. Then $X' \cap X = \emptyset$. Let $x \in X'$. Then $C_0 \cap \overline{Z}_x$ is a subset of a fiber of $\overline{\psi}_x$, hence has dimension at most $n - r$ (Lemma 4.4). This implies

$$\begin{aligned} 2n - r - 1 &= \dim C_0 \leqslant (n - r) + \dim X' \leqslant (n - r) + (n - 1) \\ &= 2n - r - 1. \end{aligned} \tag{5.2}$$

Since equality has to hold throughout, we obtain, in particular, that there exists an $x_0 \in X'$ such that $\dim C_0 \cap \overline{Z}_{x_0} = n - r$. Therefore, $C_0 \cap \overline{Z}_{x_0}$ is the union of irreducible components of a fiber of $\overline{\psi}_x$. Now choose a complement $\mathcal{A}$ of $\overline{\mathcal{B}} \subseteq \overline{\mathcal{U}}$ in a neighborhood of x_0. From Corollary 5.2, we obtain $C_0 \cap \mathcal{A}_{x_0} \neq \emptyset$. In particular, there is $\alpha \in \mathfrak{a}^*$ with $\tau(\alpha, x_0) \in C_0$.

Now let $C_1 \subseteq F_1$, $C_2 \subseteq F_2$ be irreducible components such that $C_0 \subseteq \overline{C_1} \cap \overline{C_2}$. Since $\tau(\alpha, x_0) \in \overline{C_i}$ we have $\tau(\alpha, x) \in \overline{C_i}$ for all x in a neighborhood of x_0 (Lemma 5.3). If we choose $x \in X$ we get $\tau(\alpha, x) \in C_1 \cap C_2 \subseteq F_1 \cap F_2$, a contradiction to $F_1 \cap F_2 = \emptyset$. This finishes the proof of Theorem 1.1 when X is homogeneous.

6. The proof in general

From now on, let X be any smooth G-variety. To prove the general case we use the following trivial lemma.

Lemma 6.1. *Let F be a variety and $F' \subseteq F$ an open subset. Assume*

(i) *F' is connected*;
(ii) *for every point $\eta \in F$ there are two irreducible subvarieties D and E of F such that $\eta \in D$, $D \cap E \neq \emptyset$, and $E \cap F' \neq \emptyset$.*

Then F is connected.

Fix $u \in \mathfrak{a}^*/W_X$ and let $F := \psi^{-1}(u)$. To construct F' we start with the following lemma.

Lemma 6.2. *Let X be an irreducible G-variety. Then there is a non-empty open G-stable subset $X' \subseteq X$ with the following properties*:

(i) *The orbit space $Q = X'/G$ exists, i.e., there is a G-invariant surjective morphism $\pi : X' \to Q$ such that all fibers $X'_v := \pi^{-1}(v)$ are reduced and homogeneous.*
(ii) *The spaces X' and Q are smooth.*
(iii) *All orbits X'_v share the same dimension, complexity, rank, and little Weyl group W_X.*

Proof. The existence of an open subset X' as in (i) is a well-known theorem of Rosenlicht ([Ro]; see also [Sp, Section 2]). Then (ii) can be achieved by shrinking Q to an open subset. Finally, (iii) follows from [Kn1, 2.5] (for dimension, complexity, and rank) and [Kn1, 6.5.4] (for the little Weyl group). □

Let $X' \subseteq X$ be as in the lemma. Let

$$T^*_{X'/Q} := \dot{\bigcup_{v \in Q}} T^*_{X'_v} \to X' \tag{6.1}$$

be the relative cotangent bundle. Since $W_{X'_v} = W_X$, we obtain morphisms $\psi_v : T^*_{X'_v} \to \mathfrak{a}^*/W_X$ which glue to a morphism $\psi_* : T^*_{X'/Q} \to \mathfrak{a}^*/W_X$. Let $N \subseteq T^*_{X'/Q}$ be the fiber $\psi_*^{-1}(u)$ and let $\pi : N \to Q$ be the projection. Then $\pi^{-1}(v) = \psi_v^{-1}(u)$ is connected since X'_v is homogeneous. Upon shrinking Q to an open subset we may assume that π is flat and surjective. Thus, Lemma 4.2 implies that N is connected. Next observe that there is a projection $p : T^*_{X'} \twoheadrightarrow T^*_{X'/Q}$. In fact, this is a locally trivial bundle of affine spaces and therefore flat, surjective with connected fibers. On the other hand, the morphism $\psi : T^*_{X'} \to \mathfrak{a}^*/W_X$ is just the composition $\psi_* \circ p$. Thus

$$F' := F \cap T^*_{X'} = p^{-1}(N)$$

is connected, which verifies part (i) of Lemma 6.1.

For the second part, let η be any point of F and let Y be the image of $G\eta$ in X. We need to compare the moment map for X with that for the orbit $Y \subseteq X$. The main tool to study the interrelation is the following local structure theorem.

Theorem 6.3. *There exists a parabolic subgroup P (containing B) with Levi part L (containing T) and an affine L-stable subvariety S of X such that*:

(i) *The natural morphism $P \times^L S \to X$ is an open embedding.*
(ii) *The intersection $Y_0 := Y \cap S$ is a non-empty L-variety.*
(iii) *Let L_0 be the kernel of the action of L on Y_0. Then $A_Y := L/L_0$ is a torus.*
(iv) *The action of A_Y on Y_0 is locally free, i.e., has finite isotropy groups.*

Proof. This theorem is essentially due to Brion–Luna–Vust [BLV]. We use the refinement in Sections 1–2 of [Kn2] as follows. By [Kn2, 2.1], we may replace X by a G-stable open subset which supports an ample G-linearized line bundle $\mathcal{L}$. Now we choose a section σ of a power of $\mathcal{L}$, as in [Kn2, 2.10]. Let $P = G_{\langle\sigma\rangle}$ be the stabilizer of the line $\langle\sigma\rangle = \mathbb{C}\sigma$.

Now (i) follows from [Kn2, 1.2.3], while (ii) is implied by [Kn2, 2.10.2]. Furthermore, [Kn2, 2.10.3 and 2.8.1] assert (iii). Finally, by [Kn2, 2.10.4] all orbits of A_Y in Y_0 are closed. Since Y_0 is affine, all orbits are of the same dimension. Since the generic isotropy group is trivial by construction, we see that all isotropy groups are finite. □

Let P_u be the unipotent radical of P. Since $P = P_u \times L$ (as a right L-variety), we have $P \times^L S = P_u \times S$. Thus,

$$P_u \times S \to X \tag{6.2}$$

is an open embedding. That implies that S is smooth and irreducible. Moreover, for every $x \in S$ the tangent space splits:

$$T_{X,x} = \mathfrak{p}_u x \oplus T_{S,x}. \tag{6.3}$$

That allows us to embed T_S^* into T_X^*. The relationship between the moment maps is given by the following lemma.

Lemma 6.4. *The image of T_S^* under the moment map on T_X^* is in $\mathfrak{p}_u^\perp$. Moreover, the following diagram commutes*:

$$\begin{tikzcd}
T_S^* \arrow[rr, hook] \arrow[d, "\mu_S"'] \arrow[dr] & & T_X^* \arrow[d, "\mu_X"] \\
\mathfrak{l}^* \arrow[d] & \mathfrak{p}_u^\perp \arrow[l] \arrow[r, hook] & \mathfrak{g}^* \arrow[d] \\
\mathfrak{l}^*/\!\!/L \arrow[rr] & & \mathfrak{g}^*/\!\!/G.
\end{tikzcd} \tag{6.4}$$

Proof. That image of T_S^* under the moment map on T_X^* is in $\mathfrak{p}_u^\perp$, is a reformulation of the definition of the embedding $T_S^* \hookrightarrow T_X^*$. That proves the commutativity of the quadrangle. The triangle commutes since $S \hookrightarrow X$ is L-equivariant. Finally, the commutativity of the pentagon is given by [Kn1, 6.1]. $\square$

Now choose $y \in Y_0$. Since S is affine and since the orbit Ly is closed, it has a "slice" in S, i.e., an irreducible, smooth, L_0-stable subvariety S_0 of S with $y \in S_0$ and such that $L \times^{L_0} S_0 \to S$ is étale. The Lie algebra of L decomposes as $\mathfrak{l} = \mathfrak{a}_Y \oplus \mathfrak{l}_0$ which induces a decomposition of tangent spaces: $T_{S,x} = \mathfrak{a}_Y x \oplus T_{S_0,x}$ for every $x \in S_0$. Thus, we get embeddings and a commutative diagram

$$\begin{tikzcd} \mathfrak{a}_Y^* \times S_0 \arrow[r, hook] \arrow[d] & T_S^* \arrow[r, hook] \arrow[d] & T_X^* \arrow[d] \\ \mathfrak{a}_Y^* \arrow[r] & \mathfrak{l}^* /\!\!/ L \arrow[r] & \mathfrak{g}^* /\!\!/ G. \end{tikzcd} \tag{6.5}$$

Now we look more closely at the subspace $\mathfrak{a}^0 := \mathfrak{a}_Y^* \times \{y\} \subseteq T_{X,y}^*$. We have

$$T_{X,y} = \mathfrak{p}_u y \oplus \mathfrak{a}_Y y \oplus T_{S_0,y}. \tag{6.6}$$

Since $\mathfrak{a}_Y y = \mathfrak{l} y$, we have

$$\mathfrak{p}_u y \oplus \mathfrak{a}_Y y = \mathfrak{b} y \quad \text{and} \quad \mathfrak{p}_u y = \mathfrak{u} y. \tag{6.7}$$

Thus, $\mathfrak{a}^0$ is a complement of $(\mathfrak{b}y)^\perp$ in $(\mathfrak{u}y)^\perp$.

On the other hand, since $P \times^L Y_0 \to Y$ is an open embedding, the generic P-orbit in Y is isomorphic to P/L_0. Because $Py = P/L_0 = B/(B \cap L_0) = By$ it follows that By represents a generic B-orbit in Y. Moreover, Uy is a generic U-orbit in Y.

Now consider the restriction map $\rho : T_{X,y}^* \twoheadrightarrow T_{Y,y}^* = \mathfrak{g}_y^\perp$. The moment map μ clearly factors through ρ. Moreover, the image $\mathfrak{a}' := \rho(\mathfrak{a}^0)$ is a complement of $(\mathfrak{b}y)^\perp$ in $(\mathfrak{u}y)^\perp$. It follows from Corollary 5.2 that $\mathfrak{a}'$ intersects every irreducible component of every fiber of $T_{Y,y}^* \to \mathfrak{g}^* /\!\!/ G$. The same holds for $\mathfrak{a}^0$ with respect to the morphism $T_{X,y}^* \to \mathfrak{g}^* /\!\!/ G$.

Now we are able to prove the connectedness theorem in general: by changing η in its G-orbit we may assume that $\eta \in T_{X,y}^* \cap F$. Let D be an irreducible component of the fiber of $T_{X,y}^* \to \mathfrak{g}^* /\!\!/ G$ containing η. By construction, F is a union of connected components of a fiber of $T_X^* \to \mathfrak{g}^* /\!\!/ G$. Thus $D \subseteq F$. Moreover, there is $\alpha \in \mathfrak{a}_Y^*$ with $(\alpha, y) \in D$. Let $E := \{\alpha\} \times S_0$. Then the commutative diagrams (6.4) and (6.5) show that E is contained in a fiber of $T_X^* \to \mathfrak{g}^* /\!\!/ G$. Thus $E \subseteq F$ and $D \cap E \neq \emptyset$. Finally, $G \cdot S_0$ is dense in X. Hence $S_0 \cap X' \neq \emptyset$ implies $E \cap F' \neq \emptyset$. Thus, we conclude with Lemma 6.1 that F is connected.

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