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# Large-time behavior of solutions for the one-dimensional infrarelativistic model of a compressible viscous gas with radiation

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ABSTRACT

In this paper, assuming suitable hypotheses on the transport coefficients, we prove the large-time behavior, as time tends to infinity, of solutions in $\mathcal{H}_i = H^i \times H_0^i \times H^i \times H^{i+1}$ $(i = 1, 2)$ for the one-dimensional infrarelativistic model of a compressible viscous gas with radiation.



## 1. Introduction

It is well known that radiation dynamics includes the radiative effects into the hydrodynamical framework. When equilibrium holds between the matter and the radiation, a simple way to do that is to include local radiation terms into the state functions and the transport coefficients. We know from quantum mechanics that radiation can be described by its quanta, the photons, which are massless particles traveling at the speed $c$ of light, characterized by their frequency $\nu$, their energy $E = h\nu$ (where $h$ is Planck's constant), and their momentum $\vec{p} = \frac{h\nu}{c}\vec{\Omega}$ with $\vec{\Omega}$ as a vector of the 2-unit sphere. Moreover, from statistical mechanics, we can describe macroscopically an assembly of massless photons of energy $E$ and momentum $\vec{p}$ by using a distribution function: the radiative density $I(r, t, \vec{\Omega}, \nu)$. Using this fundamental quantity, we can derive global quantities by integrating with respect to the angular and frequency variables: the spectral radiative energy density $E_R(r, t)$ per unit volume is then $E_R(r, t) := \frac{1}{c} \iint I(r, t, \vec{\Omega}, \nu)\, d\Omega\, d\nu$, and the spectral radiative flux

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$\overrightarrow{F_R} = \iint \vec{\Omega} I(r,t,\vec{\Omega},\nu)\, d\Omega\, d\nu$. If the matter is in thermodynamic equilibrium at constant temperature $T$ and if radiation is also in thermodynamic equilibrium at matter, its temperature is also $T$ and statistical mechanics tells us that the distribution function for photons is given by the Bose–Einstein statistics with zero chemical potential.

If there are no radiative effects, we know that the complete hydrodynamical system can be derived from the standard conservation laws of mass, momentum and energy by using Boltzmann's equation satisfied by the $f_m(r,\vec{v},t)$ and the Chapman–Enskog expansion [9]. Then we can get the compressible Navier–Stokes system

$$\begin{cases} \rho_t + \nabla\cdot(\rho\vec{u}) = 0, \\ (\rho\vec{u})_t + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) = -\nabla\cdot\vec{\Pi} + \vec{f}, \\ (\rho\varepsilon)_t + \nabla\cdot(\rho\varepsilon\vec{u}) = -\nabla\vec{q} - \vec{D} : \vec{\Pi} + g, \end{cases} \tag{1.1}$$

where $\vec{\Pi} = -p(\rho,T)\vec{I} + \vec{\pi}$ is the material stress tensor for a Newtonian fluid with the viscous contribution $\vec{\pi} = 2\mu\vec{D} + \lambda\nabla\cdot\vec{u}\,\vec{I}$ with $3\lambda + 2\mu \geqslant 0$ and $\mu > 0$, and the strain tensor $\vec{D}$ such that $\vec{D}_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. $\vec{q}$ is the thermal heat flux and $\vec{f}$ and $g$ are external force and source terms.

When radiation is present, Chandrasekhar [2] investigated the radiation integro-differential equation: the terms $\vec{f}$ and $g$ include the terms for the coupling between the matter and the radiation, depending on $I$, and $I$ is driven by a transport equation.

If the matter is at local thermodynamics equilibrium, the coupled system reads (see, e.g., [15,16] for details)

$$\begin{cases} \rho_t + \nabla\cdot(\rho\vec{u}) = 0, \\ (\rho\vec{u})_t + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) = -\nabla\cdot\vec{\Pi} + \overrightarrow{S_F}, \\ (\rho\varepsilon)_t + \nabla\cdot(\rho\varepsilon\vec{u}) = -\nabla\vec{q} - \vec{D} : \vec{\Pi} + S_E, \\ \dfrac{1}{c}\dfrac{\partial}{\partial t} I(r,t,\vec{\Omega},\nu) + \vec{\Omega}\cdot\nabla I(r,t,\vec{\Omega},\nu) = S_t(r,t,\vec{\Omega},\nu), \end{cases} \tag{1.2}$$

where $\rho(x,t)$, $\vec{u}(x,t)$, $\theta(x,t)$ represent the density, velocity and temperature respectively, the coupling terms are

$$\begin{aligned} S_t(r,t,\vec{\Omega},\nu) = {}& \sigma_a\left(\nu,\vec{\Omega},\rho,T,\frac{\vec{\Omega}\cdot\vec{u}}{c}\right)\big[B(\nu,T) - I(r,t,\vec{\Omega},\nu)\big] + \iint \sigma_s\big(r,t,\rho,\vec{\Omega}'\cdot\vec{\Omega},\nu'\to\nu\big) \\ & \times\left\{\frac{\nu}{\nu'} I\big(r,t,\vec{\Omega}',\nu'\big) I(r,t,\vec{\Omega},\nu)\right. \\ & \left. - \sigma_s\big(r,t,\rho,\vec{\Omega}'\cdot\vec{\Omega},\nu'\to\nu\big) I\big(r,t,\vec{\Omega}',\nu'\big) I(r,t,\vec{\Omega},\nu)\right\} d\Omega'\, d\nu', \end{aligned}$$

the radiative energy source

$$S_E(r,t) := \iint S_t(r,t,\vec{\Omega},\nu)\, d\Omega\, d\nu,$$

the radiative flux

$$\overrightarrow{S_F}(r,t) := \frac{1}{c}\iint \vec{\Omega} S_t(r,t,\vec{\Omega},\nu)\, d\Omega\, d\nu,$$

the functions $\sigma_a$ and $\sigma_s$ describe in a phenomenological way the absorption–emission and scattering properties of the photon-matter interaction, and Planck's function $B(\nu,\theta)$ describes the frequency-temperature black body distribution. We would like to mention some results in [1,4,11–14,20].

In the one-dimensional case, (1.2) can be reduced into the following system

$$\begin{cases} \rho_\tau + (\rho v)_y = 0, \\ (\rho v)_\tau + \left(\rho v^2\right)_y + p_y = \mu v_{yy} - (S_F)_R, \\ \left[\rho\left(e + \frac{1}{2}v^2\right)\right]_\tau + \left[\rho v\left(e + \frac{1}{2}v^2\right) + pv - \kappa\theta_y - \mu v v_y\right]_y = -(S_E)_R, \\ \frac{1}{c}I_t + \omega I_y = S. \end{cases} \tag{1.3}$$

Now we assume that the fluid motion is small enough with respect to the velocity of light $c$ so that we can drop all the $\frac{1}{c}$ factors in the previous formulation and then get an "infrarelativistic" model of a compressible Navier–Stokes system for a one-dimensional flow coupled to the radiative transfer equation given in the following system

$$\begin{cases} \rho_\tau + (\rho v)_y = 0, \\ (\rho v)_\tau + \left(\rho v^2\right)_y + p_y = \mu v_{yy}, \\ \left[\rho\left(e + \frac{1}{2}v^2\right)\right]_\tau + \left[\rho v\left(e + \frac{1}{2}v^2\right) + pv - \kappa\theta_y - \mu v v_y\right]_y = -(S_E)_R, \\ \omega I_y = S. \end{cases} \tag{1.4}$$

Under the Lagrangian coordinates, i.e.,

$$x = \int_0^y \rho(\xi, \tau)\, d\xi, \qquad t = \tau,$$

system (1.4) is transformed into the following system

$$\eta_t = v_x, \tag{1.5}$$

$$v_t = \sigma_x, \tag{1.6}$$

$$\left(e + \frac{1}{2}v^2\right)_t = (\sigma v - Q)_x - \eta(S_E)_R, \tag{1.7}$$

$$\omega I_x = \eta S, \tag{1.8}$$

where $x \in [0,1]$, $\eta$ is the specific volume (i.e., $\eta = \frac{1}{\rho}$), $v$ denotes the velocity, $\theta$ is the temperature, $I$ represents the radiative intensity depending on the Lagrangian mass coordinates $(x,t)$ and also on two extra variables: the radiation frequency $\nu \in R_+ = (0, +\infty)$ and the angular variable $\omega \in S^1 := [-1, 1]$.

We denote by $\sigma := -p + \mu\frac{v_x}{\eta}$ the stress and by $Q := -\kappa\frac{\theta_x}{\eta}$ the heat flux with the heat conductivity $\kappa$ and the viscosity coefficient $\mu$. The source term $S$ in the last equation is expressed as

$$\begin{aligned} S(x,t;\nu,\omega) = {} & \sigma_a(\nu,\omega;\eta,\theta)\big[B(\nu;\theta) - I(x,t;\nu,\omega)\big] \\ & + \sigma_s(\nu;\eta,\theta)\big[\tilde{I}(x,t;\nu) - I(x,t;\nu,\omega)\big], \end{aligned} \tag{1.9}$$

where $\tilde{I}(x,t,\nu) := \frac{1}{2}\int_{-1}^{1} I(x,t;\nu,\omega)\, d\omega$ and $B$ is a function of temperature and frequency describing the equilibrium state.

We define the radiative energy

$$E_R = \int_{-1}^{1} \int_{0}^{\infty} I(x,t;\nu,\omega)\,d\nu\,d\omega, \tag{1.10}$$

the radiative flux

$$F_R = \int_{-1}^{1} \int_{0}^{\infty} \omega I(x,t;\nu,\omega)\,d\nu\,d\omega, \tag{1.11}$$

and the radiative energy source

$$(S_E)_R = \int_{-1}^{1} \int_{0}^{\infty} S(x,t;\nu,\omega)\,d\nu\,d\omega. \tag{1.12}$$

We consider a typical initial boundary value problem for (1.5)–(1.8) in the reference domain $Q := \Omega \times [0,+\infty) = (0,1) \times [0,+\infty)$ under the Dirichlet–Neumann boundary conditions for the fluid unknowns

$$v(0,t) = v(1,t) = 0, \qquad Q(0,t) = Q(1,t) = 0, \quad \forall t \geqslant 0, \tag{1.13}$$

and transparent boundary conditions for the radiative intensity

$$\begin{cases} I(0,t;\nu,\omega) = 0 & \text{for } \omega \in (0,1),\ \forall t \geqslant 0, \\ I(1,t;\nu,\omega) = 0 & \text{for } \omega \in (-1,0),\ \forall t \geqslant 0, \end{cases} \tag{1.14}$$

and initial conditions

$$\eta(x,0) = \eta_0(x), \qquad v(x,0) = v_0(x), \qquad \theta(x,0) = \theta_0(x) \quad \text{on } \Omega, \tag{1.15}$$

and

$$I(x,0;\nu,\omega) = I_0(x;\nu,\omega) \quad \text{on } \Omega \times R_+ \times S^1. \tag{1.16}$$

Pressure and energy of the matter are related by the thermodynamical relation

$$e_\eta(\eta,\theta) = -p(\eta,\theta) + \theta p_\theta(\eta,\theta). \tag{1.17}$$

We assume that $e$, $p$, $\sigma$ and $\kappa$ are twice continuously differential on $0 < \eta < +\infty$ and $0 \leqslant \theta < +\infty$, and we suppose the following growth conditions:

$$\begin{cases} e(\eta,0)\geqslant 0, \quad c_1\big(1+\theta^r\big)\leqslant e_\theta(\eta,\theta)\leqslant C_1\big(1+\theta^r\big),\\ -c_2\eta^{-2}\big(1+\theta^{1+r}\big)\leqslant p_\eta(\eta,\theta)\leqslant -C_2\eta^{-2}\big(1+\theta^{1+r}\big),\\ \big|p_\theta(\eta,\theta)\big|\leqslant C_3\eta^{-1}\big(1+\theta^r\big),\\ c_4\big(1+\theta^{1+r}\big)\leqslant \eta p(\eta,\theta)\leqslant C_4\big(1+\theta^{1+r}\big), \quad p_\eta(\eta,\theta_0)\leqslant 0,\\ 0\leqslant p(\eta,\theta)\leqslant C_5\big(1+\theta^{1+r}\big),\\ c_6\big(1+\theta^q\big)\leqslant \kappa(\eta,\theta)\leqslant C_6\big(1+\theta^q\big),\\ \big|\kappa_\eta(\eta,\theta)\big|+\big|\kappa_{\eta\eta}(\eta,\theta)\big|\leqslant C_7\big(1+\theta^q\big),\\ \eta\sigma_a(\nu,\omega;\eta,\theta)B^m(\nu,\theta)\leqslant C_8|\omega|\theta^{\alpha+1}f(\nu,\omega) \quad \text{for } m=1,2,\\ 0<\sigma_a(\nu,\omega;\eta,\theta)\leqslant C_9|\omega|^2 g(\nu,\omega),\\ \big[\sigma_a+\big|(\sigma_a)_\eta\big|+\big|(\sigma_a)_\theta\big|\big](\nu,\omega;\eta,\theta)\big[1+B(\nu,\theta)+\big|B_\theta(\nu,\theta)\big|+\big|B_{\theta\theta}(\nu,\theta)\big|\big]\leqslant C_{10}|\omega|h(\nu,\omega),\\ 0<\sigma_s(\nu;\eta,\theta)\leqslant C_{11}|\omega|^2 k(\nu,\omega),\\ \big[\big|(\sigma_a)_{\eta\eta}\big|+\big|(\sigma_a)_{\eta\theta}\big|+\big|(\sigma_a)_{\theta\theta}\big|\big](\nu,\omega;\eta,\theta)\big(1+B(\nu,\theta)+\big|B_\theta(\nu,\theta)\big|\big)\leqslant C_{12}|\omega|l(\nu,\omega),\\ \big[\big|(\sigma_s)_\eta\big|+\big|(\sigma_s)_\theta\big|+\big|(\sigma_s)_{\eta\eta}\big|+\big|(\sigma_s)_{\eta\theta}\big|+\big|(\sigma_s)_{\theta\theta}\big|\big](\nu,;\eta,\theta)\leqslant C_{13}|\omega|\mathcal{M}(\nu,\omega), \end{cases} \tag{1.18}$$

where $r\in[0,1]$, $q\geqslant r+1$, $0\leqslant\alpha$, the numbers $c_i$, $C_j$ $(i=1,\dots,7,\ j=1,\dots,13)$ are positive constants and the nonnegative functions $f$, $g$, $h$, $k$, $l$, $\mathcal{M}$ are such that

$$f,g,h,k,l,\mathcal{M}\in L^1\big(R_+\times S^1\big)\cap L^\infty\big(R_+\times S^1\big).$$

We assume that the viscosity coefficient $\mu$ is a positive constant. In the following, we denote

$$\mathcal{I}(x,t):=\int_0^\infty\int_{S^1} I(x,t;\nu,\omega)\,d\omega\,d\nu$$

for the integrated radiative intensity. In particular,

$$\mathcal{I}(x,0)=\mathcal{I}_0=\int_0^\infty\int_{S^1} I(x,0;\nu,\omega)\,d\omega\,d\nu.$$

We define

$$\mathcal{H}_i=H^i(0,1)\times H_0^i(0,1)\times H^i(0,1)\times H^{i+1}(0,1) \quad (i=1,2).$$

Let us recall some previous works concerning radiative fluids. In paper [6], Ducomet and Nečasová considered the following system

$$\begin{cases} \eta_t=v_x,\\ v_t=\sigma_x-\eta(S_F)_R,\\ \left(e+\dfrac{1}{2}v^2\right)=(\sigma v-Q)_x-\eta(S_E)_R,\\ I_t+\eta^{-1}(c\omega-v)I_x=cS \end{cases} \tag{1.19}$$

with $(S_F)_R=\frac{1}{c}\int_{-1}^1\int_0^\infty \omega S(x,t;\nu,\omega)\,d\nu\,d\omega$ in the domain $(0,M)\times R_+$. They considered the Dirichlet–Neumann boundary conditions

$$v|_{x=0,M}=0, \qquad Q|_{x=0,M}=0, \tag{1.20}$$

and

$$I|_{x=0}=0 \quad \text{for } \omega\in(0,1), \qquad I|_{x=M}=0 \quad \text{for } \omega\in(-1,0). \tag{1.21}$$

Under suitable assumptions and $q\geqslant r+1$, they proved the existence and uniqueness of weak solutions. But all the estimates depended on any given time $T>0$. So they didn't show the large-time behavior of problem (1.19)–(1.21). Recently, Ducomet and Nečasová [7] investigated the problem (1.19) with the different boundary conditions from [6]

$$v|_{x=0,M}=0, \qquad Q|_{x=0,M}=0, \tag{1.22}$$

and

$$I|_{x=0}=I_b(\nu) \quad \text{for } \omega\in(0,1), \qquad I|_{x=M}=I_b(\nu) \quad \text{for } \omega\in(-1,0). \tag{1.23}$$

They proved that the unique strong solutions of (1.19) converged to a well-determined equilibrium state at exponential rate in $H^1(0,M)$ for the fluid variables $\eta$, $v$, $\theta$ and in $L^2(0,M)$ for the radiative intensity $\mathcal{I}$.

For the system (1.5)–(1.16), Ducomet and Nečasová [5] established the global existence of solutions in $\mathcal{H}_i$ $(i=1,2)$. But the estimates obtained there depend on any given time $T$, so it is impossible to establish the large-time behavior of global solutions in $\mathcal{H}_i$ $(i=1,2)$. Moreover, in [5], all the estimates hold only for $q\geqslant 2r+1$. We establish the uniform-in-time estimates of $(\eta(t),v(t),\theta(t),\mathcal{I}(t))$ in $\mathcal{H}_i$ $(i=1,2)$, which hold for $q\geqslant r+1$. Hence our results improve those in [5]. Furthermore, the system we will consider here is different from the system in [17], so our uniform-in-time estimates are also different from those in [17]. Since the large-time behavior of global solutions in $\mathcal{H}_i$ $(i=1,2)$ is open, we study this issue in this paper.

The main aim of this paper is to establish the large-time behavior of solutions to the system (1.5)–(1.16). In Section 2, we shall first establish uniform-in-time estimates which are independent of any length of time and prove the uniform upper and positive lower bounds of specific volume (i.e., $\frac{1}{\rho}$) away from 0 in Lagrangian coordinates and establish the uniform-in-time estimates in $\mathcal{H}_1$. In Section 3, we prove the large-time behavior of solutions in $\mathcal{H}_1$. In Section 4, we establish the uniform-in-time estimates in $\mathcal{H}_2$. We shall prove the large-time behavior in $\mathcal{H}_2$ in Section 5.

The notation in this paper will be as follows:

$L^q$, $1\leqslant q\leqslant+\infty$, $W^{m,q}$, $m\in N$, $H^1=W^{1,2}$, $H_0^1=W_0^{1,2}$ denote the usual (Sobolev) spaces on [0, 1]. In addition, $\|\cdot\|_B$ denotes the norm in the space $B$; we also put $\|\cdot\|=\|\cdot\|_{L^2[0,1]}$. Subscripts $t$ and $x$ denote the (partial) derivatives with respect to $t$ and $x$, respectively. We use $C_i$ $(i=1,2)$ to denote the generic positive constants depending on the $\|(\eta_0,v_0,\theta_0,\mathcal{I}_0)\|_{\mathcal{H}_i}$, $\min_{x\in[0,1]}\eta_0(x)$, $\min_{x\in[0,1]}\theta_0(x)$, but not depending on $t$.

Our main results read as follows.

**Theorem 1.1.** *Suppose that $(\eta_0,v_0,\theta_0,\mathcal{I}_0)\in\mathcal{H}_1$ and the compatibility conditions hold. Then there exists a unique global solution $(\eta(t),v(t),\theta(t),\mathcal{I}(t))\in L^\infty([0,+\infty),\mathcal{H}_1)$ to the problem* (1.5)–(1.16) *verifying that*

$$0<C_1^{-1}\leqslant\eta(x,t)\leqslant C_1, \quad \forall(x,t)\in[0,1]\times[0,+\infty) \tag{1.24}$$

*and*

$$\|\eta(t)-\bar\eta\|_{H^1}^2+\|v(t)\|_{H^1}^2+\|\theta(t)-\bar\theta\|_{H^1}^2+\|\mathcal{I}(t)\|_{H^2}^2+\int_0^t\big(\|\eta-\bar\eta\|_{H^1}^2+\|v\|_{H^2}^2$$

$$+\|\theta-\bar\theta\|_{H^2}^2+\|\theta_t\|^2\big)(s)\,ds+\int_0^t\int_0^1\int_0^\infty\int_{S^1} I_t^2\,d\omega\,d\nu\,dx\,ds\leqslant C_1,\quad \forall t>0. \tag{1.25}$$

*Moreover, as $t\to+\infty$, we have*

$$\|\eta(t)-\bar\eta\|_{H^1}\to 0,\qquad \|v(t)\|_{H^1}\to 0,\qquad \|\theta(t)-\bar\theta\|_{H^1}\to 0,\qquad \|\mathcal{I}(t)\|_{H^2}\to 0, \tag{1.26}$$

*where $\bar\eta=\int_0^1\eta(x,t)\,dx=\int_0^1\eta_0\,dx$, $\bar\theta>0$ is determined by $e(\bar\eta,\bar\theta)=\int_0^1(\frac{1}{2}v_0^2+e(\eta_0,\theta_0)+F_R(0))\,dx$.*

**Theorem 1.2.** *Suppose that $(\eta_0,v_0,\theta_0,\mathcal{I}_0)\in\mathcal{H}_2$ and the compatibility conditions hold. Then there exists a unique global solution $(\eta(t),v(t),\theta(t),\mathcal{I}(t))\in L^\infty([0,+\infty),\mathcal{H}_2)$ to the problem* (1.5)–(1.16) *satisfying for any $t>0$*

$$\|\eta(t)-\bar\eta\|_{H^2}^2+\|v(t)\|_{H^2}^2+\|\theta(t)-\bar\theta\|_{H^2}^2+\|\mathcal{I}(t)\|_{H^3}^2+\|v_t(t)\|^2+\|\theta_t(t)\|^2$$

$$+\int_0^t\big(\|v_{xt}\|^2+\|\theta_{xt}\|^2+\|\theta-\bar\theta\|_{H^3}^2+\|v\|_{H^3}^2+\|\eta-\bar\eta\|_{H^2}^2\big)(s)\,ds\leqslant C_2. \tag{1.27}$$

*Moreover, as $t\to+\infty$, we have*

$$\|\eta(t)-\bar\eta\|_{H^2}\to 0,\qquad \|v(t)\|_{H^2}\to 0,\qquad \|\theta(t)-\bar\theta\|_{H^2}\to 0,\qquad \|\mathcal{I}(t)\|_{H^3}\to 0. \tag{1.28}$$

**Remark 1.1.** Theorems 1.1–1.2 also hold for the boundary conditions (1.14) and

$$v(0,t)=v(1,t)=0,\qquad \theta(0,t)=\theta(1,t)=T_0=const.>0,$$

where $\bar\theta$ can be replaced by $T_0$.

**Remark 1.2.** The relation between $q$ and $r$ in our results also holds in more general situations as in the book of Qin [17].

## 2. Uniform-in-time estimates in $\mathcal{H}_1$

First we shall establish some uniform-in-time estimates in $\mathcal{H}_1$.

**Lemma 2.1.** *Under the assumptions in Theorem* 1.1, *the following estimates hold*

$$\theta(x,t)>0,\quad \forall(x,t)\in[0,1]\times[0,+\infty), \tag{2.1}$$

$$\int_0^1\eta(x,t)\,dx=\int_0^1\eta_0(x)\,dx\equiv\bar\eta_0,\quad \forall t>0, \tag{2.2}$$

$$\int_0^1 \big(\theta + \theta^{1+r}\big)(x,t)\,dx \leqslant C_1, \quad \forall t > 0, \tag{2.3}$$

$$\int_0^1 \big[(\theta - \log\theta - 1) + \theta^{1+r} + v^2\big](x,t)\,dx + \int_0^t \int_0^1 \left(\frac{(1+\theta^q)\theta_x^2}{\eta\theta^2} + \frac{\mu v_x^2}{\eta\theta}\right)(x,s)\,dx\,ds \leqslant C_1. \tag{2.4}$$

**Proof.** Inequality (2.1) is a consequence of the generalized maximum principle [3] applied to the following equation, which is equivalent to (1.7)

$$e_\theta(\eta,\theta)\theta_t + \theta p_\theta(\eta,\theta)v_x - \mu\frac{v_x^2}{\eta} + \eta(S_E)_R = \left(\frac{\kappa(\eta,\theta)\theta_x}{\eta}\right)_x \tag{2.5}$$

by considering the positivity of $\theta_0$.

Integrating (1.5) over $Q_t = (0,1)\times(0,t)$, and using the boundary conditions, we can easily deduce (2.2).

From (1.8), (1.11) and (1.12), we can infer

$$(F_R)_x = \eta(S_E)_R. \tag{2.6}$$

Inserting (2.6) into (1.7), we arrive at

$$\left(e + \frac{1}{2}v^2\right)_t = (\sigma v - q - F_R)_x. \tag{2.7}$$

Integrating (2.7) over $Q_t$ and using boundary conditions (1.13)–(1.14) gives

$$\int_0^1 \left(e + \frac{1}{2}v^2\right)(x,t)\,dx + \int_0^t F_R\big|_{x=0}^{x=1}\,ds = \int_0^1 \left(e_0 + \frac{1}{2}v_0^2\right)(x)\,dx. \tag{2.8}$$

Using (1.14), the contribution of the radiation term reads (see, e.g., [5])

$$\int_0^t F_R\big|_{x=0}^{x=1}\,ds = \int_0^t \left[\int_0^\infty \int_0^1 \omega I(1,t;\nu,\omega)\,d\omega\,d\nu - \int_0^\infty \int_{-1}^0 \omega I(0,t;\nu,\omega)\,d\omega\,d\nu\right] ds$$
$$\geqslant 0,$$

which, together with (2.8), implies

$$\int_0^1 \left(e + \frac{1}{2}v^2\right)(x,t)\,dx \leqslant C_1. \tag{2.9}$$

Combining (2.9) with (1.18) yields (2.3).

Noting that the radiative term $\eta(S_E)_R$ appears in (2.5), our estimate (2.9) is different from the one in [17] where there is no radiative term.

We define the free energy $\psi := e - \theta S$ with $\psi_\theta = -S$ and $\psi_\eta = -p$. Let us consider the auxiliary function

$$E(\eta,\theta) := \psi(\eta,\theta) - \psi(1,1) - (\eta-1)\psi_\eta(1,1) - (\theta-1)\psi_\theta(\eta,\theta). \tag{2.10}$$

We have the following estimate (see, e.g., [5] for details)

$$\begin{aligned}&\int_0^1 \left(E + \frac{1}{2}v^2\right) dx + \int_0^t \int_0^1 \left(\frac{\mu v_x^2}{\eta\theta} + \frac{\kappa\theta_x^2}{\eta\theta^2}\right) dx\, ds + \int_0^t \int_0^1 \frac{\eta}{\theta} \int_0^\infty \int_{S^1} \sigma_a I\, d\omega\, d\nu\, dx\, ds \\ &\quad + \int_0^t \left[\int_0^\infty \int_0^1 \omega I(1,t;\nu,\omega)\, d\omega\, d\nu - \int_0^\infty \int_{-1}^0 \omega I(0,t;\nu,\omega)\, d\omega\, d\nu\right] ds \leqslant C_1.\end{aligned} \tag{2.11}$$

Using the Taylor theorem and the definition of $E(\eta,\theta)$, we can conclude

$$\begin{aligned}&E(\eta,\theta) - \psi(\eta,\theta) + \psi(\eta,1) + (\theta-1)\psi_\theta(\eta,\theta) \\ &\quad = \psi(\eta,1) - \psi(1,1) - \psi_\eta(\eta,\theta) \\ &\quad = (\eta-1)^2 \int_0^1 (1-\xi)\psi_{\eta\eta}\big(1+\xi(\eta-1),1\big)\, d\xi \geqslant 0.\end{aligned}$$

Thus,

$$\begin{aligned}E(\eta,\theta) &\geqslant \psi(\eta,\theta) - \psi(\eta,1) - (\theta-1)\psi_\theta(\eta,\theta) \\ &= -(1-\theta)^2 \int_0^1 (1-\tau)\psi_{\theta\theta}\big(\eta,\theta+\tau(1-\theta)\big)\, d\tau \\ &\geqslant C_1^{-1}(1-\theta)^2 \int_0^1 \frac{(1-\tau)\{1+[\theta+\tau(1-\theta)]^r\}}{\theta+\tau(1-\theta)}\, d\tau \\ &= \begin{cases} C_1^{-1}(\theta - \log\theta - 1) + \frac{C_1^{-1}(1-\theta^r)}{r} + \frac{C_1^{-1}(1-\theta^{r+1})}{r+1} & \text{for } r>0, \\ 2C_1^{-1}(\theta-\log\theta-1) & \text{for } r=0, \end{cases} \\ &\geqslant C_1^{-1}(\theta-\log\theta-1) + C_1^{-1}\theta^{r+1} - C_1^{-1}.\end{aligned} \tag{2.12}$$

Combining (2.12) and (2.11), and using (1.18) yields (2.4). The proof is now complete. □

The following two lemmas concerning the uniform-in-time estimate of specific volume $\eta$ play a very crucial role in this paper. The uniform-in-time estimate is different from the one in [5], where the estimate was dependent on any given time $T > 0$.

**Lemma 2.2.** *For any $t \geqslant 0$, there exists one point $x_1 = x_1(t) \in [0,1]$ such that the solution $\eta(x,t)$ to the problem* (1.5)–(1.8), (1.13)–(1.16) *possesses the following expression*:

$$\eta(x,t)=D(x,t)Z(t)\left\{1+\frac{1}{\mu}\int_0^t \eta(x,s)p(x,s)D^{-1}(x,s)Z^{-1}(s)\,ds\right\} \tag{2.13}$$

*where*

$$D(x,t)=\eta_0(x)\exp\left\{\frac{1}{\mu}\left(\int_{x_1(t)}^x v(y,t)\,dy-\int_0^x v_0(y)\,dy+\frac{1}{\bar{\eta}_0}\int_0^1 \eta_0(x)\int_0^x v_0(y)\,dy\,dx\right)\right\}, \tag{2.14}$$

$$Z(t)=\exp\left\{-\frac{1}{\mu\bar{\eta}_0}\int_0^t\int_0^1 \left(v^2+\eta p\right)(y,s)\,dy\,ds\right\}. \tag{2.15}$$

**Proof.** See, e.g., [17]. □

**Lemma 2.3.** *There holds that*

$$0<C_1^{-1}\leqslant \eta(x,t)\leqslant C_1, \quad \forall (x,t)\in[0,1]\times[0,+\infty), \tag{2.16}$$

$$\int_0^t \left\|v(s)\right\|_{L^\infty}^2 ds\leqslant C_1, \quad \forall t>0. \tag{2.17}$$

**Proof.** Let

$$M_\eta(t)=\max_{x\in[0,1]}\eta(x,t).$$

By Young's inequality, Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}
&\left|\int_{x_1(t)}^x v(y,t)\,dy-\int_0^x v_0(y)\,dy+\frac{1}{\bar{\eta}_0}\int_0^1 \eta_0(x)\int_0^x v_0(y)\,dy\,dx\right| \\
&\quad\leqslant\left(\int_0^1 v^2\,dy\right)^{\frac{1}{2}}+\left(\int_0^1 v_0^2\,dy\right)^{\frac{1}{2}}+\frac{1}{\bar{\eta}_0}\int_0^1 \eta_0(x)\left(\int_0^1 v_0^2\,dy\right)^{\frac{1}{2}}dx \\
&\quad\leqslant C_1\|v\|^2+C_1\leqslant C_1.
\end{aligned} \tag{2.18}$$

Eqs. (2.18) and (2.14) imply that there exists some positive constant $C_1>0$ such that

$$0<C_1^{-1}\leqslant D(x,t)\leqslant C_1, \quad \forall (x,t)\in[0,1]\times[0,+\infty).$$

From (1.18) and (2.1), we deduce

$$\int_0^1 \left(v^2+\eta p\right)(x,t)\,dx\geqslant\int_0^1 \eta p(x,t)\,dx\geqslant\int_0^1 \left(c_4+\theta^{1+r}\right)dx\geqslant C_1^{-1}, \quad \forall t>0. \tag{2.19}$$

By Lemma 2.1, we conclude

$$\int_0^1 \big(v^2+\eta p\big)(x,t)\,dx \leqslant \|v\|^2 + C_4\int_0^1 \big(1+\theta^{1+r}\big)\,dx \leqslant C_1, \quad \forall t>0. \tag{2.20}$$

Therefore, from (2.19) and (2.20), we infer for $0\leqslant s\leqslant t$,

$$C_1^{-1}(t-s)\leqslant \int_s^t\int_0^1 \big(v^2+\eta p\big)(x,s)\,dx\,ds \leqslant C_1(t-s), \tag{2.21}$$

which, together with (2.15), implies that for any $0\leqslant s\leqslant t$

$$e^{-C_1(t-s)}\leqslant Z(t)Z^{-1}(s)=\exp\left\{-\frac{1}{\mu\bar{\eta}_0}\int_s^t\int_0^1 \big(v^2+\eta p\big)(y,s)\,dy\,ds\right\}\leqslant e^{-C_1^{-1}(t-s)}. \tag{2.22}$$

Now, for any $t>0$, there exists a point $a(t)\in[0,1]$ such that

$$\begin{aligned}\left|\theta^{\frac{r+1}{2}}(x,t)-\theta^{\frac{r+1}{2}}\big(a(t),t\big)\right| &= \left|\int_{a(t)}^x \big(\theta^{\frac{r+1}{2}}(x,t)\big)_x\,dy\right| \leqslant C_1\int_0^1 \theta^{\frac{r-1}{2}}|\theta_x|\,dx\\ &\leqslant C_1\left(\int_0^1 \frac{1+\theta^q}{\eta\theta^2}\theta_x^2\,dx\right)^{\frac12}\left(\int_0^1 \eta\frac{\theta^{r+1}}{1+\theta^q}\,dx\right)^{\frac12}\\ &\leqslant C_1V^{\frac12}(t)\end{aligned} \tag{2.23}$$

where $V(t)=\int_0^1 \frac{1+\theta^q}{\eta\theta^2}\theta_x^2\,dx$.

Then for any $(x,t)\in[0,1]\times[0,+\infty)$, we get

$$C_1^{-1}-C_1^{-1}V(t)\leqslant \theta^{r+1}(x,t)\leqslant C_1+C_1V(t). \tag{2.24}$$

Thus we conclude from Lemma 2.2 and (2.22)–(2.24)

$$\begin{aligned}\eta(x,t)&=D(x,t)\left[Z(t)+\frac{1}{\mu}\int_0^t \eta(x,s)p(x,s)D^{-1}(x,s)Z(t)Z^{-1}(s)\,ds\right]\\ &\leqslant C_1\left[e^{-C_1t}+\int_0^t \big(1+V(s)M_\eta(s)\big)e^{-C_1(t-s)}\,ds\right]\\ &\leqslant C_1+C_1\int_0^t M_\eta(s)V(s)\,ds\end{aligned}$$

whence

$$M_\eta(t) \leqslant C_1 + C_1 \int_0^t M_\eta(s)V(s)\,ds,$$

which, by using Gronwall's inequality and (2.4), yields

$$M_\eta(t) \leqslant C_1. \tag{2.25}$$

By (2.13) and (2.22), there exists a large time $t_0$ such that as $t \geqslant t_0$, $x \in [0,1]$

$$\begin{aligned}
\eta(x,t) &= D(x,t)Z(t)\left\{1 + \frac{1}{\mu}\int_0^t \eta(x,s)p(x,s)D^{-1}(x,s)Z^{-1}(s)\,ds\right\} \\
&\geqslant C_1^{-1}\left[e^{-C_1 t} + \int_0^t e^{-C_1(t-s)}\,ds\right] \\
&\geqslant C_1^{-1}\int_0^t e^{-C_1(t-s)}\,ds \geqslant (2C_1)^{-1}.
\end{aligned} \tag{2.26}$$

Noting that $D(x,t) \geqslant C_1^{-1}$, $Z(t) \geqslant \exp(-C_1 t)$, we infer that for any $(x,t) \in [0,1] \times [0,t_0]$,

$$\eta(x,t) \geqslant D(x,t)Z(t) \geqslant C_1^{-1}\exp(-C_1 t) \geqslant C_1^{-1}\exp(-C_1 t_0),$$

which, together with (2.26), implies that for any $(x,t) \in [0,1] \times [0,+\infty)$

$$\eta(x,t) \geqslant C_1^{-1}. \tag{2.27}$$

Combining (2.25) and (2.27), we easily get (2.16).

By Hölder's inequality, (2.16) and Lemma 2.1, we obtain for any $t \geqslant 0$

$$\int_0^t \|v(s)\|_{L^\infty}^2\,ds \leqslant \int_0^t \left(\int_0^1 |v_x|\,dx\right)^2 ds \leqslant \int_0^t \left(\int_0^1 \frac{v_x^2}{\theta}\,dx\right)\left(\int_0^1 \theta\,dx\right) ds \leqslant C_1.$$

The proof is complete. □

The following lemma is a new uniform-in-time estimate.

**Lemma 2.4.** *Under the assumptions in Theorem* 1.1, *the following estimates hold for any* $t > 0$,

$$\|\eta_x(t)\|^2 + \int_0^t \int_0^1 \left(1 + \theta^{1+r}\right)\eta_x^2\,dx\,ds \leqslant C_1, \tag{2.28}$$

$$\|v(t)\|^2 + \int_0^t \|v_x(s)\|^2\,ds \leqslant C_1. \tag{2.29}$$

**Proof.** Eq. (1.6) can be written as

$$\left(v-\mu\frac{\eta_x}{\eta}\right)_t+p_\eta\eta_x=-p_\theta\theta_x. \tag{2.30}$$

Multiplying (2.30) by $(v-\mu\frac{\eta_x}{\eta})$ and then integrating the result over $Q_t$, we have

$$\begin{aligned}&\frac{1}{2}\left\|v-\mu\frac{\eta_x}{\eta}\right\|^2+\int_0^t\int_0^1\frac{-\mu p_\eta\eta_x^2}{\eta}\,dx\,ds\\&=\frac{1}{2}\left\|v_0-\mu\frac{\eta_{0x}}{\eta_0}\right\|^2-\int_0^t\int_0^1\left[p_\eta\eta_x v+p_\theta\theta_x\left(v-\mu\frac{\eta_x}{\eta}\right)\right]dx\,ds.\end{aligned}$$

From Young's inequality, (1.18) and Lemmas 2.1–2.3, we can infer

$$\begin{aligned}&\frac{1}{2}\left\|v-\mu\frac{\eta_x}{\eta}\right\|^2+\int_0^t\int_0^1\left(1+\theta^{1+r}\right)\eta_x^2\,dx\,ds\\&\leqslant C_1+C_1\int_0^t\int_0^1\left[\left(1+\theta^{1+r}\right)|\eta_x v|+\left(1+\theta^r\right)\left|\theta_x\left(v-\mu\frac{\eta_x}{\eta}\right)\right|\right]dx\,ds\\&\leqslant C_1+\frac{\varepsilon}{2}\int_0^t\int_0^1\left(1+\theta^{1+r}\right)\eta_x^2\,dx\,ds+C(\varepsilon)\int_0^t\left\|v(s)\right\|_{L^\infty}^2\int_0^1\left(1+\theta^{1+r}\right)dx\,ds\\&\quad+C_1\int_0^t\int_0^1\frac{(1+\theta^q)\theta_x^2}{\eta\theta^2}\,dx\,ds+C_1\int_0^t\left\|v(s)\right\|_{L^\infty}^2\int_0^1\left(1+\theta^{2r+2-q}\right)dx\,ds\\&\quad+C_1\int_0^t\int_0^1\left(1+\theta^r\right)|\theta_x\eta_x|\,dx\,ds\\&\leqslant C_1+\frac{\varepsilon}{2}\int_0^t\int_0^1\left(1+\theta^{1+r}\right)\eta_x^2\,dx\,ds+C_1\int_0^t\int_0^1\left(1+\theta^r\right)|\theta_x\eta_x|\,dx\,ds.\end{aligned} \tag{2.31}$$

Now we estimate the last term of (2.31). Noting that $q\geqslant r+1$, using Young's inequality and Lemmas 2.1–2.3, we can conclude

$$\begin{aligned}\int_0^t\int_0^1\left(1+\theta^r\right)|\theta_x\eta_x|\,dx\,ds&\leqslant\frac{\varepsilon}{2}\int_0^t\int_0^1\left(1+\theta^{1+r}\right)\eta_x^2\,dx\,ds+C(\varepsilon)\int_0^t\int_0^1\frac{(1+\theta^r)^2}{1+\theta^{1+r}}\theta_x^2\,dx\,ds\\&\leqslant\frac{\varepsilon}{2}\int_0^t\int_0^1\left(1+\theta^{1+r}\right)\eta_x^2\,dx\,ds+C_1,\end{aligned} \tag{2.32}$$

which, together with (2.31), yields

$$\left\| v-\mu\frac{\eta_x}{\eta}\right\|^2+\int_0^t\int_0^1\left(1+\theta^{1+r}\right)\eta_x^2\,dx\,ds\leqslant\varepsilon\int_0^t\int_0^1\left(1+\theta^{1+r}\right)\eta_x^2\,dx\,ds+C_1.$$

Taking $\varepsilon>0$ small enough, we get (2.28).

Multiplying (1.6) by $v$, integrating the result over $Q_t$, and using Young's inequality, Lemmas 2.1–2.3 and (2.28), we derive

$$\begin{aligned}
&\frac{1}{2}\int_0^1 v^2\,dx+\int_0^t\int_0^1\mu\frac{v_x^2}{\eta}\,dx\,ds\\
&=\frac{1}{2}\int_0^1 v_0^2\,dx-\int_0^t\int_0^1(p_\eta\eta_x+p_\theta\theta_x)v\,dx\,ds\\
&\leqslant C_1+C_1\int_0^t\int_0^1\left[\left(1+\theta^{1+r}\right)|\eta_x v|+\left(1+\theta^r\right)|\theta_x v|\right]dx\,ds\\
&\leqslant C_1+C_1\int_0^t\int_0^1\left(1+\theta^{1+r}\right)\eta_x^2\,dx\,ds+C_1\int_0^t\int_0^1\left(1+\theta^{1+r}\right)v^2\,dx\,ds+C_1\int_0^t\int_0^1\frac{(1+\theta^q)\theta_x^2}{\theta^2}\,dx\,ds\\
&\leqslant C_1.
\end{aligned}$$

The proof is complete. □

**Lemma 2.5.** *Under the assumptions in Theorem* 1.1, *the following estimates hold for any* $t>0$,

$$\int_0^1\left(\theta^2+\theta^{2+2r}+v_x^2\right)(x,t)\,dx+\int_0^t\left\|v_x(s)\right\|_{L^\infty}^2 ds\leqslant C_1, \tag{2.33}$$

$$\int_0^t\left\|v_{xx}(s)\right\|^2 ds+\int_0^t\int_0^1\left(1+\theta^{q+r}\right)\theta_x^2\,dx\,ds+\int_0^t\int_0^1\left(1+\theta^{2+2r}\right)\eta_x^2\,dx\,ds\leqslant C_1. \tag{2.34}$$

**Proof.** See, e.g., [7]. □

**Lemma 2.6.** *Under the assumptions in Theorem* 1.1, *the following estimate holds for any* $t>0$,

$$\left\|v_x(t)\right\|^2+\int_0^t\left\|v_t(s)\right\|^2 ds\leqslant C_1. \tag{2.35}$$

**Proof.** Multiplying (1.6) by $v_t$, integrating the result over $Q_t$, and using Young's inequality and (2.34), we deduce

$$\begin{aligned}\|v_x(t)\|^2+\int_0^t\int_0^1 v_t^2\,dx\,ds &\leqslant C_1+C_1\int_0^t\int_0^1 |p_\eta\eta_x v_t+p_\theta\theta_x v_t|\,dx\,ds\\ &\leqslant C_1+\varepsilon\int_0^t\int_0^1 v_t^2\,dx\,ds+C_1\int_0^t\int_0^1\left(1+\theta^{2+2r}\right)\eta_x^2\,dx\,ds\\ &\quad+C_1\int_0^t\int_0^1\left(1+\theta^{q+r}\right)\theta_x^2\,dx\,ds\\ &\leqslant C_1+\varepsilon\int_0^t\int_0^1 v_t^2\,dx\,ds\end{aligned}$$

which, by taking $\varepsilon>0$ small enough, implies (2.35). The proof is complete. □

Under assumptions of Lemma 2.3, we can deduce the new uniform-in-time estimates on radiative term $I(x,t;\nu,\omega)$ given in the following lemma, which are more complicated, delicate than and quite different from those in [5], where estimates are not uniform-in-time.

**Lemma 2.7.** *The following estimates hold for any $t>0$,*

$$\int_0^t\int_0^1\left(\int_0^\infty\int_{S^1}(\sigma_a+\sigma_s)I^2\,d\omega\,d\nu\right)dx\,ds\leqslant C_1\max_{(x,s)\in Q_t}\theta^{\alpha+1}(x,s), \tag{2.36}$$

$$\int_0^t\int_0^1\left(\int_0^\infty\int_{S^1}\sigma_s(\tilde I-I)^2\,d\omega\,d\nu\right)dx\,ds\leqslant C_1\max_{(x,s)\in Q_t}\theta^{\alpha+1}(x,s), \tag{2.37}$$

$$\int_0^\infty\int_{S^1}I(x,t;\nu,\omega)\,d\omega\,d\nu\leqslant C_1+C_1\max_{x\in\Omega}\theta^{\max(\alpha-2r-1,0)}(x,t). \tag{2.38}$$

**Proof.** Multiplying (1.8) by $I$, integrating the result over $(0,1)\times S^1$ and using boundary conditions (1.13)–(1.14), we get

$$\begin{aligned}&\frac12\int_{S^1}\omega I^2(1,t;\nu,\omega)\,d\omega-\frac12\int_{S^1}\omega I^2(0,t;\nu,\omega)\,d\omega+\int_0^1\int_{S^1}\eta(\sigma_a+\sigma_s)I^2\,d\omega\,dx\\ &\quad+\int_0^1\int_{S^1}\eta\sigma_s(\tilde I-I)^2\,d\omega\,dx=\int_0^1\int_{S^1}\eta\sigma_a BI\,d\omega\,dx.\end{aligned}$$

Integrating the above equality over frequency, estimating the right-hand side by the Young inequality and using (1.18), we have

$$\frac{1}{2}\int_0^\infty\int_{S^1}\omega I^2(1,t;\nu,\omega)\,d\omega\,d\nu-\frac{1}{2}\int_0^\infty\int_{S^1}\omega I^2(0,t;\nu,\omega)\,d\omega\,d\nu$$

$$+\int_0^\infty\int_0^1\int_{S^1}\eta(\sigma_a+\sigma_s)I^2\,d\omega\,dx\,d\nu+\int_0^\infty\int_0^1\int_{S^1}\eta\sigma_s(\tilde I-I)^2\,d\omega\,dx\,d\nu$$

$$\leqslant\frac{1}{2}\int_0^\infty\int_0^1\int_{S^1}\eta(\sigma_a+\sigma_s)I^2\,d\omega\,dx\,d\nu+\frac{1}{2}\int_0^\infty\int_0^1\int_{S^1}\eta\frac{\sigma_a^2}{\sigma_a+\sigma_s}B^2\,d\omega\,dx\,d\nu$$

$$\leqslant\frac{1}{2}\int_0^\infty\int_0^1\int_{S^1}\eta(\sigma_a+\sigma_s)I^2\,d\omega\,dx\,d\nu+C_1\int_0^\infty\int_0^1\int_{S^1}|\omega|\theta^{\alpha+1}f(\nu,\omega)\,d\omega\,dx\,d\nu$$

$$\leqslant\frac{1}{2}\int_0^\infty\int_0^1\int_{S^1}\eta(\sigma_a+\sigma_s)I^2\,d\omega\,dx\,d\nu+C_1\max_{x\in[0,1]}\theta^{\alpha+1}(x,t),$$

which, together with (2.16), implies (2.36) and (2.37).

In order to derive (2.38), we consider the following integro-differential equation

$$\begin{cases}\omega\dfrac{\partial}{\partial x}I(x,t;\nu,\omega)=\eta\sigma_a(\nu,\omega;\eta,\theta)\big[B(\nu,\theta)-I(x;\nu,\omega)\big]\\ \quad+\eta\sigma_s(\nu;\eta,\theta)\big[\tilde I(x;\nu)-I(x;\nu,\omega)\big]\quad\text{on }\Omega\times[0,t]\times R_+\times S^1,\\ I(0,t;\nu,\omega)=0\quad\text{for }\omega\in(0,1),\\ I(1,t;\nu,\omega)=0\quad\text{for }\omega\in(-1,0),\\ I(x,0;\nu,\omega)=I_0(x;\nu,\omega)\quad\text{on }\Omega\times R_+\times S^1.\end{cases}\tag{2.39}$$

Solving explicitly the ordinary differential equation and using boundary condition (1.14), we arrive at (see [5] for details)

$$I(x,t;\nu,\omega)=\begin{cases}\int_0^x e^{\int_x^y\frac{\eta(\sigma_a+\sigma_s)}{\omega}dz}\frac{\eta}{\omega}(\sigma_aB+\sigma_s\tilde I)\,dy & \text{for }\omega\in(0,1),\\ -\int_x^1 e^{\int_x^y\frac{\eta(\sigma_a+\sigma_s)}{\omega}dz}\frac{\eta}{\omega}(\sigma_aB+\sigma_s\tilde I)\,dy & \text{for }\omega\in(-1,0).\end{cases}\tag{2.40}$$

Using Young's inequality, (2.36), (2.37), (2.33) and (1.18), we have for $\omega\in(0,1)$

$$\int_0^\infty\int_{S^1}I\,d\nu\,d\omega=\int_0^\infty\int_{S^1}\left(\int_0^x e^{\int_x^y\frac{\eta(\sigma_a+\sigma_s)}{\omega}dz}\frac{\eta}{\omega}(\sigma_aB+\sigma_s\tilde I)\,dy\right)d\nu\,d\omega$$

$$\leqslant\left|\int_0^\infty\int_{S^1}\int_0^1\frac{\eta}{\omega}(\sigma_aB+\sigma_s\tilde I)\,dy\,d\nu\,d\omega\right|$$

$$=\left|\int_0^1\eta\int_0^\infty\int_{S^1}\frac{1}{\omega}\sigma_aB\,d\omega\,d\nu\,dx+\int_0^1\eta\int_0^\infty\int_{S^1}\frac{1}{\omega}\sigma_s(\tilde I-I+I)\,d\omega\,d\nu\,dx\right|$$

$$\leqslant C_1 \int_0^1 \theta^{\alpha+1}\,dx + C_1 \left| \int_0^1 \eta \int_0^\infty \int_{S^1} \left[ \frac{1}{\omega^2}\sigma_s + \sigma_s(\tilde{I} - I)^2 + \sigma_s \tilde{I}^2 \right] d\omega\, d\nu\, dx \right|$$

$$\leqslant C_1 + C_1 \int_0^1 \theta^{\alpha+1}\,dx \leqslant C_1 + C_1 \max_{x\in\Omega} \theta^{\max(\alpha-2r-1,0)}(x,t),$$

where we have used the fact

$$\tilde{I}^2 \leqslant C_1 \int_{S^1} I^2\,d\omega.$$

Analogously, we have the same result for $\omega \in (-1, 0)$. The proof is complete. □

In the following lemma, we shall prove the new uniform-in-time upper bound on temperature $\theta(x,t)$. The difficulty of the proof is how to derive the uniform-in-time estimate on the last term $\int_0^t \int_0^1 \eta(S_E)_R K_t\,dx\,ds$ in (2.44) below.

**Lemma 2.8.** *Under the assumptions in Theorem* 1.1, *the following estimates hold for any* $t > 0$,

$$\int_0^1 (1+\theta^{2q})\theta_x^2(x,t)\,dx + \int_0^t \int_0^1 (1+\theta^{q+r})\theta_t^2\,dx\,ds \leqslant C_1, \tag{2.41}$$

$$\max_{(x,s)\in Q_t} \theta(x,s) \leqslant C_1. \tag{2.42}$$

**Proof.** Let

$$K(\eta,\theta) = \int_0^\theta \frac{\kappa(\eta,u)}{\eta}\,du,$$

$$X(t) = \int_0^t \int_0^1 (1+\theta^{q+r})\theta_t^2\,dx\,ds, \qquad Y(t) = \int_0^1 (1+\theta^{2q})\theta_x^2\,dx.$$

Then it is easy to verify that

$$K_t = K_\eta v_x + \frac{\kappa}{\eta}\theta_t, \qquad K_{xt} = \left(\frac{\kappa\theta_t}{\eta}\right)_t + K_{\eta\eta} v_x \eta_x + \left(\frac{\kappa}{\eta}\right)_\eta \eta_x\theta_t + K_\eta v_{xx}.$$

We know from (1.18) that

$$|K_\eta| + |K_{\eta\eta}| \leqslant C_1\big(1+\theta^{q+1}\big).$$

Eq. (1.7) can be rewritten as

$$e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta} v_x^2 = \left(\frac{\kappa\theta_x}{\eta}\right)_x - \eta(S_E)_R. \tag{2.43}$$

Multiplying (2.43) by $K_t$ and integrating the result over $Q_t$, we arrive at

$$\int_0^t\int_0^1\left(e_\theta\theta_t+\theta p_\theta v_x-\frac{\mu}{\eta}v_x^2\right)K_t\,dx\,ds+\int_0^t\int_0^1\left[\left(\frac{\kappa\theta_x}{\eta}\right)K_{tx}+\eta(S_E)_R K_t\right]dx\,ds=0. \tag{2.44}$$

Now we estimate each term in (2.44). The first two integrals lead to the same estimates as in [7], we have

$$\int_0^t\int_0^1\kappa e_\theta\theta_t^2\,dx\,ds\geqslant C_1^{-1}X(t),$$

$$\left|\int_0^t\int_0^1 e_\theta\theta_t K_\eta v_x\,dx\,ds\right|\leqslant\frac{\varepsilon}{4}X(t)+C_1\Big(1+\max_{(x,s)\in Q_t}\theta^{q+r+2}(x,s)\Big),$$

$$\left|\int_0^t\int_0^1\left(\theta p_\theta v_x-\frac{\mu}{\eta}v_x^2\right)K_t\,dx\,ds\right|\leqslant\frac{\varepsilon}{4}X(t)+C_1\Big(1+\max_{(x,s)\in Q_t}\theta^{q+r+2}(x,s)\Big),$$

$$\left|\int_0^t\int_0^1\frac{\kappa\theta_x}{\eta}\left(\frac{\kappa\theta_x}{\eta}\right)_t dx\,ds\right|\geqslant C_1^{-1}Y(t)-C_1^{-1},$$

$$\left|\int_0^t\int_0^1\frac{\kappa\theta_x}{\eta}(K_\eta v_{xx}+K_{\eta\eta}v_x\eta_x)\,dx\,ds\right|\leqslant C_1\Big(1+\max_{(x,s)\in Q_t}\theta^{1+\frac{3q}{2}}(x,s)\Big),$$

$$\left|\int_0^t\int_0^1\frac{\kappa\theta_x}{\eta}\left(\frac{\kappa}{\eta}\right)_\eta\eta_x\theta_t\,dx\,ds\right|\leqslant\frac{\varepsilon}{4}X(t)+C_1\Big(1+\max_{(x,s)\in Q_t}\theta^{2q+1}(x,s)\Big).$$

Now we estimate the last term in (2.44),

$$\begin{aligned}\left|\int_0^t\int_0^1\eta(S_E)_R K_t\,dx\,ds\right|&\leqslant\int_0^t\int_0^1\left(\int_0^\infty\int_{S^1}\eta\sigma_a(B+I)\,d\nu\,d\omega\right)|K_t|\,dx\,ds\\&\quad+\int_0^t\int_0^1\left(\int_0^\infty\int_{S^1}\eta\sigma_s|\tilde I-I|\,d\nu\,d\omega\right)|K_t|\,dx\,ds=:P+Q.\end{aligned}$$

Using (1.18) and Lemma 2.7, we derive

$$\begin{aligned}P&\leqslant C_1\int_0^t\int_0^1\big(1+\theta^{\alpha+1}\big)|K_t|\,dx\,ds\\&\leqslant C_1\int_0^t\int_0^1\big(1+\theta^{q+\alpha+2}\big)|v_x|\,dx\,ds+C_1\int_0^t\int_0^1\big(1+\theta^{q+\alpha+1}\big)|\theta_t|\,dx\,ds=:A_1+B_1.\end{aligned}$$

From (1.18) and Lemmas 2.1–2.7, and using Young's and Hölder's inequalities, we deduce

$$\begin{aligned}
A_1 &= \int_0^t \int_0^1 \left(1+\theta^{q+\alpha+2}\right)|v_x|\,dx\,ds \\
&\leqslant C_1 \int_0^t \int_0^1 \left(1+\theta^{2r+2}\right)v_x^2\,dx\,ds + C_1 \int_0^t \int_0^1 \frac{1+\theta^{2q+2\alpha+4}}{1+\theta^{2r+2}}\,dx\,ds \\
&\leqslant C_1 \int_0^t \left\|v_x(s)\right\|_{L^\infty}^2 \int_0^1 \left(1+\theta^{2r+2}\right)dx\,ds + C_1 \int_0^t \int_0^1 \frac{1+\theta^{2q+2r+4}}{1+\theta^{2r+2}}\,dx\,ds \\
&\leqslant C_1\Big(1+\max_{(x,s)\in Q_t}\theta^{2q+2}(x,s)\Big), \\
B_1 &= \int_0^t \int_0^1 \left(1+\theta^{q+\alpha+1}\right)|\theta_t|\,dx\,ds \\
&\leqslant \frac{\varepsilon}{12} \int_0^t \int_0^1 \left(1+\theta^{q+r}\right)\theta_t^2\,dx\,ds + C(\varepsilon) \int_0^t \int_0^1 \frac{1+\theta^{2q+2r+2}}{1+\theta^{q+r}}\,dx\,ds \\
&\leqslant \frac{\varepsilon}{12} X(t) + C_1\Big(1+\max_{(x,s)\in Q_t}\theta^{q+r+2}(x,s)\Big), \\
Q &= \int_0^t \int_0^1 \left(\int_0^\infty \int_{S^1} \eta\sigma_s|\tilde{I}-I|\,d\omega\,d\nu\right)|K_t|\,dx\,ds \\
&\leqslant \int_0^t \int_0^1 \left(\int_0^\infty \int_{S^1} \eta\sigma_s(\tilde{I}-I)^2\,d\omega\,d\nu\right)^{\frac{1}{2}} \left(\int_0^\infty \int_{S^1} \eta\sigma_s\,d\omega\,d\nu\right)^{\frac{1}{2}} |K_t|\,dx\,ds \\
&\leqslant C_1 \int_0^t \int_0^1 \left[\left(\int_0^\infty \int_{S^1} \eta\sigma_s(\tilde{I}-I)^2\,d\omega\,d\nu\right) + \left(\int_0^\infty \int_{S^1} \eta\sigma_s\,d\omega\,d\nu\right)\right]|K_t|\,dx\,ds \\
&\leqslant C_1 \int_0^t \int_0^1 \left(1+\theta^{\alpha+1}\right)|K_t|\,dx\,ds + C_1 \int_0^t \int_0^1 |K_t|\,dx\,ds \\
&\leqslant C_1 \int_0^t \int_0^1 \left(1+\theta^{q+\alpha+2}\right)|v_x|\,dx\,ds + C_1 \int_0^t \int_0^1 \left(1+\theta^{q+\alpha+1}\right)|\theta_t|\,dx\,ds \\
&\quad + C_1 \int_0^t \int_0^1 \left(1+\theta^{q+1}\right)|v_x|\,dx\,ds + C_1 \int_0^t \int_0^1 \left(1+\theta^{q}\right)|\theta_t|\,dx\,ds =: \sum_{i=1}^4 D_i.
\end{aligned}$$

The estimate $D_1$ is the same as the estimate $A$, that is,

$$D_1 = \int_0^t \int_0^1 \big(1+\theta^{q+\alpha+2}\big)|v_x|\,dx\,ds \leqslant C_1\Big(1+\max_{(x,s)\in Q_t}\theta^{2q+2}(x,s)\Big).$$

By Young's inequality and (2.33), we conclude

$$\begin{aligned} D_2 &= \int_0^t \int_0^1 \big(1+\theta^{q+\alpha+1}\big)|\theta_t|\,dx\,ds \\ &\leqslant \frac{\varepsilon}{12}\int_0^t \int_0^1 \big(1+\theta^{q+r}\big)\theta_t^2\,dx\,ds + C(\varepsilon)\int_0^t \int_0^1 \frac{1+\theta^{2q+2\alpha+2}}{1+\theta^{q+r}}\,dx\,ds \\ &\leqslant \frac{\varepsilon}{12}X(t) + C_1\Big(1+\max_{(x,s)\in Q_t}\theta^{q+r+2}(x,s)\Big), \\ D_3 &= \int_0^t \int_0^1 \big(1+\theta^{q+1}\big)|v_x|\,dx\,ds \leqslant C_1\Big(1+\max_{(x,s)\in Q_t}\theta^{2q+2}(x,s)\Big), \\ D_4 &= \int_0^t \int_0^1 \big(1+\theta^{q}\big)|\theta_t|\,dx\,ds \\ &\leqslant \frac{\varepsilon}{12}\int_0^t \int_0^1 \big(1+\theta^{q+r}\big)\theta_t^2\,dx\,ds + C(\varepsilon)\int_0^t \int_0^1 \frac{1+\theta^{2q+2}}{1+\theta^{q+r}}\,dx\,ds \\ &\leqslant \frac{\varepsilon}{12}X(t) + C_1\Big(1+\max_{(x,s)\in Q_t}\theta^{q-r}(x,s)\Big). \end{aligned}$$

Inserting all the previous estimates into (2.44), using Young's inequality and taking $\varepsilon > 0$ small enough, we derive

$$X(t) + Y(t) \leqslant C_1\Big(1+\max_{(x,s)\in Q_t}\theta^{2q+2}(x,s)\Big). \tag{2.45}$$

By (2.33) and Hölder's inequality, there exists a point $a(t) \in [0,1]$ such that for any $t > 0$

$$\begin{aligned} \theta^{q+r+2}(x,t) - \theta^{q+r+2}\big(a(t),t\big) &= \int_{a(t)}^x \big(\theta^{q+r+2}\big)_x\,dx \leqslant C_1\int_0^1 \theta^{q+r+1}|\theta_x|\,dx \\ &\leqslant C_1\left(\int_0^1 \big(1+\theta^{2q}\big)\theta_x^2\,dx\right)^{\frac{1}{2}}\left(\int_0^1 \theta^{2r+2}\,dx\right)^{\frac{1}{2}} \\ &\leqslant C_1 Y^{\frac{1}{2}}(t). \end{aligned}$$

Then

$$\max_{(x,s)\in Q_t} \theta(x,s) \leqslant C_1 Y^{\frac{1}{2q+2r+4}} + C_1. \tag{2.46}$$

Combining (2.46) with (2.45), and using Young's inequality yields

$$X(t) + Y(t) \leqslant C_1$$

which gives

$$\max_{(x,s)\in Q_t} \theta(x,s) \leqslant C_1.$$

The proof is now complete. □

The following lemmas are concerned with the new arguments on the uniform-in-time estimates of the radiative term $I(x,t;\nu,\omega)$.

**Lemma 2.9.** *There hold*

$$\left\| \int_0^\infty \int_{S^1} I d\omega\, d\nu \right\|_{L^\infty(Q_t)} \leqslant C_1, \quad \forall t > 0, \tag{2.47}$$

$$\left\| \int_0^\infty \int_{S^1} I^2\, d\omega\, d\nu \right\|_{L^\infty(Q_t)} \leqslant C_1, \quad \forall t > 0, \tag{2.48}$$

$$\left\| \int_0^\infty \int_{S^1} |I_x|\, d\omega\, d\nu \right\|_{L^\infty(Q_t)} \leqslant C_1, \quad \forall t > 0, \tag{2.49}$$

$$\int_0^t \int_0^1 \int_0^\infty \int_{S^1} I_t^2\, d\omega\, d\nu\, dx\, ds \leqslant C_1, \quad \forall t > 0. \tag{2.50}$$

**Proof.** By (2.38) and (2.42), we can easily get (2.47).

By the definition of $\tilde{I}$, we can derive from (2.47) that

$$\left\| \int_0^\infty \int_{S^1} \tilde{I}\, d\omega\, d\nu \right\|_{L^\infty(Q_t)} \leqslant C_1. \tag{2.51}$$

From (2.40), using Young's and Hölder's inequalities, (1.18) and Lemmas 2.7–2.8, we derive

$$\int_0^\infty \int_{S^1} I^2\, d\nu\, d\omega \leqslant \int_0^\infty \int_{S^1} \left( \int_0^1 \frac{\eta}{\omega} (\sigma_a B + \sigma_s \tilde{I})\, dx \right)^2 d\nu\, d\omega$$

$$\leqslant C_1 \int_0^\infty \int_{S^1} \left( \frac{\eta^2}{\omega^2} \sigma_a \, dx \cdot \int_0^1 \sigma_a B^2 \, dx \right) d\omega \, d\nu$$

$$+ C_1 \int_0^\infty \int_{S^1} \left( \int_0^1 \frac{\eta^2}{\omega^2} \sigma_s \, dx \cdot \int_0^1 \sigma_s \tilde{I}^2 \, dx \right) d\omega \, d\nu$$

$$\leqslant C_1 + C_1 \int_0^\infty \int_{S^1} \int_0^1 \left( \sigma_s (\tilde{I} - I)^2 + \sigma_s I^2 \right) dx \, d\omega \, d\nu \leqslant C_1, \tag{2.52}$$

which, by the definition of $\tilde{I}$, implies

$$\int_0^\infty \int_{S^1} \tilde{I}^2 \, d\omega \, d\nu \leqslant C_1. \tag{2.53}$$

From (1.8), we get

$$I_x = -\frac{\eta}{\omega} (\sigma_a + \sigma_s) I + \frac{\eta}{\omega} (\sigma_a B + \sigma_s \tilde{I}).$$

Integrating the above equality and using (1.18), we obtain

$$\int_0^\infty \int_{S^1} |I_x| \, d\omega \, d\nu \leqslant C_1 \int_0^\infty \int_{S^1} \frac{1}{|\omega|} (\sigma_a + \sigma_s) |I| \, d\omega \, d\nu + C_1 \int_0^\infty \int_{S^1} \frac{1}{|\omega|} (\sigma_a B + \sigma_s \tilde{I}) \, d\omega \, d\nu$$

$$\leqslant C_1 + C_1 \int_0^\infty \int_{S^1} |I| \, d\omega \, d\nu + C_1 \int_0^\infty \int_{S^1} |\tilde{I}| \, d\omega \, d\nu \tag{2.54}$$

which, using (2.47) and (2.51), gives (2.49).

By (2.40), we have for any $\omega \in (0, 1)$,

$$I_t = \int_0^x e^{\int_x^y \frac{\eta}{\omega} (\sigma_a + \sigma_s) \, dz} \left( \int_x^y \frac{\eta}{\omega} (\sigma_a + \sigma_s) \, dz \right)_t \frac{\eta}{\omega} (\sigma_a B + \sigma_s \tilde{I}) + \int_0^x e^{\int_x^y \frac{\eta}{\omega} (\sigma_a + \sigma_s) \, dz} \left( \frac{\eta}{\omega} (\sigma_a B + \sigma_s \tilde{I}) \right)_t dy$$

$$=: A_2 + B_2. \tag{2.55}$$

Using Young's inequality, (2.53), (1.18), Lemmas 2.4 and 2.8, we deduce

$$\int_0^t \int_0^\infty \int_{S^1} A_2^2 \, d\omega \, d\nu \, ds \leqslant C_1 \int_0^t \int_0^\infty \int_{S^1} \left[ \int_0^x \left( \int_x^y \frac{v_x}{\omega} (\sigma_a + \sigma_s) + \frac{\eta}{\omega} \left( (\sigma_a + \sigma_s)_\eta v_x \right.\right.\right.$$

$$\left.\left.\left. + (\sigma_a + \sigma_s)_\theta \theta_t \right) dz \right) \frac{\eta}{\omega} (\sigma_a B + \sigma_s \tilde{I}) \, dy \right]^2 d\omega \, d\nu \, ds$$

$$\leqslant C_1 \int_0^t \int_0^\infty \int_{S^1} \left( \int_0^1 \frac{v_x^2}{\omega^2} \big(\sigma_a + \sigma_s + (\sigma_a)_\eta + (\sigma_s)_\eta\big)^2 + \frac{\theta_t^2}{\omega^2} \big((\sigma_a)_\theta + (\sigma_s)_\theta\big)^2 \, dx \cdot \int_0^1 \frac{\eta^2}{\omega^2} \sigma_a^2 B^2 \, dx \right) d\omega \, d\nu \, ds$$

$$+ C_1 \int_0^t \int_0^\infty \int_{S^1} \left( \int_0^1 \frac{v_x^2}{\omega^2} \big(\sigma_a + \sigma_s + (\sigma_a)_\eta + (\sigma_s)_\eta\big)^2 + \frac{\theta_t^2}{\omega^2} \big((\sigma_a)_\theta + (\sigma_s)_\theta\big)^2 \, dx \cdot \int_0^1 \frac{\eta^2}{\omega^2} \sigma_s^2 \tilde{I}^2 \, dx \right) d\omega \, d\nu \, ds$$

$$\leqslant C_1 \int_0^t \int_0^1 \big(v_x^2 + \theta_t^2\big) \, dx \, ds + C_1 \int_0^t \int_0^\infty \int_{S^1} \int_0^1 \tilde{I}^2 \, dx \, d\omega \, d\nu \, ds \leqslant C_1. \tag{2.56}$$

Analogously,

$$\int_0^t \int_0^\infty \int_{S^1} B_2^2 \, d\omega \, d\nu \, ds \leqslant C_1 \int_0^t \int_0^\infty \int_{S^1} \left[ \int_0^x \left( \frac{v_x}{\omega} (\sigma_a B + \sigma_s \tilde{I}) + \frac{\eta}{\omega} \big((\sigma_a)_\eta v_x B + (\sigma_a)_\theta \theta_t B + \sigma B_\theta \theta_t + (\sigma_s)_\eta \tilde{I} v_x + (\sigma_s)_\theta \theta_t \tilde{I} + \sigma_s \tilde{I}_t\big) \right) dx \right]^2 d\omega \, d\nu \, ds$$

$$\leqslant C_1 \int_0^t \int_0^1 \big(v_x^2 + \theta_t^2\big) \, dx \, ds + C_1 \int_0^x \int_0^t \int_0^\infty \int_{S^1} I_t^2 \, d\omega \, d\nu \, ds \, dy$$

$$\leqslant C_1 + C_1 \int_0^x \int_0^t \int_0^\infty \int_{S^1} I_t^2 \, d\omega \, d\nu \, ds \, dy,$$

which, together with (2.56), implies

$$\int_0^t \int_0^\infty \int_{S^1} I_t^2 \, d\omega \, d\nu \, ds \leqslant C_1 + C_1 \int_0^x \int_0^t \int_0^\infty \int_{S^1} I_t^2 \, d\omega \, d\nu \, ds \, dy.$$

Fixing $t > 0$ and using Gronwall's inequality, we get

$$\int_0^t \int_0^\infty \int_{S^1} I_t^2 \, d\omega \, d\nu \, ds \leqslant C_1 e^{C_1 x} \leqslant C_1 e^{C_1} \leqslant C_1, \quad \forall x \in [0, 1].$$

The proof is complete. □

**Lemma 2.10.** *Under the assumptions in Theorem* 1.1, *the following estimate holds*:

$$\left\|\mathcal{I}_{xx}(t)\right\| \leqslant C_2, \quad \forall t > 0. \tag{2.57}$$

**Proof.** By virtue of the direct computation, we have

$$\begin{aligned}
\left\|\mathcal{I}_{xx}(t)\right\|^2 &= \int_0^1 \left(\int_0^\infty \int_{S^1} I_{xx}\, d\omega\, d\nu\right)^2 dx \\
&= \int_0^1 \left(\int_0^\infty \int_{S^1} \frac{1}{\omega}(\eta_x S + \eta S_x)\, d\omega\, d\nu\right)^2 dx \\
&\leqslant C_1 \int_0^1 \left[\left(\int_0^\infty \int_{S^1} \frac{1}{\omega}\eta_x S\, d\omega\, d\nu\right)^2 + \left(\int_0^\infty \int_{S^1} \frac{1}{\omega}\eta S_x\, d\omega\, d\nu\right)^2\right] dx \\
&=: G + H.
\end{aligned} \tag{2.58}$$

Using (1.18), (2.28) and Lemma 2.9, we see that

$$\begin{aligned}
G &= \int_0^1 \left(\int_0^\infty \int_{S^1} \frac{1}{\omega}\eta_x S\, d\omega\, d\nu\right)^2 dx \\
&= \int_0^1 \eta_x^2 \left(\int_0^\infty \int_{S^1} \frac{1}{\omega}\big(\sigma_a(B - I) + \sigma_s(\tilde{I} - I)\big)\, d\omega\, d\nu\right)^2 dx \\
&\leqslant C_1 \int_0^1 \eta_x^2\, dx \leqslant C_1.
\end{aligned} \tag{2.59}$$

Similarly,

$$\begin{aligned}
H &= \int_0^1 \Bigg(\int_0^\infty \int_{S^1} \frac{\eta}{\omega}\big\{\big[(\sigma_a)_\eta \eta_x + (\sigma_a)_\theta \theta_x\big](B - I) + \sigma_a(B_\theta \theta_x - I_x) \\
&\quad + \big[(\sigma_s)_\eta \eta_x + (\sigma_s)_\theta \theta_x\big](\tilde{I} - I) + \sigma_s(\tilde{I} - I)_x\big\}\, d\omega\, d\nu\Bigg)^2 dx \\
&\leqslant C_1 \int_0^1 \big(\eta_x^2 + \theta_x^2\big)\, dx + C_1 \\
&\leqslant C_1.
\end{aligned} \tag{2.60}$$

Plugging (2.59)–(2.60) into (2.58), we can get (2.57). The proof is complete. □

## 3. Large-time behavior in $\mathcal{H}_1$

In this section, we shall complete the proof of Theorem 1.1. To begin with, we introduce a differential inequality in next lemma.

**Lemma 3.1.** *Let $T$ be given with $0 < T \leqslant +\infty$. Suppose that $y$ and $h$ are nonnegative continuous functions defined on $[0, T]$ and satisfy the following conditions*

$$\frac{dy}{dt} \leqslant A_1 y^2(t) + A_2 + h(t),$$
$$\int_0^T y(s)\,ds \leqslant A_3, \qquad \int_0^T h(s)\,ds \leqslant A_4,$$

*where $A_1, A_2, A_3, A_4$ are given nonnegative constants. Then for any $r > 0$, with $0 < r < T$,*

$$y(t+r) \leqslant \left(\frac{A_3}{r} + A_2 r + A_4\right) \cdot e^{A_1 A_3}.$$

*Furthermore, if $T = +\infty$, then*

$$\lim_{t\to+\infty} y(t) = 0.$$

**Proof.** See, e.g., [19]. □

**Lemma 3.2.** *Under the assumptions in Theorem 1.1, we have*

$$\lim_{t\to+\infty} \left\|\eta(t) - \bar{\eta}\right\|_{H^1} = 0, \tag{3.1}$$
$$\lim_{t\to+\infty} \left\|v(t)\right\|_{H^1} = 0, \tag{3.2}$$

*where $\bar{\eta} = \int_0^1 \eta(y,t)\,dy = \int_0^1 \eta_0(y)\,dy$.*

**Proof.** See, e.g., [17]. □

**Lemma 3.3.** *Under the assumptions in Theorem 1.1, we have*

$$\lim_{t\to+\infty} \left\|\theta(t) - \bar{\theta}\right\|_{H^1} = 0, \tag{3.3}$$

*where $\bar{\theta} > 0$ is determined by $e(\bar{\eta}, \bar{\theta}) = \int_0^1 (\frac{1}{2} v_0^2 + e(\eta_0, \theta_0) + F_R(0))\,dx$.*

**Proof.** Eq. (1.7) can be rewritten as

$$e_\theta \theta_t + (-p + \theta p_\theta) v_x - \sigma v_x + q_x + \eta (S_E)_R = 0. \tag{3.4}$$

Multiplying (3.4) by $e_\theta^{-1}\theta_{xx}$, integrating the result over $(0, 1)$ and using Young's inequality, the interpolation inequality and Lemmas 2.1–2.8, we can conclude for $\varepsilon > 0$,

$$\frac{d}{dt}\|\theta_x(t)\|^2+2\int_0^1\frac{\kappa\theta_{xx}^2}{e_\theta\eta}=2\int_0^1\left[\frac{\theta p_\theta v_x}{e_\theta}-\frac{\mu v_x^2}{e_\theta\eta}-\frac{(\frac{\kappa}{\eta})_x\theta_x}{e_\theta}+\frac{\eta(S_E)_R}{e_\theta}\right]\theta_{xx}\,dx$$
$$\leqslant\frac{\varepsilon}{2}\|\theta_{xx}(t)\|^2+C_1\big(\|v_x\|^2+\|v_x\|_{L^4}^4+\|\theta_x\|_{L^4}^4+\|\eta_x\theta_x\|^2+\|(S_E)_R\|^2\big)$$
$$\leqslant\frac{\varepsilon}{2}\|\theta_{xx}(t)\|^2+C_1\big(\|v_x\|^2+\|v_x\|^3\|v_{xx}\|+\|v_x\|^4+\|\theta_x\|^3\|\theta_{xx}\|+\|\theta_x\|^4$$
$$+\|\theta_x\|_{L^\infty}+\|(S_E)_R\|^2\big)$$
$$\leqslant\varepsilon\|\theta_{xx}(t)\|^2+C_1\big(\|v_x\|^2+\|v_{xx}\|^2+\|\theta_x\|^2+\|(S_E)_R\|^2\big). \tag{3.5}$$

Now we need to estimate the new radiative term $\eta(S_E)_R$ in (3.5), which didn't appear in [17]. From (1.18), Young's and Hölder's inequalities and Lemmas 2.7–2.8, we derive

$$\|(S_E)_R\|^2=\int_0^1\left(\int_0^\infty\int_{S^1}\big(\sigma_a(B-I)+\sigma_s(\tilde I-I)\big)\,d\omega\,d\nu\right)^2dx$$
$$\leqslant C_1\int_0^1\left[\left(\int_0^\infty\int_{S^1}\sigma_a(B-I)\,d\omega\,d\nu\right)^2+\left(\int_0^\infty\int_{S^1}\sigma_s(\tilde I-I)\,d\omega\,d\nu\right)^2\right]dx$$
$$\leqslant C_1\int_0^1\left[\left(\int_0^\infty\int_{S^1}\sigma_a\,d\omega\,d\nu\right)\left(\int_0^\infty\int_{S^1}\sigma_a\big(B^2+I^2\big)\,d\omega\,d\nu\right)\right.$$
$$\left.+C_1\left(\int_0^\infty\int_{S^1}\sigma_s\,d\omega\,d\nu\right)\left(\int_0^\infty\int_{S^1}\sigma_s(\tilde I-I)^2\,d\omega\,d\nu\right)\right]dx$$
$$\leqslant C_1\int_0^1\theta^{\alpha+1}\,dx+C_1\leqslant C_1, \tag{3.6}$$

which, together with (3.5), yields

$$\|\theta_x(t)\|^2+\int_0^t\|\theta_{xx}(s)\|^2\,ds\leqslant C_1.$$

Plugging (3.6) into (3.5), taking $\varepsilon>0$ small enough, and using Lemmas 2.1–2.8, we have

$$\frac{d}{dt}\|\theta_x(t)\|^2+C_1\int_0^1\big(1+\theta^{q-r}\big)\theta_{xx}^2\,dx\leqslant C_1\big(\|v_{xx}\|^2+1\big), \tag{3.7}$$

which, together with Lemma 3.1 and (2.34), yields

$$\lim_{t\to+\infty}\|\theta_x(t)\|^2=0. \tag{3.8}$$

By Poincaré's inequality, we deduce

$$\left\|\theta(t)-\bar{\theta}\right\|_{H^1}\leqslant C_1\left\|\theta_x(t)\right\|,$$

which, combined with (3.8), gives (3.3). The proof is now complete. □

The following lemma is the new argument on the large-time behavior of the radiative term $I(x,t;\nu,\omega)$.

**Lemma 3.4.** *Under the assumptions in Theorem* 1.1, *we have*

$$\lim_{t\to+\infty}\left\|\mathcal{I}(t)\right\|_{H^2}=0. \tag{3.9}$$

**Proof.** By (2.48) and (1.8), the direct computation yields

$$\begin{aligned}
\frac{d}{dt}\left\|\mathcal{I}_x(t)\right\|^2&=\frac{d}{dt}\int_0^1\left(\int_0^\infty\int_{S^1}I_x\,d\omega\,d\nu\right)^2dx\\
&=2\int_0^1\left(\int_0^\infty\int_{S^1}I_x\,d\omega\,d\nu\right)\left(\int_0^\infty\int_{S^1}I_{xt}\,d\omega\,d\nu\right)dx\\
&\leqslant C_1\int_0^1\left(\int_0^\infty\int_{S^1}|I_{xt}|\,d\omega\,d\nu\right)dx\\
&\leqslant C_1\int_0^1\left(\int_0^\infty\int_{S^1}\frac{1}{\omega}|v_xS+\eta S_t|\,d\omega\,d\nu\right)dx=:A_3+B_3.
\end{aligned} \tag{3.10}$$

We denote

$$\begin{aligned}
A_3&=\int_0^1\int_0^\infty\int_{S^1}\left|\frac{1}{\omega}v_xS\right|d\omega\,d\nu\,dx\\
&=\int_0^1\int_0^\infty\int_{S^1}\left|\frac{1}{\omega}v_x\big(\sigma_a(B-I)+\sigma_s(\tilde{I}-I)\big)\right|d\omega\,d\nu\,dx=:\widehat{C}+\widehat{D}.
\end{aligned}$$

Using (1.18), Lemmas 2.7–2.9 and Young's inequality, we can derive

$$\begin{aligned}
\widehat{C}&\leqslant\int_0^1\int_0^\infty\int_{S^1}\left|\frac{1}{\omega}v_x\sigma_a(B-I)\right|d\omega\,d\nu\,dx\\
&\leqslant C_1\int_0^1|v_x|\int_0^\infty\int_{S^1}\frac{1}{|\omega|}\big(|\omega|\theta^{\alpha+1}f(\nu,\omega)+|\omega|g(\nu,\omega)I\big)\,d\omega\,d\nu\,dx
\end{aligned}$$

$$\leqslant C_1 \int_0^1 |v_x|\,dx \leqslant C_1 \int_0^1 v_x^2\,dx + C_1,$$

$$\begin{aligned}
\widehat{D} &\leqslant \int_0^1 \int_0^\infty \int_{S^1} \left| \frac{1}{\omega} v_x \sigma_s(\tilde{I} - I) \right| d\omega\,d\nu\,dx \\
&\leqslant C_1 \int_0^1 |v_x| \int_0^\infty \int_{S^1} \frac{1}{|\omega|} |\omega| k(\nu,\omega) |\tilde{I} - I|\,d\omega\,d\nu\,dx \\
&\leqslant C_1 \int_0^1 |v_x| \int_0^\infty \int_{S^1} I\,d\omega\,d\nu\,dx \leqslant C_1 \int_0^1 |v_x|\,dx \leqslant C_1 \int_0^1 v_x^2\,dx + C_1.
\end{aligned}$$

We denote

$$\begin{aligned}
B_3 &= \int_0^1 \int_0^\infty \int_{S^1} \left| \frac{1}{\omega} \eta S_t \right| d\omega\,d\nu\,dx \\
&= \int_0^1 \int_0^\infty \int_{S^1} \left| \frac{\eta}{\omega} \left\{ \left[ (\sigma_a)_\eta v_x + (\sigma_a)_\theta \theta_t \right] (B - I) + \sigma_a (B_\theta \theta_t - I_t) \right. \right. \\
&\quad \left. \left. + \left[ (\sigma_s)_\eta v_x + (\sigma_s)_\theta \theta_t \right] (\tilde{I} - I) + \sigma_s (\tilde{I} - I)_t \right\} \right| d\omega\,d\nu\,dx \\
&=: E + F,
\end{aligned}$$

where

$$E = \int_0^1 \int_0^\infty \int_{S^1} \left| \frac{\eta}{\omega} \left\{ \left[ (\sigma_a)_\eta v_x + (\sigma_a)_\theta \theta_t \right] (B - I) + \sigma_a (B_\theta \theta_t - I_t) \right\} \right| d\omega\,d\nu\,dx =: \sum_{i=1}^6 P_i,$$

$$F =: \int_0^1 \int_0^\infty \int_{S^1} \left| \frac{\eta}{\omega} \left\{ \left[ (\sigma_s)_\eta v_x + (\sigma_s)_\theta \theta_t \right] (\tilde{I} - I) + \sigma_s (\tilde{I} - I)_t \right\} \right| d\omega\,d\nu\,dx.$$

Using (1.18), Lemmas 2.7–2.9 and Young's inequality, we can conclude

$$\begin{aligned}
|P_1| &\leqslant C_1 \int_0^1 |v_x| \int_0^\infty \int_{S^1} \frac{1}{|\omega|} \left| (\sigma_a)_\eta B \right| d\omega\,d\nu\,dx \\
&\leqslant C_1 \int_0^1 |v_x| \int_0^\infty \int_{S^1} \frac{1}{|\omega|} |\omega| h(\nu,\omega)\,d\omega\,d\nu\,dx
\end{aligned}$$

$$\leqslant C_1 \int_0^1 |v_x|\,dx$$

$$\leqslant C_1 \int_0^1 v_x^2\,dx + C_1.$$

Analogously,

$$|P_2| \leqslant C_1 \int_0^1 |v_x| \int_0^\infty \int_{S^1} \frac{1}{|\omega|} \big|(\sigma_a)_\eta I\big|\,d\omega\,d\nu\,dx$$

$$\leqslant C_1 \int_0^1 |v_x|\,dx \leqslant C_1 \int_0^1 v_x^2\,dx + C_1,$$

$$|P_3| \leqslant C_1 \int_0^1 \theta_t \int_0^\infty \int_{S^1} \frac{1}{|\omega|} \big|(\sigma_a)_\theta\big| B\,d\omega\,d\nu\,dx$$

$$\leqslant C_1 \int_0^1 |\theta_t|\,dx \leqslant C_1 \int_0^1 \theta_t^2\,dx + C_1,$$

$$|P_4| \leqslant C_1 \int_0^1 |\theta_t| \int_0^\infty \int_{S^1} \frac{1}{|\omega|} \big|(\sigma_a)_\theta I\big|\,d\omega\,d\nu\,dx$$

$$\leqslant C_1 \int_0^1 |\theta_t|\,dx \leqslant C_1 \int_0^1 \theta_t^2\,dx + C_1,$$

$$|P_5| \leqslant C_1 \int_0^1 |\theta_t| \int_0^\infty \int_{S^1} \frac{1}{|\omega|} \big|\sigma_a B_\theta\big|\,d\omega\,d\nu\,dx$$

$$\leqslant C_1 \int_0^1 |\theta_t|\,dx \leqslant C_1 \int_0^1 \theta_t^2\,dx + C_1,$$

$$|P_6| \leqslant C_1 \int_0^1 \int_0^\infty \int_{S^1} \frac{1}{|\omega|} \sigma_a |I_t|\,d\omega\,d\nu\,dx$$

$$\leqslant C_1 + \int_0^1 \int_0^\infty \int_{S^1} I_t^2\,d\omega\,d\nu\,dx.$$

Now we estimate $F$.

$$F \leqslant \int_0^1 \int_0^\infty \int_{S^1} \frac{\eta}{|\omega|} \big|(\sigma_s)_\eta v_x\big| |\tilde{I} - I| \, d\omega \, d\nu \, dx + \int_0^1 \int_0^\infty \int_{S^1} \frac{\eta}{|\omega|} \big|(\sigma_s)_\theta \theta_t\big| |\tilde{I} - I| \, d\omega \, d\nu \, dx$$
$$+ \int_0^1 \int_0^\infty \int_{S^1} \frac{\eta}{|\omega|} \sigma_s \big|(\tilde{I} - I)_t\big| \, d\omega \, d\nu \, dx =: \sum_{i=1}^3 M_i.$$

By (1.18) and Lemmas 2.7–2.9, we deduce

$$M_1 \leqslant \int_0^1 |v_x| \int_0^\infty \int_{S^1} \frac{\eta}{|\omega|} \big|(\sigma_s)_\eta\big| |\tilde{I} - I| \, d\omega \, d\nu \, dx$$
$$\leqslant C_1 \int_0^1 |v_x| \int_0^\infty \int_{S^1} |I| \, d\omega \, d\nu \, dx \leqslant C_1 \int_0^1 v_x^2 \, dx + C_1,$$
$$M_2 \leqslant C_1 \int_0^1 |\theta_t| \int_0^\infty \int_{S^1} |I| \, d\omega \, d\nu \, dx \leqslant C_1 \int_0^1 \theta_t^2 \, dx + C_1,$$
$$M_3 \leqslant C_1 + C_1 \int_0^1 \int_0^\infty \int_{S^1} I_t^2 \, d\omega \, d\nu \, dx.$$

Inserting all the previous estimates into (3.10) implies

$$\frac{d}{dt} \big\|\mathcal{I}_x(t)\big\|^2 \leqslant C_1 \big(\big\|v_x(t)\big\|^2 + \big\|\theta_t(t)\big\|^2 + \|I_t\|^2_{L^2(S^1 \times R_+ \times \Omega)}\big) + C_1, \tag{3.11}$$

which, together with Lemma 3.1, (2.29), (2.41) and (2.50), yields

$$\lim_{t \to +\infty} \big\|\mathcal{I}_x(t)\big\|^2 = 0. \tag{3.12}$$

From (1.8), we derive

$$\big\|\mathcal{I}_{xx}(t)\big\|^2 = \bigg\| \int_0^\infty \int_{S^1} I_{xx} \, d\omega \, d\nu \bigg\|^2 = \bigg\| \int_0^\infty \int_{S^1} \frac{1}{\omega} (\eta_x S + \eta S_x) \, d\omega \, d\nu \bigg\|^2$$
$$\leqslant C_1 \bigg\| \int_0^\infty \int_{S^1} \frac{1}{\omega} \eta_x S \, d\omega \, d\nu \bigg\|^2 + C_1 \bigg\| \int_0^\infty \int_{S^1} \frac{1}{\omega} \eta S_x \, d\omega \, d\nu \bigg\|^2$$
$$=: N_1 + N_2. \tag{3.13}$$

Using (1.18) and Lemmas 2.4–2.9, we deduce

$$
\begin{aligned}
N_1 &\leqslant C_1 \int_0^1 \eta_x^2 \left( \int_0^\infty \int_{S^1} \frac{1}{\omega} \left[ \sigma_a (B - I) + \sigma_s (\tilde{I} - I) \right] d\omega \, d\nu \right)^2 dx \\
&\leqslant C_1 \left\| \eta_x(t) \right\|^2, \qquad (3.14)\\
N_2 &\leqslant C_1 \int_0^1 \Bigg( \int_0^\infty \int_{S^1} \frac{1}{\omega} \Big( \big( (\sigma_1)_\eta \eta_x + (\sigma_a)_\theta \theta_x \big)(B - I) + \sigma_a (B_\theta \theta_x - I) \\
&\qquad + \big( (\sigma_s)_\eta \eta_x + (\sigma_s)_\theta \theta_x \big)(\tilde{I} - I) + \sigma_s (\tilde{I} - I)_x \Big) d\omega \, d\nu \Bigg)^2 dx \\
&\leqslant C_1 \left( \left\| \eta_x(t) \right\|^2 + \left\| \theta_x(t) \right\|^2 + \left\| \mathcal{I}_x(t) \right\|^2 \right). \qquad (3.15)
\end{aligned}
$$

Inserting (3.14)–(3.15) into (3.13) and using (3.1), (3.8) and (3.12), we get

$$
\lim_{t \to +\infty} \left\| \mathcal{I}_{xx}(t) \right\| = 0. \tag{3.16}
$$

Thus (3.9) follows from (3.12), (3.16). □

**Proof of Theorem 1.1.** Combining Lemmas 2.1–2.10 and Lemmas 3.2–3.4, we complete the proof of Theorem 1.1. □

## 4. Uniform-in-time estimates in $\mathcal{H}_2$

In this section, we shall prove the uniform-in-time estimates in $\mathcal{H}_2$. The next lemma concerns the uniform-in-time global (in time) positive lower bound (independent of $t$) of the absolute temperature $\theta$.

**Lemma 4.1.** *Under the assumptions in Theorem* 1.1, *then the generalized global solution* $(\eta(t), v(t), \theta(t), \mathcal{I}(t))$ *to the problems* (1.5)–(1.8) *and* (1.13)–(1.15) *satisfies*

$$
0 < C_1^{-1} \leqslant \theta(x,t), \quad \forall (x,t) \in [0,1] \times [0,+\infty). \tag{4.1}
$$

**Proof.** See, e.g., Lemma 2.3.3 on page 85 of [17]. □

**Lemma 4.2.** *Under the assumptions in Theorem* 1.2, *the following estimates hold*:

$$
\left\| \theta_t(t) \right\|^2 + \left\| v_t(t) \right\|^2 + \int_0^t \left( \| v_{xt} \|^2 + \| \theta_{xt} \|^2 \right)(s) \, ds \leqslant C_2, \quad \forall t > 0, \tag{4.2}
$$

$$
\left\| v_{xx}(t) \right\|^2 + \left\| \theta_{xx}(t) \right\|^2 + \int_0^t \left( \| v_{xxx} \|^2 + \| \theta_{xx} \|^2 \right)(s) \, ds \leqslant C_2, \quad \forall t > 0, \tag{4.3}
$$

$$
\left\| \eta_{xx}(t) \right\|^2 + \int_0^t \left\| \eta_{xx}(s) \right\|^2 ds \leqslant C_2, \quad \forall t > 0. \tag{4.4}
$$

**Proof.** We have now the following estimates (see, e.g., Lemma 2.3.8 on page 90 of [17] for the details)

$$\|v_t(t)\|^2+\int_0^t\|v_{xt}(s)\|^2\,ds\leqslant C_2,\quad\forall t>0,\tag{4.5}$$

$$\|v_{xx}(t)\|\leqslant C_2,\qquad\|v_x(t)\|_{L^\infty}\leqslant C_2,\qquad\int_0^t\|v_{xxx}(s)\|^2\,ds\leqslant C_2,\quad\forall t>0.\tag{4.6}$$

Using Eq. (1.7), Lemmas 2.1–2.8, (3.6), the Gagliardo–Nirenberg interpolation inequality and Young's inequality, we have

$$\|\theta_{xx}(t)\|\leqslant C_1\big(\|\theta_t(t)\|+\|\eta(S_E)_R\|\big)\leqslant C_1\big(\|\theta_t(t)\|+1\big).\tag{4.7}$$

Differentiating (1.7) with respect to $t$, multiplying the result by $\theta_t$ and integrating over $(0,1)$, we infer that for any $\varepsilon>0$,

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{e_\theta}\theta_t(t)\|^2+C_1^{-1}\|\theta_{xt}(t)\|^2\\&\quad\leqslant\varepsilon\|\theta_{xt}(t)\|^2+C_1\big\{\|\theta_x(t)\|^2+\|v_x(t)\|^2+\|\theta_t(t)\|_{L^3}^3+\|\theta_t(t)\|^2\\&\qquad+\|v_{xt}(t)\|^2+\big(\|\theta_t(t)\|+\|\theta_t(t)\|^{\frac12}\|\theta_{tx}(t)\|^{\frac12}\big)\|\theta_{xt}(t)\|+C_1\|[\eta(S_E)_R]_t\|^2\big\}.\end{aligned}\tag{4.8}$$

Integrating (4.8) with respect to $t$ and using Lemmas 2.1–2.8 and Young's inequality, we derive for any $\varepsilon>0$,

$$\begin{aligned}&\|\theta_t(t)\|^2+\int_0^t\|\theta_{xt}(s)\|^2\,ds\\&\quad\leqslant C_2+\varepsilon\int_0^t\|\theta_{xt}(s)\|^2\,ds+C_1\int_0^t\big(\|\theta_t\|^{\frac52}\|\theta_{tx}\|^{\frac12}+\|\theta_t\|^3\big)(s)\,ds+C_1\int_0^t\|[\eta(S_E)_R]_t\|^2(s)\,ds\\&\quad\leqslant C_2+\varepsilon\int_0^t\|\theta_{xt}(s)\|^2\,ds+C_1\sup_{0\leqslant s\leqslant t}\|\theta_t(s)\|^{\frac43}+C_1\int_0^t\|[\eta(S_E)_R]_t\|^2(s)\,ds\\&\quad\leqslant C_2+\varepsilon\int_0^t\|\theta_{xt}(s)\|^2\,ds+\frac12\sup_{0\leqslant s\leqslant t}\|\theta_t(s)\|^2+C_1\int_0^t\|[\eta(S_E)_R]_t\|^2(s)\,ds.\end{aligned}\tag{4.9}$$

Noting that the new radiative term $\int_0^t\|[\eta(S_E)_R]_t\|^2(s)\,ds$, we need to derive the uniform-in-time estimate.

From (2.29), (2.41) and (2.50), we can derive

$$\begin{aligned}
\int_0^t \left\|\left[\eta(S_E)_R\right]_t\right\|^2(s)\,ds &= \int_0^t\int_0^1 \left[v_x(S_E)_R+\eta\left[(S_E)_R\right]_t\right]^2 dx\,ds\\
&\leqslant C_1\int_0^t\int_0^1 v_x^2\left(\int_0^\infty\int_{S^1}\sigma_a(B-I)+\sigma_s(\tilde I-I)\,d\omega\,d\nu\right)^2 dx\,ds\\
&\quad + C_1\int_0^t\int_0^1\left\{\int_0^\infty\int_{S^1}\left[(\sigma_a)_\eta v_x+(\sigma_a)_\theta\theta_t\right](B-I)+\sigma_a(B_\theta\theta_t-I_t)\right.\\
&\quad \left.+\left[(\sigma_s)_\eta v_x+(\sigma_s)_\theta\theta_t\right](\tilde I-I)+\sigma_s(\tilde I-I)_t\,d\omega\,d\nu\right\}^2 dx\,ds\\
&\leqslant C_1\int_0^t\left(\|v_x\|^2+\|\theta_t\|^2\right)(s)\,ds + C_1\int_0^t\int_0^1\int_0^\infty\int_{S^1} I_t^2\,d\omega\,d\nu\,dx\,ds\\
&\leqslant C_1.
\end{aligned}\tag{4.10}$$

Inserting (4.10) into (4.9), then taking supremum in $t$ on the left-hand side of (4.9), picking $\varepsilon > 0$ small enough, we get

$$\left\|\theta_t(t)\right\|^2+\int_0^t\left\|\theta_{xt}(s)\right\|^2 ds\leqslant C_2,\quad \forall t>0,\tag{4.11}$$

which, together with (4.7), implies

$$\left\|\theta_{xx}(t)\right\|\leqslant C_2.\tag{4.12}$$

The estimate (4.4) follows from Lemma 2.3.9 on page 91 of [17].

Differentiating (1.7) with respect to $x$, using Young's inequality, the Gagliardo–Nirenberg interpolation and Poincaré's inequality and Lemmas 2.1–2.8, (4.11), (4.12) and (4.4), we deduce

$$\begin{aligned}
\int_0^t\left\|\theta_{xxx}(s)\right\|^2 ds &\leqslant C_2\int_0^t\left(\|\eta_x\|^2+\|\eta_{xx}\|^2+\|v_x\|^2+\|v_{xx}\|^2+\|\theta_t\|^2+\|\theta_{xt}\|^2\right)(s)\,ds\\
&\quad + C_2\int_0^t\left\|\left[\eta(S_E)_R\right]_x\right\|^2(s)\,ds\\
&\leqslant C_2+C_2\int_0^t\left\|\left[\eta(S_E)_R\right]_x\right\|^2(s)\,ds.
\end{aligned}\tag{4.13}$$

The same estimate as (4.10) yields

$$\begin{aligned}\int_0^t \big\|\big[\eta(S_E)_R\big]_x\big\|^2(s)\,ds &= \int_0^t\int_0^1 \big(\eta_x(S_E)_R+\eta\big[(S_E)_R\big]_x\big)^2\,dx\,ds\\ &\leqslant C_1\int_0^t \big(\|\eta_x\|^2+\|\theta_x\|^2\big)(s)\,ds + C_1\bigg\|\int_0^\infty\int_{S^1}|I_x|\,d\omega\,d\nu\bigg\|^2_{L^\infty(Q_t)}\\ &\leqslant C_1, \end{aligned} \tag{4.14}$$

which, together with (4.13), implies

$$\int_0^t \big\|\theta_{xxx}(s)\big\|^2\,dx\,ds \leqslant C_2,\quad \forall t>0. \tag{4.15}$$

Thus (4.2)–(4.3) follow from (4.5)–(4.6), (4.11)–(4.12) and (4.15). The proof is now complete. □

The following two lemmas are the new arguments on the uniform-in-time estimates of the radiative term $I(x,t;\nu,\omega)$.

**Lemma 4.3.** *There holds*

$$\bigg\|\int_0^\infty\int_{S^1}|I_t|\,d\omega\,d\nu\bigg\|_{L^\infty(Q_t)} \leqslant C_2,\quad \forall t>0. \tag{4.16}$$

**Proof.** Using (1.18), Young's inequality, Lemmas 2.5 and 2.9, we derive from (2.55)

$$\begin{aligned}\int_0^\infty\int_{S^1}|A_2|\,d\omega\,d\nu &\leqslant C_1\int_0^\infty\int_{S^1}\bigg|\int_0^x\bigg(\int_x^y \frac{v_x}{\omega}\big(\sigma_a+\sigma_s+(\sigma_a+\sigma_s)_\eta\big)\\ &\quad+\frac{\theta_t}{\omega}(\sigma_a+\sigma_s)_\theta\,dz\bigg)\frac{\eta}{\omega}(\sigma_a B+\sigma_s\tilde{I})\,dy\bigg|\,d\omega\,d\nu\\ &\leqslant C_1\int_0^\infty\int_{S^1}\bigg|\bigg[\int_0^x\bigg(\int_x^y\frac{v_x}{\omega}\big(\sigma_a+\sigma_s+(\sigma_a+\sigma_s)_\eta\big)\\ &\quad+\frac{\theta_t}{\omega}(\sigma_a+\sigma_s)_\theta\,dz\bigg)^2 dy+\int_0^x\frac{\eta^2}{\omega^2}\sigma_a^2B^2\,dy+\int_0^x\sigma_s^2\tilde{I}^2\,dy\bigg]\bigg|\,d\omega\,d\nu\\ &\leqslant C_1\int_0^1\big(v_x^2+\theta_t^2\big)\,dx\leqslant C_2. \end{aligned} \tag{4.17}$$

Analogously,

$$\int_0^\infty \int_{S^1} |B_2|\,d\omega\,d\nu \leqslant C_1 \int_0^\infty \int_{S^1} \left| \left[ \int_0^x \left( \frac{v_x}{\omega}(\sigma_a B + \sigma_s \tilde{I}) + \frac{\eta}{\omega}\big((\sigma_a)_\eta v_x B \right.\right.\right.$$
$$\left.\left.\left. + (\sigma_a)_\theta \theta_t B + \sigma B_\theta \theta_t + (\sigma_s)_\eta \tilde{I} v_x + (\sigma_s)_\theta \theta_t \tilde{I} + \sigma_s \tilde{I}_t\big)\right) dy \right] \right| d\omega\,d\nu$$
$$\leqslant C_1 \int_0^1 \big(|v_x| + |\theta_t|\big)\,dx + C_1 \int_0^x \int_0^\infty \int_{S^1} \left| \frac{\eta}{\omega}\sigma_s \tilde{I}_t \right| d\omega\,d\nu\,dy$$
$$\leqslant C_2 + C_1 \int_0^x \int_0^\infty \int_{S^1} |I_t|\,d\omega\,d\nu\,dy,$$

which, together with (4.17) and (2.55) and using Gronwall's inequality, implies

$$\int_0^\infty \int_{S^1} |I_t|\,d\omega\,d\nu \leqslant C_2.$$

The proof is complete. □

**Lemma 4.4.** *Under the assumptions in Theorem* 1.2, *the following estimates hold*

$$\big\| \mathcal{I}_{xxx}(t) \big\| \leqslant C_2. \tag{4.18}$$

**Proof.** The elementary computation yields

$$\big\| \mathcal{I}_{xxx}(t) \big\|^2 = \int_0^1 \left( \int_0^\infty \int_{S^1} I_{xxx}\,d\omega\,d\nu \right)^2 dx$$
$$= \int_0^1 \left( \int_0^\infty \int_{S^1} \frac{1}{\omega}(\eta_{xx} S + 2\eta_x S_x + \eta S_{xx})\,d\omega\,d\nu \right)^2 dx$$
$$\leqslant C_1 \int_0^1 \left[ \left( \int_0^\infty \int_{S^1} \frac{1}{\omega}\eta_{xx} S\,d\omega\,d\nu \right)^2 + \left( \int_0^\infty \int_{S^1} \frac{1}{\omega}\eta_x S_x\,d\omega\,d\nu \right)^2 \right.$$
$$\left. + \left( \int_0^\infty \int_{S^1} \frac{1}{\omega}\eta S_{xx}\,d\omega\,d\nu \right)^2 \right] dx =: \sum_{i=1}^3 J_i. \tag{4.19}$$

Employing the Gagliardo–Nirenberg interpolation inequality and using (1.18), Lemma 2.9 and (4.4), we conclude

$$
\begin{aligned}
J_1 &\leqslant C_1 \int_0^1 \left( \int_0^\infty \int_{S^1} \frac{1}{\omega} \eta_{xx} S \, d\omega \, d\nu \right)^2 dx \\
&\leqslant C_1 \int_0^1 \eta_{xx}^2 \left( \int_0^\infty \int_{S^1} \frac{1}{\omega} \big[ \sigma_a (B - I) + \sigma_s (\tilde{I} - I) \big] \, d\omega \, d\nu \right)^2 dx \\
&\leqslant C_1 \int_0^1 \eta_{xx}^2 \, dx \leqslant C_2, \qquad (4.20)
\end{aligned}
$$

$$
\begin{aligned}
J_2 &\leqslant C_1 \int_0^1 \left( \int_0^\infty \int_{S^1} \frac{1}{\omega} \eta_x \big\{ \big[ (\sigma_a)_\eta \eta_x + (\sigma_a)_\theta \theta_x \big] (B - I) \right. \\
&\qquad \left. + \sigma_a (B_\theta \theta_x - I_x) + \big[ (\sigma_s)_\eta \eta_x + (\sigma_s)_\theta \theta_x \big] (\tilde{I} - I) + \sigma_s (\tilde{I} - I)_x \big\} \, d\omega \, d\nu \right)^2 dx \\
&\leqslant C_1 \int_0^1 \big( \eta_x^4 + \eta_x^2 \theta_x^2 + \eta_x^2 \big) \, dx \\
&\leqslant C_1 \max_\Omega \eta_x^2 \left( \int_0^1 \big( \eta_x^2 + \theta_x^2 \big) \, dx \right) + C_1 \\
&\leqslant C_1 \big( \|\eta_x\| \cdot \|\eta_{xx}\| + \|\eta_x\|^2 \big) + C_1 \\
&\leqslant C_2. \qquad (4.21)
\end{aligned}
$$

Analogously, we infer from (1.18), (4.3)–(4.4) and (2.57) that

$$
\begin{aligned}
J_3 &\leqslant C_1 \int_0^1 \left( \int_0^\infty \int_{S^1} \frac{\eta}{\omega} \big\{ \big[ (\sigma_a)_\eta \eta_x + (\sigma_a)_\theta \theta_x \big] (B - I) \right. \\
&\qquad \left. + \sigma_a (B_\theta \theta_x - I_x) + \big[ (\sigma_s)_\eta \eta_x + (\sigma_s)_\theta \theta_x \big] (\tilde{I} - I) + \sigma_s (\tilde{I} - I)_x \big\}_x \, d\omega \, d\nu \right)^2 dx \\
&\leqslant C_1 \int_0^1 \big( \eta_x^4 + \theta_x^4 + \eta_x^2 \theta_x^2 + \eta_{xx}^2 + \theta_{xx}^2 + \mathcal{I}_{xx}^2 \big) \, dx \\
&\leqslant C_1 \big( \|\eta_x\| \cdot \|\eta_{xx}\| + \|\eta_x\|^2 + \|\theta_x\| \cdot \|\theta_{xx}\| + \|\theta_x\|^2 \big) \int_0^1 \big( \eta_x^2 + \theta_x^2 \big) \, dx + C_2 \\
&\leqslant C_2,
\end{aligned}
$$

which, along with (4.19)–(4.21), gives (4.16). The proof is complete. □

## 5. Large-time behavior in $\mathcal{H}_2$

In this section, we shall derive the large-time behavior of solutions $(\eta, v, \theta, \mathcal{I})$ in $\mathcal{H}_2$.

**Lemma 5.1.** *Under the assumptions in Theorem* 1.2, *we have*

$$\lim_{t\to+\infty}\left\|\eta(t)-\bar{\eta}\right\|_{H^2}=0, \tag{5.1}$$

$$\lim_{t\to+\infty}\left\|v(t)\right\|_{H^2}=0, \tag{5.2}$$

*where* $\bar{\eta}=\int_0^1\eta(y,t)\,dy=\int_0^1\eta_0(y)\,dy$.

**Proof.** We can easily derive (5.1)–(5.2) by the same method as that in Lemmas 5.1–5.2 of [18]. The proof is complete. □

In order to show the next lemma, we have to estimate a new radiative term $\|(S_E)_R\|^2$ since the radiative term $(S_E)_R$ is present. Hence the uniform-in-time estimates are more complicated than those in [17].

**Lemma 5.2.** *Under the assumptions in Theorem* 1.2, *we have*

$$\lim_{t\to+\infty}\left\|\theta(t)-\bar{\theta}\right\|_{H^2}=0, \tag{5.3}$$

*where* $\bar{\theta}>0$ *is determined by* $e(\bar{\eta},\bar{\theta})=\int_0^1(\frac{1}{2}v_0^2+e(\eta_0,\theta_0)+F_R(0))\,dx$.

**Proof.** By (4.8), we can get

$$\frac{d}{dt}\left\|\theta_t(t)\right\|^2+\left\|\theta_{xt}(t)\right\|^2\leqslant C_2\left(\left\|\theta_x(t)\right\|^2+\left\|v_x(t)\right\|^2+\left\|\theta_t(t)\right\|^2+\left\|v_{xt}(t)\right\|^2+\left\|\left[\eta(S_E)_R\right]_t\right\|^2\right),$$

which, combined with (2.29), (2.34), (2.41), (4.2), (4.10) and Lemma 3.1, we can conclude

$$\lim_{t\to+\infty}\left\|\theta_t(t)\right\|^2=0. \tag{5.4}$$

Using (1.18) and Lemma 2.9, we can deduce

$$\begin{aligned}\frac{d}{dt}\left\|(S_E)_R\right\|^2&=2\int_0^1\left(\int_0^\infty\int_{S^1}S\,d\omega\,d\nu\right)\left(\int_0^\infty\int_{S^1}S_t\,d\omega\,d\nu\right)dx\\&\leqslant C_2\left(1+\left\|v_x(t)\right\|^2+\left\|\theta_t(t)\right\|^2\right),\end{aligned}$$

which, together with (2.29), (2.41) and Lemma 3.1, implies

$$\lim_{t\to+\infty}\left\|(S_E)_R\right\|^2=0. \tag{5.5}$$

By Eq. (1.7), we see that

$$\left\|\theta_{xx}(t)\right\|\leqslant C_2\left(\left\|v_x(t)\right\|+\left\|\theta_t(t)\right\|+\left\|\theta_x(t)\right\|+\left\|(S_E)_R\right\|\right),$$

which, along with (3.2), (3.8) and (5.4)–(5.5), gives

$$\lim_{t\to+\infty}\left\|\theta_{xx}(t)\right\|=0. \tag{5.6}$$

Thus (5.3) follows from (5.6) and (3.3). The proof is complete. □

The new large-time behavior of radiative term $I(x,t;\nu,\omega)$ in $H^3$ will be given in the following lemma.

**Lemma 5.3.** *Under the assumptions in Theorem* 1.2, *we have*

$$\lim_{t\to+\infty}\left\|\mathcal{I}(t)\right\|_{H^3}=0. \tag{5.7}$$

**Proof.** By (1.8), we denote

$$\begin{aligned}\left\|\mathcal{I}_{xxx}(t)\right\|^2 &\leqslant C_1\left\|\int_0^\infty\int_{S^1}\frac{1}{\omega}\eta_{xx}S\,d\omega\,d\nu\right\|^2+C_1\left\|\int_0^\infty\int_{S^1}\frac{1}{\omega}\eta_x S_x\,d\omega\,d\nu\right\|^2\\&\quad+\left\|\int_0^\infty\int_{S^1}\frac{1}{\omega}\eta S_{xx}\,d\omega\,d\nu\right\|^2=:R_1+R_2+R_3.\end{aligned} \tag{5.8}$$

Similarly, by (1.18), Lemma 2.9 and the delicate computation, we see that

$$R_1\leqslant C_1\left\|\eta_{xx}(t)\right\|^2, \tag{5.9}$$

$$R_2\leqslant C_1\left(\left\|\eta_x(t)\right\|^2+\left\|\theta_x(t)\right\|^2\right), \tag{5.10}$$

$$R_3\leqslant C_1\left(\left\|\eta_{xx}(t)\right\|^2+\left\|\theta_{xx}(t)\right\|^2+\left\|\mathcal{I}_{xx}(t)\right\|^2\right). \tag{5.11}$$

Plugging (5.9)–(5.11) into (5.8) and using (5.1), (5.6) and (3.16), we obtain

$$\lim_{t\to+\infty}\left\|\mathcal{I}_{xxx}(t)\right\|^2=0,$$

which, combined with (3.9), yields (5.7). The proof is now complete. □

**Proof of Theorem 1.2.** Combining Lemmas 4.1–4.4 and Lemmas 5.1–5.3, we can complete the proof of Theorem 1.2. □

**Remark.** The multi-dimensional viscous situation has been poorly understood even at the formal level. Since the one-dimensional model possesses the special constitutive state equations, which the multi-dimensional model don't have, to our knowledge, we have not found any results on the global existence and asymptotic behavior of solutions to system (1.2), i.e., the multi-dimensional case of (1.5)–(1.8). Moreover, some Sobolev embedding inequalities and interpolation inequalities involved in our arguments heavily depend on the dimension, hence this may bring about some difficulties in deriving uniform-in-time estimates. In a word, the method we deal with the one-dimensional case cannot be applied directly to the multi-dimensional case, which depends on the special constitutive relations of state functions, and so on. However, we can refer to [10] for a macroscopic treatment of radiation in the astrophysical context, and [8] for the associated mathematical treatment.

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