On Degeneracy in Linear Complementarity Problems

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ABSTRACT

Let $\mathbf{M}$ be an $n \times n$ matrix and $\mathbf{q}$ an nth order vector. Then the linear complementarity problem LCP($\mathbf{q},\mathbf{M}$) is defined as follows: determine $\mathbf{x} \geqslant \mathbf{0}$ such that $\mathbf{w} = \mathbf{Mx}+\mathbf{q} \geqslant \mathbf{0}$ and $\mathbf{x}^T\mathbf{w} = 0$. A vector $\mathbf{x}$ which satisfies these conditions is called a solution of the problem, and a solution for which $x_i = w_i = 0$ for at least one value of i is termed degenerate. If the solutions of LCP($\mathbf{q},\mathbf{M}$) are nondegenerate and their number is odd (even), we say that the solution set has odd (even) parity, and Murty has shown that this parity is determined uniquely by $\mathbf{M}$. In this paper the idea of parity is extended to degenerate solutions and, through these, to solution sets containing both degenerate and nondegenerate solutions. These results are then used to give a generalization of Lemke's method and to analyse the stability of certain degenerate solutions of linear complementarity problems.

1. INTRODUCTION

Let $\mathbf{M}$ be an $n \times n$ matrix and $\mathbf{q}$ an nth order vector. Then the linear complementarity problem LCP($\mathbf{q},\mathbf{M}$) is defined as follows: determine $\mathbf{x} \geqslant \mathbf{0}$ such that

$$\mathbf{w} = \mathbf{Mx}+\mathbf{q} \geqslant \mathbf{0} \tag{1}$$

and

$$\mathbf{x}^T\mathbf{w} = 0. \tag{2}$$

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The nonnegativity conditions together with the complementarity condition (2) imply that $x_i = 0$ if $w_i > 0$, and $w_i = 0$ if $x_i > 0$; a vector $\mathbf{x}$ which satisfies these conditions is called a *solution* of the problem. A solution for which $x_i = w_i = 0$ for at least one value of i is termed *degenerate*, and it is the properties of such degenerate solutions that are the concern of this paper.

We first review some results pertaining to the number of solutions of an LCP. A sufficient condition for this number to be finite for particular values of $\mathbf{M}$ and $\mathbf{q}$, given by Lemke [6], is that no solution of LCP($\mathbf{q}, \mathbf{M}$) is degenerate. Subsequently Murty [8] showed that a necessary and sufficient condition for the number of solutions of LCP($\mathbf{q}, \mathbf{M}$) to be finite for an arbitrary choice of $\mathbf{q}$ is that $\mathbf{M}$ is *nondegenerate*, i.e. that no principal minor of $\mathbf{M}$ is equal to zero. This result is true even if some solutions of the LCP are degenerate, as they must necessarily be for certain values of $\mathbf{q}$. A similar result was obtained by Mangasarian [7]. All these results hinge on the fact that if a principal submatrix of $\mathbf{M}$ is singular, it is possible to find a vector $\mathbf{q}$ for which LCP($\mathbf{q}, \mathbf{M}$) has two distinct solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ such that any convex combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ is also a solution of the LCP. Conversely, if no principal submatrix of $\mathbf{M}$ is singular, then any strictly convex combination of two solutions of an LCP fails to satisfy the complementarity condition, and a continuum of solutions is thus not possible. Since this paper is concerned only with finite solution sets, it will be assumed throughout that $\mathbf{M}$ is nondegenerate.

Sufficient conditions for every solution of LCP($\mathbf{q}, \mathbf{M}$) to be nondegenerate were considered by, among others, Eaves [2] and Murty [8]. In particular Eaves showed that "almost any $\mathbf{q}$" gives rise to nondegenerate solutions. The importance of nondegeneracy was demonstrated by Murty [8], who proved that if all the solutions of LCP($\mathbf{q}_1, \mathbf{M}$) and LCP($\mathbf{q}_2, \mathbf{M}$) are nondegenerate, then the number of solutions of LCP($\mathbf{q}_1, \mathbf{M}$) differs from that of LCP($\mathbf{q}_2, \mathbf{M}$) by a (possibly zero) multiple of two. We shall subsequently refer to this result as *Murty's theorem*. It follows from this that if LCP($\mathbf{q}_1, \mathbf{M}$) has an odd (even) number of nondegenerate solutions and no degenerate ones, then if all the solutions of LCP($\mathbf{q}, \mathbf{M}$) are nondegenerate, the number of such solutions is odd (even). As a natural consequence of this, Murty defined the *parity* of a set of nondegenerate solutions to be odd if the set consisted of an odd number of solutions, and even otherwise.

In this paper the idea of parity is extended to a set which may contain both nondegenerate and degenerate solutions. It is shown that a degenerate solution itself possesses a unique parity (nondegenerate solutions are odd) and behaves, from the standpoint of enumerating the number of solutions, either like an odd or like an even number of nondegenerate solutions. This result is achieved by applying Murty's theorem to the auxiliary problems described by Broyden [1].

2. DEGENERATE SOLUTIONS

Murty's theorem states that if $\mathbf{M}$ is nondegenerate and the solutions of LCP($\mathbf{q}, \mathbf{M}$) are also nondegenerate, then the parity of the solution set is independent of $\mathbf{q}$. It is thus a property of $\mathbf{M}$ itself, and we thus offer the following formal definition.

DEFINITION 1. A nondegenerate matrix $\mathbf{M}$ will be said to be odd (even), or to have odd (even) parity, if for every vector $\mathbf{q}$ for which every solution of LCP($\mathbf{q}, \mathbf{M}$) is nondegenerate, the total number of solutions is odd (even).

Note that this definition is always valid, since if $\mathbf{M}$ is nondegenerate it is always possible, as was shown by Eaves [2], to find a $\mathbf{q}$ such that every solution of LCP($\mathbf{q}, \mathbf{M}$) is nondegenerate.

The importance of the idea of parity, as was recognized by Murty [8, Corollary 6.9], is that if $\mathbf{M}$ is odd, then it is a Q-matrix [a matrix $\mathbf{M}$ for which LCP($\mathbf{q}, \mathbf{M}$) has a solution for any $\mathbf{q}$]. LCPs involving even matrices, on the other hand, may have no solution at all for certain values of $\mathbf{q}$, but if they have one nondegenerate solution, they must have at least one other solution, which may or may not be degenerate. This possibility of further solutions is the reason why, unless an LCP is known to have only one solution, Lemke's method should never be terminated when a solution is found but should be allowed to continue until it terminates naturally on a ray. It is possible that other solutions may be discovered en route.

LEMMA 1. *Let $\mathbf{Q}$ be a permutation matrix. Then the parities of $\mathbf{M}$ and $\mathbf{Q}^T\mathbf{M}\mathbf{Q}$ are equal.*

Proof. Follows immediately from the fact that $\mathbf{x}$ is a solution of LCP($\mathbf{q}, \mathbf{M}$) if and only if $\mathbf{Q}^T\mathbf{x}$ is a solution of LCP($\mathbf{Q}^T\mathbf{x}, \mathbf{Q}^T\mathbf{M}\mathbf{Q}$). ■

We now show how the notion of parity may be extended to a degenerate solution of a linear complementarity problem.

DEFINITION 2. Let LCP($\mathbf{p}, \mathbf{M}$) have a degenerate solution $\mathbf{x}$, so that Equation (1) may be written, after suitably permuting the rows and columns of $\mathbf{M}$ (see [1]),

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{w}_3 \end{bmatrix}, \tag{3}$$

where $\mathbf{x}_1 > \mathbf{0}$ and $\mathbf{w}_3 > \mathbf{0}$; and let

$$\mathbf{G} = \mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{M}_{12}. \tag{4}$$

Then $\mathbf{G}$ is called an *auxiliary matrix* of $\mathbf{x}$.

We see that $\mathbf{G}$ depends only on the degeneracy pattern of $\mathbf{x}$, being determined by those values of i for which $x_i = w_i = 0$ [it is trivial to show that $\mathbf{G}$ is invariant under identical permutations to the first block row and block column of $\mathbf{M}$, i.e. permutations affecting only the elements of $\mathbf{x}_1$ and those of $\mathbf{p}_1$ in Equation (3)]. It is not unique, since any matrix $\mathbf{Q}^T\mathbf{G}\mathbf{Q}$, where $\mathbf{Q}$ is a permutation matrix, is also an auxiliary matrix of $\mathbf{x}$. Its parity, however, is unique from Lemma 1, and it is this property that makes the following definition possible.

DEFINITION 3. The *parity* of a degenerate solution is defined to be the parity of its auxiliary matrices.

We now use these ideas to generalize Murty's theorem.

THEOREM 1. *Let* $\mathbf{M}$ *be nondegenerate, and let* $\mathbf{p}$ *be arbitrary. Define the parity of a nondegenerate solution of a linear complementarity problem to be odd, and let that of a degenerate solution be given by Definition* 3. *Let* ω *be the number of solutions of* LCP$(\mathbf{p},\mathbf{M})$ *having odd parity. Then* $\mathbf{M}$ *is odd if* ω *is odd, and even otherwise.*

Proof. Choose $\mathbf{d}$ to be some vector satisfying the nondegeneracy conditions of Lemma 1 of [1] [these guarantee that if ε_0 is sufficiently small, then every solution of LCP$(\mathbf{p}+\varepsilon\mathbf{d},\mathbf{M})$, $0<\varepsilon\leqslant\varepsilon_0$, is nondegenerate], and assume that ε_0 is sufficiently small for this to occur. It was shown in [1] that if the solutions of LCP$(\mathbf{p}+\varepsilon\mathbf{d},\mathbf{M})$ are regarded as functions of ε, they are distinct as long as they remain nondegenerate, but may coalesce into, and branch out from, degenerate solutions at particular values of ε. It thus follows from the assumptions above that the number of solutions of LCP$(\mathbf{p}+\varepsilon\mathbf{d},\mathbf{M})$ remains constant for $0<\varepsilon\leqslant\varepsilon_0$, and is equal to the number of nondegenerate solutions at $\varepsilon=0$ together with the total number of branches emanating from the degenerate solutions at $\varepsilon=0$ in the direction of increasing ε. Moreover, since all solutions of LCP$(\mathbf{p}+\varepsilon_0\mathbf{d},\mathbf{M})$ are nondegenerate, the parity of the solution set is equal, from Definition 1, to the parity of $\mathbf{M}$.

Now it was shown in [1] that the number of branches emanating from a degenerate solution is identical to the number of solutions of LCP$(\mathbf{h},\mathbf{G})$, where $\mathbf{G}$ is an auxiliary matrix of the degenerate solution and $\mathbf{h}$ is some vector derived from $\mathbf{M}$ and $\mathbf{d}$. Moreover, if $\mathbf{d}$ satisfies the assumptions of the theorem, each such solution is nondegenerate. It therefore follows from Definitions 1 and 3 (above) that the number of branches emanating from each degenerate solution at $\varepsilon = 0$ is odd if the parity of that solution is odd, and even otherwise. Now the parity of the solution set of LCP($\mathbf{p} + \varepsilon_0 \mathbf{d}, \mathbf{M}$) remains unchanged if the branches emanating from the even degenerate solutions are deleted and those emanating from the odd degenerate solutions are replaced by a single branch. The parity of $\mathbf{M}$ is thus the parity of this modified solution set, and the theorem follows from the observation that this in turn, since nondegenerate solutions are odd by definition, is determined by the number of odd solutions of LCP($\mathbf{p}, \mathbf{M}$). ■

COROLLARY. *If* $\mathbf{M}$ *is odd, then* LCP($\mathbf{q}, \mathbf{M}$) *has at least one odd solution.*

Proof. Straightforward. ■

We note that, if all solutions of LCP($\mathbf{p}, \mathbf{M}$) are nondegenerate (and hence odd), Theorem 1 reduces essentially to Murty's theorem.

3. APPLICATIONS AND CONCLUSIONS

Lemke's method of solving a linear complementarity problem consists essentially of tracking a particular solution of LCP($\mathbf{q} + \vartheta \mathbf{d}, \mathbf{M}$) as ϑ varies, and it is usually assumed (e.g. [1] and [2]) that $\mathbf{d}$ is such that this solution is nondegenerate except for a finite number of values of ϑ. This restriction on $\mathbf{d}$ is based on purely practical considerations, i.e. the difficulty of computing degenerate solutions, but it is possible in principle to use the same ideas to track along arcs of the solution graph (see [1]) on which the solution is degenerate. By way of illustration, we solve LCP($\mathbf{q}, \mathbf{M}$) by solving LCP($\mathbf{q} + \vartheta \mathbf{d}, \mathbf{M}$), where

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 1 \\ 2 & 3 & 2 \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \quad \text{and} \quad \mathbf{d} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix},$$

the unorthodox value of $\mathbf{d}$ being chosen for purpose of illustration.

Clearly, for $\vartheta > 1$, $\mathbf{x} = \mathbf{0}$ and $\mathbf{w} = \mathbf{q} + \vartheta \mathbf{d}$ is a nondegenerate solution of LCP($\mathbf{q} + \vartheta \mathbf{d}, \mathbf{M}$), but when ϑ is reduced to unity $\mathbf{x}$ becomes doubly degenerate, a situation that would normally be resolved by perturbation. Now the auxiliary problems for this solution (see [1]) are LCP($\pm$**h**,**G**) where

$$\mathbf{h} = \begin{bmatrix} d_1 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{G} = \begin{bmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix},$$

and it is readily shown that $\mathbf{x} = [0,0]^T$ is the only solution of LCP(**h**,**G**). Since this solution is nondegenerate, it follows that **G** is odd. LCP($-$**h**,**G**), however, has two solutions, namely $[0,1]^T$ and $[1,0]^T$, the second of which is degenerate. Since **G** is odd, this solution must be even, a deduction which may be confirmed by computing its auxiliary matrix. This, from Equation (4), is

$$g_{22} - g_{21}g_{11}^{-1}g_{12} = m_{33} - m_{31}m_{11}^{-1}m_{13} = -2.$$

Now each solution of LCP($-$**h**,**G**) corresponds to a branch of the solution graph of LCP($\mathbf{q} + \vartheta\mathbf{d}, \mathbf{M}$) for ϑ decreasing, and taking these in turn gives the two solutions of LCP($\mathbf{q} + \vartheta\mathbf{d}, \mathbf{M}$), $\vartheta \leqslant 1$, to be $\mathbf{x} = [0,0,1-\vartheta]^T$ and $\mathbf{x} = [1-\vartheta,0,0]^T$, for which the corresponding values of **w** are $[1-\vartheta,2,0]^T$ and $[0,1+\vartheta,0]^T$. The second of these two solutions is degenerate for $\forall\vartheta$, since $x_3 = w_3 = 0$. Thus if an auxiliary problem of a degenerate solution itself has a degenerate solution, the solution on the corresponding arc of the solution graph is also degenerate. Since the supports of both **x** and **w** are constant on this arc, the auxiliary matrices, and hence the parity, of the solution as ϑ varies are constant. In the case quoted the auxiliary matrix is equal to $m_{33} - m_{31}m_{11}^{-1}m_{13} = -2$. The degenerate solution is thus even for all ϑ, and putting $\vartheta = 0$ gives two solutions of LCP(**q**,**M**) to be $[0,0,1]^T$ and $[1,0,0]^T$. Note that the possibility of further solutions is not excluded.

Another use of the ideas of this paper lies in their application to the stability of the solution of a linear complementarity problem, i.e. the sensitivity of the solutions with respect to perturbations of either or both of **q** and **M**. Ha [3] gave a definition of stability, and Jansen and Tijs [4] introduced the idea of robustness, and showed that a solution is stable in the sense of Ha if and only if it is both robust and isolated. Since the solutions of linear complementarity problems involving nondegenerate matrices are always isolated [7], the two definitions are, for our purposes, equivalent, so we give the simpler one of Jansen and Tijs and say that a solution **x** of LCP(**q**,**M**) is *stable* if for every neighborhood V of **x** there is a neighborhood U of (**q**,**M**) such that for $\forall(\mathbf{q}',\mathbf{M}') \in U$, LCP($\mathbf{q}',\mathbf{M}'$) has a solution in V. Moreover, following Ha, we shall say that the solution is *strongly stable* if it is unique in V.

Ha showed that a nondegenerate solution of a linear complementarity problem is strongly stable iff it is isolated. If we consider only perturbations to $\mathbf{q}$, it follows from [1, Theorem 3] that if $\mathbf{M}$ is nondegenerate, a degenerate solution of LCP($\mathbf{q},\mathbf{M}$) is stable iff its auxiliary matrices are Q-matrices and strongly stable iff its auxiliary matrices are P-matrices (matrices $\mathbf{M}$ for which LCP($\mathbf{q},\mathbf{M}$) has a unique solution for any $\mathbf{q}$; see [8]). Thus odd solutions are stable, and even solutions may or may not be stable but can never be strongly stable.

If we consider perturbations to $\mathbf{M}$ as well as to $\mathbf{q}$, the situation is far more complex, and the author is indebted to the referee for drawing his attention both to this aspect of the problem and to the associated references ([5], [9], and [10]). It was shown by Tamir [9] that if LCP($\mathbf{0},\mathbf{M}$) has only one solution (and LCPs involving nondegenerate matrices certainly have this property), then if $\mathbf{M}$ is odd, its parity is invariant under small perturbations. We therefore conjecture that a degenerate solution of LCP($\mathbf{q},\mathbf{M}$) is stable if it is odd, and indeed would go further to suggest that if the auxiliary matrix of such a solution is a P-matrix, then the solution is strongly stable. However, an earlier conjecture of the author, that if $\mathbf{M}$ is nondegenerate then a degenerate solution of LCP($\mathbf{q},\mathbf{M}$) is stable if the corresponding auxiliary matrix is a Q-matrix, was demolished by the quite remarkable example of Kelly and Watson [5], who showed that if

$$\mathbf{M}=\begin{bmatrix} 21 & 25 & -27 & -36 \\ 7 & 3 & -9 & 36 \\ 12 & 12 & -20 & 0 \\ 4 & 4 & -4 & 8 \end{bmatrix}, \tag{5}$$

$\mathbf{c}=[-1,1,0,0]^T$, and $\mathbf{e}_4=[0,0,0,1]^T$, then although $\mathbf{M}$ is a Q-matrix, $\mathbf{M}+\varepsilon\mathbf{c}\mathbf{e}_4^T$ is not a Q-matrix for $0<\varepsilon<1$, since LCP($\mathbf{q}+\varepsilon\mathbf{d},\mathbf{M}+\varepsilon\mathbf{c}\mathbf{e}_4^T$), where $\mathbf{q}=[0,0,32,0]^T$ and $\mathbf{d}=[0.26,-0.02,-1.20,-0.08]^T$, has no solution for $0<\varepsilon<1$. If such an $\mathbf{M}$ were the auxiliary matrix of some degenerate solution, that solution would not be stable. This example is so surprising that we examine it in some detail to see just what it is that makes it tick.

We first look at a related problem

$$\mathrm{P1}=\mathrm{LCP}(\mathbf{q}+\varepsilon\mathbf{d},\mathbf{M}), \tag{6}$$

where $\mathbf{M}$, $\mathbf{q}$, and $\mathbf{d}$ are defined by Equation (5) et seq., i.e. the example of Kelly and Watson but with the matrix of the problem held constant. This has no fewer than three solutions at $\varepsilon = 0$, all degenerate and all even. These are

Solution A: $\mathbf{x} = [0,0,0,0]^T$, $\mathbf{w} = [0,0,32,0]^T$,

Solution B: $\mathbf{x} = [9,0,7,0]^T$, $\mathbf{w} = [0,0,0,8]^T$,

Solution C: $\mathbf{x} = [0,9,7,1]^T$, $\mathbf{w} = [0,0,0,0]^T$.

The variation of these solutions with ε in the neighborhood at $\varepsilon = 0$ is shown schematically in Figure 1, and it can be seen that, for $\varepsilon < 0$ and increasing, four (nondegenerate) solutions merge into two degenerate ones at $\varepsilon = 0$ and then vanish completely. The Q-ness of $\mathbf{M}$ is maintained as ε increases by the appearance of a totally unrelated solution at $\varepsilon = 0$, which becomes two solutions as ε increases further. It would seem therefore that the Q-ness of this particular $\mathbf{M}$ is somewhat fragile and that a small perturbation could distort the solutions, opening up a "gap" between them leaving a small interval of ε for which no solution exists. Such changes might be expected when P1 is replaced by the Kelly-Watson example, but the changes that in fact do occur are far more dramatic.

If then

$$\mathrm{P2} = \mathrm{LCP}(\mathbf{q} + \varepsilon\mathbf{d}, \mathbf{M} + \varepsilon\mathbf{c}\mathbf{e}_4^T), \tag{7}$$

the Kelly-Watson example, and we trace the solutions in the region of $\varepsilon = 0$, we obtain Figure 2. Solution C has "flipped" and now appears when $\varepsilon < 0$ instead of when $\varepsilon > 0$. To explain this behavior we need to consider the

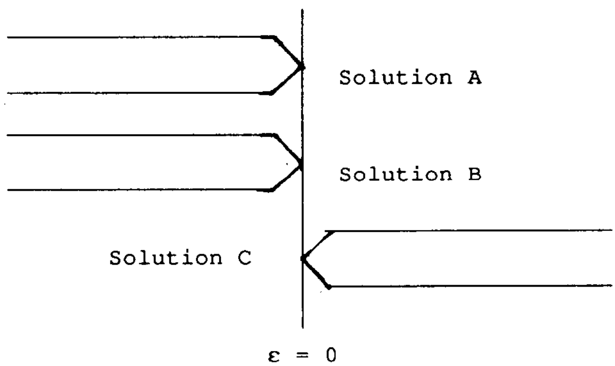

FIG. 1.

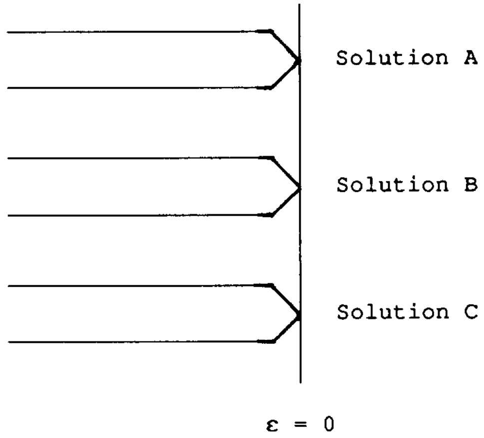

FIG. 2.

auxiliary problem of solution C, but this is complicated by the fact that $\mathbf{M} + \varepsilon \mathbf{c}\mathbf{e}_4^T$ varies with ε. We need, therefore, to take this variation into account when carrying out the analysis.

For problem P2 it follows from Equations (1) and (7) that

$$\left(\mathbf{M} + \varepsilon \mathbf{c}\mathbf{e}_4^T\right)\mathbf{x} + \mathbf{q} + \varepsilon \mathbf{d} = \mathbf{w}. \tag{8}$$

Since $\mathbf{x}$ is a function of ε, we may write its fourth element as $x_4(\varepsilon)$, so that Equation (8) becomes

$$\mathbf{M}\mathbf{x} + \mathbf{q} + \varepsilon\left[\mathbf{c}x_4(\varepsilon) + \mathbf{d}\right] = \mathbf{w}. \tag{9}$$

Now since $\mathbf{M}$ is nondegenerate, it follows from standard perturbation theory (see e.g. [11, p. 189 et seq.]) that for $|\varepsilon|$ sufficiently small, $x_4(\varepsilon) = x_4(0) + O(\varepsilon)$. Now the value of $x_4(0)$ for solution C is unity, and substituting this value in Equation (9) and ignoring terms of order ε^2 then gives problem

$$\text{P3} = \text{LCP}(\mathbf{q} + \varepsilon \mathbf{d}_e, \mathbf{M}), \tag{10}$$

where $\mathbf{d}_e$, the "equivalent value" of $\mathbf{d}$, is given by

$$\mathbf{d}_e = \mathbf{c} + \mathbf{d} = [-0.74, 0.98, -1.20, -0.08]^T.$$

Thus, for $|\varepsilon|$ sufficiently small, solution C of P2 behaves virtually identically to solution C of P3. Note that solutions A and B do not correspond, since $\mathbf{d}_e$ involves the value of $x_4(0)$ specific to solution C. Note further that all three problems are the same if $\varepsilon = 0$.

Now for P1 and P3 the behavior of solution C as ε varies is governed by its auxiliary problem (see [1]). This, for P1, is LCP($\mathbf{h},\mathbf{G}$), where $\mathbf{G}$ is defined by Equation (4) above and $\mathbf{h}$ is given (see [1]) by

$$\mathbf{h} = \mathbf{d}_2 - \mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{d}_1, \tag{11}$$

with obvious notation, but where both $\mathbf{M}$ and $\mathbf{d}$ are assumed to be permuted and partitioned as in Equation (3) above. The comparable result for problem P3 is obtained by substituting $\mathbf{d}_e$ for $\mathbf{d}$ in Equation (11). Now for solution C, $x_i = w_i = 0$ for only one value of i, so both $\mathbf{G}$ and $\mathbf{h}$ are scalars, $\mathbf{G}$ being the same for both P1 and P3. For these problems $\mathbf{G} = -5.333$, and its negativity implies that solution C branches either into two solutions for $\varepsilon > 0$ and none for $\varepsilon < 0$ or vice versa, depending on the sign of $\mathbf{h}$. For P1, $\mathbf{h} = 0.746$ and is positive, so that solution C branches into two solutions for $\varepsilon > 0$. For P3, $\mathbf{h} = -0.586$, so the solutions branch out in the reverse direction.

The behavior of all the solutions of P3 in the neighborhood of $\varepsilon = 0$ is shown in Figure 3. Note also that, although the perturbation to $\mathbf{M}$ may be arbitrarily small, the crucial difference between $\mathbf{d}$ and $\mathbf{d}_e$ is quite substantial and is, moreover, independent of ε.

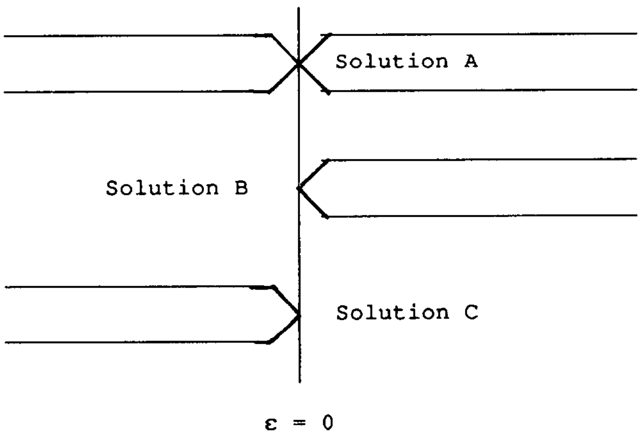

FIG. 3.

Although the Kelly-Watson example destroys the supposition that nondegenerate matrices are stable, it leaves unscathed the conjecture that even-parity nondegenerate matrices form an open set. Comparison of Figures 1 and 2 shows that not only is the total number of solutions even in each case, but the even degenerate solutions remain even. Thus not only the matrix itself, but three of its auxiliary matrices, remain even under the perturbation. We thus conjecture that provided $\mathbf{M}$ is nondegenerate, small perturbations alter the parity neither of $\mathbf{M}$ nor of any of its principal submatrices or Schur complements. Indeed, we conjecture that these parities remain constant until a crucial principal minor of $\mathbf{M}$ changes sign. If true, this would suggest that the parity of a matrix is determined in some way by the signs of its principal minors, a suggestion that is reinforced by the fact that if all these are positive, then $\mathbf{M}$ is a P-matrix whose parity, and that of all possible auxiliary matrices, is odd (see [1]). However, if $\mathbf{M}$ is not a P-matrix, no simple relationship appears to exist.

Finally, since parity has been identified as the property of a matrix, it is only natural to ask how this relates to the parities of other matrices. It is trivial to show that the parities of $\mathbf{M}$ and $\mathbf{M}^{-1}$ are identical [merely compare the solutions of LCP($\mathbf{q},\mathbf{M}$) and LCP($-\mathbf{M}^{-1}\mathbf{q},\mathbf{M}^{-1}$)], but even so simple an operation as negating $\mathbf{M}$ gives rise to complications. If $\mathbf{M}$ is the 2×2 unit matrix, then $\mathbf{M}$ is odd and $-\mathbf{M}$ is even, but if one of the diagonal elements of $\mathbf{M}$ is negated, then both $\mathbf{M}$ and $-\mathbf{M}$ are even. It is hoped to deal more fully with these matters in a subsequent paper.

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REFERENCES

1 C. G. Broyden, Lemke's method—a recursive approach, *Linear Algebra Appl.* to appear.

2 B. Curtis Eaves, The linear complementarity problem, *Management Sci.* 17:612–634 (1971).

3 C. D. Ha, Stability of the linear complementarity problem at a solution point, *Math. Programming* 31:327–338 (1985).

4 M. J. M. Jansen and S. H. Tijs, Robustness and nondegenerateness for linear complementarity problems, *Math. Programming* 37:293–308 (1987).

5 L. M. Kelly and L. T. Watson, Erratum: Some perturbation theorems for Q-matrices, *SIAM J. Appl. Math.* 34(2):320–321 (1978).

6 C. E. Lemke, Bimatrix equilibrium points and mathematical programming, *Management Sci.* 11:681–689 (1965).

7 O. L. Mangasarian, Locally unique solutions of quadratic programs, linear and nonlinear complementarity problems, *Math. Programming* 19:200–212 (1980).
8 K. G. Murty, On the number of solutions to the complementarity problem and spanning properties of complementary cones, *Linear Algebra Appl.* 5:65–108 (1972).
9 A. Tamir, The Complementarity Problem of Mathematical Programming, Doctoral Dissertation, Case Western Reserve Univ. Cleveland, 1973.
10 L. T. Watson, Some perturbation theorems for Q-matrices, *SIAM J. Appl. Math.* 31:379–384 (1976).
11 J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon, Oxford 1965.

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