A Meshless Method for Solving the Cauchy Problem in Three-Dimensional Elastostatics

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Abstract—The application of the method of fundamental solutions to the Cauchy problem in three-dimensional isotropic linear elasticity is investigated. The resulting system of linear algebraic equations is ill-conditioned and therefore, its solution is regularized by employing the first-order Tikhonov functional, while the choice of the regularization parameter is based on the L-curve method. Numerical results are presented for both under- and equally-determined Cauchy problems in a piece-wise smooth geometry. The convergence, accuracy, and stability of the method with respect to increasing the number of source points and the distance between the source points and the boundary of the solution domain, and decreasing the amount of noise added into the input data, respectively, are analysed.

Keywords—Meshless method, Method of fundamental solutions, Cauchy problem, Elastostatics, Regularization, Inverse problem.

1. INTRODUCTION

The method of fundamental solutions (MFS) was originally introduced by Kupradze and Alek-sidze [1], whilst its numerical formulation was first given by Mathon and Johnston [2]. The main idea of the MFS consists of approximating the solution of the problem by a linear combination of fundamental solutions with respect to some singularities/source points which are located outside the domain. Then, the original problem is reduced to determining the unknown coefficients of the fundamental solutions and the coordinates of the source points by requiring the approximation to satisfy the boundary conditions and hence solving a nonlinear problem. If the source points are a priori fixed then the coefficients of the MFS approximation are determined by solving a linear problem. An excellent survey of the MFS and related methods over the past three decades...
has been presented in [3]. The advantages of the MFS over domain discretisation methods, such as the finite-difference method (FDM) and the finite-element method (FEM), are very well documented [3]. In addition, the MFS has all the advantages of boundary methods, such as the boundary element method (BEM), as well as several advantages over other boundary methods. For example, the MFS does not require an elaborate discretisation of the boundary, integrations over the boundary are avoided, the solution in the interior of the domain is evaluated without extra quadratures, its implementation is very easy and only little data preparation is required. The most arguable issue regarding the MFS is still the location of the source points. However, this problem can be overcome by employing a nonlinear least-squares minimisation procedure. Alternatively, the source points can be prescribed a priori, see [4-6], and the post-processing analysis of the errors can indicate their optimal location.

The MFS has been successfully applied for solving a wide variety of boundary value problems. Karageorghis and Fairweather [7] have solved numerically the biharmonic equation using the MFS and later their method has been modified in order to take into account the presence of boundary singularities in both the Laplace and the biharmonic equations by Poullikkas et al. [8]. Furthermore, Poullikkas et al. [9] have investigated the numerical solution of the inhomogeneous harmonic and biharmonic equations by reducing these problems to the homogeneous corresponding cases and subtracting a particular solution of the governing equation. The MFS has been formulated for three-dimensional Signorini boundary value problems and it has been tested on a three-dimensional electropainting problem related to the coating of vehicle roofs in [10]. Karageorghis and Fairweather [11] have studied the use of the MFS for the approximate solution of three-dimensional isotropic materials with axisymmetrical geometry and both axisymmetrical and arbitrary boundary conditions. The application of the MFS to two-dimensional problems of steady-state heat conduction and elastostatics in isotropic and anisotropic bimaterials has been addressed by Berger and Karageorghis [12, 13]. Karageorghis [14] has investigated the calculation of the eigenvalues of the Helmholtz equation subject to homogeneous Dirichlet boundary conditions for circular and rectangular geometries by employing the MFS, whilst Redekop and Cheung [15] and Poullikkas et al. [16] have successfully applied the MFS for solving three-dimensional elastostatics problems. The MFS, in conjunction with singular value decomposition, has been employed by Ramachandran [17] in order to obtain numerical solutions of the Laplace and the Helmholtz equations. Recently, Balakrishnan et al. [18] have proposed an operator splitting-radial basis function method as a generic solution procedure for transient nonlinear Poisson problems by combining the concepts of operator splitting, radial basis function interpolation, particular solutions and the MFS.

In most boundary value problems in solid mechanics, the governing system of partial differential equations, i.e., the equilibrium, constitutive, and kinematics equations, have to be solved with the appropriate initial and boundary conditions for the displacement and/or traction vectors, i.e., Dirichlet, Neumann, or mixed boundary conditions. These problems are called direct problems and their existence and uniqueness are well-established, see for example [19]. However, there are other engineering problems which do not belong to this category. For example, when the material properties and/or the external sources are unknown, the geometry of a portion of the boundary is not determined or the boundary conditions are incomplete, either in the form of underspecified and overspecified boundary conditions on different parts of the boundary or the solution is prescribed at some internal points in the domain. These are inverse problems, and it is well known that they are generally ill-posed, i.e., the existence, uniqueness, and stability of their solutions are not always guaranteed, see [20].

A classical example of an inverse problem in elasticity is the Cauchy problem in which both displacement and traction boundary conditions are prescribed only on a part of the boundary of the solution domain, whilst no information is available on the remaining part of the boundary. This problem has been studied by many authors, see [21–28], but only in the two-dimensional case. Yeih et al. [21] have analysed its existence, uniqueness, and continuous dependence on the
data and have proposed an alternative regularization procedure, namely, the fictitious boundary indirect method, based on the simple or double layer potential theory. The numerical implementation of the aforementioned method has been undertaken by Koya et al. [22], who have used the BEM and the Nyström method for discretising the integrals. However, this formulation has not yet removed the problem of multiple integrations. Marin et al. [23] have determined the approximate solutions to the Cauchy problem in two-dimensional linear elasticity using an alternating iterative BEM which reduced the problem to solving a sequence of well-posed boundary value problems and they have later extended this numerical method to singular Cauchy problems, see [24]. Huang and Shih [25], and Marin et al. [26] have both used the conjugate gradient method combined with the BEM, in order to solve the same problem. The Tikhonov regularization method and the singular value decomposition, in conjunction with the BEM, have been employed by Marin and Lesnic [27,28] to solve the two-dimensional Cauchy problem in linear elasticity. Recently, noniterative meshless numerical methods based on the radial basis functions and the MFS have been employed to solve inverse problems. Hon and Wu [29] have determined an unknown boundary of a two-dimensional domain from Cauchy data using the radial interpolation for Hermite-Birkhoff data and the shift invariability of the harmonic space. A radial basis meshless method for solving the Cauchy problem for second-order linear elliptic equations with space-dependent coefficients has been proposed by Li [30]. Marin and Lesnic [31,32] have solved the the Cauchy problem for two-dimensional isotropic linear elasticity and Helmholtz-type equations, respectively, by employing the MFS in conjunction with the Tikhonov regularization method.

To our knowledge, the numerical solution of the Cauchy problem in three-dimensional linear elasticity has not been investigated yet. The MFS discretised system of equations is ill-conditioned and hence, it is solved by employing the first-order Tikhonov regularization method, see e.g., [33], whilst the choice of the regularization parameter is based on the L-curve criterion, see [34]. Two benchmark examples for the three-dimensional isotropic linear elasticity are investigated and the convergence and stability of the method with respect to the location and the number of source points and the amount of noise added into the Cauchy input data, respectively, are analysed.

2. MATHEMATICAL FORMULATION

Consider an isotropic linear elastic material which occupies an open bounded domain $\Omega \subset \mathbb{R}^3$ and assume that $\Omega$ is bounded by a piecewise smooth surface $\Gamma \equiv \partial \Omega$, such that $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1, \Gamma_2 \neq \emptyset$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. In the absence of body forces, the equilibrium equations with respect to the displacement vector $u(x)$, also known as the Lamé or Navier equations, are given by, see e.g., [35],

$$\sum_{j=1}^{3} \left( G \frac{\partial^2 u_i (x)}{\partial x_j \partial x_j} + \frac{G}{1-2\nu} \frac{\partial^2 u_j (x)}{\partial x_i \partial x_j} \right) = 0, \quad x \in \Omega, \quad i = 1, 2, 3, \quad (1)$$

where $G$ and $\nu$ are the shear modulus and Poisson ratio, respectively. The strains $\varepsilon_{ij}(x), \; i,j = 1, 2, 3$, are related to the displacement gradients by the kinematic relations,

$$\varepsilon_{ij}(x) = \frac{1}{2} \left( \frac{\partial u_i (x)}{\partial x_j} + \frac{\partial u_j (x)}{\partial x_i} \right), \quad x \in \Omega, \quad i,j = 1, 2, 3, \quad (2)$$

while the stresses $\sigma_{ij}(x), \; i,j = 1, 2, 3$, are related to the strains through the constitutive law (Hooke's law), namely,

$$\sigma_{ij}(x) = 2G \left( \varepsilon_{ij} (x) + \frac{\nu}{1-2\nu} \delta_{ij} \sum_{k=1}^{3} \varepsilon_{kk} (x) \right), \quad x \in \Omega, \quad i,j = 1, 2, 3, \quad (3)$$
with $\delta_{ij}$ the Kronecker delta tensor. Now, we let $n(x)$ be the outward normal vector at $\Gamma$ and $t(x)$ be the traction vector at a point $x \in \Gamma$ whose components are defined by

$$t_i(x) = \sum_{j=1}^{3} \sigma_{ij}(x) n_j(x), \quad x \in \Gamma, \quad i = 1, 2, 3. \quad (4)$$

In the direct problem formulation, the knowledge of the displacement and/or traction vectors on the whole boundary $\Gamma$ gives the corresponding Dirichlet, Neumann, or mixed boundary conditions which enables us to determine the displacement vector in the domain $\Omega$. Then, the strain tensor $\varepsilon_{ij}$ can be calculated from the kinematic relations (2) and the stress tensor is determined using the constitutive law (3). If it is possible to measure both the displacement and traction vectors on a part of the boundary $\Gamma$, say $\Gamma_1$, then this leads to the mathematical formulation of an inverse problem consisting of equations (1) and the boundary conditions,

$$u_i(x) = \bar{u}_i(x), \quad t_i(x) = \bar{t}_i(x), \quad x \in \Gamma_1, \quad i = 1, 2, 3, \quad (5)$$

where $\bar{u}$ and $\bar{t}$ are prescribed vector valued functions. In the above formulation of the boundary conditions (5), it can be seen that the boundary $\Gamma_1$ is overspecified by prescribing both the displacement $u|_{\Gamma_1} = \bar{u}$ and the traction $t|_{\Gamma_1} = \bar{t}$ vectors, whilst the boundary $\Gamma_2$ is underspecified since both the displacement $u|_{\Gamma_2}$ and the traction $t|_{\Gamma_2}$ vectors are unknown and have to be determined. This problem, known as the Cauchy problem, is much more difficult to solve both analytically and numerically than the direct problem, since the solution does not satisfy the general conditions of well-posedness. Although the problem may have a unique solution, it is well known, see e.g., Hadamard [20], that this solution is unstable with respect to small perturbations in the data on $\Gamma_1$. Thus, the problem is ill-posed and we cannot use a direct approach, such as the Gauss elimination method, in order to solve the system of linear equations which arises from the discretisation of the partial differential equations (1) and the boundary conditions (5). Therefore, regularization methods are required in order to solve accurately the Cauchy problem in linear elasticity.

3. THE METHOD OF FUNDAMENTAL SOLUTIONS (MFS)

The fundamental solutions $U_j(x, y)$, $j = 1, 2, 3$, of the Lamé system (1) in the three-dimensional case are given by, see e.g., [16],

$$U_j(x, y) = \sum_{i=1}^{3} U_{ij}(x, y) e_i, \quad x \in \tilde{\Omega}, \quad y \in \mathbb{R}^2 \setminus \tilde{\Omega}, \quad j = 1, 2, 3, \quad (6)$$

where $y$ is a source point, $e_i, i = 1, 2, 3$, is the unit vector along the $x_i$-axis and

$$U_{ij}(x, y) = \frac{1}{16\pi G(1-\nu)} \frac{1}{r(x, y)} \left( (3 - 4\nu) \delta_{ij} + \frac{(x_i - y_i)(x_j - y_j)}{r^2(x, y)} \right), \quad i, j = 1, 2, 3. \quad (7)$$

Here, $r(x, y) = \left( \sum_{i=1}^{3} (x_i - y_i)^2 \right)^{1/2}$ represents the distance between the domain point $x = (x_1, x_2, x_3)$ and the source point $y = (y_1, y_2, y_3)$ and $\delta_{ij}$ is the Kronecker delta tensor.

Justified by a density result, see [36], the main idea of the MFS consists of the approximation of the displacement vector in the solution domain by a linear combination of fundamental solutions with respect to $M$ source points $y_j$ in the form,

$$u(x) \approx u^M(a, b, c, Y; x)$$

$$= \sum_{j=1}^{M} \left( a_j U^{(1)}(x, y^j) + b_j U^{(2)}(x, y^j) + c_j U^{(3)}(x, y^j) \right), \quad x \in \tilde{\Omega}, \quad (8)$$
where \( a = (a_1, \ldots, a_M), \ b = (b_1, \ldots, b_M), \ c = (c_1, \ldots, c_M), \) and \( Y \) is a \( 3M \)-vector containing the coordinates of the source points \( y^j, \ j = 1, \ldots, M. \) On taking into account the kinematic relations (2), the constitutive law (3), the definitions of the components of the traction vector (4), and the fundamental solutions (6) then the traction vector can be approximated on the boundary \( \Gamma \) by

\[
\mathbf{t}(x) \approx \mathbf{t}^M(a, b, c, Y; x, n) = \sum_{j=1}^{M} \left( a_j \mathbf{T}^{(1)}(x, y^j; n) + b_j \mathbf{T}^{(2)}(x, y^j; n) + c_j \mathbf{T}^{(3)}(x, y^j; n) \right), \tag{9}
\]

where

\[
\mathbf{T}^{(j)}(x, y; n) = \sum_{i=1}^{3} T_{ij}(x, y; n)e_i, \quad x \in \Gamma, \ y \in \mathbb{R}^3 \setminus \bar{\Omega}, \ j = 1, 2, 3, \tag{10}
\]

and

\[
T_{ij}(x, y; n) = \frac{2G}{1 - 2\nu} \left( (1 - \nu) \frac{\partial U_{ij}(x, y)}{\partial x_1} + \nu \frac{\partial U_{ij}(x, y)}{\partial x_2} + \nu \frac{\partial U_{ij}(x, y)}{\partial x_3} \right) n_i(x) + G \left( \frac{\partial U_{ij}(x, y)}{\partial x_3} + \frac{\partial U_{ij}(x, y)}{\partial x_2} \right) n_3(x), \ j = 1, 2, 3, \tag{11}
\]

If \( N \) collocation points \( x^t, t = 1, \ldots, N, \) are chosen on the overspecified boundary \( \Gamma_1 \) and the locations of the source points \( y^j, j = 1, \ldots, M, \) are set in \( \mathbb{R}^3 \setminus \bar{\Omega}, \) then equations (5), (8), and (9) generate a system of \( 6N \) linear algebraic equations with \( 3M \) unknowns which can be written as

\[
A\mathbf{X} = \mathbf{F}, \tag{12}
\]

where the MFS matrix \( A, \) the unknown vector \( \mathbf{X} \) and the right-hand side vector \( \mathbf{F} \) are given by

\[
A_{(k-1)N+1,(l-1)M+1} = U_{kl}(x^t, y^j), \quad A_{(k+2)N+1,(l-1)M+1} = T_{kl}(x^t, y^j),
\]

\[
X_j = a_j, \quad X_{M+j} = b_j, \quad X_{2(M+j)} = c_j, \quad F_{(k-1)N+i} = u_k(x^t), \quad F_{(k+2)N+i} = t_k(x^t),
\]

\[
t = 1, \ldots, N, \quad j = 1, \ldots, M, \quad k, l = 1, 2, 3.
\]

It should be noted that in order to uniquely determine the solution \( \mathbf{X} \) of the system of linear algebraic equations (12), i.e., the coefficients \( a_j, b_j, \) and \( c_j, j = 1, \ldots, M, \) in the approximations (8) and (9), the number \( N \) of boundary collocation points and the number \( M \) of source points must satisfy the inequality \( M \leq 2N. \) However, the system of linear algebraic equations (12) cannot
be solved by direct methods, such as the least-squares method, since such an approach would produce a highly unstable solution. Most of the standard numerical methods cannot achieve good accuracy in the solution of the system of linear algebraic equations (12) due to the large value of the condition number of the matrix $A$ which increases dramatically as the number of boundary collocation points and source points increases. It should be mentioned that for inverse problems, the resulting systems of linear algebraic equations are ill-conditioned, even if other well-known numerical methods (FDM, FEM, or BEM) are employed. Although the MFS system of linear algebraic equations (12) is ill-conditioned even when dealing with direct problems, the MFS has no longer this disadvantage in comparison with other numerical methods and, in addition, it preserves its advantages, such as the fact that it is meshless, the high accuracy of the numerical results, etc.

4. REGULARIZATION

Several regularization procedures have been developed to solve such ill-conditioned problems, see, for example, [37]. However, we only consider the Tikhonov regularization method, see e.g., [33], in our study since it is simple, noniterative, and it provides an explicit solution, see equation (17) below. Moreover, the Tikhonov regularization method is feasible to apply for large systems of equations unlike the singular value decomposition which may become prohibitive for such large systems, see e.g., [34].

4.1. The Tikhonov Regularization Method

The Tikhonov regularized solution to the system of linear algebraic equations (12) is sought as

$$X_\lambda: T_\lambda (X_\lambda) = \min_{X \in \mathbb{R}^{3M}} T_\lambda (X),$$

where $T_\lambda$ represents the $k^{th}$-order Tikhonov functional given by

$$T_\lambda (\cdot): \mathbb{R}^{3M} \rightarrow [0, \infty), \quad T_\lambda (X) = \|AX - F\|_2^2 + \lambda \|R^{(k)}X\|_2^2,$$

the matrix, $R^{(k)} \in \mathbb{R}^{(3M-k) \times 3M}$, induces a $C^k$-constraint on the solution $X$ and $\lambda > 0$ is the regularization parameter to be chosen. For example, in the case of zeroth-, first- and second-order Tikhonov regularization method, the matrix $R^{(k)}$, i.e., $k = 0, 1, 2$, is given by

$$R^{(0)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{M \times M},$$

$$R^{(1)} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \in \mathbb{R}^{(M-1) \times M},$$

$$R^{(2)} = \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix} \in \mathbb{R}^{(M-2) \times M}.$$ 

Formally, the Tikhonov regularized solution $X_\lambda$ of problem (14) is given as the solution of the regularized system of equations,

$$\left(A^T A + \lambda R^{(k)T} R^{(k)}\right) X = A^T F.$$  

(17)
Regularization is necessary when solving ill-conditioned systems of linear equations because the simple least-squares solution, i.e., $\lambda = 0$, is completely dominated by contributions from data errors and rounding errors. By adding regularization, we are able to dampen out these contributions and maintain the norm $\|R(k)X\|_2$ to be of reasonable size.

4.2. Choice of the Regularization Parameter

The choice of the regularization parameter in equation (17) is crucial for obtaining a stable solution and this is discussed next. If too much regularization, or damping, i.e., $\lambda$ is large, is imposed on the solution then it will not fit the given data $F$ properly and the residual norm $\|AX - F\|_2$ will be too large. If too little regularization is imposed on the solution, i.e., $\lambda > 0$ is small, then the fit will be good, but the solution will be dominated by the contributions from the data errors, and hence, $\|R(k)X\|_2$ will be too large. It is quite natural to plot the norm of the solution as a function of the norm of the residual parametrised by the regularization parameter $\lambda$, i.e.,

$$\{\log \|AX - F\|_2, \log \|R(k)X\|_2, \lambda > 0\}.$$ 

Hence, the L-curve is really a trade-off curve between two quantities that both should be controlled and, according to the L-curve criterion, the optimal value $\lambda_{\text{opt}}$ of the regularization parameter $\lambda$ is chosen at the "corner" of the L-curve, see [34,37].

As with every practical method, the L-curve has its advantages and disadvantages. There are two main disadvantages or limitations of the L-curve criterion. The first disadvantage is concerned with the reconstruction of very smooth exact solutions, see [38]. For such solutions, Hanke [39] showed that the L-curve criterion will fail, and the smoother the solution, the worse the regularization parameter $\lambda$ computed by the L-curve criterion. However, it is not clear how often very smooth solutions arise in applications. The second limitation of the L-curve criterion is related to its asymptotic behaviour as the problem size $3M$ increases. As pointed out by Vogel [40], the regularization parameter $\lambda$ computed by the L-curve criterion may not behave consistently with the optimal parameter $\lambda_{\text{opt}}$ as $3M$ increases. However, this ideal situation in which the same problem is discretised for increasing $3M$ may not arise so often in practice. Often the problem size $3M$ is fixed by the particular measurement setup given by $N$, and if a larger $3M$ is required then a new experiment must be undertaken since the inequality $M \leq 2N$ must be satisfied. Apart from these two limitations, the advantages of the L-curve criterion are its robustness and ability to treat perturbations consisting of correlated noise, for more details, see [34].

5. NUMERICAL RESULTS AND DISCUSSION

Although neither convergence nor estimate proofs are as yet available for the MFS applied to linear elasticity, the numerical results presented in this section for the Cauchy problem in three-dimensional isotropic linear elasticity indicate that the proposed method is feasible and efficient. In order to assess the performance of the MFS in conjunction with the first-order Tikhonov regularization method, i.e., $k = 1$ in (15), we consider an isotropic linear elastic medium characterised by the material constants $G = 3.35 \times 10^{10} \text{N/m}^2$ and $\nu = 0.34$ corresponding to a copper alloy and we solve the Cauchy problem (1) and (5) for two typical examples in a piecewise smooth geometry, such as the cube $\Omega = (-0.5, 0.5)^3 \subset \mathbb{R}^3$, see Figure 1.

EXAMPLE 1. We consider the following analytical solution for the displacements,

$$u_t^{(\text{an})}(x) = x_i, \quad i = 1, 2, 3,$$

in the domain $\Omega$, which corresponds to a uniform hydrostatic stress given by

$$\sigma^{(\text{an})}_{ij}(x) = \frac{2G(1 + \nu)}{1 - 2\nu} \delta_{ij}, \quad i, j = 1, 2, 3.$$
Here, $\Gamma_1 = \bigcup_{j=1}^6 \Gamma^{(j)}$ and $\Gamma_2 = \Gamma \setminus \Gamma_1 = \Gamma^{(6)}$, where $\Gamma = \bigcup_{j=1}^6 \Gamma^{(j)}$, $\Gamma^{(j)} = \{x \in \Gamma \mid x_j = -0.5\}$, $j = 1, 2, 3$, and $\Gamma^{(6+3)} = \{x \in \Gamma \mid x_3 = 0.5\}$, $j = 1, 2, 3$.

**Example 2.** We consider the following analytical solution for the displacements,

$$
\begin{align*}
    u_1^{\text{an}}(x) &= \frac{\sigma_0}{2G}x_1, \\
    u_2^{\text{an}}(x) &= -\frac{\sigma_0\nu}{2G(1-2\nu)}x_2, \\
    u_3^{\text{an}}(x) &= -\frac{\sigma_0\nu}{2G(1-2\nu)}x_3,
\end{align*}
$$

in the domain $\Omega$, which corresponds to a uniform traction stress given by

$$
\sigma_{ij}^{\text{an}}(x) = \begin{cases} 
    \sigma_0, & \text{if } i = j = 1, \\
    0, & \text{else},
\end{cases} \quad \sigma_0 = 1.0 \times 10^{10} \text{ N/m}^2.
$$

Here, $\Gamma_1 = \bigcup_{j=1}^3 \Gamma^{(j)}$ and $\Gamma_2 = \Gamma \setminus \Gamma_1 = \bigcup_{j=4}^6 \Gamma^{(j)}$.

It should be noted that for the examples considered, the Cauchy data is available on a portion $\Gamma_1$ of the boundary $\Gamma$, such that $\text{meas}(\Gamma_1)/5 = \text{meas}(\Gamma_2) = \text{meas}(\Gamma)/6$ in the case of Example 1.
and \( \text{meas} (\Gamma_1) = \text{meas} (\Gamma_2) = \text{meas} (\Gamma)/2 \) in the case of Example 2. The Cauchy problems investigated in this study have been solved using a uniform distribution of both the boundary collocation points \( x^i, i = 1, \ldots, N, \) and the source points \( y^j, j = 1, \ldots, M, \) with the mention that the later were located on the boundary of the cube \( \Omega = (-R, R)^3 \subset \mathbb{R}^3, \) where \( R > 0 \) was chosen, such that \( \Omega \subset \tilde{\Omega}, \) i.e., \( R > 0.5. \) Furthermore, the number of boundary collocation points on the overspecified boundary \( \Gamma_1 \) was set to \( N = 500 \) and \( N = 300 \) for the Examples 1 and 2, respectively. More precisely, we consider \( N^{(i)} = 100 \) collocation points uniformly distributed on each part \( \Gamma^{(i)} \subset \Gamma_1, \) so that \( N = 5N^{(i)} \) in the case of Example 1 and \( N = 3N^{(i)} \) in the case of Example 2.

5.1. Stability of the Method

In order to investigate the stability of the MFS, the displacement vector \( u_{\Gamma_1} = u^{(an)}|_{\Gamma_1} \) has been perturbed as

\[
\tilde{u}_i = u_i + \delta u_i, \quad i = 1, 2, 3,
\]

where \( \delta u_i \) is a Gaussian random variable with mean zero and standard deviation \( \sigma_i = \max_{x | u_i|} (p_{u}/100), \) generated by the NAG subroutine G05DDF, and \( p_{u} \% \) is the percentage of additive noise included in the input data \( u_{\Gamma_1} \) to simulate the inherent measurement errors.

Figure 2a presents the L-curves obtained for the Cauchy problem given by Example 1 using the first-order Tikhonov regularization method to solve the MFS system of equations (12), \( M = 600 \) source points, \( R = 10.0 \) and with various levels of noise added into the input displacement data. From this figure it can be seen that for each amount of noise considered the “corner” of the corresponding L-curve can be clearly determined and \( \lambda = \lambda_{\text{opt}} = 1.0 \times 10^{-5}, \lambda = \lambda_{\text{opt}} = 1.0 \times 10^{-4}, \) and \( \lambda = \lambda_{\text{opt}} = 1.0 \times 10^{-3} \) for \( p_{u} = 1, p_{u} = 3, \) and \( p_{u} = 5, \) respectively.

In order to analyse the accuracy of the numerical results obtained, we introduce the errors \( e_u \) and \( e_t \) given by

\[
e_u (\lambda) = \frac{1}{L} \left[ \sum_{i=1}^{L} \| u^{(an)} (x^i) - u^{(\lambda)} (x^i) \|^2 \right]^{1/2}, \quad e_t (\lambda) = \frac{1}{L} \left[ \sum_{i=1}^{L} \| t^{(an)} (x^i) - t^{(\lambda)} (x^i) \|^2 \right]^{1/2},
\]

where \( x^i, i = 1, \ldots, L, \) are \( L \) uniformly distributed points on the underspecified boundary \( \Gamma_2, \) \( u^{(an)} \) and \( t^{(an)} \) are the analytical displacement and traction vectors, respectively, and \( u^{(\lambda)} \) and \( t^{(\lambda)} \) are the numerical displacement and traction vectors, respectively, obtained for the value \( \lambda \) of the regularization parameter. Figures 2b and 2c illustrate the accuracy errors \( e_u \) and \( e_t, \) respectively, given by relation (23), as functions of the regularization parameter \( \lambda, \) obtained with various levels of noise added into the input displacement data \( u_{\Gamma_1}, \) for the Cauchy problem given by Example 1. From this figure, it can be seen that both errors \( e_u \) and \( e_t \) decrease as the level of noise added into the input displacement data decreases for all the values of the regularization parameter \( \lambda. \) Also, \( e_u < e_t \) for all \( \lambda \) and a fixed amount \( p_{u} \) of noise added into the input displacement data, i.e., \( \lambda, \) the numerical results obtained for the displacements are more accurate than those retrieved for the tractions on the underspecified boundary \( \Gamma_2. \) Furthermore, by comparing Figures 2a–2c, it can be seen, for various levels of noise, that the “corner” of the L-curve occurs at about the same value of the regularization parameter \( \lambda \) where the minimum in the accuracy errors \( e_u \) and \( e_t \) is attained. Hence, the choice of the optimal regularization parameter \( \lambda_{\text{opt}} \) according to the L-curve criterion is fully justified. Similar results have been obtained for the Cauchy problem given by Example 2 and therefore, they are not presented here.

Figures 3 and 4 illustrate the numerical results for the displacement \( u_2 \) and the traction \( t_3, \) respectively, obtained on the underspecified boundary \( \Gamma_2 = \Gamma^{(6)} \) using the optimal regularization parameter \( \lambda = \lambda_{\text{opt}} \) chosen according to the L-curve criterion, \( M = 600 \) source points, \( R = 10.0, \) and various levels of noise added into the input displacement data \( u_{\Gamma_1}, \) namely, \( p_{u} \in \{1, 3, 5\}, \) for the Cauchy problem given by Example 1. Similar results have been obtained for the other
Figure 2  (a) The L-curves, and the accuracy errors (b) $\varepsilon_u$ and (c) $\varepsilon_t$ as functions of the regularization parameter $\lambda$, obtained for various levels of noise added into the displacement $u|_{T_1}$, namely, $p_u = 1\%$ (---), $p_u = 3\%$ (---), and $p_u = 5\%$ (---), with $M = 600$ source points, $N = 500$ boundary collocation points, and $R = 10.0$ for Example 1.
Figure 3. The numerical displacement \( u_2^{(\lambda)} \) retrieved on the underspecified boundary \( \Gamma_2 = \Gamma'(6) \) with \( M = 600 \) source points, \( N = 500 \) boundary collocation points, \( R = 10 \), \( \lambda = \lambda_{\text{opt}} \), and various levels of noise added into the displacement \( u|_{\Gamma_1} \), namely, (a) \( p_u = 1\% \), (b) \( p_u = 3\% \), and (c) \( p_u = 5\% \), for Example 1.
Figure 4. The numerical traction $t^{(3)}$ retrieved on the underspecified boundary $\Gamma_2 = \Gamma^{(6)}$ with $M = 600$ source points, $N = 500$ boundary collocation points, $R = 10.0$, $\lambda = \lambda_{opt}$, and various levels of noise added into the displacement $u|_{\Gamma_1}$, namely (a) $p_u = 1\%$, (b) $p_u = 3\%$, and (c) $p_u = 5\%$, for Example 1.
Figure 5. The numerical displacement $u_1^{(4)}$ retrieved on the underspecified boundary $\Gamma' \subset \Gamma_2$ with $M = 600$ source points, $N = 300$ boundary collocation points, $R = 100$, $\lambda = \lambda_{\text{opt}}$, and various levels of noise added into the displacement $u|_{\Gamma_1}$, namely (a) $p_u = 1\%$, (b) $p_u = 3\%$, and (c) $p_u = 5\%$, for Example 2.
Figure 6. The numerical traction \( t^{(1)} \) retrieved on the underspecified boundary \( \Gamma_1^{(4)} \subset \Gamma_2 \) with \( M = 600 \) source points, \( N = 300 \) boundary collocation points, \( R = 10.0 \), \( \lambda = \lambda_{\text{opt}} \), and various levels of noise added into the displacement \( u_1^{(1)} \), namely (a) \( p_u = 1\% \), (b) \( p_u = 3\% \), and (c) \( p_u = 5\% \), for Example 2.
components of the displacement and traction vectors on the underspecified boundary $\Gamma_2$ and therefore, they are not presented. From these figures, we can conclude that the numerical solutions retrieved for Example 1 are stable with respect to the amount of noise $p_u$ added into the input displacement data $u|_{\Gamma_1}$. Moreover, a similar conclusion can be drawn from Figures 5 and 6 which present the numerical results for the displacement $u_4$ and the traction $t_1$ on the underspecified boundary $\Gamma_4 \subset \Gamma_2$, respectively, obtained using $\lambda = \lambda_{\text{opt}}$ chosen according to the L-curve criterion, $M = 600$ source points, $R = 10.0$, and various levels of noise added into the input displacement data $u|_{\Gamma_4}$, namely, $p_u \in \{1, 3, 5\}$, for the Cauchy problem given by Example 2. Although not presented here, it should be noted that analogous results have been obtained for the other components of the displacement and traction vectors on $\Gamma_4$, as well as for the displacements and the tractions on the boundaries $\Gamma_5$ and $\Gamma_6$ in the case of Example 2.

In order to describe quantitatively the stability of the numerical method employed, we define the normalised errors

$$
\text{err}_i(u_i(x)) = \frac{|u_i^{(\text{an})}(x) - u_i^{(\lambda)}(x)|}{\max_{y \in \Gamma} |u_i^{(\text{an})}(y)|}, \quad \text{err}_i(t_i(x)) = \frac{|t_i^{(\text{an})}(x) - t_i^{(\lambda)}(x)|}{\max_{y \in \Gamma} |t_i^{(\text{an})}(y)|}, \quad i = 1, 2, 3,
$$

$\text{err}_i(u_i(x))$ and $\text{err}_i(t_i(x))$ retrieved on the underspecified boundary $\Gamma_2 = \Gamma^{(6)}$ with $M = 600$ source points, $N = 500$ boundary collocation points, $R = 10.0$, $\lambda = \lambda_{\text{opt}}$, and various levels of noise added into the displacement $u|_{\Gamma_1}$, namely, $p_u \in \{1, 3, 5\}$%,
Figure 8. The normalised errors (a) \( \text{err}\left(u_2(x)\right) \) and (b) \( \text{err}\left(t_3(x)\right) \) retrieved on the underspecified boundary \( \Gamma^{(4)} \subset \Gamma_2 \) with \( M = 600 \) source points, \( N = 300 \) boundary collocation points, \( R = 10.0, \lambda = \lambda_{opt} \), and various levels of noise added into the displacement \( u|_{\Gamma_1} \), namely, \( p_n \in \{1, 3, 5\}\% \), for Example 2.

5.2. Convergence and Accuracy of the Method

Although not illustrated, it is reported that the convergence of the numerical solutions for the displacement and the traction vectors on the underspecified boundary \( \Gamma_2 \) towards their analytical solutions, respectively, as \( M \) or \( R \) increases, has been obtained when \( p_n = 0 \) for both examples.
considered in this study. In order to investigate the influence of the number $M$ of source points on the accuracy and stability of the numerical solutions for the displacement and the traction vectors on the underspecified boundary $\Gamma_2$, we set $R = 5.0$ and $p_u = 5$ for the Cauchy problem given by Example 1. In Figures 9a and 9b, we present the accuracy errors $e_u$ and $e_t$, respectively, as functions of the number $M$ of source points, obtained using the optimal values of $\lambda = \lambda_{opt}$ given by the L-curve criterion for each value of $M$. It can be seen from these figures that both accuracy errors tend to zero as the number $M$ of source points increases and, in addition, these errors do not decrease substantially for $M \geq 54$. These results indicate the fact that accurate numerical solutions for the displacement and the traction vectors on the underspecified boundary $\Gamma_2$ can be obtained using a relatively small number $M$ of source points. Similar results have been obtained for the Cauchy problem given by Example 2 and therefore, they are not presented here. From Figures 9a and 9b, it can be seen that the MFS, in conjunction with the first-order Tikhonov regularization method and the L-curve criterion, provides accurate numerical solutions with respect to increasing the number of source points, $M$, with the mention that even with a small number of source points a high accuracy of the numerical displacements and tractions is achieved.
Next, we analyse the accuracy of the numerical method proposed with respect to the position of the source points. To do so, we set \( M = 600 \) and \( p_u = 5 \) for the Cauchy problem given by Example 1, while at the same time varying the length \( R \). Figures 10a and 10b illustrate the accuracy errors \( e_u \) and \( e_t \), respectively, as functions of \( R \), obtained using the optimal values of \( \lambda = \lambda_{\text{opt}} \) given by the L-curve criterion for each value of \( R \). From these figures, it can be seen that the larger the distance from the source points to the boundary \( \Gamma \) of the solution domain \( \Omega \), i.e., the larger \( R \), the better the accuracy in the numerical displacements and tractions. It should be noted that the value \( R = 2.0 \) was found to be sufficiently large such that any further increase of the distance between the source points and the boundary \( \Gamma \) did not significantly improve the accuracy of the numerical solutions for both examples tested in this paper.

6. CONCLUSIONS

In this paper, the Cauchy problem in three-dimensional isotropic linear elasticity has been investigated by employing the MFS. The resulting ill-conditioned system of linear algebraic equations has been regularized by using the first-order Tikhonov regularization method, while the choice of the optimal regularization parameter was based on the L-curve criterion. Two examples in-
volving both under- and equally-determined Cauchy problems in a piecewise smooth geometry have been analysed. The numerical results obtained show that the proposed method is convergent and accurate with respect to increasing the number of source points and the distance from the source points to the boundary of the solution domain and stable with respect to decreasing the amount of noise added into the input data. Moreover, the method is efficient and easy to adapt to three-dimensional Cauchy problems in more complex and irregular domains, but these investigations are deferred to future work.

REFERENCES