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# An Inverse Problem for a Linear Diffusion Equation With Non-Linear Boundary Condition

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**Abstract.** In this paper we consider the determination of an unknown radiation term in the non-linear boundary condition of a linear diffusion equation from an overspecified condition. It is shown that the solutions of this inverse problem is unique and stable.

## 1. INTRODUCTION

In this paper, we consider the problem of identifying an unknown function p(u) that depends only on the temperature u(x,t) in a radiative-diffusion equation.

We shall determine a pair of function (u,p) from the following inverse problem

$$\partial_t u - \Delta u = 0 \quad x \varepsilon \Omega, \quad t > 0 \tag{1.1}$$

$$u(x,0) = 0, \quad x \varepsilon \Omega, \tag{1.2}$$

$$\partial_\nu u - p(u) = g(x,t) \quad x \varepsilon \partial\Omega, \quad t > 0 \tag{1.3}$$

$$u|_{x_n=0} = h(x',t) \quad x' \varepsilon \Omega', \quad t > 0. \tag{1.4}$$

Here $\Omega$ is a bounded open subset of $R^n$, with boundary $\partial\Omega$ lying on $\Omega'$, one side of $\Omega$. We will use the notation $x = (x_1, x_2, \dots, x_n) = (x', x_n)$ and $\partial_\nu u(x,t) = \nabla u \cdot \hat{a}_\nu(x)$, where $\hat{a}_\nu(x) = \sum_{i=1}^n f_i(x)\hat{a}_{x_i}$ denotes the inner unit normal to $\partial\Omega$. The linear heat equation with the non-linear conditions has been discussed by many authors, (see e.g., [1], [2], [3], [4], [7]).

If we determine a unique solution to the inverse problem (1.1)-(1.4), then we have an obvious physical meaning, which asserts that if the initial temperature in $\Omega$ is zero, and it is heated at one side of the boundary, then temperature in $\Omega$ and flux on $\partial\Omega$ are given at any time.

If $p(u)$ is known, then (1.1)-(1.4) will not, generally, provide a solution. However, for any given $p(u)$ there may be a unique solution to the problem (1.1)-(1.3), and under certain conditions this solution will satisfy the overspecified condition (1.4). In such case, the solution to $(p, u)$ is called a solution to the inverse problem (1.1)-(1.4).

In the next section, we consider the direct problem (1.1)-(1.3), and its uniqueness solution. In Section 3 we will determine p in terms of g and h. The final section describes some unicity and stability results for the solution p.

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## 2. ASSUMPTIONS

We shall assume $\partial\Omega$ is of class $c^{1+\beta}(0 < \beta < 1)$ i.e., $\partial\Omega$ can be locally represented in the form,

$$x_i = \gamma(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \tag{2.1}$$

with $\gamma$ in the class $c^{1+\beta}$.

We need also the following assumptions:
$A_1 \cdot g(x,t)$ is a continuous function for $x \varepsilon \partial\Omega, t > 0$.
$A_2 \cdot p(w)$ is a continuous and strictly monotone increasing function.
$A_3 \cdot p(w) \to \infty$ as $w \to \infty$.

Under the above condition on g and p the direct problem (1.1)-(1.3) has a unique solution u(x,t) which can be written as [5],

$$u(x,t) = \int_0^t \int_{\partial\Omega} \Gamma(x-\xi,\ t-\tau)\rho(\xi,\tau)d\Omega_\xi d\tau \tag{2.2}$$

where

$$\Gamma(x,t) = (4\pi t)^{-\frac{n}{2}} \exp\{-\frac{x^2}{4t}\};\ x^2 = \sum_{i=1}^{n} x_i^2 \tag{2.3}$$

is the fundamental solution of the heat equation, $d\Omega_\xi$ is the surface element on $\partial\Omega$, and $\rho$ is a solution of the integral equation,

$$\rho(x,t) = 2\left\{\int_0^t \int_{\partial\Omega} \frac{\partial\Gamma}{\partial\nu}(x-\xi,\ t-\tau)\rho(\xi,\tau)d\Omega_\xi d\tau - g(x,t) - p(u)\right\}. \tag{2.4}$$

The overposed data $u|_{x_n=0} = h(x',t)$ will be required the following conditions:
$B_1 \cdot s = h(x',t)$ is a continuous function.
$B_2 \cdot s = h(x',t)$ is invertible with respect to $t$.

By a solution pair to (1.1)-(1.4) we mean that,

(1) $p$ satisfies $A_2, A_3$.
(2) $u$ solve the direct problem (1.1-1.3).
(3) $u$ satisfies the overspecified condition (1.4).

## 3. EXISTENCE AND UNIQUENESS

By demonstrating the following result, we shall identify the function $p$, when $(p,u)$ is a solution to the inverse problem.

THEOREM 3.1.. *Let the data $h(x',t)$ satisfy $B_1$ and $B_2$, and let $p$ and $g$ satisfiy $A_2$, $A_3$ and $A_1$ respectively, then there exists a unique solution to (1.1)-(1.4) on $x \in \Omega, t > 0$.*

**Proof -** We begin with the auxilliary problem.

$$\partial_t u - \Delta u = 0, \quad x \in \Omega, \quad t > 0 \tag{3.1}$$

$$u(x,0) = 0, \quad x \in \Omega \tag{3.2}$$

$$u|_{x_n=0} = h(x',t), \quad x' \in \Omega', \quad t > 0 \tag{3.3}$$

The solution of this problem can be written as [6]:

$$u(x,t) = -2\int_0^t \int_{\Omega'} \frac{\partial \Gamma'(x'-\xi, x_n, t-\tau)}{\partial x_n} h(\xi,\tau)d\xi d\tau, \tag{3.4}$$

where

$$\Gamma'(x'-\xi, x_n, t-\tau) = (4\pi(t-\tau))^{-\frac{n}{2}} \exp\left\{\frac{-1}{4(t-\tau)}\left(\sum_{i=1}^{n-1}(x_i-\xi_i)^2 + x_n^2\right)\right\} \tag{3.5}$$

Now, by definition of normal derivatives (3.4) implies that

$$\partial_\nu u(x_0,t) = -2\int_0^t \int_{\Omega}' \sum_{i=1}^n \frac{\partial^2 \Gamma'(x'-\xi, x_n, t-\tau)}{\partial x_i \partial x_n}|_{x=x_0} f_i(x_0) h(\xi,\tau)d\xi d\tau \tag{3.6}$$

where $x_0 \in \Omega'$.

From (1.3) and (3.6), we have

$$p(u(x_0,t)) = -g(x_0,t) + \\ -2\int_0^t \int_{\Omega'} \sum_{i=1}^n \frac{\partial^2 \Gamma'(x'-\xi, x_n, t-\tau)}{\partial x_i \partial x_n}|_{x=x_0} f_i(x_0) h(\xi,\tau)d\xi d\tau. \tag{3.7}$$

Since $s = h(x',t)$ satisfies $B_2$ then (3.7) may be written as,

$$p(s) = -g\left(x_0, h^{-1}(x_0', s)\right) + \\ -2\int_0^{h^{-1}(x_0',s)} \int_{\Omega'} \sum_{i=1}^n \frac{\partial^2 \Gamma'(x'-\xi, x_n, h^{-1}(x_0',s)-\tau)}{\partial x_i \partial x_n}|_{x=x_0} f_i(x_0) h(\xi,\tau)d\xi d\tau. \tag{3.8}$$

This solution for $p$ to the inverse problem (1.1)-(1.4) is unique. In fact, if $(p,u)$ and $(Q,z)$ solve the inverse problem (1.1)-(1.4) then from (2.2),(2.4), $u(x_0,t) = h(x_0',t)$ and $Z(x_0,t) = h(x_0',t)$, we find

$$\int_0^t \int_{\partial\Omega} \Gamma(x_0-\xi, t-\tau)(\rho_1-\rho_2)d\Omega_\xi d\tau = 0. \tag{3.9}$$

since (3.9) is satisfied for all $t > 0$ and all surface $\partial\Omega$ of class $c^{1+\beta}$ $(0 < \beta < 1)$, hence from this equality one may deduce that,

$$\Gamma(x_0-\xi, t-\tau)(\rho_1-\rho_2) = 0 \tag{3.10}$$

whence

$$\rho_1 = \rho_2 \tag{3.11}$$

From (3.11) and (2.4) we find $p = Q$, and the proof is complete.

## 4. Stability and Uniqueness Results

In this section, we consider the stability and unicity solution for $p$. Firstly by demonstrating the following statement we shall show that the solution for $p$ is stable.

THEOREM 4.1. *If $h(x',t) = h(t)$ then the solution for $p$ is stable.*

**Proof -** If $p_1$ and $p_2$ correspond to the given data $(g_1, h_1)$ and $(g_2, h_2)$, respectively, then

$$|p_1 - p_2| = |g_2 - g_1| + |\int_0^t \int_{\Omega'} \sum_{i=1}^n \frac{\partial^2 \Gamma'}{\partial x_i \partial x_n} f_i(x)(h_1 - h_2) d\xi d\tau| \tag{4.1}$$

and hence

$$|p_1 - p_2| \leq |g_2 - g_1| + \int_0^t \int_{\Omega'} |\sum_{i=1}^n \frac{\partial^2 \Gamma'}{\partial x_i \partial x_n}| \cdot |f_i(x)| \cdot |h_1 - h_2| d\xi d\tau \tag{4.2}$$

If we assume, $|f_i(x)| \leq c_1$ for all $x \in \partial\Omega$, $1 \leq i \leq n$, and choose,

$$M = \sup_{t>0} \left\{ \int_{\Omega'} |\sum_{i=1}^n \frac{\partial^2 \Gamma'}{\partial x_i \partial x_n}| d\xi \right\}, \tag{4.3}$$

and $c_2 = Mc_1$, then

$$|p_1 - p_2| \leq |g_2 - g_1| + c_2 \int_0^t |h_1 - h_2| d\tau, \tag{4.4}$$

and therefore by mean value theorem,

$$|p_1 - p_2| \leq |g_2 - g_1| + c_2 T |h_1(c) - h_2(c)|, \tag{4.5}$$

where $0 < c < T$. If

$$\inf |h_1(t) - h_2(t)| \geq \epsilon > 0. \tag{4.6}$$

Then there exists a constant $N$ such that,

$$|p_1 - p_2| \leq |g_2 - g_1| + TN \in c_2 < |g_2 - g_1| + \lambda |h_1 - h_2| \tag{4.7}$$

where $\lambda = TNc_2$.

Finally by giving the following statement, we shall prove the unicity of solution for $p$ related to different data.

THEOREM 3.2. *Let $p$,$g_1$ and $g_2$, $h_1$ and $h_2$, satisfy $A_2$, $A_1$, $B_1$ and $B_2$, respectively, and let $(g_1,\ h_1)$ and $(g_2,\ h_2)$ be two given data which provide the same $p$, then $g_1 = g_2$ and $h_1 = h_2$.*

**Proof -** If $p(h_1) = p(h_2)$, then by strictly monotonicity of $p$ we deduce that $h_1 = h_2$. Using (3.8) we find $g_1 = g_2$ and the proof is complete.

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