Mod 2 group algebras with metabelian unit groups

Donald B. Coleman\(^a\),\(^*\), Robert Sandling\(^b\),\(^*\)

\(^a\) Mathematics Department, University of Kentucky, Lexington, KY 40506, USA
\(^b\) Mathematics Department, The University, Manchester M13 9PL, UK

Communicated by J. Huebschmann; received 7 May 1994

Abstract

The unit group of the group algebra of a finite group is solvable only rarely. Those groups for which it is metabelian were determined by Aner Shalev except when the field of coefficients is of characteristic 2. We resolve the characteristic 2 case here which completes the classification.

\(\copyright\) 1998 Elsevier Science B.V. All rights reserved.

\textit{AMS Classification:} Primary 16U60; Secondary 16S34, 20C05, 20C20, 20D15

1. Introduction

A group is metabelian if its commutator subgroup is abelian. The identification of those group algebras \(FG\) whose unit groups \(U(FG)\) are metabelian was initiated by Shalev [12]. He found that, for \(F\) a field of characteristic \(p > 2\) and \(G\) a finite group, \(U(FG)\) has derived length 2 if and only if \(p = 3\) and \(G' \cong C_3\). He did not include his findings for the \(p = 2\) case in [12] but noted: "The delicate case char(\(F\)) = 2 seems to require a separate discussion". This is indeed the case and that discussion is presented here. The methods used differ from those of [12] although some of its results are used.

We remark that, if \(G\) is nonabelian and \(F\) has characteristic 0, then \(U(FG)\) is not solvable. For noncommutative division rings have nonsolvable unit groups, and general linear groups of degree \(\geq 2\) in characteristic 0 are not solvable.

In Section 3 we use Bateman's work [1] to treat the case in which \(G\) is not a 2-group. Section 2 deals with the main case, namely that in which \(G\) is a 2-group. Experimentation with computers played an important role in this section. Both authors

\(^*\) E-mail: mtbbo@ms.uky.edu, rsandling@manchester.ac.uk.
have used computers in group ring computations [3, 10]. The second author has given presentations for unit groups using CAYLEY [2], SOGOS [6] and FORTRAN. His student Rao has extended this work [8]. The presentations so obtained made available the derived series for the unit groups for the groups of order dividing 32. The first author has developed programs in C that manipulate group ring elements in a fairly interactive manner. Here also one generates a multiplication table for the group $G$ using CAYLEY.

These experimental results, which were seen to be consistent with Shalev's Corollary 2.2 [12], prompted the conjecture, established below, that a nonabelian 2-group having a group algebra with a metabelian unit group must have nilpotency class 2 and elementary abelian commutator subgroup of order 2 or 4. We used our programs by trying to determine noncommuting commutators in groups that failed in these properties. It was fairly easy to come up with such by choosing units at random. What was more interesting was to try to generate examples in a certain form so that reasonable proofs could be found. We had available O'Brien's list [5] of the 2-groups of order up through 128 in quite a usable form from the CAYLEY library. This accessibility of these groups is astonishing, considering the history of their classification.

To state our theorem in a succinct manner, we introduce the following term to describe a type of group which enters into our classification and which first arose in this context in Bateman's characterization of solvable unit groups.

**Definition.** A finite group will be called a **B-group** if it is the split extension of a nontrivial elementary abelian 3-group by an involution acting as inversion.

**Theorem 1.1.** Let $G$ be a finite group and $F$ a field of characteristic 2. Then the group $U(FG)$ of units of the group algebra $FG$ is metabelian if and only if one of the following conditions holds:

(i) $G$ is a nilpotent group of class $\leq 2$ and $G' \leq C_2 \times C_2$;

(ii) $F = GF(2)$ and $G$ is a B-group.

Combining this with Shalev's theorems we have the following complete classification of group algebras with metabelian unit groups; in it only statements (ii) and (iii) are new.

**Theorem 1.2.** Let $G$ be a finite group and $F$ a field of characteristic $p \geq 0$. Then the group $U(FG)$ of units of the group algebra $FG$ is metabelian if and only if $G$ is abelian or one of the following conditions holds:

(i) $p = 3$, $G$ is nilpotent of class 2 and $G' \cong C_3$;

(ii) $p = 2$, $G$ is nilpotent of class 2 and $G' \cong C_2$ or $C_2 \times C_2$;

(iii) $F = GF(2)$ and $G$ is a B-group.

It is proved in [7] that, except for (iii), these properties characterize $p$-groups whose modular group algebras are metabelian as Lie algebras. (The authors thank Gerhard
Rosenberger for calling this to our attention.) Also for $p$-groups, these same properties are equivalent to the condition that, in characteristic $p$, $U(FG)$ has nilpotency class $\leq 3$, as shown in [9]. These results, along with Theorem 1.2, give the following theorem.

**Theorem 1.3.** Let $G$ be a finite $p$-group and $F$ a field of characteristic $p$. The following statements are equivalent:

1. $FG$ is a metabelian Lie algebra.
2. $U(FG)$ has nilpotency class $\leq 3$.
3. $U(FG)$ is a metabelian group.

Note that properties (1) and (2) fail for a $B$-group over any field.

We end this section with an indication of the notation used. Commutators in the unit group $U(FG)$ are denoted by square brackets, viz., $[v,u] = v^{-1}u^{-1}vu$, while those in the ring $FG$ viewed as a Lie algebra are denoted by round brackets, viz., $(x,\beta) = x\beta + \beta x$ in characteristic 2. For units $u$ and $v$, $u^v = v^{-1}uv$. We denote the cyclic group of order $n$ by $C_n$. The center of a group $X$ is denoted $Z(X)$, its commutator subgroup $X'$, the $n$th term of its lower central series $X_n$ and its largest normal 2-subgroup $O_2(X)$; $d(X)$ is the minimal number of generators of $X$. If $G$ is a finite $p$-group and $F$ is a field of characteristic $p$, then the group $U(FG)$ of units of the modular group algebra $FG$ is a direct product $U(F) \times V$, where $V = V(FG)$ is the subgroup of normalized units. As $V = 1 + I$ where $I = I(FG)$ is the augmentation ideal, $V$ is also a $p$-group, finite if $F$ is.

2. The 2-group case

In this section Theorem 1.1 is proved for the case in which $G$ is a finite 2-group and $F$ a field of characteristic 2. Our main task in this section is to show that the group $V = V(FG)$ is not metabelian under certain hypotheses on $G$. This is accomplished by explicit construction of nontrivial double commutators.

Next to trivial units the simplest units for ease of calculation are those which have support of size 3. We have made as much use of such units as possible. (This was suggested by the computer experimentation.) Those having 1 in their support play a particularly important role. For them we introduce the following notation.

**Notation.** For $g, h \in G$, denote the unit $1 + g + h$ in $FG$ by $\gamma_{g,h}$.

Note that $\gamma_{g,h}^2 = \gamma_{g^{-1},h^{-1}} + (g,h) = \gamma_{g^{-1},h^{-1}} + (c + 1)gh$, where $c = [h^{-1},g^{-1}]$. In determining whether a commutator $[[u,v],w] = [u^{-1}u^w,w] = 1$, it is sufficient to use $u^{-1}$ modulo the center of $V$. For a unit $u$ such as $u_{g,h}$, $u$ is a 2-element as it is a unit of augmentation 1. Its inverse is thus a product of the form $uu^2uu^4\ldots$. Where $u^{-1}$ enters into a commutator as above, it is thus necessary only to include those factors up to the
power of \( u \) which lies in \( \zeta(V) \). In many of our applications, \( u^4 \in \zeta(V) \) so that \( u^{-1} \equiv u^3 \) mod \( \zeta(V) \).

The first of our lemmas addresses the situation in which \( G \) has elements of order 4 in its commutator subgroup.

**Lemma 2.1.** Let \( F \) be a field of characteristic 2 and let \( G \) be a finite 2-group of class 2. Let \( x, y \in G \). Then \([([x, [x, y]], [z, y]]) = 1\) if and only if \([y, x]^2 = 1\).

**Proof.** Write \( v = [x, y] \) and \( e = [y, x] \). Note that \( e \) is central. Then

\[
[x, v] = (x^{-1} v x)^{-1} v = \gamma_{x, y}^{-1} v
\]

while

\[
[v, y] = v^{-1} \gamma_{x e^{-1}, y}.
\]

Now \([([x, v], [v, y])] = 1\) if and only if \([x, v][v, y] = [v, y][x, v]\), that is,

\[
\gamma_{x, y}^{-1} \gamma_{x e^{-1}, y} = v^{-1} \gamma_{x e^{-1}, y} \gamma_{x, y}^{-1} v.
\]

But \( x + ye \) commutes with \( xc^{-1} + y = (x + ye)c^{-1} \), so that \( \gamma_{x, ye} \) commutes with \( \gamma_{x e^{-1}, y} \).

Therefore, \( \gamma_{x, ye}^{-1} \) commutes with \( \gamma_{x e^{-1}, y} \). It follows that \([([x, v], [v, y]]) = 1\) if and only if \( v \) commutes with \( \gamma_{x, ye}^{-1} \gamma_{x e^{-1}, y} \).

But

\[
\gamma_{x, ye}^{-1} \gamma_{x e^{-1}, y} = \gamma_{x, ye}^{-1} (1 + c) + \gamma_{x, ye}^{-1} c^{-1} = \gamma_{x, ye}^{-1} (1 + c^{-1}) + c^{-1}.
\]

Hence, \([([x, v], [v, y]]) = 1\) if and only if \( v \) commutes with \( \gamma_{x, ye}^{-1} (1 + c^{-1}) \), that is, if and only if \( v \) commutes with \( \gamma_{x, ye} (1 + c^{-1}) \). It is easy to see that this is the case if and only if \( xy^e + ye^{-1} = ye c + xe^{-1} \). By the linear independence of elements of \( G \), this is so if and only if \( c^2 = 1 \). \( \square \)

Our next lemmas deal with the case in which \( G \) is a group of class 2 and \( G' \) is elementary abelian. In this case \( G^2 \leq \zeta(G) \).

**Lemma 2.2.** Let \( F \) be a field of characteristic 2 and let \( G \) be a finite 2-group of class 2 in which \( d(G) = 3 \). Suppose that \( G' \) is elementary abelian of rank 3. Then \( V(FG)^{''} \neq 1 \).

**Proof.** To fix notation, let \( G = \langle x, y, z \rangle \). The hypotheses imply that \( G' \) is a central subgroup generated by the independent commutators \( a = [x, y] = [y, x], b = [y, z] = [z, y], c = [x, z] = [z, x] \).

Let \( v = \gamma_{x, y}^{-1} \) and \( w = \gamma_{z, y}^{-1} \). We claim that \([([x, v], [z, w]]) \neq 1\). Since \( \gamma_{x, y}^2 = \gamma_{x^2, y^2} + (a + 1)xy \) with \( \gamma_{x^2, y^2} \) central and \((a + 1)^2 = 0\), we see that \( \gamma_{x, y}^4 = \gamma_{x^4, y^4} \) is a central element. Thus \( v \equiv \gamma_{x, y}^3 \) mod \( \zeta(V) \), and similarly \( w \equiv \gamma_{z, y}^3 \).
It follows that \([x, v] = (v^{-1})^xy \equiv A\) and \([z, w] \equiv B\) modulo \(\zeta(V)\), where

\[
A = \gamma_{x,y}^3 \gamma_{x,y}^3 (\gamma_{x,y}^x, \gamma_{x,y}^y) \gamma_{x,y}^2 \\
= \gamma_{x,y} \gamma_{x,y}^x (\gamma_{x,y}^x, y^2 + (a + 1)xy) \\
= [\gamma_{x,y}^x, \gamma_{x,y}^x + (a + 1)y][\gamma_{x,y}^x, y^2 + (a + 1)xy] \\
= \gamma_{x,y} \gamma_{x,y} (\gamma_{x,y}^x, y^2 + (a + 1)[y \gamma_{x,y}^x, y^2 + xy \gamma_{x,y}^x])
\]

and, similarly,

\[
B = \gamma_{x,y} \gamma_{x,y} (\gamma_{x,y}^x, y^2 + (b + 1)[y \gamma_{x,y}^x, y^2 + yz \gamma_{x,y}^x]).
\]

We want to show that \((A, B) \neq 0\).

Since all the \(T\) terms in \(A\) and \(B\) are central, we have

\[
(A, B) = (a + 1)(b + 1)[\gamma_{x,y}^x, y^2 (y, yz) + \gamma_{x,y}^x, y^2 (xy, y)] \\
+ \gamma_{x,y}^x, y^2 (y, yz)]
\]

Note that \((y, yz) = (b+1)y^2z\) and \((b+1)^2 = 0\); also \((xy, y) = (a+1)xy^2\) and \((a+1)^2 = 0\). Thus

\[
(A, B) = (a + 1)(b + 1) \gamma_{x,y}^x, y^2 (xy, yz) \\
= (a + 1)(b + 1)(c + 1) \gamma_{x,y}^x, y^2 (x, y) \\
= (a + 1)(b + 1)(c + 1) \gamma_{x,y}^x, y^2 (xy, yz).
\]

That is, \((A, B)\) is a unit times the nonzero element \((a + 1)(b + 1)(c + 1)\), so \((A, B) \neq 0\) as required. \(\Box\)

The following lemma prepares for the case in which \(G\) is a group of class 2, \(G'\) is elementary abelian of rank 3 and \(G\) has no abelian maximal subgroup.

**Lemma 2.3.** Let \(F\) be a field of characteristic 2 and let \(G\) be a finite 2-group of class 2 in which \(G'\) is elementary abelian of rank 3. Suppose that \(G = \langle x, y, z, g \rangle\) where \(G'\) is generated by the independent commutators \(a = [x, g], b = [y, g], c = [z, g]\). Assume that \([x, y] = 1\) and that neither \(\langle x, y, z \rangle\)' nor \(\langle x, y, zg \rangle\)' is contained in \(\langle c \rangle\). Then \(V(FG)' \neq 1\).

**Proof.** Letting \(u = \gamma_{x,g}\) so that \(u^2 = \gamma_{x,g}^x + (c + 1)zg\), we have \(u^4 = \gamma_{x,g}^x + (c + 1)zg \in \zeta(V)\) so that \(u^{-1} \equiv u^3 \mod \zeta(V)\).

Let \(v = \gamma_{x,y}\) so that \(v^2 = \gamma_{x,y}^x \in \zeta(V)\) and \(v^{-1} \equiv v \mod \zeta(V)\). Thus \([v, g], [u, g] = [v^g, u^3 u^g]\).
But
\[ uu^\vartheta = \gamma_{z,g} \gamma_{v',g} = \gamma_{z,g} \gamma_{cz,g} \]
\[ = \gamma_{cz^2,g^2} + (c + 1)z. \]
As \( \gamma_{cz^2,g^2} = (c + 1)z^2 + \gamma_{z^2,g^2} \), \( (c + 1) \gamma_{cz^2,g^2} = (c + 1) \gamma_{z^2,g^2} \) since \( (c + 1)^2 = 0 \). It follows that
\[ u^3 u^\vartheta = (\gamma_{z^2,g^2} + (c + 1)z)(\gamma_{cz^2,g^2} + (c + 1)z) \]
\[ = \gamma_{z^2,g^2} \gamma_{cz^2,g^2} + \gamma_{z^2,g^2}(c + 1)z + \gamma_{cz^2,g^2}(c + 1)zg \]
\[ = \gamma_{z^2,g^2}(c + 1)z(g + 1) \mod \zeta(FG). \]

On the other hand
\[ vv^\vartheta = \gamma_{x,y} \gamma_{v',y'} = \gamma_{x,y} \gamma_{ax,by} \]
\[ = (a + 1)x(y + 1) + (b + 1)(x + 1)y \mod \zeta(FG). \]
This last element, A say, commutes with g as is readily checked using \( (a + 1)x^\vartheta = (a + 1)ax - (a + 1)x \) and \( (b + 1)y^\vartheta = (b + 1)y \).

Now assume that \([ [v, g], [u, g]] = 1\). This is equivalent to \( (vug, u^3u^\vartheta) = 0 \), which gives
\[ 0 = (A, \gamma_{z^2,g^2}(c + 1)z(g + 1)) = \gamma_{z^2,g^2}(A, z)(c + 1)(g + 1). \]

Since \( \gamma_{z^2,g^2} \) is a unit, it follows that
\[ 0 = ((a + 1)((xy,z) + (x,z)) + (b + 1)((xy,z) + (y,z)))(c + 1)(g + 1). \quad (1) \]

Write \([x,z] = a'bc^k \) and \([y,z] = a'b^m c^n \). Then \( (x,z) = xz(1 + [z,x]) = xz(a'b^j c^k + 1) \).

As \( (a + 1)(ah + 1) = (a + 1)(h + 1) \) for any \( h \in G \),
\[ (a + 1)(x,z) = xz(a + 1)(b^j c^k + 1), \]
\[ (b + 1)(y,z) = yz(b + 1)(a^l c^n + 1), \]
\[ (a + 1)(xy,z) = xyz(a + 1)(b^{j+m} c^{k+n} + 1), \]
\[ (b + 1)(xy,z) = xyz(b + 1)(a^{i+l} c^{k+n} + 1). \]

An argument of the same sort shows that (1) reduces to
\[ 0 = ((a + 1)(b^{j+1} + 1)zx + (a^{l+1} + 1)(b + 1)yz \]
\[ + ((a + 1)(b^{i+m} + 1) + (a^{i+l} + 1)(b + 1))xyz)(c + 1)(g + 1). \]

By the linear independence of the elements of \( G \) and the fact that \( g \notin \langle x, y, z \rangle \), we conclude that \( j = 0 = l \) so that
\[ 0 = ((a + 1)(b^{m+1} + 1) + (a^{l+1} + 1)(b + 1))xyz(c + 1)(g + 1). \]
This forces \( i = m \). If their common value is 0, then \( \langle x, y, z \rangle' \leq \langle c \rangle \); if 1, then \( \langle x, y, zg \rangle' \leq \langle c \rangle \) since \( [x, zg] = [x, z][x, g] = a^2c^k = c^k \) and \( [y, zg] = c^n \). In either case we obtain a contradiction to our hypothesis so that \( V'' \neq 1 \). □

**Lemma 2.4.** Let \( F \) be a field of characteristic 2 and let \( G \) be a finite 2-group of class 2 in which \( G' \) is elementary abelian of rank 3. Suppose that \( G = \langle x, y, z, g \rangle \) where \( G' \) is generated by the independent commutators \( a = [x, g], b = [y, g], c = [z, g] \). Assume that \( G \) has no abelian maximal subgroup. Then \( V(FG)' \neq 1 \).

**Proof.** Let \( H = \langle x, y, z \rangle \). If \( H \) is abelian, then the maximal subgroup \( HG^2 \) is abelian contrary to hypothesis. If \( H' \) is of order 8, \( V(FH)' \neq 1 \) by Lemma 2.2. Thus \( 2 \leq |H'| \leq 4 \). With a possible change in the generators of \( H \), we may assume that \( [x, y] = 1 \). By the previous lemma, we may assume that \( H' = \langle c \rangle \). We may assume without loss of generality that \( [x, z] = c \). If \( [y, z] = 1 \), we may apply the previous lemma to \( G \) viewed as generated by \( y, z, x, g \) in that order to conclude that \( H' = \langle x, y, z \rangle' \leq \langle a \rangle \) which is false, or that \( \langle y, z, yg \rangle' \leq \langle a \rangle \), which is also false since \( b = [y, xg] \not\in \langle a \rangle \). Thus \( [y, z] = c \).

It follows that \( [xy, z] = 1 \). But an argument as above applied to \( G \) viewed as generated by \( xy, z, x, g \) in that order again leads to a contradiction. □

**Lemma 2.5.** Let \( F \) be a field of characteristic 2 and let \( G \) be a finite 2-group of class 2 in which \( G' \) is elementary abelian of rank 3. Suppose that \( G = \langle x, y, z, g \rangle \) where \( G' \) is generated by the independent commutators \( a = [x, g], b = [y, g], c = [z, g] \). Assume that \( \langle x, y, z \rangle \) is abelian. Then \( V(FG)' \neq 1 \).

**Proof.** We show that \([[[T_{x,g}, yg], T_{z,g}, xg]] \neq 1.\)

Since \( G' \) is central, \( T_{x,g}, yg \) is central while \( T_{x,g}^2 = T_{x,g}^2 + (a+1)yg \). So \( T_{x,g}^3 = T_{x,g}^3 + (a+1)yg \) which is central. Therefore \( T_{x,g}^{-1} \equiv T_{x,g}^3 \mod \zeta(V) \). Similarly, \( T_{x,g}^{-1} \equiv T_{x,g}^3 \mod \zeta(V) \).

Now write the two commutators modulo \( \zeta(V) \):

\[
[T_{x,g}, yg] = T_{x,g}^{-1} T_{x,g}^y = T_{x,g}^{-1} T_{ax,bg} = T_{x,g}^3 T_{ax,bg} \mod \zeta(V);
\]

\[
[T_{z,g}, xg] = T_{z,g}^{-1} T_{z,g}^x = T_{z,g}^{-1} T_{cz,ag} = T_{z,g} T_{cz,ag} \mod \zeta(V).
\]

It suffices to show that the Lie commutator \( (T_{x,g}^3 T_{ax,bg}, T_{z,g}^3 T_{cz,ag}) \) is not zero. We write each part modulo the additive subgroup \( \zeta(FG) \) of \( FG \).

To this end, note that \( (c+1)^2 = 0 \), that \( (a+1)x \) and \( (c+1)z \) are central, and that \( (c+1)(cz+1) = (c+1)(z+1) \). Then

\[
T_{x,g}^3 T_{ax,bg} = T_{x,g}^2 T_{x,g} T_{ax,bg} = [T_{x,g}^2 + (a+1)xg][T_{ax,bg}^2 + (a+1)x + (b+1)(x+1)g] = T_{x,g}^3 T_{ax,bg}(b+1)(x+1)g + T_{ax,bg}^3(a+1)xg \mod \zeta(FG).
\]

Denote the term on the last line by \( A \).
By symmetry,
\[ T_{x,g}^2 T_{x^2,g}^2 = T_{x^2,g}^2(a + 1)(z + 1)g + T_{x^2,g}^2(c + 1)zg \mod \zeta(FG). \]

Let \( B \) denote the right-hand expression.

It suffices to show that \((A,B) \neq 0\). Since \((a + 1)^2 = 0\), we have
\[
(A,B) = T_{x^2,g}^2 T_{x^2,g}^2(a + 1)(b + 1)((x + 1)g,(z + 1)g) + T_{x^2,g}^2 T_{x^2,g}^2(b + 1)(c + 1)((x + 1)g, zg) + T_{x^2,g}^2 T_{x^2,g}^2(a + 1)(c + 1)(xg, zg).
\]
The last term is zero since \((xg, zg) = (c + a)xzg^2\) and \((a + 1)(c + 1)(c + a) = 0\). Again, since \(((x + 1)g, g) = (xg, g) = (a + 1)xg^2\),
\[
(A,B) = [T_{x^2,g}^2 T_{x^2,g}^2(a + 1)(b + 1) + T_{x^2,g}^2 T_{x^2,g}^2(b + 1)(c + 1)((x + 1)g, zg)].
\]
Use the identities
\[
((x + 1)g, zg) = (c + a)xzg^2 + (c + 1)zg^2,
\]
\[
(a + 1)(b + 1)(c + a) = (a + 1)(b + 1)(c + 1) = (b + 1)(c + 1)(c + a),
\]
\[
T_{x^2,g}^2 T_{x^2,g}^2(a + 1)(c + 1) = T_{x^2,g}^2(a + 1)(c + 1)
\]
to get
\[
(A,B) = [T_{x^2,g}^2 T_{x^2,g}^2(a + 1)(b + 1)(c + 1)zg^2].
\]
Since \(T_{x^2,g}^2 T_{x^2,g}^2 zg^2\) is a unit and since \((a + 1)(b + 1)(c + 1) \neq 0\), it follows that \((A,B) \neq 0\). \( \Box \)

The last lemma of this section turns to the case of 2-groups of class 3.

**Lemma 2.6.** Let \( F \) be a field of characteristic 2 and let \( G \) be a finite 2-group of nilpotency class 3. Suppose that \( G' \) is of order 4. Then \( V(FG)' \neq 1 \).

**Proof.** By hypothesis, \( G_2/G_3 \cong C_2 \) and \( G_2 \cong C_4 \) or \( C_2 \times C_2 \). In either case Aut \( G_2 \) has a Sylow 2-subgroup of order 2 so that \(|G: C| = 2\), where \( C = C_0(G_2) \). There are elements \( x \in G \setminus C \) and \( y \in G \setminus G_2 \) such that \( c = [y, x] \) generates \( G_2 \mod G_3 \). Thus \( G_3 = \langle z \rangle \) where \( z = [c, x] = [y, x, x] \). We may assume that \( G = \langle x, y \rangle \) and that \( \zeta(G) \) is cyclic.

Suppose first that \( G_2 = \langle c \rangle \cong C_4 \), and so \( G_3 = \langle c^2 \rangle \). Then \([x^2, y] = 1\) and we may assume that \([y^2, x] = 1\). Thus \( x^2, y^2 \in \zeta(G) \) and \([c, y] = c^2 \).

Since \( G_2 \) is generated by \( c \) and \( G/G_2 \) by the cosets of \( x \) and \( y \), each element of \( G \) can be expressed as a product \( x^iy^jc^k \) where \( i, j, k \) are integers. As \( x^2, y^2, c^2 \in \zeta(G) \), each element is equivalent modulo \( \zeta(G) \) to such a product in which \( 0 \leq i, j, k \leq 1 \). Moreover these exponents are unique because \( G/\zeta(G) \) is nonabelian so that the index
\[ G : \zeta(G) \geq 8. \] It follows that such an element is central if and only if \( i, j, k \) are even; thus, \( \zeta(G) = \langle x^2, y^2, c^2 \rangle \).

If \( |G| = 16 \), then \( G \) is one of the three such groups of maximal class. In each of the three cases, \( V(FG) \) is known not to be metabelian for \( F = GF(2) \). This was computed using SOGOS based on the presentations given in [11]; it was also computed in [8] using CAYLEY. (SOGOS has subsequently been incorporated into the GAP package, and CAYLEY into MAGMA.) We may assume, then, that \( |G| > 16 \). It follows that \( \zeta(G) > \langle c^2 \rangle \). As \( \zeta(G) \) is cyclic, it is generated by \( x^2 \) or \( y^2 \). We may assume without loss of generality that \( \zeta(G) = \langle y^2 \rangle \).

Let \( v = \gamma_{x,y} \) so that \( v^{-1} \equiv v \mod \zeta(V) \) and so \([v, x], c] = 1 \) if and only if \([v, x], c] = 1 \). Now

\[
v^sv = \gamma_{y,z} \gamma_{y,z} \gamma_{x,y} = 1 + y + y^3 + y^4 + (y + y^2 + y^3)c
\]

and so

\[
(v^sv)^c = 1 + ye^2 + y^3c^2 + y^4 + (ye^2 + y^2 + y^3c^2)c
\]

since \([y, c] = [y^3, c] = c^2 \). Thus,

\[
v^sv - (v^sv)^c = y + y^2 + y^3c^2 + yc + ye^3 + y^3c + y^3c^3.
\]

As the elements of \( G \) are independent, we see that, if this is 0, then \( y \) must be equal to one of the other elements in this expression which is impossible. Thus \([v, x], c] \neq 1 \) and so \( V(FG)^{s'} \neq 1 \) as desired.

Lastly suppose that \( G_2 \cong C_2 \times C_2 \). We may assume that \( y \in C \setminus G^2 \). Now \([x^2, y] = [x, y]^{2}[x, y, x] = z \) while \([y^2, x] = 1 = [x^4, y] \).

Let \( v = \gamma_{x,z} \) so that \( v^{-1} \equiv v^3 \mod \zeta(V) \) and so \([v, y], c] = 1 \) if and only if \([v^3, v^c], c] = 1 \). Now \( v^3 = 1 + x + x^3 + x^5 + x^6 \) while \( v^y = \gamma_{x,c'}, z \). Thus

\[
v^3v^y = 1 + xc + x^2z + x + x^2c + x^3z + x^3 + x^4c + x^5z
\]

\[ + x^5 + x^6c + x^7z + x^6 + x^7c + x^8z \]

and, since \( x^c = x[x, c] = xc \) and \([x^2, c] = 1 \) and since \( z^2 = 1 \),

\[
(v^3v^y)^c = 1 + xc + x^2z + xz + x^2c + x^3 + x^3z + x^4c + x^5
\]

\[ + x^5z + x^6c + x^7 + x^6 + x^7c + x^8z. \]

Subtracting we find that

\[
v^3v^y - (v^3v^y)^c = xc + 1 + x^6z + x^6c + cz + z + x^6 + x^6cz. \]

As before, we see that, if this is 0, then \( x^6 \) must be equal to one of the other elements in the second factor, which is impossible since \( x^2 \notin G_2 \). Thus \([v, y], c] \neq 1 \) and so \( V(FG)^{s'} \neq 1 \) as desired. \( \square \)
All the lemmas requisite for the proof of Theorem 1.1 in the case of a 2-group $G$ are now established. Let $G$ be a 2-group of minimal order which contradicts the conclusion of the theorem, i.e., $V(FG)^n = 1$ but $1 < G' \neq C_2, C_2 \times C_2$.

Suppose first that $G$ is of class 2. By Lemma 2.1, $G'$ is elementary abelian and, by assumption, of rank $\geq 3$. As $G$ is a minimal counterexample, $d(G') = 3$. By [4, III.1.11], $d(G) \geq 3$. If $d(G) = 3$, then Lemma 2.2 shows that $G$ is not a counterexample.

We may thus assume that $G$ satisfies $d(G) \geq 4$. In fact, as seen in the remarks after case (I) of [9], $d(G) = 4$. Moreover it can also be deduced as follows from the analysis of the cases given there that $G$ has a conjugacy class of size 8. By a result of Knoche [4, p. 309], there are elements of $G$ whose centralisers are of index 4. We show that there is an element of $G$ with centraliser of index 8 by examining the two configurations in [9] in which there is an element $g$ with $C := C_G(g)$ of index 4 (we make free use here of the notation and conclusions of that paper).

In the first, $G = \langle x, y, u, g \rangle$ with $[x, g], [y, g]$ and $[x, u]$ generators of $G_2$. Here $|G : C_G(gx)| = 8$. In the second, $G = \langle tx, y, u, g \rangle$ with $[x, g], [y, g]$ and $[tx, u]$ generators of $G_2$. In this case $[v, y] = [x, g]^i[y, g]^j$ for some $i, j$. If $j = 0$, $|G : C_G(gux)| = 8$, while, if $j = 1$, $|G : C_G(vx)| = 8$.

Assume that $g \in G$ is such that $|G : C_G(g)| = 8$. Then $G^2 \leq \zeta(G) \leq C_G(g)$ while $|G : G^2| = 16$. If $G$ has an abelian maximal subgroup, $M$ say, then $g \notin M$ so that $G = \langle M, g \rangle$. Let $x, y, z$ generate $M$ modulo $G^2$ so that $G = \langle x, y, z, g \rangle$ and $G'$ is generated by the independent commutators $[x, g], [y, g]$ and $[z, g]$. Lemma 2.5 shows that $G$ is not a counterexample.

We may conclude that $G$ has no abelian maximal subgroups. Choosing elements $x, y, z$ so that $G = \langle x, y, z, g \rangle$, we again see that $G'$ is generated by the independent commutators $[x, g], [y, g]$ and $[z, g]$. This time Lemma 2.4 shows that $G$ is not a counterexample.

We may now assume that $d(G) \geq 3$. Then $G_3$ is its unique central subgroup of order 2. As $G/G_3$ is not a counterexample, $G_2/G_3$ is elementary abelian of rank at most 2. The case in which $G_2/G_3$ has rank 2 is dealt with as in [9]. That is, let $x, y \in G$; as $d(G) \geq 3$, $\langle x, y \rangle < G$ so that $[y, x]$ commutes with $x$ and $y$ while $1 = [y, x]^2 = [y^2, x]$; thus, $\zeta(G) \geq G^2 \geq G_2$, contrary to hypothesis.

The case in which $G_2/G_3 \cong C_2$ is all that remains. But here $|G'| = 4$ and so Lemma 2.6 applies to finish the proof of Theorem 1.1 for a 2-group $G$.

3. The general case

In this section we will assume that $G$ is not a 2-group. Suppose then that $U := U(FG)$ is metabelian. Since $\text{rad} FG$ is a nil ideal, $U(FG/\text{rad} FG)$ is an epimorphic image of $U$ and hence is metabelian; and $W := 1 + \text{rad} FG$ is a 2-subgroup of $U$. Note that $I(O_2(G))FG \subseteq \text{rad} FG$ and that $O_2(G) = G \cap W$. Thus, $G/O_2(G) \leq U(FG/\text{rad} FG)$ which is a direct product of general linear groups over division rings over $F$. 

Suppose that $G/O_2(G)$ is not abelian. Then at least one of the general linear groups is of degree $\geq 2$. As the only such soluble group in characteristic 2 is $GL(2,2)$, we conclude that $F = GF(2)$. Moreover, since $\overline{U} := U/W \cong U(FG/\text{rad } FG)$ is a direct power of $GL(2,2)$, $W = O_2(U)$.

The group $GL(2,2)$ is metabelian, its Sylow 3-subgroup, cyclic of order 3, being its commutator subgroup. Let $T$ be a Sylow 3-subgroup of $U$ so that $T = \overline{U}'$. As $|U : U'| = |U : WU'| |W : W \cap U'|$ is a power of 2, $T \leq U'$ which is abelian by hypothesis, so that $T$ is characteristic in $U'$ and $T < U$.

We show next that $O_3(G) = 1$ in this case. Let $H = T \cap G$, the normal Sylow 3-subgroup of $G$, which is not trivial as $G/O_2(G) \neq 1$. As $W$ is a normal 2-subgroup of $U$, $[T, W] = 1$ so that $H$ centralises $1 + I(O_2(G))FG$. Let $g \in O_2(G)$. As $H$ is not central in $G$, there are elements $x \in G$ and $h \in H$ such that $h' = h^{-1} \neq h$. Then

$$1 + (g + 1)x = (1 + (g + 1)x)^h = 1 + (g + 1)x^h$$

$$= 1 + (g + 1)h^{-1}xh^{-1}x = 1 + (g + 1)h^{-1}h'x.$$ 

It follows that $(g + 1)(h^{-1}h' + 1) = 0$ whence $g = 1$ so that $O_2(G) = 1$.

It now follows from one of Bateman's main theorems that $G$ is a $B$-group but it is not difficult to draw this conclusion directly in this special case. All nontrivial 2-elements of $G$ are of order 2 since, as $G \cap W = 1$, $G$ is isomorphic to a subgroup of a direct power of $GL(2,2)$. Let $x$ be such an element. Since $O_2(G) = 1$, $H$ is self-centralising and so $x$ acts nontrivially on $H$. We next show that $x$ acts by inversion. This fact suffices to show that a Sylow 2-subgroup $P$ of $G$ is of order 2 and so that $G$ is a $B$-group since, if $y, z \in P$, $y, z \neq 1$, then $y^{-1}z \in C_G(H) \cap P = 1$ whence $y = z$.

Suppose then that $x$ does not act on $H$ by inversion so that there is an element $h \in H$ such that $h^x \notin \langle h \rangle$. We may assume that $G = \langle h, x \rangle$ with $H = \langle h, h^x \rangle \cong C_3 \times C_3$ and $\langle x \rangle \cong C_2$ a Sylow 2-subgroup of $G$. For this group $G$, the metabelian group $U$ involves the nonsolvable group $GL(2,4)$ which is a contradiction.

Now suppose that $G/O_2(G)$ is abelian. In this case $P := O_2(G)$ is the Sylow 2-subgroup of $G$. Let $H$ be a Hall 2'-subgroup of $G$ so that $H$ is abelian. By [12, Proposition 3.1], $G \cong P \times H$. Lastly $U(FP)$ is metabelian so that $P' \leq C_2 \times C_2$ by the results of the previous section. The desired conclusion that $G$ is a nilpotent group of class $\leq 2$ and $G' \leq C_2 \times C_2$ is immediate.

We now turn to the converse statement. If $G$ is nilpotent such that $G' \leq C_2 \times C_2$, then, as noted in [12, Corollary 2.2], $U(FG)$ is metabelian. If $G$ is not nilpotent but a $B$-group and $F = GF(2)$, $U(FG)$ is also metabelian. This can be shown by decomposing $FG$ as a direct sum of ideals and concluding that $U \cong C_2 \times GL(2,2)^n$ for some $n$. However a straightforward proof by induction is available. It was noted in [1] that, for $G = GL(2,2)$, the smallest $B$-group, $U \cong C_2 \times G$ which is metabelian. Let $G = H \langle x \rangle$ where $H$ is an elementary abelian 3-group of order $> 9$ and $x$ is an involution acting on $H$ by inversion. Let $K \leq H$, $|K| = 9$, so that $K$ has 4 subgroups of order 3, $A_i = \langle a_i \rangle$, $i = 1, 2, 3, 4$. Then, for each $i$, $e_i = a_i + a_i^2$ is a central idempotent.
It follows that
\[ e_1FG \cap e_2FG \cap e_3FG \cap e_4FG = e_1e_2e_3e_4FG, \]
which is 0 in characteristic 2. Thus, \( FG \) embeds in the direct product of the quotient rings \( FG/e_iFG \). But \( e_iFG = I(A_i)FG \), so that \( U(FG/e_iFG) \cong U(F(G/A_i)) \) is metabelian by induction. Consequently \( U(FG) \) is also metabelian. The proof of Theorem 1.1 is now complete.

References