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Divergence theorems in path space III: Hypoelliptic diffusions and beyond

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Abstract

Let x denote a diffusion process defined on a closed compact manifold. In an earlier article, the author introduced a new approach to constructing admissible vector fields on the associated space of paths, under the assumption of ellipticity of x. In this article, this method is extended to yield similar results for degenerate diffusion processes. In particular, these results apply to non-elliptic diffusions satisfying Hörmander's condition.

Keywords: Diffusion process; Manifold; Admissible vector field; Divergence

1. Introduction

Let $X_1, \ldots, X_n$ and V denote smooth vector fields on a closed compact manifold M such that V lies within the span of the vectors $X_1, \ldots, X_n$ at every point in M. Fix a point $o \in M$ and a positive time T and consider the Stratonovich stochastic differential equation (SDE)

$$dx_t = \sum_{i=1}^{n} X_i(x_t) \circ dw_i + V(x_t)\,dt, \quad t \in [0, T],$$
$$x_0 = o, \tag{1.1}$$

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where $w = (w_1, \dots, w_n)$ is a standard Wiener process in $\mathbf{R}^n$. Then the solution process x is a random variable taking values in the space of paths

$$C_o(M) = \{\sigma : [0, T] \mapsto M \mid \sigma(0) = o\},$$

an infinite-dimensional manifold with tangent bundle consisting of fibers

$$T_\sigma C_o(M) = \{r : [0, T] \mapsto TM \mid r_0 = 0,\ r_t \in T_{\sigma_t} M\ \forall t \in [0, T]\}.$$

The law γ of x, as a measure on $C_o(M)$, can be considered as a generalized version of Wiener measure on $C_0(\mathbf{R}^n)$. A major goal in stochastic analysis is to extend the rich body of results that have been developed for the Wiener measure to this more general setting.

The *Cameron–Martin space*, i.e. the space of paths $\{\sigma : [0, T] \mapsto \mathbf{R}^n,\ \sigma_0 = 0\}$ with finite energy

$$\int_0^T \|\dot{\sigma}_t\|^2 \, dt$$

provides a geometrical framework for the Wiener measure and plays a central role in its analysis. Therefore, in addressing the problem raised above, it is natural to seek an analogue of the Cameron–Martin space for the measure γ. A reasonable candidate for such an analogue is the set of vector fields on the space $C_o(M)$ that admit an "integration by parts" formula of the type described in the following definition.

Definition 1.1. A vector field η on $C_o(M)$ is admissible (with respect to γ) if there exists an L^1 function $\mathrm{Div}(\eta)$ such that the relation

$$\int_{C_o(M)} \eta(\Phi) \, d\gamma = \int_{C_o(M)} \Phi \, \mathrm{Div}(\eta) \, d\gamma \tag{1.2}$$

holds for a dense class of smooth functions Φ on $C_o(M)$.

The construction of admissible vector fields is an important problem that has been extensively studied in the last three decades. A breakthrough in the problem was achieved by Driver [6] in 1992, following important partial results by Bismut [5]. Driver proved that stochastic parallel translation along x of Cameron–Martin paths in $T_o M$ produces admissible vector fields on $C_o(M)$. A fundamental innovation in [6] is the use of the rotation-invariance of the Wiener process. This property also plays a crucial role in the present work.

The work of Bismut and Driver stimulated a great deal of activity in this area and the problem is still being widely studied (cf., e.g. Driver [7], Hsu [10,11], Enchev and Stroock [9], Elworthy, Le Jan and Li [8]). Much of this work has dealt with the *elliptic* case, where the vector fields $X_1, \dots, X_n$ in (1.1) are assumed to span TM at all points of M. In [1], the author introduced a new approach to the problem of constructing admissible vector fields on the space of paths defined by the diffusion process (1.1), again in the elliptic setting. The purpose of the present article, the third in a series of papers on this theme (cf. [1,2]), is to extend this approach to *degenerate* (i.e. non-elliptic) diffusions.

The central object of study in the author's approach is the *Itô map* $g : w \mapsto x$ defined by Eq. (1.1). This is used to lift the problem from the manifold M to $\mathbf{R}^n$, where classical integration by parts theorems can be applied. The lifting method had previously been used by Malliavin in his probabilistic approach to the hypoellipticity problem [12]. "Lifting" is defined as follows.

Definition 1.2. A process r taking values in $\mathbf{R}^n$ is said to be a lift of η to $C_0(\mathbf{R}^n)$ (via the Itô map) if the following diagram commutes:

$$\begin{array}{ccc} TC_0(\mathbf{R}^n) & \xrightarrow{dg} & TC_o(M) \\ {\scriptstyle r}\uparrow & & \uparrow{\scriptstyle \eta} \\ C_0(\mathbf{R}^n) & \xrightarrow[g]{} & C_o(M) \end{array}$$

Since g is non-differentiable in the classical sense the derivative dg must be interpreted in the extended sense of the Malliavin calculus. (As this type of regularity is now generally well-understood by stochastic analysts, this point will not be emphasized in the paper. See e.g. the monographs [3,13–15] for an introduction to the Malliavin calculus.) The idea in [1] is to simultaneously construct a vector field η on $C_o(M)$ and an *admissible* lift r of η to $C_0(\mathbf{R}^n)$. In particular (cf. Theorems 2.1 and 2.2), this requires that r take the form

$$r_t = \int_0^t A(s)\, dw_s + \int_0^t B(s)\, ds$$

where A and B are continuous adapted processes taking values in so(n) (the space of skew-symmetric $n \times n$ matrices) and $\mathbf{R}^n$, respectively. Processes of this form thus comprise the tangent bundle $TC_0(\mathbf{R}^n)$ in the above diagram.

For test (i.e. smooth cylindrical) functions Φ on $C_o(M)$, one then has

$$E\big[(\eta\Phi)(x)\big] = E\big[r(\Phi \circ g)(w)\big] = E\big[\Phi \circ g(w)\,\mathrm{Div}(r)\big] = E\big[\Phi(x)E\big[\mathrm{Div}(r)/x\big]\big]$$

where Div denotes the divergence operator in the classical Wiener space. Thus η is admissible with divergence

$$\mathrm{Div}(\eta)(x) = E\big[\mathrm{Div}(r)/x\big].$$

An important consequence of the ellipticity assumption is the fact that every non-anticipating vector field on $C_o(M)$ can be written in the form

$$\eta_t = \sum_{i=1}^n h_i(t) X_i(x_t) \tag{1.3}$$

where $h_i, i = 1, \ldots, n$, are real-valued processes, adapted to the filtration of x. In the highly non-generic situation where the vector fields $\{X_i\}$ *commute*, then for every $t > 0$, x_t becomes a function of w_t and the problem trivializes. The argument in [1] sets up a duality between the processes h and r, the lift of η, in which (in the non-commuting case) the commutators $[X_i, X_j]$ play an explicit role.

It was shown in [2] that in the *hypoelliptic* case (where the diffusion process (1.1) is degenerate but Hörmander's condition holds), generic vector fields of the form (1.3) do not admit lifts to $C_0(\mathbf{R}^n)$ under the Itô map. In particular, admissible vector fields on $C_o(M)$ consisting of linear combinations of $X_1, \ldots, X_n$ cannot be constructed by the author's method in this case. The point of departure for the present work is the a priori selection of an additional collection of vector fields $\{V_I\colon I \in \mathcal{I}\}$ on M such that

$$\left\{V_I(x)\colon I \in \mathcal{I}\right\} \text{ span } T_x M, \quad \forall x \in M. \tag{1.4}$$

Thus in the elliptic case $\{V_I\}$ can be taken to be the set $\{X_1, \ldots, X_n\}$, whereas in the hypoelliptic case, one can choose $\{V_I\} = \mathrm{Lie}(X_1, \ldots, X_n)$, the Lie algebra generated by the vector fields $X_1, \ldots, X_n$. We construct admissible vector fields on $C_o(M)$ in the form

$$\eta_t = \sum_{I \in \mathcal{I}} h_I(t) V_I(x_t).$$

Somewhat surprisingly, it proves to be possible to trade ellipticity in $\{X_1, \ldots, X_n\}$ for condition (1.4). This enables us to establish our results under very general hypotheses.

The layout of the paper is as follows. Section 2 contains background material. The results here are well known, for the most part. Theorem 2.1 asserts that Riemann integrals of continuous adapted paths have divergence given by an Itô integral, while Theorem 2.2 states that Itô integrals with continuous adapted skew-symmetric integrands are divergence-free. The former result follows easily from the Girsanov theorem, the latter from the infinitesimal rotation-invariance of the Wiener measure. Theorem 2.6 gives a relationship between a vector field η along the path x and the lift of η to the Wiener space. This relationship, expressed in terms of the derivative of the stochastic flow of the SDE (1.1) and the inverse flow, plays a key role throughout. The required geometric machinery and notations are also introduced in this section of the paper.

Section 3 contains the main results. Theorem 3.1 produces a class of admissible vector fields on $C_o(M)$, under hypotheses that allow the SDE (1.1) to be degenerate. The proof of Theorem 3.1 follows the above outline and is an extension of the argument in [1]. An essential step in the proof is the decomposition of non-tensorial terms in the lift obtained from Theorem 2.6, into *tensorial* plus *skew-symmetric* parts.

Theorem 3.2 is a variation on Theorem 3.1 that exhibits a vector field on $C_o(M)$ with *given* divergence. In particular, we obtain a class of vector fields with divergence expressed in terms of Ricci curvature. The interest of this result lies in the fact that formulae of this type appear in the work of other authors, e.g Driver [6] and Elworthy, Le Jan and Li [8], where they are obtained using different methods. In Example 3.3, Theorem 3.2 is applied to yield vector fields on $C_o(M)$ with divergence having no extraneous dependence on the Wiener path w. This property is important in applications of the theorem that require a degree of regularity of the divergence such as the study of quasi-invariance. Theorem 3.4 is an intrinsic formulation of Theorem 3.1 that does not depend on the choice of a basis $\{V_I\}$. We assume here that M is a Riemannian manifold. The proof of Theorem 3.4 requires the introduction of a tensor that enables us to express the Levi-Civita connection on M in terms of a connection intrinsic to the diffusion process (1.1). In Theorem 3.6, we apply our theory to gradient systems. As a consequence (Corollary 3.7), we obtain Driver's result cited above.

In Section 4, we consider the special case where the vector fields $X_1, \dots, X_n$ are linearly independent. In this case, the problem under consideration simplifies considerably and our argument simplifies accordingly. We conclude with an example where the SDE (1.1) takes values in the Heisenberg group G. In this case we obtain explicit formulae for a class of admissible vector fields on the path space $C_o(G)$.

2. Background material

2.1. Divergence theorems for Wiener space

We present two such results. These concern the transformation of Wiener measure under Euclidean motions; the first under translations, the second under rotations.

Let Ω denote the measure space for the Wiener process, equipped with the filtration

$$\mathcal{F}_t = \sigma\{w_s \mid s \leqslant t\}.$$

Theorem 2.1. *Let $h : \Omega \times [0, T] \mapsto \mathbf{R}^n$ be a continuous adapted path. Then the process $\int_0^\cdot h$ is admissible (with respect to the Wiener measure) and*

$$\mathrm{Div}\left[\int_0^\cdot h_s\, ds\right] = \int_0^T h_s \cdot dw_s$$

where $\cdot$ on the right-hand side of the equation denotes the Euclidean inner product.

Proof. The result follows easily from the Girsanov theorem, which implies that for $\Phi \in C_b^\infty(C_0(\mathbf{R}^n))$ and $\epsilon \in \mathbf{R}$,

$$E\left[\Phi\left(w + \epsilon \int_0^\cdot h_s\, ds\right)\right] = E\big[\Phi(w) G_\epsilon(w)\big] \tag{2.1}$$

where

$$G_\epsilon(w) \equiv \epsilon \int_0^T h_s \cdot dw_s - \frac{\epsilon^2}{2} \int_0^T \|h_s\|^2\, ds.$$

Differentiating each side of (2.1) with respect to ϵ and setting $\epsilon = 0$ gives the theorem. □

Theorem 2.2. *Let $A : \Omega \times [0, T] \mapsto \mathrm{so}(n)$ be a continuous adapted process. Then the process $\int_0^\cdot A\, dw$ is admissible and*

$$\mathrm{Div}\left[\int_0^\cdot A\, dw\right] = 0.$$

Proof. Define a process $\theta_t^\epsilon = \exp \epsilon(A_t)$ where exp denotes matrix exponentiation. Then θ_t^ϵ is an adapted $O(n)$-valued matrix process with $\theta_t^0 = I$. It follows from the infinitesimal rotation-invariance of the Wiener measure that the law of the process

$$w^\epsilon \equiv \int_0^\cdot \theta_t^\epsilon \, dw_t$$

is invariant under ϵ. Hence for $\Phi \in C_{\mathrm{b}}^\infty(C_0(\mathbf{R}^n))$, we have

$$E\left[\Phi\left(w^\epsilon\right)\right] = E\left[\Phi(w)\right].$$

As before, differentiating in ϵ and setting $\epsilon = 0$ gives the result. □

2.2. Geometric preliminaries

In this section we introduce the geometric machinery that will be needed in Section 3. We adopt the summation convention throughout the paper: whenever an index in a product (or a bilinear form) is repeated, it will be assumed to be summed on.

First, let $[g_{jk}]$ be the Riemannian metric defined on M by

$$g^{jk} = a_I^j a_I^k$$

where

$$V_I = a_I^j \frac{\partial}{\partial x_j}, \quad I \in \mathcal{I},$$

is the expression of the vector fields in local coordinates (note that the matrix $[g^{jk}]$ is non-degenerate by the spanning condition (1.4)).

Let $(\cdot,\cdot)$ denote the inner product structure on TM defined by the metric $[g_{jk}]$. Then we have

$$V = (V, V_I)V_I, \quad \forall V \in TM. \tag{2.2}$$

To see this, let $V = b_J V_J$ and write $V_J = a_J^j \frac{\partial}{\partial x_j}$ for each J, as above. Then

$$\begin{aligned}(V, V_I)V_I &= \left(b_J a_J^j \frac{\partial}{\partial x_j}, a_I^k \frac{\partial}{\partial x_k}\right) a_I^l \frac{\partial}{\partial x_l} = b_J a_J^j g^{kl} g_{jk} \frac{\partial}{\partial x_l} \\ &= b_J a_J^j \delta_{jl} \frac{\partial}{\partial x_l} = b_J a_J^l \frac{\partial}{\partial x_l} = V\end{aligned}$$

as claimed.

We denote the Levi-Civita covariant derivative associated with this metric by $\tilde{\nabla}$.

The following constructions were introduced by Elworthy, Le Jan and Li (cf. [8]). Assume the set of vectors $\{X_1(x), \dots, X_n(x)\}$ span a subspace E_x of T_xM of constant dimension as x varies in M and define E to be the subbundle of TM

$$E = \bigcup_{x \in M} E_x.$$

Then E becomes a Riemannian bundle under the inner product $\langle \cdot, \cdot \rangle$ induced on E by the linear maps

$$X(x) : (h_1, \dots, h_n) \in \mathbf{R}^n \mapsto h_i X_i(x) \tag{2.3}$$

from the Euclidean space $\mathbf{R}^n$.

There is a metric connection ∇ on E compatible with the metric $\langle \cdot, \cdot \rangle$. This connection (termed the *Le Jan–Watanabe connection* in [8]), is defined by

$$\nabla_V Z = X(x) d_V (X^* Z), \quad Z \in \Gamma(E),\ V \in T_x M,$$

where d is the standard derivative, applied the function

$$x \in M \mapsto X(x)^* Z(x) \in \mathbf{R}^n.$$

Lemma 2.3. *For all $x \in M$ and V and W in T_xM, we have*

$$\langle \nabla_V X_j, W \rangle X_j = 0.$$

Proof. Let $P = P(x)$ denote orthogonal projection in $\mathbf{R}^n$ onto the subspace $\operatorname{Ker} X(x)^{\perp}$ and $\{e_1, \dots, e_n\}$ the standard orthonormal basis of $\mathbf{R}^n$. Then

$$X^* \big[\langle \nabla_V X_j, W \rangle X_j \big] = X^* \big[\langle X d_V P e_j, W \rangle X_j \big] = \langle d_V P e_j, X^* W \rangle P e_j$$

(where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product)

$$= \langle e_j, d_V P X^* W \rangle P e_j = P (d_V P) X^* W = P (d_V P) P X^* W.$$

On the other hand, differentiating the relation $P^2 = P$ gives

$$d_V P P + P \, d_V P = d_V P.$$

Thus

$$d_V P P = d_V P - P \, d_V P = Q \, d_V P$$

where $Q = I - P$. Hence

$$P \, d_V P P = P Q \, d_V P = 0$$

and we have

$$X^*\big[\langle \nabla_V X_j, W\rangle X_j\big] = 0.$$

The lemma now follows from the fact that $XX^* = I$. □

The *Riemann curvature* tensor R corresponding to this connection is defined in the usual way, by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The *Ricci tensor* is defined by

$$\mathrm{Ric}(X) = R(X, e_i)e_i$$

where $\{e_i\}$ is a (locally defined) orthonormal frame in E.

The next result shows that the vector fields $\{X_i\}$ play the role of a (generalized) orthonormal basis of E and, in particular, the Ricci tensor can be computed using these vector fields.

Lemma 2.4.

(i) $\langle Y, X_i\rangle X_i = Y$, $\forall Y \in E$.
(ii) $\mathrm{Ric}(Y) = R(Y, X_i)X_i$, $\forall Y \in TM$.

We omit the proofs of these statements since they are elementary. (The proof of (i) is similar to that of (2.2) above. A proof of (ii) can be found in [1, Section 2].)

2.3. Flow-related theorems

Lemma 2.5. *Let $g_t : M \mapsto M$ denote the stochastic flow $x_0 \mapsto x_t$ defined by the SDE* (1.1). *Define $Y_t : T_{x_0}M \mapsto T_{x_t}M$ and $Z_t : T_{x_t}M \mapsto T_{x_0}M$ by $Y_t \equiv dg_t$ and $Z_t \equiv Y_t^{-1}$. Let B denote a vector field on M and d the stochastic time differential. Then*

$$d\big[Z_t B(x_t)\big] = Z_t\big([X_i, B](x_t) \circ dw_i + [V, B](x_t)\, dt\big).$$

Proof. Let D_t denote the stochastic covariant differential along the path x_t, with respect to the Levi-Civita $\tilde{\nabla}$ connection defined above. Then differentiating with respect to the initial point o in (1.1) gives

$$D_t Y = \tilde{\nabla}_{Y_t} X_i \circ dw_i + \tilde{\nabla}_{Y_t} V\, dt.^2$$

We then have

$$D_t Z = D_t\big(Y_t^{-1}\big) = -Z_t D_t Y Z_t = -Z_t(\tilde{\nabla}_{Id_t} X_i \circ dw_i + \tilde{\nabla}_{Id_t} V\, dt)$$

[2] Here and in the sequel, we assume that all vector fields appearing in the equations are evaluated at x_t.

where Id_t denotes the identity map on $T_{x_t}M$. Thus

$$\begin{aligned}
d(Z_t B) &= D_t Z B + Z_t \tilde{\nabla}_{dx_t} B \\
&= -Z_t(\tilde{\nabla}_B X_i \circ dw_i + \tilde{\nabla}_B V\,dt) + Z_t(\tilde{\nabla}_{X_i} B \circ dw_i + \tilde{\nabla}_V B\,dt), \\
d\big[Z_t B(x_t)\big] &= Z_t\big([X_i, B](x_t) \circ dw_i + [V, B](x_t)\,dt\big)
\end{aligned}$$

as required. □

Theorem 2.6. *Let $r : \Omega \times [0, T] \mapsto \mathbf{R}^n$ be an Itô process. Then the path $\eta \equiv dg(w)r$ is given by*

$$\eta_t = Y_t \int_0^t Z_s X_i(x_s) \circ dr_i. \tag{2.4}$$

Proof. Note that η is a vector field along the path x. Let $U_s : T_oM \mapsto T_{x_s}M$ denote stochastic parallel translation along x.

Differentiating in (1.1) with respect to w gives the following covariant equation for η

$$\begin{aligned}
&D_t\eta = \tilde{\nabla}_\eta X_i(x_t) \circ dw_i + X_i(x_t) \circ dr_i + \tilde{\nabla}_\eta V(x_t)\,dt, \\
&\eta_0 = 0.
\end{aligned} \tag{2.5}$$

We write (2.5) as

$$d\big(U_t^{-1}\eta\big) = U_t^{-1}\tilde{\nabla}_\eta X_i(x_t) \circ dw_i + U_t^{-1} X_i(x_t) \circ dr_i + U_t^{-1}\tilde{\nabla}_\eta V(x_t)\,dt.$$

Denoting the path $t \mapsto U_t^{-1}\eta_t$ by y, we note that the equation for y has the form

$$dy = M_i(t)y_t \circ dw_i + M_0(t)y_t + U_t^{-1} X_i(x_t) \circ dr_i \tag{2.6}$$

where $M_j(t)$, $j = 1, \ldots, n$, are linear operators on T_oM.

On the other hand, differentiation in (1.1) with respect the the initial point o gives the following equation for $\tilde{Y}_t \equiv U_t^{-1}Y_t$:

$$\begin{aligned}
&d\tilde{Y} = M_i(t)\tilde{Y}_t \circ dw_i + M_0(t)\tilde{Y}_t\,dt, \\
&\tilde{Y}_0 = I.
\end{aligned} \tag{2.7}$$

Equation (2.6) can be solved in terms of $\tilde{Y}$ using an operator version of the familiar "integrating factor" method used to solve first-order linear ODE's. Noting, then, that $\tilde{Y}^{-1}$ is an integrating factor for (2.6) and using this to solve for y gives

$$y_t = \tilde{Y}_t \int_0^t \tilde{Y}_s^{-1} U_s^{-1} X_i(x_s) \circ dr_i. \tag{2.8}$$

Writing (2.8) in terms of η and Y, we obtain (2.4). □

Remarks. (1) Theorem 2.6 gives an alternative proof of the "lifting" equation (Eq. (3.2)) in [1].

(2) Suppose η in (2.4) has the form $\eta_t = X_i(x_t)h_i(t)$ for an $\mathbf{R}^n$-valued process $h = (h_1, \dots, h_n)$. Then, writing

$$X = [X_1 \dots X_n]$$

and solving for dr in (2.4), we have

$$Z_t X(x_t) \circ dr = d\big[Z_t X(x_t) h_t\big].$$

This equation suggests that r can be considered as a type of "covariant derivative" of h along x, where the operator $Z_t X(x_t)$ plays the role of backward parallel translation.

3. Divergence theorems

3.1. First result

Let X be as defined in (2.3). Then the SDE (1.1) may be written

$$dx = X(x_t) \circ d\tilde{w}$$

where

$$d\tilde{w} = dw + X(x_t)^* V(x_t)\, dt$$

and the adjoint map is defined using the metric $\langle \cdot, \cdot \rangle$ on E (so $X(x)^*$ is a right inverse for $X(x)$). By the Girsanov theorem, the law $\tilde{\nu}$ of $\tilde{w}$ is equivalent to the law ν of w, with Radon–Nikodym derivative $\frac{d\tilde{\nu}}{d\nu}$ given by

$$G(w) = \exp\left(\int_0^T X(x_t)^* V(x_t) \cdot dw - \frac{1}{2} \int_0^T \big\| X(x_t)^* V(x_t) \big\|^2 dt \right).$$

Suppose that r is an admissible lift for the vector field η under the map $\tilde{g} : \tilde{w} \mapsto x$. Then

$$\begin{aligned} E\big[\eta\phi(x)\big] &= E\big[G(w) \cdot r(\Phi \circ \tilde{g})(w)\big] \\ &= E\big[\Phi \circ \tilde{g}(w)\, \mathrm{Div}(G \cdot r)\big] = E\big[\Phi \circ \tilde{g}(w)\big\{G \cdot \mathrm{Div}(r) - r(G)\big\}\big]. \end{aligned}$$

Thus η is admissible.

In view of this discussion, there is no loss in generality in assuming $V = 0$ and we shall assume in the sequel that this is the case.

We introduce the following tensors $\{T_I\}$ associated to the vector fields $\{V_I\}$:

$$T_I(X) = \nabla_{V_I} X + [X, V_I], \quad X \in E.$$

Theorem 3.1. *Let $r = (r_1, \ldots, r_n)$ be a path in the Cameron–Martin space of $\mathbf{R}^n$ and define $\{h_I\colon I \in \mathcal{I}\}$ by the linear stochastic system*

$$\begin{aligned} dh_I &= (X_i, V_I)\dot{r}_i\,dt - \big(T_J(\circ dx), V_I\big)h_J, \\ h_I(0) &= 0. \end{aligned} \tag{3.1}$$

Then the vector field $\eta_t \equiv h_I(t)V_I(x_t)$, $t \in [0, T]$, is admissible on $C_o(M)$.

Proof. We first note that Theorem 2.6 implies that r is lift of η if r satisfies

$$X_i\,dr_i = Y_t\,d[Z_t\eta_t]. \tag{3.2}$$

Substituting $\eta_t = h_I(t)V_I(x_t)$ into (3.2) and using Lemma 2.5, we have

$$X_i\,dr_i = V_I \circ dh_I + [X_j, V_I]h_I \circ dw_j. \tag{3.3}$$

Writing the Lie bracket term involving X_j in terms of the connection ∇ and using Lemma 2.4(i) gives

$$[X_j, V_I] = T_I(X_j) - \nabla_{V_I}X_j = T_I(X_j) - \langle\nabla_{V_I}X_j, X_i\rangle X_i.$$

Denote

$$G_I^{ij} = \langle\nabla_{V_I}X_i, X_j\rangle - \langle\nabla_{V_I}X_j, X_i\rangle. \tag{3.4}$$

Combining the previous two lines with Lemma 2.3, we have

$$[X_j, V_I] = G_I^{ij}X_i + T_I(X_j).$$

Substituting this into (3.3) gives

$$X_i\,dr_i = V_I \circ dh_I + G_I^{ij}h_I X_i \circ dw_j + T_I(\circ dx)h_I. \tag{3.5}$$

We note that, more generally, a semimartingale path $\tilde{r}$ is a lift of $h_I V_I$ if Eq. (3.5) holds with the left-hand side replaced by the Stratonovich differential $X_i \circ d\tilde{r}_i$.

Suppose now the coefficient functions $\{h_I\}$ satisfy the system

$$\begin{aligned} X_i dr_i &= V_I \circ dh_I + T_I(\circ dx)h_I, \\ h_I(0) &= 0. \end{aligned} \tag{3.6}$$

Then

$$X_i\big[dr_i + G_I^{ij}h_I \circ dwj\big] = V_I \circ dh_I + G_I^{ij}X_i h_I \circ dw_j + T_I(X_j)h_I \circ dw_j.$$

So if we define

$$\tilde{r}_i = r_i + \int_0^{\cdot} G_I^{ij} h_I \circ dw_j \tag{3.7}$$

then (3.3) holds with r replaced by $\tilde{r}$. It follows that $\tilde{r}$ is a lift of η, where

$$\eta_t = h_I(t) V_I(x_t). \tag{3.8}$$

Furthermore, the skew-symmetry of the functions G_I^{ij} in the upper indices and Theorem 2.2 imply that the Stratonovich integral in (3.7) can be written as a Riemann integral plus a divergence-free Itô integral. It follows from Theorems 2.1 and 2.2 that $\tilde{r}$ is admissible. Note also that by (2.2), the processes h_I defined by (3.1) satisfy Eq. (3.3).

We have thus shown that $\tilde{r}$ is an admissible lift to the Wiener space of the vector field η in (3.8). In view of Definition 1.2, we have for any test function Φ on $C_o(M)$

$$\begin{aligned} E\big[(\eta\Phi)(x)\big] &= E\big[r(\Phi \circ g)(w)\big] = E\big[\Phi \circ g(w)\,\mathrm{Div}(r)\big] \\ &= E\big[\Phi(x) E\big[\mathrm{Div}(r)/x\big]\big]. \end{aligned}$$

Thus η is admissible and

$$\mathrm{Div}(\eta)(x) = E\big[\mathrm{Div}(r)/x\big]. \qquad \square$$

3.2. Computation of the divergence

In order to compute the divergence of the vector field η in Theorem 3.1, it is necessary to convert the Stratonovich integral in (3.7) into Itô form. The relation between the Stratonovich and Itô differentials is formally

$$G_I^{ij} h_I \circ dw_j = G_I^{ij} h_I\, dw_j + \frac{1}{2} d\big(G_I^{ij} h_I\big)\, dw_j. \tag{3.9}$$

Write

$$\begin{aligned} \alpha_I^{kij} &= \langle \nabla_{X_k} \nabla_{V_I} X_i, X_j \rangle + \langle \nabla_{V_I} X_i, \nabla_{X_k} X_j \rangle \\ &\quad - \langle \nabla_{X_k} \nabla_{V_I} X_j, X_i \rangle - \langle \nabla_{V_I} X_j, \nabla_{X_k} X_i \rangle \end{aligned} \tag{3.10}$$

and

$$\beta_I^k = -\big(T_J(X_k), V_I\big) h_J. \tag{3.11}$$

Then by (3.1) and (3.4)

$$dG_I^{ij} = \alpha_I^{kij}\, dw_k + \{\ldots\}\, dt$$

and

$$dh_I = \beta_I^k\, dw_k + \{\ldots\}\, dt.$$

Substituting these into (3.9) and using the Itô rules

$$dw_i\, dw_j = \delta_{ij}\, dt, \qquad dw_i\, dt = 0$$

we see that the Itô–Stratonovich correction term in (3.9) is

$$\frac{1}{2}\left(\alpha_I^{kik} h_I + G_I^{ik} \beta_I^k\right) dt. \tag{3.12}$$

Thus (3.7) becomes

$$\tilde{r}_i = r_i + \int_0^{\cdot} G_I^{ij} h_I dw_j + \frac{1}{2} \int_0^{\cdot} \left(\alpha_I^{kik} h_I + G_I^{ik} \beta_I^k\right) dt.$$

As remarked in the proof of Theorem 3.1, the Itô integral has divergence zero and using Theorem 2.1 we obtain

$$\operatorname{Div}(\tilde{r}) = \int_0^T \left(\dot{r}_i + \frac{1}{2}\left(\alpha_I^{kik} h_I + G_I^{ik} \beta_I^k\right)\right) dw_i.$$

Hence

$$\operatorname{Div}(\eta) = E\left[\int_0^T \left(\dot{r}_i + \frac{1}{2}\left(\alpha_I^{kik} h_I + G_I^{ik} \beta_I^k\right)\right) dw_i \Big/ x\right] \tag{3.13}$$

where the α's and β's are given in (3.10) and (3.11).

By adjusting the right-hand side in Eq. (3.1) by the addition of a suitably chosen drift term, the above argument can easily be modified to give

Theorem 3.2. *Let $\gamma : \Omega \times [0, T] \mapsto \mathbf{R}^n$ be a C^1 adapted process and define $\{h_I\}$ by $h_I(0) = 0$ and*

$$dh_I = \left(\left(d\gamma_i - \frac{1}{2} G_J^{ik} \beta_J^k\, dt\right) X_i + \left(T_J(\circ dx) - \frac{1}{2} \alpha_J^{kik} X_i\, dt\right) h_J, V_I\right).$$

Then the vector field $\eta = h_I V_I$ is admissible and for every test function Φ on $C_o(M)$, we have

$$E\left[(\eta \Phi)(x)\right] = E\left[\Phi(x) \int_0^T \dot{\gamma}_i\, dw_i\right]. \tag{3.14}$$

The proof of Theorem 3.2 is an easy modification of the argument above, where we replace r by the path

$$\tilde{r}_i = \gamma_i - \frac{1}{2}\int_0^{\cdot} \left(\alpha_I^{kik} h_I + G_I^{ik}\beta_I^k\right) dt.$$

The essential point is that the correction term (3.12) in the computation of the divergence does not *explicitly* involve the path r.

Corollary. *Given any path r in the Cameron–Martin space of $\mathbf{R}^n$, we can construct an admissible vector field η on $C_o(M)$ such that*

$$E\big[(\eta\Phi)(x)\big] = E\left[\Phi(x)\int_0^T \left(\dot{r}_i + \frac{1}{2}\big\langle \mathrm{Ric}(\eta), X_i\big\rangle(x_t)\right) dw_i\right]. \tag{3.15}$$

Remarks. (1) Formula (3.12) is similar to those appearing in the work of Driver [6,7] and Elworthy, Le Jan and Li [8].

(2) Choosing $\gamma = 0$ in Theorem 3.2, we see that the path $\tilde{\eta} \equiv V_I h_I$, where

$$\begin{aligned} dh_I &= \left(h_J T_J(\circ dx) - \frac{1}{2} X_i\big(G_J^{ik}\beta_J^k + \alpha_J^{kik} h_J\big)\, dt, V_I\right), \\ h_I(0) &= 0 \end{aligned}$$

is *divergence-free* with respect to the law of x. In this sense $\tilde{\eta}$ is analogous to a vector field on Wiener space of the form $\int_0^{\cdot} A\, dw$, where A is a continuous adapted $\mathrm{so}(n)$-valued process.

(3) The appearance of the conditional expectation in (3.13) entails a loss of information concerning the regularity of the function $\mathrm{Div}(\eta)$. This point is crucial in certain applications of the results presented here. For example, the regularity of $\mathrm{Div}(\eta)$ plays a major role in recent work of the author [4] in which the admissibility of η is used, in the elliptic setting, to establish quasi-invariance of the law of x under the flow generated by η on $C_o(M)$.

With this in mind, we note that by choosing the process γ in (3.14) appropriately, we can eliminate the extraneous dependence of the stochastic integrals on w and thus circumvent this problem. The next example illustrates this point.

Example 3.3. Suppose B is a smooth vector field on M, ρ is a deterministic C^1 real-valued function, and define

$$\gamma_i(t) = \int_0^{\cdot} \rho_t(B, X_i)(x_t)\, dt$$

so

$$\int_0^T \dot{\gamma}_i\, dw_i = \int_0^T \rho_t(B, X_i)\, dw_i.$$

Using the Levi-Civita connection $\tilde{\nabla}$ to write this in Stratonovich form we have

$$\begin{aligned}\int_0^T \rho_t(B, X_i)\,dw_i &= \int_0^T \rho_t(B, X_i)\circ dw_i - \frac{1}{2}\int_0^T \rho_t\big((\tilde{\nabla}_{X_i}B, X_i) + (B, \tilde{\nabla}_{X_i}X_i)\big)\,dt \\ &= \int_0^T \rho_t(B, \circ dx) - \frac{1}{2}\int_0^T \rho_t\big((\tilde{\nabla}_{X_i}B, X_i) + (B, \tilde{\nabla}_{X_i}X_i)\big)\,dt. \end{aligned} \tag{3.16}$$

Since (3.16) is measurable with respect to x, (3.14) becomes

$$\mathrm{Div}(\eta) = \int_0^T \rho_t(B, \circ dx) - \frac{1}{2}\int_0^T \rho_t\big((\tilde{\nabla}_{X_i}B, X_i) + (B, \tilde{\nabla}_{X_i}X_i)\big)\,dt.$$

In particular, $\mathrm{Div}(\eta)$ is an explicit function of the path x.

3.3. A basis-free formulation of the argument

Assume now that M is a *Riemannian* manifold. In this case we can formulate the preceding argument *intrinsically*, i.e. in a way that does not depend on the choice of a basis $\{V_I\}$.

Let $\tilde{\nabla}$ denote the Levi-Civita covariant derivative with respect to the Riemannian metric on M and $\tilde{D}$ the corresponding covariant stochastic differential. As before, $\langle\cdot,\cdot\rangle$ and ∇ will denote the inner product and the connection on the subbundle E introduced in Section 2.2.

We define

$$T(X, Y) = \tilde{\nabla}_Y X - \nabla_Y X, \quad Y \in TM,\ X \in E, \tag{3.17}$$

noting that T is tensorial in *both* arguments.

Let $r : [0, T] \times \Omega \mapsto \mathbf{R}^n$ be an Itô semimartingale

$$dr_k(t) = b^{kj}(t)\,dw_j + c^k(t)\,dt$$

where b^{kj} and c^k are adapted continuous processes. Then differentiation in Eq. (1.1) gives the following covariant equation for the path $\eta \equiv dg(w)r$:

$$\begin{aligned}\tilde{D}_t\eta &= \tilde{\nabla}_\eta X_i \circ dw_i + X_i \circ dr_i \\ &= \nabla_\eta X_i \circ dw_i + T(X_i, \eta)\circ dw_i + X_i \circ dr_i \\ &= \langle\nabla_\eta X_i, X_j\rangle X_j \circ dw_i + T(X_i, \eta)\circ dw_i + X_i \circ dr_i \\ &= \langle\nabla_\eta X_j, X_i\rangle X_j \circ dw_i + G_\eta^{ij} X_j \circ dw_i + T(X_i, \eta)\circ dw_i + X_i \circ dr_i\end{aligned}$$

where

$$G_V^{ij} \equiv \langle\nabla_V X_i, X_j\rangle - \langle\nabla_V X_j, X_i\rangle.$$

In view of Lemma 2.3, we have

$$\tilde{D}_t\eta = G_\eta^{ij} X_j \circ dw_i + T(X_i, \eta) \circ dw_i + X_i \circ dr_i.$$

Thus

$$\tilde{D}_t\eta = T(X_i, \eta) \circ dw_i + X_i\big(\circ dr_i + G_\eta^{ji} \circ dw_j\big). \tag{3.18}$$

Theorem 3.4. *Let r be any Cameron–Martin path in $\mathbf{R}^n$ and define a vector field η along x by the covariant SDE*

$$\begin{aligned} \tilde{D}_t\eta &= T(\circ dx, \eta) + X_i \dot{r}_i\, dt, \\ \eta(0) &= 0. \end{aligned} \tag{3.19}$$

Then η is an admissible vector field on $C_o(M)$. Define the differential operator

$$L_{Y,X} \equiv \nabla_Y \nabla_X - \nabla_{\tilde{\nabla}_Y X}$$

acting on vector fields X and Y on M. Then for test functions Φ on $C_o(M)$,

$$E\big[(\eta\Phi)(x)\big] = E\left[\Phi(x) \int_0^T \left(\dot{r}_i + \frac{1}{2}\alpha_i\right) dw_i\right] \tag{3.20}$$

where

$$\begin{aligned} \alpha_i(t) = {} & \langle L_{\eta,X_j} X_i, X_j\rangle - \langle L_{\eta,X_j} X_j, X_i\rangle + \langle \nabla_\eta X_i, \nabla_{X_j} X_j\rangle \\ & - \langle \nabla_\eta X_j, \nabla_{X_j} X_i\rangle + \langle \nabla_{T(X_j,\eta)} X_i, X_j\rangle - \langle \nabla_{T(X_j,\eta)} X_j, X_i\rangle. \end{aligned}$$

Proof. Note that Eq. (3.18) implies $\tilde{r}$ is a lift of η, where

$$\tilde{r}_i = r_i - \int_0^{\cdot} G_\eta^{ji} \circ dw_j. \tag{3.21}$$

Since the functions G_η^{ji} are skew-symmetric in the indices j and i, Theorems 2.1 and 2.2 imply that $\tilde{r}$ is an admissible vector field on the Wiener space. As before, for any test function Φ on $C_o(M)$, we have

$$E\big[D\Phi(x)\eta\big] = E\big[\Phi(x)\,\mathrm{Div}(\tilde{r})\big]$$

and it follows that η is admissible as claimed.

We now derive the formula for the divergence of the vector field η. As before, this requires the computation of the Stratonovich–Itô correction term in (3.21). We now proceed to do this.

Note that the operator-valued map $(X, Y) \mapsto L_{Y,X}$ is tensorial in both X and Y. We have

$$\begin{aligned}\nabla_X \nabla_Y &= R(X,Y) + \nabla_Y \nabla_X + \nabla_{[X,Y]}\\ &= R(X,Y) + \nabla_Y \nabla_X - \nabla_{\tilde{\nabla}_Y X} + \nabla_{\tilde{\nabla}_X Y}\\ &= R(X,Y) + L_{Y,X} + \nabla_{\tilde{\nabla}_X Y}.\end{aligned}$$

In particular

$$D_t \nabla_\eta X_i = \big[R(\circ dx_t, \eta) + L_{\eta, \circ dx_t} + \nabla_{\tilde{D}_t \eta}\big] X_i.$$

Thus, neglecting differentials of terms of bounded variation (which will not affect the present calculation)

$$D_t \nabla_\eta X_i = \big[R(X_k, \eta) + L_{\eta, X_k}\big] X_i \, dw_k + \nabla_{\tilde{D}_t \eta} X_i.$$

This yields

$$\begin{aligned} d_t G_\eta^{ij} &= \langle D_t \nabla_\eta X_i, X_j\rangle - \langle D_t \nabla_\eta X_j, X_i\rangle + \langle \nabla_\eta X_i, D_t X_j\rangle - \langle \nabla_\eta X_j, D_t X_i\rangle\\ &= \big\{\big\langle \big[R(X_k,\eta) + L_{\eta,X_k}\big] X_i, X_j\big\rangle - \big\langle \big[R(X_k,\eta) + L_{\eta,X_k}\big] X_j, X_i\big\rangle\\ &\quad + \langle \nabla_\eta X_i, \nabla_{X_k} X_j\rangle - \langle \nabla_\eta X_j, \nabla_{X_k} X_i\rangle\big\} dw_k + \langle \nabla_{\tilde{D}_t \eta} X_i, X_j\rangle - \langle \nabla_{\tilde{D}_t \eta} X_j, X_i\rangle.\end{aligned}$$

Substituting for $\tilde{D}_t \eta$ from Eq. (3.19) and using Lemma 2.4(b) and the symmetry of the Ricci tensor, we obtain

$$\begin{aligned} d_t G_\eta^{ij}\, dw_j &= \big\{\langle L_{\eta, X_j} X_i, X_j\rangle - \langle L_{\eta, X_j} X_j, X_i\rangle + \langle \nabla_\eta X_i, \nabla_{X_j} X_j\rangle\\ &\quad - \langle \nabla_\eta X_j, \nabla_{X_j} X_i\rangle + \langle \nabla_{T(X_j,\eta)} X_i, X_j\rangle - \langle \nabla_{T(X_j,\eta)} X_j, X_i\rangle\big\}\, dt\\ &= \alpha_i(t)\, dt.\end{aligned}$$

Thus (3.9) gives

$$\tilde{r}_i = r_i + \int_0^\cdot G_\eta^{ij}\, dw_j + \frac{1}{2}\int_0^\cdot \alpha_i\, dt.$$

Formula (3.21) now follows from Theorems 2.1 and 2.2, as before. □

Remark 3.5. It is clear that the argument used to prove Theorem 3.4 is valid in more generality, with the deterministic Cameron–Martin path r replaced by an x-measurable random path of the form

$$r = \int_0^\cdot A(s)\, dw_s + \int_0^\cdot B(s)\, ds \tag{3.22}$$

where $A : \Omega \times [0,T] \mapsto \mathrm{so}(n)$ and $B : \Omega \times [0,T] \mapsto \mathbf{R}^n$ are continuous adapted processes. We note that it is easy to construct examples of x-measurable processes of the form (3.22). A large class of such examples is obtained by choosing a 2-form λ on M and a deterministic continuous real-valued function f and defining $r = r_1, \dots, r_n)$ where

$$r_i = \int_0^{\cdot} f(s)\lambda\bigl(X_i(x_s), X_j(x_s)\bigr)\, dw_j.$$

In view of Theorems 2.1 and 2.2, it is natural to consider the Wiener space $C_0(\mathbf{R}^n)$ as a manifold with tangent bundle $\bigcup_w T_w C_0(\mathbf{R}^n)$, where each fiber $T_w C_0(\mathbf{R}^n)$ consists of paths of the form (3.22).

For each such path $r = r(x)$, Eq. (3.19) produces a vector field η on $C_o(M)$ that is then lifted to a vector field $\tilde{r}$ on $C_0(\mathbf{R}^n)$ by Eq. (3.21). We summarize these constructions as follows.

Define

$$H(r) = (r, \eta), \quad r \in TC_0\bigl(\mathbf{R}^n\bigr)$$

and let

$$\pi : TC_0\bigl(\mathbf{R}^n\bigr) \mapsto C_0\bigl(\mathbf{R}^n\bigr)$$

denote the bundle projection.

Then the chain of maps in Theorem 3.4 and its proof is illustrated by the following commutative diagram:

$$\begin{tikzcd}
 & TC_0\bigl(\mathbf{R}^n\bigr) \times TC_o(M) \arrow[dr, "(3.21)"] & \\
TC_0\bigl(\mathbf{R}^n\bigr) \arrow[ur, "H"] \arrow[r, "(3.19)"] \arrow[d, "\pi"'] & TC_o(M) & TC_0\bigl(\mathbf{R}^n\bigr) \arrow[l, "dg"'] \\
C_0\bigl(\mathbf{R}^n\bigr) \arrow[r, "g"'] & C_o(M) \arrow[u, "\eta"] \arrow[ul, "r"] & C_0\bigl(\mathbf{R}^n\bigr) \arrow[l, "g"] \arrow[u, "\tilde{r}"']
\end{tikzcd}$$

3.4. Gradient systems

Suppose M is an isometrically embedded submanifold of a Euclidean space $\mathbf{R}^N$ (by Nash's embedding theorem, every finite-dimensional Riemannian manifold can be realized this way). Define $X_i = Pe_i$, $1 \leqslant i \leqslant N$ where $e_1, \dots, e_N$ is the standard orthonormal basis of $\mathbf{R}^N$ and $P(x)$ is orthogonal projection onto the tangent space $T_x M$. Then the infinitesimal generator of the process x defined by

$$dx_t = \sum_{i=1}^{n} X_i(x_t) \circ dw_i$$

is $1/2\,\Delta_B$, where Δ_B is the Laplace–Beltrami operator on M. Thus x is a Brownian motion in M.

In this case the connection ∇ coincides with the Levi-Civita connection on M (cf. [8]), hence the tensor T defined in (3.17) is zero and Eq. (3.19) reduces to

$$\tilde{D}_t \eta = X_i \dot{r}_i \, dt. \tag{3.23}$$

This yields the following.

Theorem 3.6. *If r is any (random, x-adapted) path such that $\dot{r} \in L^2[0,T]$ then the vector field η defined by* (3.24) *is admissible.*

In particular, let h be any path in the Cameron–Martin space of $T_o(M)$ and define

$$r_i = \int_0^{\cdot} \langle U_t \dot{h}_t, X_i \rangle \, dt, \quad i = 1, \ldots, N,$$

where $U.$ denotes stochastic parallel translation along the path x. Then the integral in (3.24) becomes h_t and we obtain the following result of Driver (cf. [6]).

Corollary 3.7. *For every path h in the Cameron–Martin space of $T_o(M)$, the vector field $\eta_t \equiv U_t h_t$ is admissible.*

Finally, we note that every adapted vector field on $C_o(M)$ with an admissible lift to the Wiener space is obtained from Theorem 3.4. Denote the process η in Theorem 3.4 by η^r. Then we have

Proposition 3.8. *Suppose η is an adapted vector field on $C_o(M)$ such that*

$$\eta = dg(w)r$$

for some $r \in TC_0(\mathbf{R}^n)$. Then there exists $\bar{r} \in TC_0(\mathbf{R}^n)$ such that $\eta = \eta^{\bar{r}}$.

Proof. This follows immediately from Eqs. (3.18) and (3.19). We define $\tilde{r}$ by

$$\tilde{r}_i = r_i + \int_0^{\cdot} G_\eta^{ji} \circ dw_j, \quad i = 1, \ldots, n. \qquad \square$$

4. Linearly independent diffusion coefficients

In this section we consider the special case where the vectors $X_1(x), \ldots, X_n(x)$ are linearly independent at every point $x \in M$. (In the *elliptic* case there is a topological obstruction to this condition, i.e. if M has non-zero Euler characteristic then it is impossible. However, the condition is reasonable in the non-elliptic case.) As we shall see, this implies that the Wiener path w is a function of the solution x of the SDE (1.1) i.e.

$$w = \Theta(x)$$

where Θ is a measurable function on $C_o(M)$. In this case the following simplified version of the method used in Section 3 produces admissible vector fields on $C_o(M)$.

Choose r to be any process of the form

$$r_t = \int_0^t A(s)\,dw_s + \int_o^t B(t)\,dt, \quad t \in [0, T], \tag{4.1}$$

where A and B are continuous adapted processes with values in so(n) and $\mathbf{R}^n$ and define η by (2.4), i.e.

$$\eta_t = Y_t \int_0^t Z_s X_i(x_s) \circ dr_i.$$

By Theorems 2.1, 2.2 and 2.5, r is an admissible lift of η, hence $\eta(w) = \eta(\Theta(x))$ is an admissible vector field on $C_o(M)$.

We now study how the formulae in Section 3 reduce in the linearly independent case. As before, define $X(x): \mathbf{R}^n \mapsto T_x M$ by

$$X(x)(h_1, \ldots, h_n) = X_i(x)h_i.$$

We will need the following result.

Lemma 4.1. *The vectors $X_1(x), \ldots, X_n(x)$ are linearly independent if and only if*

$$X(x)^* X(x) = I_{\mathbf{R}^n}.$$

Since Lemma 4.1 is elementary, the proof will be omitted.

Assume now that $\{X_1, \ldots, X_n\}$ are linearly independent. Then Lemma 4.1 enables us to solve the SDE (1.1) for w in terms of x and obtain

$$dw = X(x_t)^* \circ dx,$$

thus $w = \theta(x)$, as claimed above. We also have

Corollary to Lemma 4.1. *For $a_i \in C^\infty(M)$, $i = 1, \ldots, n$ and $V \in TM$,*

$$\nabla_V (a_i X_i) = V(a_i) X_i.$$

In particular

$$\nabla_V X_i = 0, \quad i = 1, \ldots, n.$$

The corollary implies that the functions G_I^{ij} in (3.4) are all zero. Furthermore, the tensors T_I in Section 3 take the form

$$T_I(aX_i) = a[X_i, V_I], \quad i = 1, \ldots, n,$$

for $a \in C^\infty(M)$. Theorem 3.1 then becomes

Theorem 4.2. *Suppose the process r is defined as in* (4.1) *and the functions h_I are chosen to satisfy*

$$\begin{aligned} dh_I &= (X_i, V_I) \circ dr_i - \big([X_i, V_J], V_I\big) h_J \circ dw_i, \\ h_I(0) &= 0. \end{aligned} \tag{4.2}$$

Then the vector field $\eta = h_I V_I$ is admissible and

$$\mathrm{Div}(\eta) = \int_0^T B_i(t)\, dw_i.$$

Example 4.3. Let M be the *Heisenberg group*, i.e. the Lie group $\mathbf{R}^3$ with group multiplication

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = \left(a_1 + b_1, a_2 + b_2, a_3 + b_3 + \frac{1}{2}(a_1 b_2 - b_1 a_2)\right).$$

Let

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z}$$

and define $V_1 = X_1$, $V_2 = X_2$, and

$$V_3 = [V_1, V_2] = \frac{\partial}{\partial z}.$$

Then

$$[X_1, V_2] = V_3, \qquad [X_2, V_1] = -V_3, \qquad [X_i, V_j] = 0, \quad i + j \neq 3.$$

Thus Eq. (4,2), which we write in the form

$$V_I \circ dh_I = X_i \circ dr_i - [X_i, V_I] h_I \circ dw_i$$

becomes

$$\begin{aligned} &V_1 \circ dh_1 + V_2 \circ dh_2 + V_3 \circ dh_3 \\ &\quad = X_1 \circ dr_1 + X_2 \circ dr_2 + V_3(h_1 \circ dw_2 - h_2 \circ dw_1). \end{aligned} \tag{4.3}$$

Since the vectors $\{V_1, V_2, V_3\}$ are linearly independent, Eq. (4.3) has a unique solution, given by

$$h_1 = r_1, \qquad h_2 = r_2, \qquad h_3 = \int_0^{\cdot} r_1 \circ dw_2 - r_2 \circ dw_1. \tag{4.4}$$

As point of interest, we note that if (w_1, w_2) is substituted for (r_1, r_2) then the preceding integral becomes the *Levy area* (it should be noted, however, that in the present context (w_1, w_2) is not an allowable choice for r).

References

[1] D. Bell, Divergence theorems in path space, J. Funct. Anal. 218 (1) (2005) 130–149.
[2] D. Bell, Divergence theorems in path space II: Degenerate diffusions, C. R. Math. Acad. Sci. Paris 342 (2006) 869–872.
[3] D. Bell, The Malliavin Calculus, 2nd ed., Dover, Mineola, NY, 2006.
[4] D. Bell, Quasi-invariant measures on the path space of a diffusion, C. R. Math. Acad. Sci. Paris 343 (2006) 197–200.
[5] J.-M. Bismut, Large Deviations and the Malliavin Calculus, Progr. Math., vol. 45, Birkhäuser Boston, Boston, MA, 1984.
[6] B. Driver, A Cameron–Martin type quasi-invariance theorem for Brownian motion on a compact manifold, J. Funct. Anal. 109 (1992) 272–376.
[7] B. Driver, A Cameron–Martin type quasi-invariance theorem for pinned Brownian motion on a compact Riemannian manifold, 1994.
[8] K.D. Elworthy, Y. Le Jan, X.-M. Li, On the Geometry of Diffusion Operators and Stochastic Flows, Lecture Notes in Math., vol. 1720, Springer-Verlag, Berlin, 1999.
[9] O. Enchev, D. Stroock, Towards a Riemannian geometry on the path space over a Riemannian manifold, J. Funct. Anal. 134 (2) (1995) 392–416.
[10] E.P. Hsu, Quasi-invariance of the Wiener measure on the path space over a compact Riemannian manifold, J. Funct. Anal. 134 (1995) 417–450.
[11] E.P. Hsu, Quasi-invariance of the Wiener measure on path spaces: Noncompact case, J. Funct. Anal. 193 (2) (2002) 278–290.
[12] P. Malliavin, Stochastic calculus of variations and hypoelliptic operators, in: Proceedings of the International Conference on Stochastic Differential Equations, Kinokuniya/Wiley, Kyoto, 1976, pp. 195–263.
[13] P. Malliavin, Stochastic Analysis, Grundlehren Math. Wiss., vol. 313, Springer-Verlag, Berlin, 1997.
[14] D. Nualart, The Malliavin Calculus and Related Topics, Prob. Appl. (N.Y.), Springer-Verlag, New York, 1995.
[15] S. Watanabe, Lectures on Stochastic Differential Equations and Malliavin Calculus, Tata Inst. Fund. Res. Lect. Math. Phys., vol. 73, Springer-Verlag, Berlin, 1984. Notes by M. Gopalan Nair and B. Rajeev.