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Convergence of the Adomian method applied to a class of nonlinear integral equations

I.L. El-Kalla

Physics and Engineering Mathematics Department, Faculty of Engineering, Mansoura University, PO 35516 Mansoura, Egypt

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Abstract

In this work, a reliable approach for convergence of the Adomian method when applied to a class of nonlinear Volterra integral equations is discussed. Convergence analysis is reliable enough to estimate the maximum absolute truncated error of the Adomian series solution.

Keywords: Nonlinear Volterra integral equations; Convergence; Adomian method

1. Introduction

The Adomian Decomposition Method (ADM) solves successfully different types of linear and nonlinear equations in deterministic or stochastic fields [1–4]. Application of ADM to different types of integral equations has been discussed by many authors, for example [5–8]. In this work, the nonlinear Volterra integral equation of the second kind

$$y(t) = x(t) + \int_0^t k(t,\tau) f(y(\tau)) \mathrm{d}\tau \tag{1}$$

is considered where $x(t)$ is assumed to be bounded $\forall t \in J = [0, T]$ and $|k(t,\tau)| \le M \ \forall 0 \le \tau \le t \le T$. The nonlinear term $f(y)$ is Lipschitz continuous with $|f(y) - f(z)| \le L\,|y - z|$ and has the Adomian polynomial representation

$$f(y) = \sum_{n=0}^{\infty} A_n(y_0, y_1, \ldots, y_n), \tag{2}$$

where the traditional formula for A_n is

$$A_n = (1/n!)(\mathrm{d}^n/\mathrm{d}\lambda^n)\left[f\left(\sum_{n=0}^{\infty} \lambda^i y_i \right)\right]_{\lambda=0}. \tag{3}$$

E-mail addresses: i.alkalla@wlv.ac.uk, al_kalla@mans.edu.eg.

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The author in [9] deduced another programmable formula for the Adomian polynomials:

$$A_n = f(S_n) - \sum_{j=0}^{n-1} A_j, \tag{4}$$

where the partial sum is $S_n = \sum_{i=0}^{n} y_i(t)$. Application of ADM to (1) yields

$$y(t) = \sum_{i=0}^{\infty} y_i(t), \tag{5}$$

where

$$y_0(t) = x(t), \tag{6}$$

$$y_i(t) = \int_0^t k(t,\tau) A_{i-1} \mathrm{d}\tau, \quad i \geq 1. \tag{7}$$

The contribution of the work reported here can be summarized in the following four points:

- Introducing the sufficient condition that guarantees a unique solution to problem (1) (see Theorem 1).
- On the basis of the above point and formula (4), convergence of ADM is discussed (see Theorem 2).
- Using point two, the maximum absolute truncated error of the Adomian series solution (5) is estimated (see Theorem 3).
- Preparation of an algorithm using the MATHEMATICA package to generate the two types of Adomian polynomials, make a comparative study and solve the related numerical examples.

2. Convergence analysis

2.1. Uniqueness theorem

Theorem 1. *Problem* (1) *has a unique solution whenever* $0 < \alpha < 1$, *where,* $\alpha = LMT$.

Proof. Let y and $\overset{*}{y}$ be two different solutions to (1) then

$$\begin{aligned} |y - \overset{*}{y}| &= \left| \int_0^t k(t,\tau)[f(y) - f(\overset{*}{y})]\, \mathrm{d}\tau \right| \\ &\leq \int_0^t |k(t,\tau)|\, |f(y) - f(\overset{*}{y})|\, \mathrm{d}\tau \\ &\leq LM|y - \overset{*}{y}| \int_0^t \mathrm{d}\tau \\ &\leq \alpha |y - \overset{*}{y}| \end{aligned}$$

from which we get $(1-\alpha)\,|y - \overset{*}{y}| \leq 0$. Since $0 < \alpha < 1$, then $|y - \overset{*}{y}| = 0$, implies $y = \overset{*}{y}$ and this completes the proof. ■

2.2. Convergence theorem

Theorem 2. *The series solution* (5) *of problem* (1) *using ADM converges if* $0 < \alpha < 1$ *and* $|y_1| < \infty$.

Proof. Denote as $(C[J], \|.\|)$ the Banach space of all continuous functions on J with the norm $\|f(t)\| = \max_{\forall t \in J} |f(t)|$. Define the sequence of partial sums $\{S_n\}$; let S_n and S_m be arbitrary partial sums with $n \geq m$. We are going to prove that $\{S_n\}$ is a Cauchy sequence in this Banach space:

$$\begin{aligned} \|S_n - S_m\| &= \max_{\forall t \in J} |S_n - S_m| \\ &= \max_{\forall t \in J} \left| \sum_{i=m+1}^{n} y_i(t) \right| \end{aligned}$$

$$= \max_{\forall t \in J} \left| \sum_{i=m+1}^{n} \int_0^t k(t,\tau) A_{i-1}\, \mathrm{d}\tau \right|$$
$$= \max_{\forall t \in J} \left| \int_0^t k(t,\tau) \sum_{i=m}^{n-1} A_i\, \mathrm{d}\tau \right|.$$

From (4) we have $\sum_{i=m}^{n-1} A_i = f(S_{n-1}) - f(S_{m-1})$ so

$$\begin{aligned} \|S_n - S_m\| &= \max_{\forall t \in J} \left| \int_0^t k(t,\tau) \left[f(S_{n-1}) - f(S_{m-1}) \right] \mathrm{d}\tau \right| \\ &\le \max_{\forall t \in J} \int_0^t |k(t,\tau)|\, |f(S_{n-1}) - f(S_{m-1})|\, \mathrm{d}\tau \\ &\le \alpha \, \|S_{n-1} - S_{m-1}\|. \end{aligned}$$

Let $n = m + 1$; then

$$\|S_{m+1} - S_m\| \le \alpha \|S_m - S_{m-1}\| \le \alpha^2 \|S_{m-1} - S_{m-2}\| \le \cdots \le \alpha^m \|S_1 - S_0\|.$$

From the triangle inequality we have

$$\begin{aligned} \|S_n - S_m\| &\le \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + \cdots + \|S_n - S_{n-1}\| \\ &\le [\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}] \|S_1 - S_0\| \\ &\le \alpha^m [1 + \alpha + \alpha^2 + \cdots + \alpha^{n-m-1}] \|S_1 - S_0\| \\ &\le \alpha^m \left(\frac{1 - \alpha^{n-m}}{1-\alpha} \right) \|y_1(t)\|. \end{aligned}$$

Since $0 < \alpha < 1$ we have $(1 - \alpha^{n-m}) < 1$; then

$$\|S_n - S_m\| \le \frac{\alpha^m}{1-\alpha} \max_{\forall t \in J} |y_1(t)|. \tag{8}$$

But $|y_1| < \infty$ (since $x(t)$ is bounded); so, as $m \to \infty$, then $\|S_n - S_m\| \to 0$. We conclude that $\{S_n\}$ is a Cauchy sequence in $C[J]$, so the series converges and the proof is complete. ■

2.3. Error estimate

Theorem 3. *The maximum absolute truncation error of the series solution (5) to problem (1) is estimated to be* $\max_{\forall t \in J} \left| y(t) - \sum_{i=0}^{m} y_i(t) \right| \le \frac{K\alpha^{m+1}}{L(1-\alpha)}$ *where* $K = \max_{\forall t \in J} |f(x(t))|$.

Proof. From Theorem 2 inequality (8) we have

$$\|S_n - S_m\| \le \frac{\alpha^m}{1-\alpha} \max_{\forall t \in J} |y_1(t)|.$$

As $n \to \infty$ then $S_n \to y(t)$ and $\max_{\forall t \in J} |y_1(t)| \le TM \max_{\forall t \in J} |f(y_0)|$, so

$$\|y(t) - S_m\| \le \frac{\alpha^{m+1}}{L(1-\alpha)} \max_{\forall t \in J} |f(x(t))|.$$

Finally, the maximum absolute truncation error in the interval J is

$$\max_{\forall t \in J} \left| y(t) - \sum_{i=0}^{m} y_i(t) \right| \le \frac{K\alpha^{m+1}}{L(1-\alpha)}. \tag{9}$$

This completes the proof. ■

3. Numerical experiments

The Adomian polynomials can be generated using formula (3) or formula (4). Formula (4) is programmable and the Adomian series solution can be converged faster when using it. For example, if $f(y) = y^2$ the first four polynomials using formulas (3) and (4) are computed to be:

Using formula (3):

$$A_0 = y_0^2$$
$$A_1 = 2y_0y_1$$
$$A_2 = y_1^2 + 2y_0y_2$$
$$A_3 = 2y_1y_2 + 2y_0y_3$$
$$A_4 = y_2^2 + 2y_1y_3 + 2y_0y_4.$$

Using formula (4):

$$A_0 = y_0^2$$
$$A_1 = 2y_0y_1 + y_1^2$$
$$A_2 = 2y_0y_2 + 2y_1y_2 + y_2^2$$
$$A_3 = 2y_0y_3 + 2y_1y_3 + 2y_2y_3 + y_3^2$$
$$A_4 = 2y_0y_4 + 2y_1y_4 + 2y_2y_4 + 2y_3y_4 + y_4^2.$$

Clearly, the first four polynomials computed using formula (4) include the first four polynomials computed using formula (3) in addition to other terms which should appear in $A_5, A_6, A_7, \ldots$ using formula (3). Thus, the solution using formula (4) forces many terms to be entered into the calculation processes earlier, yielding a faster convergence. In order to verify the conclusions of Theorems 2 and 3 consider the following numerical example:

$$y(t) = \frac{1}{20}(300 + 315t^2 + 5t^4 + t^6) - \frac{1}{150}\int_0^t (t-\tau)y^2(\tau)\mathrm{d}\tau, \quad 0 \le t \le 1,$$

with exact solution $y(t) = 15(t^2 + 1)$. Table 1 shows the exact absolute truncation error $\triangle = \left|y(t) - \sum_{i=0}^{m} y_i(t)\right|_{t=1}$ and the maximum absolute truncation error $\overset{*}{\triangle} = \frac{K\alpha^{m+1}}{L(1-\alpha)}$ for different values of m where $T = 1$, $M = \frac{1}{150}$, $L = 60$, $\alpha = \frac{2}{5}$ and $K = \frac{385641}{400}$.

Table 1

m	$\triangle$	$\overset{*}{\triangle}$
5	1.23333×10^{-3}	0.109693
10	6.27671×10^{-9}	0.00112326
15	2.60089×10^{-14}	0.0000115022
20	9.79976×10^{-19}	1.17782×10^{-7}

4. Conclusion

The sufficient condition that guarantees a unique solution to the given problem is obtained. The contraction mapping principles can be employed successfully to prove the convergence of ADM. The convergence study is reliable enough to estimate the maximum absolute truncated error of the Adomian series solution.

References

[1] G. Adomian, Stochastic System, Academic Press, 1983.
[2] G. Adomian, Nonlinear Stochastic Operator Equations, Academic Press, San Diego, 1986.

[3] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, 1995.
[4] M. El-Tawil, M. Saleh, I.L. El-kalla, Decomposition solution of stochastic nonlinear oscillator, Int. J. Differ. Equ. Appl. 6 (4) (2002) 411–422.
[5] J. Biazar, E. Babolian, Solution of a system of nonlinear Volterra integral equations of the second kind, Far East J. Math. Sci. 2 (6) (2000) 935–945.
[6] E. Babolian, J. Biazar, A.R. Vahidi, The decomposition method applied to systems of Fredholm integral equations of the second kind, Appl. Math. Comput. 148 (2004) 443–452.
[7] H. Sadeghi Goghary, Sh. Javadi, E. Babolian, Restarted Adomian method for system of nonlinear Volterra integral equations, Appl. Math. Comput. 161 (2005) 745–751.
[8] S. Abbasbandy, Numerical solution of the integral equations: Homotopy perturbation method and Adomian decomposition method, Appl. Math. Comput. 173 (2006) 393–500.
[9] I.L. El-kalla, Convergence of Adomian's method applied to a class of Volterra type integro-differential equations, Int. J. Differ. Equ. Appl. 10 (2) (2005) 225–234.