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FINITE-DIMENSIONAL HOPF ALGEBRAS ARE FREE OVER GROUPLIKE SUBALGEBRAS

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Let H be a finite-dimensional Hopf algebra over a field k, and let G be a subgroup of the group of grouplikes of H. Then every left (H, kG)-Hopf module is free as a left kG-module. If k is algebraically closed and C is a simple subcoalgebra of H with $gC=C$ for all $g \in G$, then the exponent of G and (in characteristic $p>0$) the p-part of $|G|$ divide n, where n^2 is the dimension of C.

Introduction

Let H be a finite-dimensional Hopf algebra over a field k, and let B be a Hopf subalgebra. Kaplansky conjectured [5, Appendix 2] that H is free as a module over B. This result is known and easy if B contains the coradical of H [8, Corollary 1]; deeper results are that the result holds if H is commutative or has a cocommutative coradical [9, Theorem 1, Proposition 3]. (See [12, Section 2] for two additional cases involving more technical commutativity assumptions.) Progress in determining the structure of finite-dimensional Hopf algebras has been hindered by a lack of freeness results which hold for all H. Recently the second author has shown [13, Theorem 6] that H is a free kG-module when G is a subgroup of the group $G(H)$ of grouplikes of H, provided that kG is semisimple. In this paper we remove the restriction on kG. We have worked in a generality which allows us to include here a proof of the semisimple case.

After some preparatory results from representation theory, we consider the situation in which H is a finite-dimensional Hopf algebra over an algebraically closed field k and G is a subgroup of $G(H)$ which stabilizes a simple subcoalgebra C of H. We show that C is free as a kG-module and that the exponent of G and the p-part of $|G|$ (in characteristic $p>0$) divide n, where n^2 is the dimension of C. We then prove our main result, which is that for H a finite-dimensional Hopf algebra over a field k and G a subgroup of $G(H)$, every left (H, kG)-Hopf module is free as a

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left kG-module. (In particular, H is free as a left kG-module.) This result was proved in the cases mentioned above by the authors previously cited. We then conclude the paper with two simple applications of the main theorem.

1. Preliminaries

Let G be a finite group, and let k be a field. The first result we require is a deep property of the group ring kG. The result seems to have first appeared in [3, Lemme 1].

Proposition 1.1. *Let k be a field, and let G be a finite group. Then two finitely generated projective kG-modules are isomorphic if and only if they have the same composition factors.*

Proof. In characteristic zero, the result holds since kG is semisimple. So assume characteristic $k=p>0$. By [2, Corollary 21.23], the result holds for any field k for which one can find a complete discrete valuation ring R of characteristic zero whose residue class field is isomorphic to k. Using an elementary argument, it is shown in [10, p. 48] that such an R can be found for any k. This completes the proof. □

Remark. We shall use Proposition 1.1 only for the case k algebraically closed. In that case, the Witt ring $W(k)$ provides a familiar example of a ring R of the type required.

Proposition 1.2. *Let k be a field, G a finite group, and M a finitely generated left kG-module. Suppose that M is a free $k\langle g\rangle$-module for each $g\in G$ for which $k\langle g\rangle$ is semisimple. If characteristic $k=p>0$, assume in addition that M is a free kP-module for some p-Sylow subgroup P of G. Then M is a free kG-module.*

Proof (cf. [13, Proposition 1]). By the Noether–Deuring Theorem we may assume that k is algebraically closed.

We have that M is projective as a kG-module. This is clear in characteristic zero, and follows in characteristic $p>0$ via [2, Proposition 19.5, (ix) and (viii)] from the fact that M is free as a left kP-module for some p-Sylow subgroup P of G.

Let $s=\gcd(|G|, \dim M)$. Let $M_1=M^{(|G|/s)}$, $M_2=(kG)^{((\dim M)/s)}$. Since for each $g\in G$ for which $k\langle g\rangle$ is semisimple we have that M_1 and M_2 are free $k\langle g\rangle$-modules of the same rank, we have by [1, Theorem 30.14] (in characteristic zero) or by Proposition 1.1 and [1, proof of Theorem 82.3] (in characteristic $p>0$) that $M_1\cong M_2$ as kG-modules.

Let Q be a q-Sylow subgroup of G. (In characteristic $p>0$, take $q\neq p$.) The number of times that the trivial kQ-module k occurs in a decomposition of M_1 as a direct sum of indecomposable kQ-modules is $c|G|/s$, for some integer c. Since k

occurs $[G:Q](\dim M)/s$ times in the decomposition of M_2, we see that s must contain the entire q-part of $|G|$. In characteristic $p>0$, s must contain the entire p-part of $|G|$, since M is free as a kP-module for some p-Sylow subgroup P of G. Thus $s=|G|$, and $M\cong M_2$ is a free kG-module, as required. □

A well-known elementary result [2, Exercise 10.18] is that if W is a finitely generated free left kG-module and M any finitely generated left kG-module, then $W\otimes M$ is a free left kG-module. (Here G acts on $W\otimes M$ via $g(w\otimes m)=gw\otimes gm$ for all $g\in G$, $w\in W$, $m\in M$.) Thus in this case we have $W\otimes M\cong W^{(t)}$ as kG-modules, where $t=\dim M$. Our next result is a partial converse.

Proposition 1.3. *Let k be a field, G a finite group, and W a finitely generated left kG-module. Suppose that there exists a finitely generated faithful left kG-module M with $W\otimes M\cong W^{(t)}$ as kG-modules, where $t=\dim M$. Then W is free as a left kG-module.*

Proof. We may assume that k is algebraically closed. By Proposition 1.2, it suffices to show that W is free as a kG_1-module for each subgroup G_1 such that either G_1 is cyclic and kG_1 is semisimple, or G_1 is a p-Sylow subgroup (in the case characteristic $k=p>0$).

Let F be the span of a maximal kG_1-independent subset $\mathscr{B}$ of W. Then F is free, hence injective [1, Theorem 62.3], so we may write $W=F\oplus E$ as a kG_1-module. By the maximality of $\mathscr{B}$, each element of E is annihilated by a nonzero element of kG_1. This implies that we can find $0\neq x\in kG_1$ with $xE=(0)$; this follows in the first case from the fact that kG_1 splits into a product of fields, and follows in the case G_1 a p-group, p the characteristic of k, from the fact that kG_1 has a unique minimal left ideal [2, Exercise 18.2].

Now suppose that $W\otimes M\cong W^{(t)}$, $t=\dim M$. Then $F^{(t)}\oplus E^{(t)}\cong(F\oplus E)^{(t)}\cong W^{(t)}\cong W\otimes M\cong(F\otimes M)\oplus(E\otimes M)\cong F^{(t)}\oplus(E\otimes M)$. By the Krull–Schmidt–Azumaya Theorem, we have $E^{(t)}\cong E\otimes M$. Let us decompose M in the same manner as we decomposed W: $M=F'\oplus E'$ as a kG_1-module, where F' is a free kG_1-module and $\mathrm{Ann}(E')\neq(0)$. Since M is faithful, we must have $F'\neq(0)$. The free kG_1-module $E\otimes F'$ is isomorphic to a direct summand of $E\otimes M\cong E^{(t)}$; since $xE^{(t)}=(0)$, we conclude that $E=(0)$, and thus that $W=F$ is free, as required. □

2. Main results

Theorem 2.1. *Let H be a finite-dimensional Hopf algebra over an algebraically closed field k. Let C be a simple subcoalgebra of H, and let G be a subgroup of $G(H)$ such that $gC=C$, all $g\in G$. Then C is free as a left kG-module. Moreover, for each subgroup G_2 of G which either is cyclic or which (in characteristic $p>0$) is a p-Sylow subgroup, C is isomorphic to a kG_2-module of the form* $\mathrm{End}_k(W)^*$, *where*

W is a free left kG_2-module. Thus the exponent of G and (in characteristic $p>0$) the p-part of $|G|$ divide n, where n^2 is the dimension of C.

Proof. Fix a minimal left ideal W of C^*. Since C is simple, there is an algebra isomorphism $L: C^* \to \mathrm{End}_k(W)$, given by $L_{c^*}(w) = c^*w$ for all $c^* \in C^*$, $w \in W$.

We shall use the letter L to denote various 'left multiplication' operators, we hope without causing any confusion.

For each $g \in G$, the map $L_g: C \to C$ is a coalgebra automorphism, so $(L_g)^*: C^* \to C^*$ is an algebra automorphism. It is given by $(L_g)^*(c^*) = c^*g$ for all $c^* \in C^*$. (Here c^*g is given by $\langle c^*g | c\rangle = \langle c^* | gc\rangle$ for all $c^* \in C^*$, $c \in C$.) The corresponding algebra automorphism of $\mathrm{End}_k(W)$ is then $L_{c^*} \mapsto L_{c^*g}$, all $c^* \in C^*$. By the Noether–Skolem Theorem, for each $g \in G$ there exists $U(g) \in \mathrm{GL}(W)$ with $L_{c^*g} = U(g)^{-1} L_{c^*} U(g)$, all $c^* \in C^*$. We select $U(1)$ to be id_W.

For each $g, h \in G$ we have $L_{(c^*g)h} = L_{c^*(gh)}$ for all $c^* \in C^*$. A simple computation leads to the conclusion that $U(g)U(h)U(gh)^{-1}$ lies in the center of $\mathrm{End}_k(W)$. Thus we have $U(g)U(h) = \alpha(g, h)U(gh)$ for some $\alpha(x, y) \in k^{\cdot}$, the multiplicative group of nonzero elements of k.

Thus $U: G \to \mathrm{GL}(W)$ is a projective representation of G. So $\alpha: G \times G \to k^{\cdot}$ is a cocycle – i.e., $\alpha \in Z^2(G, k^{\cdot})$.

Let us now restrict our attention to a fixed subgroup G_2 of G which either is cyclic or (in characteristic $p>0$) is a p-Sylow subgroup of G. It is proved in [6, Lemma 5.8.13] that in this case $H^2(G_2, k^{\cdot}) = 1$. Thus we can find $\beta: G_2 \to k^{\cdot}$ with $\beta(1) = 1$ and $\alpha(g, h) = \beta(g)\beta(h)\beta(gh)^{-1}$ for all $g, h \in G_2$. Define $U_2: G_2 \to \mathrm{GL}(W)$ by $U_2(g) = \beta(g)^{-1}U(g)$, all $g \in G_2$. Then U_2 is a linear representation of G_2. We shall write $gw = U_2(g)w$ for $g \in G_2$, $w \in W$, and consider W to be a left kG_2-module.

For $c^* \in C^*$, $g \in G_2$ we have $L_{c^*g} = U_2(g)^{-1} L_{c^*} U_2(g)$. Thus the right kG_2-module structure on $\mathrm{End}_k(W)$ which comes from W via the definition $(T \cdot g)(w) = g^{-1}T(gw)$ for $T \in \mathrm{End}_k(W)$, $g \in G$, $w \in W$ is the right kG_2-module structure transported from C^* by L. We have $C \cong \mathrm{End}_k(W)^*$ as a left kG_2-module.

We shall show that W is free as a left kG_2-module. This will allow us to complete the proof of the theorem, as follows. If $w_1, \dots, w_m$ is a kG_2-basis of W, then the elements $\{T_{g,i,j}: 1 \le i, j \le m\}$ of $\mathrm{End}_k(W)$ given by $(T_{g,i,j})(hw_r) = \delta_{g,h}\delta_{i,r}w_j$ form a kG_2-basis of $\mathrm{End}_k(W)$ as a right kG_2-module. Since kG_2 is a Frobenius algebra [1, Theorem 62.1], this yields that $C \cong \mathrm{End}_k(W)^*$ is free as a left kG_2-module. Then C is free as a left kG-module by Proposition 1.2.

The algebra isomorphism $L: C^* \to \mathrm{End}_k(W)$ gives rise to a sequence of algebra isomorphisms (which we shall explain) $\mathrm{Hom}_k(C, H) \cong C^* \otimes H \cong \mathrm{End}_k(W) \otimes H \cong \mathrm{End}_H(W \otimes H)$.

Here $\mathrm{Hom}_k(C, H)$ is the convolution algebra (see [11]). The map $C^* \otimes H \to \mathrm{Hom}_k(C, H)$ is given by $(c^* \otimes x)(c) = \langle c^* | c\rangle x$ for $c^* \in C^*$, $c \in C$, $x \in H$. The isomorphism from $C^* \otimes H$ to $\mathrm{End}_k(W) \otimes H$ is $L \otimes \mathrm{id}$. We are considering $W \otimes H$ as a right H-module via $(w \otimes a)b = w \otimes ab$ for $w \in W$, $a, b \in H$. The map $\mathrm{End}_k(W) \otimes H \to \mathrm{End}_H(W \otimes H)$ is given by $(T \otimes a)(w \otimes b) = T(w) \otimes ab$ for $T \in \mathrm{End}_k(W)$, $w \in W$, $a, b \in H$.

The coalgebra automorphism $L_g: C \to C$ also induces an algebra automorphism L_g^* of $\mathrm{Hom}_k(C, H)$, given by $L_g^*(T) = T \circ L_g$ for all $T \in \mathrm{Hom}_k(C, H)$. This corresponds, along our sequence, to the automorphism of $C^* \otimes H$ sending $c^* \otimes x$ to $c^*g \otimes x$ ($c^* \in C^*, x \in H$), the automorphism of $\mathrm{End}_k(W) \otimes H$ sending $L_{c^*} \otimes x$ to $L_{c^*g} \otimes x = U_2(g)^{-1} L_{c^*} U_2(g) \otimes x$ ($c^* \in C^*, x \in H$), and thus to the automorphism of $\mathrm{End}_H(W \otimes H)$ sending $T \in \mathrm{End}_H(W \otimes H)$ to $(U_2(g) \otimes \mathrm{id})^{-1} T(U_2(g) \otimes \mathrm{id})$.

Write '$\hat{\ }$' to denote extension by applying the functor $\otimes H$. Then we have that the automorphism L_g^* of $\mathrm{Hom}_k(C, H)$ corresponds to the automorphism of $\mathrm{End}_H(\hat{W})$ given by conjugation by $\hat{U}_2(g)$.

Now $\mathrm{Hom}_k(C, H)$ is a left kG_2-module via $(gT)(c) = gT(c)$ for $g \in G_2$, $T \in \mathrm{Hom}(C, H)$, $c \in C$. The corresponding kG_2-module structure on $C^* \otimes H$ is given by $g \cdot (c^* \otimes x) = c^* \otimes gx$ for $g \in G_2$, $c^* \in C^*$, $x \in H$. In $\mathrm{End}_H(\hat{W})$ this becomes $g \cdot T = (\mathrm{id} \otimes L_g)T$ for $T \in \mathrm{End}_H(\hat{W})$, where here $L_g: H \to H$.

Let $\iota: C \to H$ be the inclusion map. Let A denote the element of $\mathrm{End}_H(\hat{W})$ corresponding to ι under our sequence of isomorphisms.

The identity element of the algebra $\mathrm{Hom}_k(C, H)$ is the map $\eta\varepsilon: C \to H$ given by $(\eta\varepsilon)(c) = \varepsilon(c)1$. (Here $\varepsilon: C \to k$ is the counit of C, and $\eta: k \to H$ is the unit of H.) In $\mathrm{Hom}_k(C, H)$ we have an identity $(g \cdot \eta\varepsilon) * \iota = L_g^*(\iota)$; both maps send $c \in C$ to $gc \in H$. The corresponding identity in $\mathrm{End}_H(\hat{W})$ is $(g \cdot \mathrm{id}_{\hat{W}})A = \hat{U}_2(g)^{-1} A \hat{U}_2(g)$. Thus $A\hat{U}_2(g) = \hat{U}_2(g)(g \cdot \mathrm{id}_{\hat{W}})A = (U_2(g) \otimes \mathrm{id}_H)(\mathrm{id}_W \otimes L_g)A = (U_2(g) \otimes L_g)A$.

Now $\iota \in \mathrm{Hom}(C, H)$ is invertible, with inverse $S \circ \iota$ where S is the antipode of H. Thus A is invertible. Thus the equation $(U_2(g) \otimes L_g)A = A(U_2(g) \otimes \mathrm{id}_H)$ gives us that the kG_2-module $W \otimes H$ is isomorphic to the kG_2-module $W \otimes H_0$, where H_0 is the vector space H, given the trivial kG_2-module structure $gh = h$ for all $g \in G_2$, $h \in H$.

Thus $W \otimes H \cong W^{(t)}$ as kG_2-modules, where $t = \dim H$. Applying Proposition 1.3, we obtain that W is free as a left kG_2-module, and we are done. □

Remark. There exists an example [4] of an 8-dimensional Hopf algebra H over the complex numbers with $G = G(H) \cong Z_2 \times Z_2$ and containing a simple subcoalgebra C of dimension 4 with $gC = C$ for all $g \in G$. Thus, in the situation of Theorem 2.1, it is not true in general that $|G|$ divides n.

We now come to our main theorem.

Let H be a bialgebra over a field k, and let B be a sub-bialgebra. A left (H, B)-Hopf module is a left H-comodule M which is a left B-module in such a way that the comodule structure map $\omega: M \to H \otimes M$ is a B-module homomorphism. (Here B acts 'diagonally' on $H \otimes M$: $b \cdot (a \otimes m) = \sum b_{(1)} a \otimes b_{(2)} m$ for $b \in B$, $a \in H$, $m \in M$.) For example, any subcoalgebra C of H for which $BC \subseteq C$ is a left (H, B)-Hopf module. Our main result is the following:

Theorem 2.2. *Let H be a finite-dimensional Hopf algebra over a field k. Let G be a subgroup of $G(H)$, the grouplike elements of H. Then every left (H, kG)-Hopf module is free as a left kG-module.*

Proof. By [9, Proposition 5] or the discussion in [13], the result follows immediately from Theorem 2.1 if k has characteristic zero. So assume that k has characteristic $p>0$.

By [9, Proposition 1] it is enough to show that every finite-dimensional (H, kG)-Hopf module M is free. We may thus assume that k is algebraically closed. By [9, Proposition 1], it is enough to show that every M of the form $M=kG\cdot V$, where V is a simple left subcomodule of H, is free. Let C be the simple subcoalgebra of H for which $\Delta(V)\subseteq C\otimes V$. Let $G'=\{g\in G: gC=C\}$, and let $M'=\Delta^{-1}(C\otimes M)$. By [9, Proposition 2], M' is an (H, kG')-Hopf module, and if M' is a free left kG'-module, then M is a free left kG-module.

If $g'\in G'$ has order relatively prime to p, then M' is a free left $k\langle g'\rangle$-module by Theorem 2.1 and [9, Proposition 5]. Thus by Proposition 1.2, it is enough to show that M' is a free kP-module, where P is a p-Sylow subgroup of G'.

The comultiplication map $\Delta: M'\to C\otimes M'$ is a homomorphism of left kP-modules, with kP-module inverse $\varepsilon\otimes\mathrm{id}$. Thus M' is a direct summand of $C\otimes M'$ as a kP-module. By Theorem 2.1, C is a free kP-module. This gives us that $C\otimes M'$ is a free kP-module. Thus M' is a projective kP-module, hence free [2, Theorem 5.24] and we are done. □

3. Applications

Proposition 3.1. *Let H be a finite-dimensional Hopf algebra over a field k. Then*

(i) *The number of one-dimensional ideals of H divides* $\dim H$.

(ii) *The order of the antipode divides* $4\cdot\dim H$.

Proof. (i) By [7] the number of one-dimensional ideals of H is $|G(H^*)|$. Since H^* is a free $kG(H^*)$-module by Theorem 2.2, the result follows.

(ii) Radford showed in [7] that the order of the antipode divides $4\cdot\mathrm{lcm}\{|G(H^*)|, |G(H)|\}$. By Theorem 2.2, both $|G(H^*)|$ and $|G(H)|$ divide $\dim H$, and we are done.

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