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Continuous extensions of functions defined on subsets of products ☆

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Respectfully dedicated to Dikran Dikranjan, mathematician and educator, on the occasion of his 60th birthday

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ABSTRACT

A subset Y of a space X is *G_δ-dense* if it intersects every nonempty G_δ-set. The *G_δ-closure* of Y in X is the largest subspace of X in which Y is G_δ-dense.
The space X has a *regular G_δ-diagonal* if the diagonal of X is the intersection of countably many regular-closed subsets of $X \times X$.
Consider now these results: (a) (N. Noble, 1972 [18]) every G_δ-dense subspace in a product of separable metric spaces is C-embedded; (b) (M. Ulmer, 1970 [22], 1973 [23]) every Σ-product in a product of first-countable spaces is C-embedded; (c) (R. Pol and E. Pol, 1976 [20], also A.V. Arhangel'skiĭ, 2000 [3]; as corollaries of more general theorems), every dense subset of a product of completely regular, first-countable spaces is C-embedded in its G_δ-closure.
The present authors' Theorem 3.10 concerns the continuous extension of functions defined on subsets of product spaces with the κ-box topology. Here is the case $\kappa = \omega$ of Theorem 3.10, which simultaneously generalizes the above-mentioned results.

Theorem. *Let $\{X_i: i \in I\}$ be a set of T_1-spaces, and let Y be dense in an open subspace of $X_I := \prod_{i\in I} X_i$. If $\chi(q_i, X_i) \leqslant \omega$ for every $i \in I$ and every q in the G_δ-closure of Y in X_I, then for every regular space Z with a regular G_δ-diagonal, every continuous function $f : Y \to Z$ extends continuously over the G_δ-closure of Y in X_I.*

Some examples are cited to show that the hypothesis $\chi(q_i, X_i) \leqslant \omega$ cannot be replaced by the weaker hypothesis $\psi(q_i, X_i) \leqslant \omega$.

1. Notation and terminology

Topological spaces considered here are not subjected to any standing separation properties. Additional hypotheses are imposed as required. Throughout this paper, ω is the least infinite cardinal, κ and α are infinite cardinals. For I a set we define $[I]^{<\kappa} := \{J \subseteq I: |J| < \kappa\}$, the symbol $[I]^{\leqslant\kappa}$ is defined analogously. For X a space and $x \in X$, a set $U \subseteq X$ is a *neighborhood* of x in X if x is in the interior of U in X. For $A \subseteq X$ we denote by $\mathcal{N}_X(A)$, or simply by $\mathcal{N}(A)$ when ambiguity is unlikely, the set of open sets in X containing A. A point $x \in X$ is a *$P(\kappa)$-point* of X if $\bigcap \mathcal{V}$ is a neighborhood of x whenever $\mathcal{V} \subseteq \mathcal{N}(x)$ and $|\mathcal{V}| < \kappa$; X is a *$P(\kappa)$-space* provided each point $x \in X$ is a $P(\kappa)$-point. Clearly, every topological space is a $P(\omega)$-space. The $P(\omega^+)$-spaces are often called *P-spaces* (cf. [14] and sources cited there).

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For X a space, $x \in X$ and $A \subseteq X$, x belongs to the *G_κ-closure* of A in X if $(\bigcap \mathcal{V}) \cap A \neq \emptyset$ whenever $\mathcal{V} \subseteq \mathcal{N}(x)$ and $|\mathcal{V}| < \kappa$. A set $A \subseteq X$ is a *G_κ-set* [respectively, a *$\overline{G_\kappa}$-set*] in X if there exists $\mathcal{V} \subseteq \mathcal{N}_X(A)$ such that $|\mathcal{V}| < \kappa$ and $A = \bigcap \mathcal{V}$ [respectively, $A = \bigcap\{\overline{V}\colon V \in \mathcal{V}\}$]. (Thus, the familiar G_δ-sets are exactly the G_{ω^+}-sets.) V is a *G_κ-neighborhood of* x if there exists a G_κ-set U such that $x \in U \subseteq V$.

The symbol $\chi(x, X)$ denotes the character (i.e., the local weight) of the point x in the space X; $\chi(X) := \sup\{\chi(x, X)\colon x \in X\}$; for $x \in X$ with X a T_1-space, $\psi(x, X)$ denotes the pseudocharacter of the point x in X; and $\psi(X) := \sup\{\psi(x, X)\colon x \in X\}$.

For spaces X and Z and $Y \subseteq Z$, the symbol $C(Y, Z)$ denotes the set of all continuous functions $f : Y \to Z$. The set $C(Y, \mathbb{R})$ is denoted by $C(Y)$. The subspace Y of X is *$C(Z)$-embedded* in X provided each function $f \in C(Y, Z)$ extends continuously over X. When $Z = \mathbb{R}$, Y is said to be *C-embedded*.

Below we use the simple fact (which we will not mention again explicitly) that when Y is dense in X and Z is a regular T_1-space, a function $f \in C(Y, Z)$ extends continuously over X if and only if f extends continuously to each point of $X\backslash Y$ (restated: Y is $C(Z)$-embedded in X if and only if Y is $C(Z)$-embedded in $Y \cup \{q\}$ for each $q \in X\backslash Y$); in this connection see [5,17].

For a set $\{X_i\colon i \in I\}$ of sets and $J \subseteq I$, we write $X_J := \prod_{i\in J} X_i$; and for every *generalized rectangle* $A = \prod_{i\in I} A_i \subseteq X_I$ the *restriction set of* A, denoted $R(A)$, is the set $R(A) = \{i \in I\colon A_i \neq X_i\}$. When each $X_i = (X_i, \mathcal{T}_i)$ is a space, the symbol $(X_I)_\kappa$ denotes X_I with the *κ-box topology*; this is the topology for which $\{U\colon U = \prod_{i\in I} U_i,\ U_i \in \mathcal{T}_i,\ |R(U)| < \kappa\}$ is a base. Thus the ω-box topology on X_I is the usual product topology. We note that even when κ is regular, the intersection of fewer than κ-many sets, each open in $(X_I)_\kappa$, may fail to be open in $(X_I)_\kappa$.

For X_I as above and $q \in X_I$, the *Σ-product in X_I based at q* is the set

$$\Sigma(q) := \bigl\{x \in X_I\colon |i \in I\colon x_i \neq q_i| \leqslant \omega\bigr\}.$$

For additional topological definitions not given here see [13,14], or [10].

2. Introduction

The problem of determining conditions on a space X and a proper, dense subspace Y under which Y is C-embedded in X has generated considerable attention in the literature. H. Corson [11], I. Glicksberg [15], R. Engelking [12], N. Noble [18], N. Noble and M. Ulmer [19], M. Ulmer [22,23], M. Hušek [16], R. Pol and E. Pol [20], A.V. Arhangel'skiĭ [3] and many others have achieved nontrivial results for the case where X is a product space. As is indicated in [10,8], and [7], many of their results admit generalizations to product spaces with the κ-box topology.

The present paper continues that initiative. We remark on two features of our principal result, Theorem 3.10: (a) the basic case $\kappa = \omega$, subsumes simultaneously the results of N. Noble [18], M. Ulmer [23], R. Pol and E. Pol [20], and A.V. Arhangel'skiĭ [3] cited in the abstract; and (b) unlike those results it is *local* in flavor in the sense that it gives conditions on points $q \in X_I\backslash Y$ sufficient to ensure that functions $f \in C(Y, Z)$ extend continuously to q-conditions which in natural circumstances may fail for other points $q' \in X_I\backslash Y$ (to which $f \in C(Y, Z)$ may fail to extend).

So far as we know, Ulmer was the first to show [22,23] that not every Σ-product in every (Tychonoff) product space is C-embedded. Modifying his examples, we showed in [8] that in a (suitably constructed) product space X_I, even a G_δ-dense set of the form $X_I\backslash\{q\}$ need not be C-embedded, even when $\psi(X_i) \leqslant \omega$ for each $i \in I$.

3. Main results

Definition 3.1. Let $\kappa \geqslant \omega$ and let Z be a space. Then

(a) $\Delta(Z) := \{(z, z) \in Z \times Z\colon z \in Z\}$ is the *diagonal* of Z; and
(b) Z has a *G_κ-diagonal* [respectively, a *$\overline{G_\kappa}$-diagonal*] if $\Delta(Z)$ is a G_κ-set [respectively, a $\overline{G_\kappa}$-set] in $Z \times Z$.

Remark 3.2. (a) Spaces with a G_{ω^+}-diagonal are those with a *G_δ-diagonal*. The spaces with a $\overline{G_{\omega^+}}$-diagonal are also called spaces with *regular G_δ-diagonal*.

(b) Clearly every space Z with a $\overline{G_\kappa}$-diagonal has a G_κ-diagonal but the converse is not true in general. Indeed D. Shakhmatov [21] has shown that there are Tychonoff c.c.c. spaces of arbitrarily large cardinality with a G_{ω^+}-diagonal, while R.Z. Buzyakova [6], answering a question of A.V. Arhangel'skiĭ [2], showed that every regular c.c.c. space with a $\overline{G_{\omega^+}}$-diagonal has cardinality at most $\mathfrak{c}$.

We begin with two lemmas.

Lemma 3.3. *Let α be an infinite cardinal and let X, Y and Z be spaces such that $Y \subseteq X$ and Z has a G_{α^+}-diagonal* [*respectively, a $\overline{G_{\alpha^+}}$-diagonal*]*—say $\Delta(Z) = \bigcap\{O_\eta\colon \eta < \alpha\}$* [*respectively, $\Delta(Z) = \bigcap\{\overline{O_\eta}\colon \eta < \alpha\}$*] *with each $O_\eta \in \mathcal{N}(\Delta(Z))$. Let $f \in C(Y, Z)$ and $q \in X \setminus Y$ belong to the G_{α^+}-closure of Y in X. If for each $\eta < \alpha$ there is $U_\eta \in \mathcal{N}(q)$ such that $(f(y), f(y')) \in O_\eta$* [*respectively, $(f(y), f(y')) \in \overline{O_\eta}$*] *whenever $y, y' \in U_\eta \cap Y$, then the set $U := \bigcap_{\eta<\alpha} U_\eta$ is a G_{α^+}-neighborhood of q in X such that f is constant on $U \cap Y$.*

Proof. Suppose there are $y, y' \in Y \cap U$ such that $f(y) = z \neq z' = f(y')$. Since $(z, z') \notin \Delta(Z)$ there is $\eta < \alpha^+$ such that $(z, z') \notin O_\eta$ [respectively, $(z, z') \notin \overline{O_\eta}$]. But since $y, y' \in Y \cap U_{O_\eta}$ we have $(z, z') = (f(y), f(y')) \in O_\eta$ [respectively, $(z, z') = (f(y), f(y')) \in \overline{O}_\eta$], a contradiction. □

Lemma 3.4. *Let α be an infinite cardinal and let X, Y and Z be spaces such that $Y \subseteq X$. Let $f \in C(Y, Z)$ and $q \in X \setminus Y$ belong to the G_{α^+}-closure of Y in X and there are a G_{α^+}-neighborhood U of q in X and $z \in Z$ such that $f(y) = z$ for all $y \in U \cap Y$. If X is a T_1-space and for each $O \in \mathcal{N}(\Delta(Z))$ there is $U_O \in \mathcal{N}(q)$ such that $(f(y), f(y')) \in O$ whenever $y, y' \in U_O \cap Y$, then $\overline{f} : Y \cup \{q\} \to Z$ defined by the rule*

$$\overline{f}|Y = f, \qquad \overline{f}(q) = z$$

is continuous.

Proof. Since Y is open in $Y \cup \{q\}$, the function $\overline{f}$ remains continuous at each $y \in Y$. To show that $\overline{f}$ is continuous at q it is enough to show that for every $W \in \mathcal{N}(z)$ there is $O \in \mathcal{N}(\Delta(Z))$ such that $\overline{f}[U_O \cap (Y \cup \{q\})] \subseteq W$. If that fails then for every $O \in \mathcal{N}(\Delta(Z))$ there is $y_O \in U_O \cap Y$ such that $f(y_O) \notin W$. Then

$$(W \times W) \cap \overline{\{(z, f(y_O)): O \in \mathcal{N}(\Delta(Z))\}} = \emptyset,$$

hence

$$\Delta(Z) \cap \overline{\{(z, f(y_O)): O \in \mathcal{N}(\Delta(Z))\}} = \emptyset$$

and therefore with

$$\widetilde{O} := (Z \times Z) \setminus \overline{\{(z, f(y_O)): O \in \mathcal{N}(\Delta(Z))\}}$$

we have $\widetilde{O} \in \mathcal{N}(\Delta(Z))$ and

$$\widetilde{O} \cap \overline{\{(z, f(y_O)): O \in \mathcal{N}(\Delta(Z))\}} = \emptyset.$$

Since $U_{\widetilde{O}} \in \mathcal{N}_X(q)$ the set $U_{\widetilde{O}} \cap U$ is a G_{α^+}-neighborhood of q in X, so there is $\widetilde{y} \in U_{\widetilde{O}} \cap U \cap Y$. Then $f(\widetilde{y}) = z$ and we have the contradiction

$$(f(\widetilde{y}), f(y_{\widetilde{O}})) = (z, f(y_{\widetilde{O}})) \in \widetilde{O} \cap \overline{\{(z, f(y_O)): O \in \mathcal{N}(\Delta(Z))\}} = \emptyset. \qquad \square$$

The following definition enunciates a strictly set-theoretic (non-topological) condition. The concept is used in Theorem 3.8 where we find conditions sufficient to ensure that the hypotheses of Lemma 3.4 are satisfied.

Definition 3.5. Let $\alpha \geqslant \omega$ and let $\{X_i: i \in I\}$ be a family of sets and $Y \subseteq X_I$. Then Y *α-duplicates* $q \in X_I$ if for every $J \in [I]^{<\alpha}$ there exists a point $p \in Y$ such that $p_J = q_J$.

Remark 3.6. (a) To help the reader fix ideas, we note that if in Definition 3.5 X_I is a space which is the product of discrete spaces then Y α-duplicates q if and only if q belongs to the G_α-closure of Y in X_I. In what follows we are going to use the following more general fact, the elementary proof of which we leave to the reader: Let $\kappa \leqslant \alpha^+$ and q be a point in X_I such that $\psi(q_i, X_i) \leqslant \alpha$ for each $i \in I$. Then Y α^+-duplicates q in X_I if and only if q belongs to the G_{α^+}-closure of Y in $(X_I)_\kappa$.

(b) The reader familiar with the works [8, 2.3], [9, 2.1], [7, 2.2] will observe structural parallels between the proofs given there and in Theorem 3.7 below.

Theorem 3.7. *Let $\omega \leqslant \kappa \leqslant \alpha$ with either $\kappa < \alpha$ or α regular, let $\{X_i: i \in I\}$ be a set of spaces, and let Y be dense in an open subset U of $(X_I)_\kappa$. Let $q \in X_I \setminus Y$ be a point such that Y α^+-duplicates q in X_I. Then for every space Z and every $O \in \mathcal{N}_{Z \times Z}(\Delta(Z))$ and every $f \in C(Y, Z)$, there is $J \in [I]^{<\alpha}$ such that $(f(y), f(y')) \in \overline{O}$ whenever $y, y' \in Y$ satisfy $y_J = y'_J = q_J$.*

Proof. We suppose the result fails.

Let $Y \subseteq U \subseteq \overline{Y}$, U open in $(X_I)_\kappa$. For each $\xi < \alpha$ we define $y(\xi), y'(\xi) \in Y$, disjoint basic open neighborhoods $U(\xi) \subseteq U$ and $V(\xi) \subseteq U$ in $(X_I)_\kappa$ of $y(\xi)$ and $y'(\xi)$, respectively, and $J(\xi), A(\xi) \subseteq I$ such that:

(i) $(f(y), f(y')) \notin \overline{O}$ if $y \in U(\xi) \cap Y$, $y' \in V(\xi) \cap Y$;
(ii) $A(\xi) := \{i \in R(U(\xi)) \cup R(V(\xi)): y(\xi)_i \neq q_i \text{ or } y'(\xi)_i \neq q_i\}$;
(iii) $U(\xi)_i = V(\xi)_i$ if $i \in I \setminus A(\xi)$;
(iv) $y(\xi)_i = y'(\xi)_i = q_i$ for $i \in J(\xi)$; and with
(v) $J(0) = \emptyset$, $J(\xi) = \bigcup_{\eta < \xi} A(\eta)$ for $0 < \xi < \alpha$.

To begin, we choose $y(0) \in Y$ and $y'(0) \in Y$ such that $(f(y(0)), f(y'(0))) \notin \overline{O}$, and open neighborhoods $W_y(0)$ and $W_{y'}(0)$ in Z of $f(y(0))$ and $f(y'(0))$, respectively, such that $(W_y(0) \times W_{y'}(0)) \cap \overline{O} = \emptyset$. Then $(W_y(0) \times W_{y'}(0)) \cap \Delta_Z = \emptyset$, so $W_y(0) \cap W_{y'}(0) = \emptyset$. It follows from the continuity of f that there are disjoint, basic open neighborhoods $\widetilde{U(0)} \subseteq U$ and $\widetilde{V(0)} \subseteq U$ in $(X_I)_\kappa$ of $y(0)$ and $y'(0)$, respectively, such that $(f(y), f(y')) \notin \overline{O}$ for all $y \in \widetilde{U(0)} \cap Y$ and $y' \in \widetilde{V(0)} \cap Y$. Then, define $A(0) := \{i \in R(\widetilde{U(0)}) \cup R(\widetilde{V(0)})\colon\ y(0)_i \neq q_i$ or $y'(0)_i \neq q_i\}$ and define (basic open) neighborhoods $U(0)$ and $V(0)$ in $(X_I)_\kappa$ of $y(0)$ and $y'(0)$, respectively, as follows:

$$U(0)_i = V(0)_i = X_i \quad \text{if } i \in I \backslash \big(R\big(\widetilde{U(0)}\big) \cup R\big(\widetilde{V(0)}\big)\big);$$
$$U(0)_i = V(0)_i = \widetilde{U(0)}_i \cap \widetilde{V(0)}_i \quad \text{if } i \in \big(R\big(\widetilde{U(0)}\big) \cup R\big(\widetilde{V(0)}\big)\big) \backslash A(0); \quad \text{and}$$
$$U(0)_i = \widetilde{U(0)}_i, \qquad V(0)_i = \widetilde{V(0)}_i \quad \text{if } i \in A(0).$$

Then $U(0) \subseteq U$ and $V(0) \subseteq U$, and (i)–(v) hold for $\xi = 0$.

Suppose now that $0 < \xi < \alpha$ and that $y(\eta), y'(\eta) \in Y$, $U(\eta) \subseteq U$, $V(\eta) \subseteq U$, and $A(\eta), J(\eta) \subseteq I$ have been defined for $\eta < \xi$ satisfying (the analogues of) (i)–(v). Since $J(\xi)$, defined by (v), satisfies $|J(\xi)| < \alpha$, there are $y(\xi)$ and $y'(\xi)$ in Y such that (iv) holds and $(f(y(\xi)), f(y'(\xi))) \notin \overline{O}$, and open neighborhoods $W_y(\xi)$ and $W_{y'}(\xi)$ in Z of $f(y(\xi))$ and $f(y'(\xi))$, respectively, such that $(W_y(\xi) \times W_{y'}(\xi)) \cap \overline{O} = \emptyset$. Then $(W_y(\xi) \times W_{y'}(\xi)) \cap \Delta_Z = \emptyset$, so $W_y(\xi) \cap W_{y'}(\xi) = \emptyset$. It follows from the continuity of f that there are disjoint, basic open neighborhoods $\widetilde{U(\xi)} \subseteq U$ and $\widetilde{V(\xi)} \subseteq U$ in $(X_I)_\kappa$ of $y(\xi)$ and $y'(\xi)$, respectively, such that $(f(y), f(y')) \notin \overline{O}$ for all $y \in \widetilde{U(\xi)} \cap Y$, $y' \in \widetilde{V(\xi)} \cap Y$. Then, define $A(\xi) := \{i \in R(\widetilde{U(\xi)}) \cup R(\widetilde{V(\xi)})\colon\ y(\xi)_i \neq q_i$ or $y'(\xi)_i \neq q_i\}$ and define (basic open) neighborhoods $U(\xi)$ and $V(\xi)$ in $(X_I)_\kappa$ of $y(\xi)$ and $y'(\xi)$, respectively, as follows:

$$U(\xi)_i = V(\xi)_i = X_i \quad \text{if } i \in I \backslash \big(R\big(\widetilde{U(\xi)}\big) \cup R\big(\widetilde{V(\xi)}\big)\big);$$
$$U(\xi)_i = V(\xi)_i = \widetilde{U(\xi)}_i \cap \widetilde{V(\xi)}_i \quad \text{if } i \in \big(R\big(\widetilde{U(\xi)}\big) \cup R\big(\widetilde{V(\xi)}\big)\big) \backslash A(\xi); \quad \text{and}$$
$$U(\xi)_i = \widetilde{U(\xi)}_i, \qquad V(\xi)_i = \widetilde{V(\xi)}_i \quad \text{if } i \in A(\xi).$$

Then $U(\xi) \subseteq U$ and $V(\xi) \subseteq U$, and (i)–(v) hold. The recursive definitions are complete.

We note that if $\eta < \xi < \alpha$ and $i \in A(\eta)$ then $y(\xi)_i = y'(\xi)_i = q_i$ and hence $i \notin A(\xi)$. That is: the sets $A(\xi)$ $(\xi < \alpha)$ are pairwise disjoint.

Let $J(\alpha) := \bigcup_{\eta<\alpha} A(\eta)$. Since $|J(\alpha)| = \alpha$ and Y α^+-duplicates q in X_I, there is $\bar{p} \in Y \subseteq U$ such that $q_{J(\alpha)} = \bar{p}_{J(\alpha)}$. Notice that each basic open neighborhood $W \subseteq U$ of $\bar{p}$ in $(X_I)_\kappa$ satisfies $|\{\xi < \alpha\colon\ W \cap U(\xi) \neq \emptyset\}| \geqslant \kappa$. Fix such W and choose $\bar{\xi} < \alpha$ such that $W \cap U(\bar{\xi}) \neq \emptyset$ and no $i \in R(W)$ is in $A(\bar{\xi})$. (This is possible since $|R(W)| < \kappa$ and each $i \in R(W)$ is in at most one of the sets $A(\xi)$.) For each such $\bar{\xi}$ by (iii) we have $U(\bar{\xi})_i = V(\bar{\xi})_i$ for all $i \in R(W)$, so also $W \cap V(\bar{\xi}) \neq \emptyset$.

Since Y is dense in $U \subseteq (X_I)_\kappa$, the previous paragraph shows this: For each neighborhood W in $U \subseteq (X_I)_\kappa$ of $\bar{p}$ there is $\bar{\xi}$ such that $W \cap U(\bar{\xi}) \cap Y \neq \emptyset$ and $W \cap V(\bar{\xi}) \cap Y \neq \emptyset$. Let $G \in \mathcal{N}_Z(f(\bar{p}))$ satisfy $G \times G \subseteq O$. Since f is continuous at $\bar{p}$ there is a basic open neighborhood $W' \subseteq U$ of $\bar{p}$ such that $f[W' \cap Y] \subseteq G$. Then there is $\bar{\xi}'$ such that $W' \cap U(\bar{\xi}') \cap Y \neq \emptyset$ and $W' \cap V(\bar{\xi}') \cap Y \neq \emptyset$, and with $y \in W' \cap U(\bar{\xi}') \cap Y$ and $y' \in W' \cap V(\bar{\xi}') \cap Y$ we have $(f(y), f(y')) \in G \times G \subseteq O$. This contradicts (i), completing the proof. □

Theorem 3.8. *Let $\omega \leqslant \kappa \leqslant \alpha$ with either $\kappa < \alpha$ or α regular, let $\{X_i\colon\ i \in I\}$ be a set of spaces, and let Y be dense in an open subset U of $(X_I)_\kappa$. Let $q \in X_I \setminus Y$ be a point such that Y α^+-duplicates q in X_I, and Z be a space with $\overline{G_{\alpha^+}}$-diagonal. Then for every $f \in C(Y, Z)$, there is $J \in [I]^{\leqslant\alpha}$ such that $f(y) = f(y')$ whenever $y, y' \in Y$ are such that $y_J = y'_J = q_J$.*

Proof. Let $\{O_\eta\colon\ \eta < \alpha\} \subseteq \mathcal{N}_{Z\times Z}(\Delta(Z))$ satisfy $\Delta(Z) = \bigcap\{\overline{O}_\eta\colon\ \eta < \alpha\}$. For each $\eta < \alpha$ there is (by Theorem 3.7) $J_\eta \in [I]^{<\alpha}$ such that $(f(y), f(y')) \in \overline{O}_\eta$ whenever $y, y' \in Y$ satisfy $y_{J_\eta} = y'_{J_\eta} = q_{J_\eta}$. We set $J := \bigcup\{J_\eta\colon\ \eta < \alpha\}$. Then $|J| \leqslant \alpha$, and $f(y) = f(y')$ whenever $y, y' \in Y$ with $y_J = y'_J = q_J$. □

To prove our principal result, Theorem 3.10, we need a simple observation.

Lemma 3.9. *Let X be a space, p be a $P(\kappa)$-point in X and $\chi(p, X) \leqslant \kappa$. Then there is a local base $\{U_\alpha\colon\ \alpha < \kappa\}$ at p such that $U_{\alpha'}(p) \subseteq U_\alpha(p)$ whenever $\alpha < \alpha' < \kappa$.*

Proof. Let $\mathcal{V} = \{V_\beta\colon\ \beta < \chi(p)\}$ be a base at p. If $\chi(p) < \kappa$ we take $U_\alpha(p) := \bigcap \mathcal{V}$ for all $\alpha < \kappa$. If $\chi(p) = \kappa$ we set $U_0 = V_0$ and recursively, if $\alpha < \chi(p)$ and U_β has been defined for all $\beta < \alpha$, we choose β' so that $V_{\beta'} \subseteq V_\alpha \cap (\bigcap_{\gamma<\beta} U_\gamma)$ and we set $U_\alpha := V_{\beta'}$. □

As indicated in our introduction, it is a feature of the following theorem that the conditions shown sufficient to ensure the extension of certain continuous functions are local at the hypothesized point q.

Theorem 3.10. *Let $\omega \leqslant \kappa \leqslant \alpha$ with either $\kappa < \alpha$ or α regular, let $\{X_i: i \in I\}$ be a set of T_1-spaces, and let Y be dense in an open subset of $(X_I)_\kappa$. Let $q \in X_I \setminus Y$ be a point from the G_{α^+}-closure of Y in $(X_I)_\kappa$ such that q_i is a $P(\alpha)$-point in X_i with $\chi(q_i, X_i) \leqslant \alpha$ for each $i \in I$. Then Y is $C(Z)$-embedded in $Y \cup \{q\}$ for each regular space Z with a $\overline{G_{\alpha^+}}$-diagonal.*

Proof. Let $Y \subseteq U \subseteq \overline{Y}$, U open in $(X_I)_\kappa$, Z be as hypothesized and $f \in C(Y, Z)$.

According to Remark 3.6(a), Y α^+-duplicates q in X_I. Hence it follows from Theorem 3.8 that there exist $z \in Z$ and $J \in [I]^{\leqslant\alpha}$ such that $f(y) = z$ for all $y \in Y$ satisfying $y_J = q_J$. We define $\overline{f} : Y \cup \{q\} \to Z$ by the rule

$$\overline{f}|Y = f, \qquad \overline{f}(q) = z.$$

We must show $\overline{f} \in C(Y \cup \{q\}, Z)$. Since Y is open in $Y \cup \{q\}$, the function $\overline{f}$ remains continuous at each $y \in Y$.

If $\overline{f}$ is not continuous at q then from Lemma 3.4 it follows that there is $O \in \mathcal{N}_{Z\times Z}(\Delta(Z))$ such that for every $U_q \in \mathcal{N}_{X_I}(q)$ there exist points $y_{U_q}, y'_{U_q} \in Y \cap U_q$ such that $(f(y_{U_q}), f(y'_{U_q})) \notin O$. Let V be an open neighborhood of z such that $\overline{V} \times \overline{V} \subseteq O$ (since Z is regular such neighborhood V exists). It follows from the continuity of f at y_{U_q} and y'_{U_q} that there are basic open neighborhoods $V_{y_{U_q}}$ of y_{U_q} and $V_{y'_{U_q}}$ of y'_{U_q} such that $V_{y_{U_q}} \subseteq U \cap U_q$, $V_{y'_{U_q}} \subseteq U \cap U_q$, and such that $(f(y), f(y')) \notin \overline{V} \times \overline{V}$ whenever $y \in V_{y_{U_q}} \cap Y$ and $y' \in V_{y'_{U_q}} \cap Y$.

It follows from the hypotheses that for $i \in I$ there is a local base $\{U_\beta(q_i): \beta < \alpha\}$ at q_i such that $U_{\beta'}(q_i) \subseteq U_\beta(q_i)$ whenever $\beta < \beta' < \alpha$ (see Lemma 3.9). (To avoid ambiguity in this choice we assume without loss of generality that $X_{i'} \cap X_i = \emptyset$ for $i, i' \in I$ with $i \neq i'$.)

For $\beta < \alpha$ we define $J(\beta)$, $U(\beta)$, $y(\beta)$, and $y'(\beta)$ as follows:

(i) $J(0) = \emptyset$;
(ii) $U(0) = X_I$;
(iii) $y(0) = y_{U(0)}$, $y'(0) = y'_{U(0)}$;
(iv) $J(\beta + 1) = J(\beta) \cup R(V_{y_{U(\beta)}}) \cup R(V_{y'_{U(\beta)}})$;
(v) $J(\beta) = \cup_{\gamma<\beta} J(\gamma)$ for β limit ordinal;
(vi) $U(\beta) = \{z \in X_I: z_i \in U_\beta(q_i) \text{ for } i \in J(\beta)\}$; and
(vii) $y(\beta) = y_{U(\beta)}$, $y'(\beta) = y'_{U(\beta)}$.

For every β we have $|J(\beta)| < \alpha$ since $\kappa < \alpha$ or α is regular. Hence $U(\beta)$ is a G_α-neighborhood of q in $(X_I)_\kappa$. Let $J_1 = \bigcup_{\beta<\alpha} J(\beta)$ and $J_2 = J \cup J_1$. Let $U'(\beta) = \{z \in X_I: z_i \in U_\beta(q_i), i \in J_2\}$ for $\beta < \alpha$. The set $U' := \bigcap_{\beta<\alpha} U'(\beta)$ is a nonempty G_{α^+}-set in X_I such that $q \in U'$ and $|R(U')| < \alpha^+$, so (again by Remark 3.6(a)) there is $w \in Y \cap U'$ with $w_i = q_i$ for each $i \in J_2$. Hence we have $w_J = q_J$ and therefore $f(w) = z$. Since f is continuous at w there is a basic open neighborhood $W \subseteq U$ of w such that $f[Y \cap W] \subseteq V$. Let $\beta < \alpha$ be such that $\pi_{J_1}[U(\beta)] \subseteq \pi_{J_1}[W]$ (such β exists since $\pi_{J_1}(w) = \pi_{J_1}(q)$ and $\kappa < \alpha$ or α is regular). Then $\pi_{J_1}[V_{y_{U(\beta)}}] \subseteq \pi_{J_1}[W]$, $\pi_{J_1}[V_{y'_{U(\beta)}}] \subseteq \pi_{J_1}[W]$ and since $R(V_{y_{U(\beta)}}) \subseteq J_1$ and $R(V_{y'_{U(\beta)}}) \subseteq J_1$ we have $W \cap V_{y_{U(\beta)}} \neq \emptyset$ and $W \cap V_{y'_{U(\beta)}} \neq \emptyset$. Therefore there exist $y \in W \cap V_{y_{U(\beta)}} \cap Y$ and $y' \in W \cap V_{y'_{U(\beta)}} \cap Y$. It follows from the choice of $V_{y_{U(\beta)}}$ and $V_{y'_{U(\beta)}}$ that $(f(y), f(y')) \notin \overline{V} \times \overline{V}$ and at the same time $(f(y), f(y')) \in V \times V \subseteq O$. This contradiction completes the proof. □

4. Corollaries of Theorem 3.10

In this section we state some special cases of Theorem 3.10 and some results that follow directly from it.

Theorem 4.1. *Let $\omega \leqslant \kappa \leqslant \alpha$ with either $\kappa < \alpha$ or α regular. Let $\{X_i: i \in I\}$ be a set of T_1-spaces, and let Y be dense in an open subset of $(X_I)_\kappa$. If every $q \in X_I \setminus Y$ in the G_{α^+}-closure of Y in $(X_I)_\kappa$ satisfies*

$$i \in I \quad \Rightarrow \quad \chi(q_i, X_i) \leqslant \alpha \quad \text{and} \quad q_i \text{ is a} P(\alpha)\text{-point in } X_i,$$

then the space Y is $C(Z)$-embedded in its G_{α^+}-closure in $(X_I)_\kappa$ for each regular space Z with a $\overline{G_{\alpha^+}}$-diagonal.

The countable cases of Theorem 3.10 and Theorem 4.1 are the following:

Theorem 4.2. *Let $\{X_i: i \in I\}$ be a set of T_1-spaces, let Y be dense in some open subset of X_I, and let $q \in X_I \setminus Y$ belong to the G_δ-closure of Y and be such that $\chi(q_i, X_i) \leqslant \omega$ for every $i \in I$. Then Y is $C(Z)$-embedded in $Y \cup \{q\}$ for every regular T_1-space Z with a regular G_δ-diagonal.*

Theorem 4.3. *Let $\{X_i: i \in I\}$ be a set of T_1-spaces and let Y be dense in an open subset of X_I. If every $q \in X_I \setminus Y$ in the G_δ-closure of Y in X_I satisfies $\chi(q_i, X_i) \leqslant \omega$ for every $i \in I$, then Y is $C(Z)$-embedded in its G_δ-closure in X_I for every regular space Z with a regular G_δ-diagonal.*

Let X be a space with a subspace Y such that, for a family $\{Z_j\colon j \in J\}$ of spaces, Y is $C(Z_j)$-embedded in X for each j. If Z is a closed subspace of the product space $\prod_{j\in J} Z_j$, then Y is also $C(Z)$-embedded in X. (Indeed for $f \in C(Y, Z)$ we have $\pi_j \circ f \in C(Y, Z_j)$ for each j, and then the hypothesized family $\{\overline{\pi_j \circ f}\colon j \in J\}$ of continuous extensions furnishes $\overline{f}$ such that $f \subseteq \overline{f} \in C(X, \prod_{j\in J} Z_j)$, with $\overline{f}[X] \subseteq Z$ since Z is closed in $\prod_{j\in J} Z_j$.)

Familiar applications of this argument show (a) if Y is C-embedded in X then Y is $C(Z)$-embedded in X for each space Z closed in a space of the form $\mathbb{R}^J$ (these are the so-called *realcompact* spaces); (b) if Y is $C(M)$-embedded in X for each metrizable space M then Y is $C(Z)$-embedded in X for each space Z closed in a product of metrizable spaces (such spaces are called *topologically complete* or, by some authors, *Dieudonné-complete*). For alternative definitions and characterizations of realcompact and of topologically complete spaces, see [14].

For the case when Z is a metric space, we have the following:

Theorem 4.4. *Let $\kappa \geqslant \omega$ be regular, $\{X_i\colon i \in I\}$ be a set of T_1-spaces and let Y be dense in an open subset of $(X_I)_\kappa$. Let $q \in X_I \setminus Y$ be a point in the G_{κ^+}-closure of Y in $(X_I)_\kappa$ such that*

$$i \in I \quad \Rightarrow \quad \chi(q_i, X_i) \leqslant \kappa \quad \text{and} \quad q_i \text{ is a } P(\kappa)\text{-point in } X_i.$$

Then Y is $C(Z)$-embedded in $Y \cup \{q\}$ for every metric space Z, hence for every topologically complete space Z.

Theorem 4.5. *Let $\kappa \geqslant \omega$ be regular, $\{X_i\colon i \in I\}$ be a set of T_1-spaces and let Y be dense in $(X_I)_\kappa$. If every $q \in X_I \setminus Y$ in the G_{κ^+}-closure of Y in $(X_I)_\kappa$ satisfies*

$$i \in I \quad \Rightarrow \quad \chi(q_i, X_i) \leqslant \kappa \quad \text{and} \quad q_i \text{ is a } P(\kappa)\text{-point in } X_i,$$

then Y is $C(Z)$-embedded in its G_{κ^+}-closure in $(X_I)_\kappa$ for every metric space Z, hence for every topologically complete space Z.

The countable case of Theorem 4.5 is this.

Theorem 4.6. *Let $\{X_i\colon i \in I\}$ be a set of T_1-spaces and let Y be a dense subspace of X_I. If every $q \in X_I \setminus Y$ in the G_δ-closure of Y in X_I satisfies $\chi(q_i, X_i) \leqslant \omega$ for every $i \in I$, then Y is $C(Z)$-embedded in its G_δ-closure in X_I for every metric space Z, hence for every topologically complete space Z.*

The theorem of R. Pol and E. Pol [20] and A.V. Arhangel'skiĭ [3, 1.9, 2.21] (see also [4]) mentioned in the abstract is a special case of the case $\mathbb{R} = Z$ of Theorem 4.6 (see also Corollary 4.12).

Theorem 4.6 generalizes the following theorem of M. Ulmer [23]:

Theorem 4.7. *Let $\{X_i\colon i \in I\}$ be a set of T_1-spaces such that $\chi(X_i) \leqslant \omega$ for every $i \in I$. Then each space of the form $\Sigma(q) \subseteq X_I$ is C-embedded in X_I.*

Another generalization of Ulmer's theorem is the following:

Corollary 4.8. *Let $\kappa \geqslant \omega$ be a regular cardinal and $\{X_i\colon i \in I\}$ be a set of T_1, $P(\kappa)$-spaces such that $\chi(X_i) \leqslant \kappa$ for every $i \in I$. Then each G_{κ^+}-dense subspace of $(X_I)_\kappa$ is C-embedded in $(X_I)_\kappa$.*

Corollary 4.8 has in addition the following consequence:

Theorem 4.9. *([18]) In a product of separable metric spaces every G_δ-dense subset is C-embedded.*

The attention which the Σ-products $\Sigma(p) \subseteq X_I$ have attracted with respect to questions of C-embedded subspaces of product spaces might lead one to believe that every G_δ-dense C-embedded subspace must contain such a space. In [8, 2.7] it is shown that this is by no means the case. Indeed, in appropriate circumstances a G_δ-dense, C-embedded subspace of a product space X_I may meet each Σ-product $\Sigma(p) \subseteq X_I$ in at most one point.

Ulmer, in [22] and [23], constructed an example showing that not every Σ-product in every (Tychonoff) product space is C-embedded. Extending his ideas, the following is shown in [8, 3.2]:

Example 4.10. For every $\kappa \geqslant \omega$ there are a set $\{X_i\colon i \in I\}$ of Tychonoff spaces, with $|I| = \kappa$, $q \in X_I$ and $f \in C((X_I \setminus \{q\}), \{0, 1\})$, such that no continuous function from X_I to $\{0, 1\}$ extends f. One may arrange further that either

(i) there is $i_0 \in I$ such that $\psi(X_{i_0}) = \omega$, while for $i_0 \neq i \in I$ the space X_i is the one-point compactification of a discrete space with cardinality κ; or
(ii) the spaces X_i are pairwise homeomorphic, with $\psi(X_i) = \omega$ and either
  (a) all but one point in each space X_i is isolated; or
  (b) each space X_i is dense-in-itself.

We note for emphasis that examples as in Example 4.10(ii) show that in Theorem 3.10 and in its corollaries in this section the hypothesized inequality about χ cannot be replaced by its ψ analogue.

Theorem 4.11. *Let X and Z be spaces such that each dense subset of X is $C(Z)$-embedded in its G_δ-closure in X. Let Y be dense in an open subset U of X. Then Y is $C(Z)$-embedded in its G_δ closure in U. Indeed for each $f \in C(Y, Z)$ there is a function $\overline{f} \in C(Y', Z)$ such that $f \subseteq \overline{f}$ and Y' is G_δ-closed in a dense open subset of X.*

Proof. Let $f \in C(Y, Z)$. Fix $z \in Z$, set $V := X \setminus \mathrm{cl}_X U$, and let Y' be the G_δ-closure in X of $Y \cup V$. Define $f' : Y \cup V \to Z$ by: $f'(y) = f(y)$ when $y \in Y$, $f'(x) = z$ when $x \in V$. Since $f' \in C(Y \cup V, Z)$ and $Y \cup V$ is dense in X, there is $\overline{f} \in C(Y', Z)$ such that $f' \subseteq \overline{f}$. From $f \subseteq f'$ follows $f \subseteq \overline{f}$, as required. □

According to a definition of A.V. Arhangel'skiĭ [1] a space is a *Moscow space* if the closure of each of its open subsets is the union of G_δ sets. Among Tychonoff spaces, the Moscow spaces are characterized [4] as those for which each dense set is C-embedded in its own G_δ-closure. The case $Z = \mathbb{R}$ of the preceding theorem simply duplicates or confirms the obvious assertion that an open subset of a Moscow space is a Moscow space. Clearly, another corollary of the special case $Z = \mathbb{R}$ of Theorem 4.6 can be stated efficiently using the notion of Moscow spaces.

Corollary 4.12. *Let $\kappa \geqslant \omega$ be a regular cardinal and $\{X_i\colon i \in I\}$ be a set of T_1, $P(\kappa)$-spaces such that $\chi(X_i) \leqslant \kappa$ for every $i \in I$. Then $(X_I)_\kappa$ is a Moscow space.*

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