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Equivalence and Duality of Quotient Categories

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1. INTRODUCTION AND PRELIMINARIES

1.1. *Introduction*

Let R and S be associative rings with identity, and R-Mod, Mod-R, S-Mod, and Mod-S denote respectively the categories of unital left R-, right R-, left S-, and right S-modules. M consistently denotes, unless otherwise specified, a left R-module and N a left S-module.

Let $({}_RU_S, {}_SV_R; I, J)$ be a Morita context with the trace ideals I and J, $\mathbf{L}(R)$ the lattice of all the Gabriel topologies on R-Mod containing the trace ideal I, and $\mathbf{L}(S)$ the lattice of all Gabriel topologies on S-Mod containing the trace ideal J.

In 1980, A. I. Kašu [3, Theorem 1] proved that

THEOREM A. *Between* $\mathbf{L}(R)$ *and* $\mathbf{L}(S)$, *there exists a lattice isomorphism*

$$H\colon \mathbf{L}(R) \ni \tau \mapsto H(\tau) = \tau' \in \mathbf{L}(S).$$

From now on, we write $(\tau, \tau') \in (\mathbf{L}(R), \mathbf{L}(S))$ for this case. In the process of the proof, he also got the following key and significant result [3, Lemma 9].

THEOREM B. (1) *If* M *is* τ_I*-free and injective in* R-Mod, *then* $\mathrm{Hom}_R(U, M)$ *is also injective in* S-Mod, *and*

(2) *if* N *is* τ_J*-free and injective in* S-Mod, *then* $\mathrm{Hom}_S(V, N)$ *is also injective in* R-Mod,

where τ_I, τ_J *denote respectively the Gabriel topologies determined by* I *and* J (*cf.* [2]).

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In Section 2 of this paper, a more general result is obtained, which can be regarded as a generalization of Theorem B and is stated as follows:

THEOREM 2.8. *Let* (τ_1, τ_1'), $(\tau_2, \tau_2') \in (\mathbf{L}(R), \mathbf{L}(S))$.

(1) *If* M *is* τ_1*-free, then* $\mathrm{Hom}_R(U, E_{\tau_2}(M)) \cong E_{\tau_2'}(\mathrm{Hom}_R(U, M))$ *and is* τ_1'*-free, and*

(2) *if* N *is* τ_1'*-free, then* $\mathrm{Hom}_S(V, E_{\tau_2'}(N)) \cong E_{\tau_2}(\mathrm{Hom}_S(V, N))$ *and is* τ_1*-free,*

where $E_{\tau_2}(M)$, $E_{\tau_2}(\mathrm{Hom}_S(V, N))$ *denote the* τ_2*-injective envelopes of* M, $\mathrm{Hom}_S(V, N)$, *resp., and* $E_{\tau_2'}(N)$, $E_{\tau_2'}(\mathrm{Hom}_R(U, M))$ *the* τ_2'*-injective envelopes of* N, $\mathrm{Hom}_R(U, M)$, *resp.*

In 1974, B. J. Müller proved the following result, which generalized the well-known Morita Theorem [2, Theorem 3]:

THEOREM C. *The functors* $\mathrm{Hom}_R(U, -)$ *and* $\mathrm{Hom}_S(V, -)$ *induce an equivalence between categories*

$$_{\tau_I}\mathbf{L} \simeq {}_{\tau_J}\mathbf{L},$$

where $_{\tau_I}\mathbf{L}$ *and* $_{\tau_J}\mathbf{L}$ *denote respectively the quotient categories with respect to* τ_I *and* τ_J.

Also in Section 2, this result is utilized and extended (see 2.5 and 2.6).

See T. Kato [1] for the original versions of Theorems B and C.

In 1979, T. Kato and K. Ohtake got a dual version of Theorem C [4, Theorem 2.5]:

THEOREM D. *The functors* $-\otimes_R U$ *and* $-\otimes_S V$ *induce a category equivalence*

$$\mathbf{K}_I \simeq \mathbf{K}_J,$$

where $\mathbf{K}_I = \{C \mid C \in \text{Mod-}R,\ C \otimes_R I \cong C_R \text{ canonically}\}$, $\mathbf{K}_J = \{D \mid D \in \text{Mod-}S,\ S \otimes_S J \cong D \text{ canonically}\}$.

In Section 3, first the author succeeds in defining a new concept of a dual full subcategory $\mathbf{K}_\tau$ in Mod-R of $_\tau\mathbf{L}$, proving that it is just a generalization of the concept of $\mathbf{K}_I$ and $\mathbf{K}_J$. Then the following fact, which generalizes Theorem D, is obtained.

THEOREM 3.11. *Let*

$$K_{[\tau_1, \tau_2]} = \{C_R \mid C \text{ is } \tau_1\text{-divisible and } \tau_2\text{-flat}\},$$

$$K_{[\tau_1', \tau_2']} = \{D_S \mid D \text{ is } \tau_1'\text{-divisible and } \tau_2'\text{-flat}\};$$

then the functors $-\otimes_R U$, $-\otimes_S V$ *induce an equivalence*

$$K_{[\tau_1,\tau_2]} \simeq K_{[\tau_1',\tau_2']}$$

for any (τ_1, τ_1'), $(\tau_2, \tau_2') \in (L(R), L(S))$.

1.2. *Preliminaries*

We introduce some concepts, definitions, and necessary knowledge for this paper as follows.

DEFINITION 1.1. Let ${}_RU_S$, ${}_SV_R$ be bimodules. A Morita context is a set $({}_RU_S, {}_SV_R; I, J)$ with the following conditions:

(1) There exist bimodule homomorphisms (called pairings)

$$(-,-): U\otimes_S V \to R,$$

$$[-,-]: V\otimes_R U \to S,$$

with the image of $(-,-)$ being the ideal I and that of $[-,-]$ the ideal J.

(2) For all u, $u' \in U$, $v, v' \in V$, $(u,v)u' = u[v,u']$, $[v,u]v' = v(u,v')$ hold. I and J are called the trace ideals of the context.

DEFINITION 1.2. A nonempty set τ of left ideals of R is called a Gabriel topology on R if it satisfies conditions T1, T2, T3, and T4 (for details, cf. [5]).

DEFINITION 1.3. A hereditary torsion theory on R-Mod is a pair $(\mathbf{T}, \mathbf{F})$ of classes of modules of R-Mod with the following conditions:

(1) $\mathbf{T}$ is closed under submodules, quotient modules, direct sums, and extensions.

(2) $\mathbf{F} = \{F \mid F \in R\text{-Mod}, \operatorname{Hom}_R(T,F) = 0, \text{ for all } T \in \mathbf{T}\}$.

PROPOSITION 1.4. *There is a bijective correspondence between Gabriel topologies on R and hereditary torsion theories on* R-Mod *given by*

$$\tau \mapsto (\mathbf{T}_\tau, \mathbf{F}_\tau), \qquad (\mathbf{T}, \mathbf{F}) \mapsto \tau_{(T,F)},$$

where $\mathbf{F}_\tau = \{F \mid F \in R\text{-Mod}, \operatorname{Hom}_R(R/\mathfrak{a}, F) = 0 \text{ for all } \mathfrak{a} \in \tau\}$, *and* $\mathbf{T}_\tau = \{T \mid T \in R\text{-Mod}; \forall t \in T\, \exists \mathfrak{a} \in \tau, \mathfrak{a}t = 0\}$, $\tau_{(T,F)} = \{\mathfrak{a} \mid R/\mathfrak{a} \in \mathbf{T}\}$.

By the correspondence, we consistently write $\tau = (\mathbf{T}, \mathbf{F})$ or $\tau = (\mathbf{T}_\tau, \mathbf{F}_\tau)$ for both τ and the corresponding hereditary torsion theory $(\mathbf{T}, \mathbf{F})$.

PROPOSITION 1.5. *If* $(\mathbf{T}, \mathbf{F})$ *is a hereditary torsion theory, then* $\mathbf{F}$ *is closed under submodules, direct products, extensions, and injective envelopes.*

PROPOSITION 1.6. *A pair* $(\mathbf{T}, \mathbf{F})$ *of classes of modules of* R-Mod *is a hereditary torsion theory if and only if it can be cogenerated by an injective module* E; *i.e.*, $\mathbf{T} = \{T \mid \mathrm{Hom}_R(T, E) = 0\}$, $\mathbf{F} = \{F \mid F \hookrightarrow \prod E\}$.

DEFINITION 1.7. (1) M is said to be τ-torsion if $M \in \mathbf{T}_\tau$,

(2) M is said to be τ-free if $M \in \mathbf{F}_\tau$,

(3) M is said to be τ-injective if $\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(\mathfrak{a}, M) \to 0$ is exact under the canonical homomorphism for all $\mathfrak{a} \in \tau$,

(4) M is said to be τ-closed if it is both τ-free and τ-injective.

PROPOSITION 1.8. *For any* $M \in R$-Mod, *there is a largest submodule* $T_\tau(M)$ *of* M *such that* $T_\tau(M) \in \mathbf{T}_\tau$, *and* $M/T_\tau(M) \in \mathbf{F}_\tau$.

PROPOSITION 1.9. (1) *For any* $M \in R$-Mod, *we can get a* τ-*closed module* $\bar{\tau}(M)$, *called the module of quotient of* M, *and also it can be considered as a* $\bar{\tau}(R)$-*module.*

(2) *There is a natural* R-*homomorphism* $\Phi_M: M \to \bar{\tau}(M)$ *with* $\ker \Phi_M = T_\tau(M)$, $\mathrm{Cok}\, \Phi_M \in \mathbf{T}_\tau$, *and* M *is* τ-*closed if and only if* Φ_M *is an isomorphism.*

(3) *The full subcategory* ${}_\tau\mathbf{L}$ *of all* τ-*closed modules is called the quotient category with respect to* τ, *and it also can be considered as a full subcategory of* $\bar{\tau}(R)$-Mod.

(4) *For any* $M \in R$-Mod, $\bar{\tau}(M) = \bar{\tau}(M/T_\tau(M))$.

DEFINITION 1.10. (1) $\tau(M) = \{M' \mid M'$ is a submodule of M, and M/M' is τ-torsion$\}$.

(2) A τ-injective envelope of M is an essential monomorphism $M \to M_1$ such that M_1 is τ-injective and $M \in \tau(M_1)$; from now on, the τ-injective envelope of M is denoted by $E_\tau(M)$.

PROPOSITION 1.11. (1) *If* M *is* τ-*free, then* $E_\tau(M) \cong \bar{\tau}(M)$.

(2) $E_\tau(M)$ *can be considered as a submodule of* $E(M)$, *the injective envelope of* M, *and* $E_\tau(M)/M = T_\tau(E(M)/M)$.

2. EQUIVALENCE OF QUOTIENT CATEGORIES

In Theorem A, the lattice isomorphism H is defined as follows: If $\tau = \tau_E$, then $H: \tau = \tau_E \mapsto \tau_{\mathrm{Hom}_R(U, E)} = \tau'$, where τ_E denotes the Gabriel topology cogenerated by the injective module E, and $\tau_{\mathrm{Hom}_R(U,E)}$ by the injective module $\mathrm{Hom}_R(U, E)$. H^{-1} is defined similarly by the symmetry of a Morita context.

Now we start our main work of this section with the following useful lemmas.

LEMMA 2.1. *Let* $(\tau, \tau') \in (\mathbf{L}(R), \mathbf{L}(S))$; *then*

(1) *a left ideal* $\mathfrak{b}$ *of* $S \in \tau'$ *if and only if* $U'\mathfrak{b} \in \tau({}_R U)$ *for any* $U' \in \tau({}_R U)$, *and*

(2) *a left ideal* $\mathfrak{a}$ *of* $R \in \tau$ *if and only* $V'\mathfrak{a} \in \tau'({}_S V)$ *for any* $V' \in \tau'({}_S V)$.

Proof. (1) $\mathfrak{b} \in \tau'$ if and only if $S/\mathfrak{b}$ is τ'-torsion, i.e., $\mathrm{Hom}_S(S/\mathfrak{b}, \mathrm{Hom}_R(U, E_\tau)) = 0$ by Theorem B, where E_τ denotes an injective R-module cogenerating τ. But $E_\tau \in {}_\tau\mathbf{L}$, $U'S \in \tau({}_R U)$, $\mathrm{Hom}_S(S/\mathfrak{b}, \mathrm{Hom}_R(U, E_\tau)) \cong \mathrm{Hom}_S(S/\mathfrak{b}, \mathrm{Hom}_R(U'S, E_\tau)) \cong \mathrm{Hom}_R(U'S \otimes_S S/\mathfrak{b}, E_\tau) \cong \mathrm{Hom}_R(U'S/U'S\mathfrak{b}, E_\tau) = \mathrm{Hom}_R(U'S/U'\mathfrak{b}, E_\tau)$.

Hence $\mathrm{Hom}_S(S/\mathfrak{b}, \mathrm{Hom}_R(U, E_\tau)) = 0 \Leftrightarrow \mathrm{Hom}_S(U'S/U'\mathfrak{b}, E_\tau) = 0 \Leftrightarrow U'\mathfrak{b} \in \tau({}_R U'S) \Leftrightarrow U'\mathfrak{b} \in \tau({}_R U)$ since $U'S \in \tau({}_R U)$.

(2) By the symmetry of a Morita context.

LEMMA 2.2. *Let* $(\tau, \tau') \in (\mathbf{L}(R), \mathbf{L}(S))$; *then*

(1) *if* M *is* τ*-free, then* $\mathrm{Hom}_R(U, M)$ *is* τ'*-free, and*

(2) *if* N *is* τ'*-free, then* $\mathrm{Hom}_S(V, N)$ *is* τ*-free.*

Proof. (1) For any $\mathfrak{b} \in \tau'$, $\mathrm{Hom}_S(S/\mathfrak{b}, \mathrm{Hom}_R(U, M)) \cong \mathrm{Hom}_R(U/U\mathfrak{b}, M)$, but M is τ-free and $U\mathfrak{b} \in \tau({}_R U)$ by Lemma 2.1, hence $0 = \mathrm{Hom}_R(U/U\mathfrak{b}, M) \cong \mathrm{Hom}_S(S/\mathfrak{b}, \mathrm{Hom}_R(U, M))$; i.e., $\mathrm{Hom}_R(U, M)$ is τ'-free.

(2) By the symmetry.

We also need to note the fact that $\tau_I(\tau_J)$ is the least element in $\mathbf{L}(R)(\mathbf{L}(S))$ and $\tau^R = \{\mathfrak{a} \mid \mathfrak{a}$ is a left ideal of $R\}$ ($\tau^S = \{\mathfrak{b} \mid \mathfrak{b}$ is a left ideal of $S\}$) is the greatest element in $\mathbf{L}(R)(\mathbf{L}(S))$, so if M is τ-free (or τ-injective) for some $\tau \in \mathbf{L}(R)$, then M is τ_I-free (τ_I-injective); if N is τ'-free (τ'-injective) for some $\tau' \in \mathbf{L}(S)$, then N is τ_J-free (τ_J-injective).

Now, we prove the generalization of Theorem B and Theorem C.

THEOREM 2.3. *Let* $(\tau_1, \tau_1'), (\tau_2, \tau_2') \in (\mathbf{L}(R), \mathbf{L}(S))$.

(1) *If* M *is* τ_1*-free and* τ_2*-injective, then* $\mathrm{Hom}_R(U, M)$ *is* τ_1'*-free and* τ_2'*-injective.*

(2) *If* N *is* τ_1'*-free and* τ_2'*-injective, then* $\mathrm{Hom}_S(V, N)$ *is* τ_1*-free and* τ_2*-injective.*

Proof. By Lemma 2.2 and the symmetry, it suffices to prove that $\mathrm{Hom}_R(U, M)$ is τ_2'-injective.

Let f be an S-homomorphism from $\mathfrak{b}$ to $\mathrm{Hom}_R(U, M)$, where $\mathfrak{b} \in \tau_2'$. From f, we can get an R-homomorphism G' from $U\mathfrak{b}$ to M, defined by $G'(ub) = f(b)(u)$, where $ub \in U\mathfrak{b}$.

G' is clearly R-linear, and also G' is well-defined, for if $ub = 0$, then $(u', v')\, G'(ub) = G'((u', v')\, ub) = f(b)((u', v')u) = f(b)(u'[v', u]) = ([v', u]f(b))(u') = f([v', u]b)(u') = f([v', ub])(u') = 0$, where $u' \in U$, $v \in V$, i.e., $IG'(ub) = 0$, but ${}_RM$ is τ_1-free, hence τ_I-free, so $G'(ub) = 0$.

On the other hand, $\mathfrak{b} \in \tau_2'$, so $U\mathfrak{b} \in \tau_2({}_RU)$ by Lemma 2.1, and since M is τ_2-injective, G' can be extended to an R-homomorphism G from U to M.

Now define an S-homomorphism g from S to $\mathrm{Hom}_R(U, M)$ by $s \mapsto sG$ for any $s \in S$; then g is a desired extension of f.

COROLLARY 2.4. *Let* $\tau_1 = \tau_I$, $\tau_2 = \tau^R$; *then* $\tau_1' = \tau_J$, $\tau_2' = \tau^S$. *From the theorem above, we get Theorem* B *again.*

In particular, if $I = R$, $J = S$, then any R-module ${}_RM$ is τ_I-free and any S-module ${}_SN$ is τ_J-free, and the result is just the well-known fact that the equivalence between module categories preserves the property of injectivity of a module.

Combining Theorem 2.3 with Theorem C, we have the following Corollary 2.5 and Theorem 2.6.

COROLLARY 2.5. *Let* $\tau_1 = \tau_2 = \tau$, *then we get*: *The functors* $\mathrm{Hom}_R(U, -)$, *and* $\mathrm{Hom}_S(V, -)$ *induce an equivalence*:

$$ {}_{\tau}\mathbf{L} \cong {}_{\tau'}\mathbf{L} $$

for any $(\tau, \tau') \in (\mathbf{L}(R), \mathbf{L}(S))$.

See T. Kato [1, Theorem 2] for the original version of Corollary 2.5.

In particular, take $\tau = \tau_I$; then $\tau' = \tau_J$. This is just Theorem C. More generally, we have:

THEOREM 2.6. *Let*

$$ {}_{[\tau_1, \tau_2]}\mathbf{L} = \{ {}_RM \mid M \textit{ is } \tau_1\textit{-free and } \tau_2\textit{-injective} \}, $$

$$ {}_{[\tau_1', \tau_2']}\mathbf{L} = \{ {}_SN \mid N \textit{ is } \tau_1'\textit{-free and } \tau_2'\textit{-injective} \}; $$

then the functors $\mathrm{Hom}_R(U, -)$, $\mathrm{Hom}_S(V, -)$ *induce an equivalence*

$$ {}_{[\tau_1, \tau_2]}\mathbf{L} \simeq {}_{[\tau_1', \tau_2']}\mathbf{L} $$

for any (τ_1, τ_1'), $(\tau_2, \tau_2') \in (\mathbf{L}(R), \mathbf{L}(S))$.

In [3], A. I. Kašu has also proved the following lemma (cf. T. Kato [1, Lemma 5] for the original version).

LEMMA 2.7. (1) *If* M *is* τ_I*-free, and* $e: M \to M_1$ *is an essential monomorphism, then so is* $\mathrm{Hom}_R(U, e)$: $\mathrm{Hom}_R(U, M) \to \mathrm{Hom}_R(U, M_1)$.

(2) *If* N *is* τ_J*-free, and* $e': N \to N_1$ *is an essential monomorphism, then so is* $\mathrm{Hom}_S(V, e')$: $\mathrm{Hom}_S(V, N) \to \mathrm{Hom}_S(V, N_1)$.

By the lemma above, Theorem A is equivalent to the following:

(1) If M is τ_I-free, $E(M)$ is the injective envelope of ${}_RM$; then

$$\mathrm{Hom}_R(U, E(M)) \cong E(\mathrm{Hom}_R(U, M)),$$

where the latter is the injective envelope of $\mathrm{Hom}_R(U, M)$ in S-Mod.

(2) If N is τ_J-free, $E(N)$ is the injective envelope of ${}_SN$; then

$$\mathrm{Hom}_S(V, E(N)) \cong E(\mathrm{Hom}_S(V, N)),$$

where the latter is the injective envelope of $\mathrm{Hom}_S(V, N)$ in R-Mod. But we claim that the following more general fact is also true.

THEOREM 2.8. *For any* (τ_1, τ_1'), $(\tau_2, \tau_2') \in (\mathbf{L}(R), \mathbf{L}(S))$,

(1) *if* M *is* τ_1*-free, then* $\mathrm{Hom}_R(U, E_{\tau_2}(M)) \cong E_{\tau_2'}(\mathrm{Hom}_R(U, M))$, *and*

(2) *if* N *is* τ_1'*-free, then* $\mathrm{Hom}_S(V, E_{\tau_2'}(N)) \cong E_{\tau_2}(\mathrm{Hom}_S(V, N))$, *where* E_{τ_2}, $E_{\tau_2'}$ *denote the* τ_2*-injective,* τ_2'*-injective envelopes, resp.*

First of all, we prove the following useful lemmas.

LEMMA 2.9. *Let* $(\tau, \tau') \in (\mathbf{L}(R), \mathbf{L}(S))$; *then*

(1) *if* U' *is a submodule of* ${}_RU$, *then* $U' \in \tau({}_RU) \Leftrightarrow [V, U'] \in \tau' = \tau'(S)$, *and*

(2) *if* V' *is a submodule of* ${}_SV$, *then* $V' \in \tau'({}_SV) \Leftrightarrow (U, V') \in \tau = \tau(R)$.

Proof. (1) $[V, U'] \in \tau' \Leftrightarrow S/[V, U']$ is τ'-torsion $\Leftrightarrow \mathrm{Hom}_S(S/[V, U'], \mathrm{Hom}_R(U, E_\tau)) = 0 \Leftrightarrow \mathrm{Hom}_R(U/U[V, U'], E_\tau) = \mathrm{Hom}_R(U/IU', E_\tau) = 0$ $(U[V, U'] = (U, V)\, U' = IU') \Leftrightarrow IU' \in \tau(U) \Leftrightarrow U' \in \tau(U)$ since $IU' \in \tau(U')$ for any $\tau \in \mathbf{L}(R)$, where E_τ denotes the injective R-module cogenerating τ.

(2) By the symmetry.

LEMMA 2.10. *For any* $(\tau, \tau') \in (\mathbf{L}(R), \mathbf{L}(S))$,

(1) *if* $M \in \tau(M_1)$, *then* $\mathrm{Hom}_R(U, M) \in \tau'(\mathrm{Hom}_R(U, M_1))$, *and*

(2) *if* $N \in \tau'(N_1)$, *then* $\mathrm{Hom}_S(V, N) \in \tau(\mathrm{Hom}_S(V, N_1))$.

Proof. (1) $\mathrm{Hom}_R(U, M)$ is clearly a submodule of $\mathrm{Hom}_R(U, M_1)$. Let $f \in \mathrm{Hom}_R(U, M_1)$, $f^{-1}(M) = U'$; then $If(U') = f(IU') = f((U, V)\, U') = f(U\,[V, U']) = ([V, U']f)(U) \subseteq M$, i.e., $[V, U']f \subseteq \mathrm{Hom}_R(U, M)$. But M_1/M is τ-torsion, so U/U' is τ-torsion, and by Lemma 2.9, $[V, U'] \in \tau'$, i.e., $\mathrm{Hom}_R(U, M_1)/\mathrm{Hom}_R(U, M)$ is τ'-torsion, and hence $\mathrm{Hom}_R(U, M) \in \tau'(\mathrm{Hom}_R(U, M_1))$.

(2) By the symmetry.

Proof of Theorem 2.8. (1) M is essential in $E_{\tau_2}(M)$, so $\mathrm{Hom}_R(U, M)$ is essential in $\mathrm{Hom}_R(U, E_{\tau_2}(M))$ by Lemma 2.7. M is τ_1-free, and $E(M)$ and $E_{\tau_2}(M)$, as submodules of $E(M)$, are also τ_1-free. Therefore, by Theorem 2.3 and Lemma 2.10, $\mathrm{Hom}_R(U, E_{\tau_2}(M))$ is τ_2'-injective and $\mathrm{Hom}_R(U, M) \in \tau_2'(\mathrm{Hom}_R(U, E_{\tau_2}(M)))$. So $\mathrm{Hom}_R(U, E_{\tau_2}(M)) \cong E_{\tau_2'}(\mathrm{Hom}_R(U, M))$ by the definition.

(2) By the symmetry.

If M is τ_2-free, then $\bar{\tau}_2(M) \cong E_{\tau_2}(M)$ and $\mathrm{Hom}_R(U, M)$ is also τ_2'-free, and hence $E_{\tau_2'}(\mathrm{Hom}_R(U, M)) \cong \bar{\tau}_2'(\mathrm{Hom}_R(U, M))$, so we have

COROLLARY 2.11. *For any* $(\tau, \tau') \in (\mathbf{L}(R), \mathbf{L}(S))$,

(1) *if M is τ-free, then* $\mathrm{Hom}_R(U, \bar{\tau}(M)) \cong \bar{\tau}'(\mathrm{Hom}_R(U, M))$, *and*

(2) *if N is τ'-free, then* $\mathrm{Hom}_S(V, \bar{\tau}'(N)) \cong \bar{\tau}(\mathrm{Hom}_S(V, N))$.

3. DUALITY OF QUOTIENT CATEGORY

In this section, from any quotient category ${}_\tau\mathbf{L}$ on R-Mod, we define its dual, which is a full subcategory $\mathbf{K}_\tau$ on Mod-R, and it is proved that if $(\tau, \tau') \in (\mathbf{L}(R), \mathbf{L}(S))$, then the functors $- \otimes_R U$, and $- \otimes_S V$ induce an equivalence between $\mathbf{K}_\tau$ and $\mathbf{K}_{\tau'}$, which generalizes the work of T. Kato and K. Ohtake in [4].

We recall that for any Gabriel topology τ on R-Mod, the corresponding quotient category is

$$ {}_\tau\mathbf{L} = \{ {}_R M \mid M \text{ is both } \tau\text{-free and } \tau\text{-injective} \}. $$

By forming a "Hom-Tensor" dual contrast to the ${}_\tau\mathbf{L}$, we can define the following:

DEFINITION 3.1 [5]. A $C \in$ Mod-R is said to be τ-divisble if $C \otimes_R R/\mathfrak{a} = 0$, i.e., $C = C\mathfrak{a}$ for any $\mathfrak{a} \in \tau$.

DEFINITION 3.2. A $C \in \text{Mod-}R$ is said to be τ-flat if $C \otimes_R f$ is a monomorphism for any $f \in \tau\text{-Mon}$, where

$$\tau\text{-Mon} = \{f \mid f \text{ is a monomorphism in } R\text{-Mod, and } \operatorname{Cok} f \text{ is } \tau\text{-torsion}\}.$$

DEFINITION 3.3. $\mathbf{K}_\tau = \{M_R \mid M \text{ is both } \tau\text{-divisible and } \tau\text{-flat}\}$ is called the dual full subcategory of ${}_\tau\mathbf{L}$ in Mod-R.

About the three concepts above, we have the following facts.

LEMMA 3.4. *The following conditions on a bimodule ${}_SC_R$ are equivalent*:

(1) *C_R is τ-divisible.*

(2) *For any $f \in \tau\text{-Mon}|_\tau$, $C \otimes_R f$ is an epimorphism, where $\tau\text{-Mon}|_\tau = \{f \mid f$ is an injection from $\mathfrak{a}$ to the ring R, $\mathfrak{a} \in \tau\}$.*

(3) *$C \otimes_R M = 0$ for any $M \in \mathbf{T}_\tau$.*

(4) *For any $f \in \tau$-Mon, $C \otimes_R f$ is an epimorphism.*

(5) *For any $N \in \text{Mod-}S$, $N \otimes_S C$ is τ-divisible.*

(6) *For any $N \in S$-Mod, $\operatorname{Hom}_S(C, N) \in \mathbf{F}_\tau$.*

(7) *$\operatorname{Hom}_S(C, E) \in \mathbf{F}_\tau$, where E is an injective cogenerator of S-Mod.*

Proof. We only prove that $(1) \Leftrightarrow (7)$ and omit the others. If E is an injective cogenerator of S-Mod, then for any $\mathfrak{a} \in \tau$, $C \otimes_R R/\mathfrak{a} = 0 \Leftrightarrow \operatorname{Hom}_S(C \otimes_R R/\mathfrak{a}, E) = 0 \Leftrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Hom}_S(C, E)) = 0 \Leftrightarrow \operatorname{Hom}_S(C, E) \in \mathbf{F}_\tau$.

LEMMA 3.5. *The following conditions on a bimodule ${}_SC_R$ are equivalent*:

(1) *C_R is τ-flat.*

(2) *For any $f \in \tau\text{-Mon}|_\tau$, $C \otimes_R f$ is a monomorphism.*

(3) *$\operatorname{Hom}_S(C, E)$ is τ-injective, where E denotes an injective cogenerator of S-Mod.*

Proof. $(1) \Rightarrow (2)$ obviously.

$(2) \Rightarrow (3)$ If $0 \to C \otimes_R \mathfrak{a} \to C \otimes_R R \to C \otimes_R R/\mathfrak{a} \to 0$ is exact for $\mathfrak{a} \in \tau$, then $0 \to \operatorname{Hom}_S(C \otimes_R R/\mathfrak{a}, E) \to \operatorname{Hom}_S(C \otimes_R R, E) \to \operatorname{Hom}_S(C \otimes_R \mathfrak{a}, E) \to 0$ is also exact, i.e., $0 \to \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Hom}_S(C, E)) \to \operatorname{Hom}_R(R, \operatorname{Hom}_S(C, E)) \to \operatorname{Hom}_R(\mathfrak{a}, \operatorname{Hom}_S(C, E)) \to 0$ is exact. Hence $\operatorname{Hom}_S(C, E)$ is τ-injective.

$(3) \Rightarrow (1)$ If $\operatorname{Hom}_S(C, E)$ is τ-injective, then by the generalized Bear criterion, for any $f \in \tau$-Mon, from an exact sequence $0 \to M' \xrightarrow{f} M \to \operatorname{Cok} f \to 0$, we get another exact sequence $0 \to \operatorname{Hom}_R(\operatorname{Cok} f, \operatorname{Hom}_S(C, E)) \to \operatorname{Hom}_R(M, \operatorname{Hom}_S(C, E)) \to \operatorname{Hom}_R(M', \operatorname{Hom}_S(C, E)) \to 0$, i.e., $0 \to \operatorname{Hom}_S(C \otimes_R \operatorname{Cok} f, E) \to \operatorname{Hom}_S(C \otimes_R M, E) \to \operatorname{Hom}_S(C \otimes_R M', E) \to 0$, so we have $0 \to C \otimes_R M' \to C \otimes_R M \to C \otimes_R \operatorname{Cok} f \to 0$ exact, i.e., C_R is τ-flat.

LEMMA 3.6. *The following conditions on a bimodule* ${}_S C_R$ *are equivalent*:

(1) $C_R \in \mathbf{K}_\tau$.

(2) $C \otimes_R f$ *is an isomorphism for any* $f \in \tau$-Mon.

(3) $C \cong C \otimes_R \mathfrak{a}$ *canonically for any* $\mathfrak{a} \in \tau$.

(4) *For any* $N \in$ Mod-S, $N \otimes_S C \in \mathbf{K}_\tau$.

(5) *For any* $N \in S$-Mod, $\mathrm{Hom}_S(C, N) \in {}_\tau\mathbf{L}$.

(6) $\mathrm{Hom}_S(C, E) \in {}_\tau\mathbf{L}$, *where* E *is an injective cogenerator of* S-Mod.

Proof. From Lemmas 3.4 and 3.5, we can easily get all the results above.

Now we start to prove that $\mathbf{K}_I = \mathbf{K}_{\tau_I}$, $\mathbf{K}_J = \mathbf{K}_{\tau_J}$.

LEMMA 3.7.

$$
{}_{\tau_I}\mathbf{L} = \{ {}_R M \mid \mathrm{Hom}_R(I, M) \cong M \textit{ canonically} \},
$$
$$
{}_{\tau_J}\mathbf{L} = \{ {}_S N \mid \mathrm{Hom}_S(J, N) \cong N \textit{ canonically} \}.
$$

Proof. See [2].

LEMMA 3.8. $\mathbf{K}_I = \mathbf{K}_{\tau_I}$, $\mathbf{K}_J = \mathbf{K}_{\tau_J}$.

Proof. Obviously, $\mathbf{K}_{\tau_I} \subseteq \mathbf{K}_I$ from Lemma 3.6(3). If $C \in \mathbf{K}_I$, then $C \otimes_R I \cong C$ canonically, and therefore $\mathrm{Hom}_Z(C \otimes_R I, W) \cong \mathrm{Hom}_Z(C, W)$, where W is an injective cogenerator of Z-Mod. Hence $\mathrm{Hom}_R(I, \mathrm{Hom}_Z(C, W)) \cong \mathrm{Hom}_Z(C, W)$ canonically. This means $\mathrm{Hom}_Z(C, W) \in {}_{\tau_I}\mathbf{L}$ by Lemma 3.7 and $C \in \mathbf{K}_{\tau_I}$ by Lemma 3.6 (6).

Now we are able to show our main result in this section.

THEOREM 3.9. *Let* (τ_1, τ_1'), $(\tau_2, \tau_2') \in (\mathbf{L}(R), \mathbf{L}(S))$; *then*

(1) *if* C *is* τ_1*-divisible and* τ_2*-flat, then* $C \otimes_R U$ *is* τ_1'*-divisible and* τ_2'*-flat, and*

(2) *if* D *is* τ_1'*-divisible and* τ_2'*-flat, then* $D \otimes_S V$ *is* τ_1*-divisible and* τ_2*-flat.*

Proof. (1) If C is τ_1-divisible, then $\mathrm{Hom}_Z(C, W)$ is τ_1-free, and if C is τ_2-flat, then $\mathrm{Hom}_Z(C, W)$ is τ_2-injective, and by Theorem 2.3, $\mathrm{Hom}_R(U, \mathrm{Hom}_Z(C, W))$ is τ_1'-free and τ_2'-injective, i.e., $\mathrm{Hom}_Z(C \otimes_R U, W)$ is τ_1'-free and τ_2'-injective. Hence $C \otimes_R U$ is τ_1'-divisible and τ_2'-flat.

(2) By the symmetry.

THEOREM 3.10. *Let* $(\tau, \tau') \in (\mathbf{L}(R), \mathbf{L}(S))$; *then the functors* $-\otimes_R U$, $-\otimes_S V$ *induce an equivalence*

$$\mathbf{K}_\tau \simeq \mathbf{K}_{\tau'}.$$

Proof. Let $\tau = \tau_I$; then $\tau' = \tau_J$. Define

$$\phi\colon (-\otimes_R U \otimes_S V) \to 1 \qquad \text{by} \quad \phi_C\left(\sum c_i \otimes u_i \otimes v_i\right) = \sum c_i(u_i, v_i),$$

where $\sum c_i \otimes u_i \otimes v_i \in C \otimes U \otimes V$ and

$$\psi\colon (-\otimes_S V \otimes_R U) \to 1 \qquad \text{by} \quad \psi_D\left(\sum d_i \otimes v_i \otimes u_i\right) = \sum d_i[v_i, u_i],$$

where $\sum d_i \otimes v_i \otimes u_i \in D \otimes V \otimes U$.

These are both natural transformations. It suffices to show that ϕ_C is an isomorphism if $C \in K_{\tau_I}$ and ψ_D is an isomorphism if $D \in K_{\tau_J}$ since for any $(\tau, \tau') \in (L(R), L(S))$, $-\otimes U\colon K_\tau \to K_{\tau'}$, $-\otimes V\colon K_{\tau'} \to K_\tau$ by Theorem 3.9. By the symmetry, however, we only need to prove the former.

Now $C \in K_{\tau_I} \Rightarrow \mathrm{Hom}_z(C, W) \in {}_{\tau_I}L \Leftrightarrow \mathrm{Hom}_z(C, W) \cong \mathrm{Hom}_R(U \otimes_S V, \mathrm{Hom}_z(C, W)) \cong \mathrm{Hom}_z(C \otimes_R U \otimes_S V, W) \Rightarrow C \cong C \otimes_R U \otimes_S V$ by ϕ_c.

Finally, for any other $(\tau, \tau') \in (L(R), L(S))$, since $K_\tau \subseteq K_{\tau_I}$ and $K_{\tau'} \subseteq K_{\tau_J}$ the equivalence is obtained immediately from Theorem 3.9.

Thus we get again T. Kato and K. Ohtake's result and more in a different way.

However, this can also be proved by combining Theorem 3.9 and their Theorem D.

THEOREM 3.11. *Let*

$$\mathbf{K}_{[\tau_1, \tau_2]} = \{C_R \mid C \text{ is } \tau_1\text{-divisible and } \tau_2\text{-flat}\},$$

$$\mathbf{K}_{[\tau_1', \tau_2']} = \{D_S \mid D \text{ is } \tau_1'\text{-divisible and } \tau_2'\text{-flat}\};$$

then the functors $-\otimes_R U$, $-\otimes_S V$ *induce an equivalence*

$$\mathbf{K}_{[\tau_1, \tau_2]} \simeq \mathbf{K}_{[\tau_1', \tau_2']}$$

for any (τ_1, τ_1'), $(\tau_2, \tau_2') \in (\mathbf{L}(R), \mathbf{L}(S))$.

We know that if $\tau > \tau_1$, then ${}_{\tau_1}\mathbf{L} \supset {}_\tau\mathbf{L}$. About $\mathbf{K}$, we know it is unlikely that $\mathbf{K}_\tau = \mathbf{K}_{\tau_1}$. Besides, we have

THEOREM 3.12. *If each* $\mathfrak{a} \in \tau$ *is finitely generated projective and* $\tau > \tau_1$, *then* $\mathbf{K}_{\tau_1} \supset \mathbf{K}_\tau$.

First we need to prove the following

LEMMA 3.13. *If each* $\mathfrak{a} \in \tau$ *is finitely generated projective,* ${}_R M_S \in {}_\tau\mathbf{L}$, *then for any* $N \in$ Mod-S, $\mathrm{Hom}_S(M, N) \in \mathbf{K}_\tau$.

Proof. If $\mathfrak{a} \in \tau$, then $\mathfrak{a}$ is finitely generated projective. Hence $\mathrm{Hom}_S(M, N) \otimes_R \mathfrak{a} \cong \mathrm{Hom}_S(\mathrm{Hom}_R(\mathfrak{a}, M), N)$ (cf. [6]) $\cong \mathrm{Hom}_S(M, N)$, i.e., $\mathrm{Hom}_S(M, N) \in \mathbf{K}_\tau$.

Proof of Theorem 3.12. *If* $\tau > \tau_1$, then we can get an $E_{\tau_1} \notin {}_\tau\mathbf{L}$, where E_{τ_1} is an injective module cogenerating τ_1. By Lemma 3.13, $\mathrm{Hom}_Z(E_{\tau_1}, W) \in \mathbf{K}_{\tau_1}$, where W is an injective cogenerator of Mod-Z. We claim that $\mathrm{Hom}_Z(E_{\tau_1}, W) \notin \mathbf{K}_\tau$. If $\mathrm{Hom}_Z(E_{\tau_1}, W) \in \mathbf{K}_\tau$, then $\mathrm{Hom}_Z(\mathrm{Hom}_R(\mathfrak{a}, E_{\tau_1}), W) \cong \mathrm{Hom}_Z(E_{\tau_1}, W) \otimes_R \mathfrak{a} \cong \mathrm{Hom}_Z(E_{\tau_1}, W)$ for any $\mathfrak{a} \in \tau$. So we get $\mathrm{Hom}_Z(\mathrm{Hom}_R(\mathfrak{a}, E_{\tau_1}), W) \cong \mathrm{Hom}_Z(E_{\tau_1}, W) \Rightarrow \mathrm{Hom}_R(\mathfrak{a}, E_{\tau_1}) \cong E_{\tau_1}$ for any $\mathfrak{a} \in \tau$, i.e., $E_{\tau_1} \in {}_\tau\mathbf{L}$, contradicting $E_{\tau_1} \notin {}_\tau\mathbf{L}$.

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