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Strong solutions of the compressible nematic liquid crystal flow

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ABSTRACT

We study strong solutions of the simplified Ericksen–Leslie system modeling compressible nematic liquid crystal flows in a domain $\Omega \subset \mathbb{R}^3$. We first prove the local existence of a unique strong solution provided that the initial data ρ_0, u_0, d_0 are sufficiently regular and satisfy a natural compatibility condition. The initial density function ρ_0 may vanish on an open subset (i.e., an initial vacuum may exist). We then prove a criterion for possible breakdown of such a local strong solution at finite time in terms of blow up of the quantities $\|\rho\|_{L_t^\infty L_x^\infty}$ and $\|\nabla d\|_{L_t^3 L_x^\infty}$.

1. Introduction

Nematic liquid crystals are aggregates of molecules which possess same orientational order and are made of elongated, rod-like molecules. The continuum theory of liquid crystals was developed by Ericksen [10] and Leslie [19] during the period of 1958 through 1968, see also the book by de Gennes [9]. Since then there have been remarkable research developments in liquid crystals from both theoretical and applied aspects. When the fluid containing nematic liquid crystal materials are at rest, we have the well-known Ossen–Frank theory for static nematic liquid crystals, see Hardt, Kinderlehrer and Lin [13] on the analysis of energy minimal configurations of nematic liquid crystals. In general, the motion of fluid always takes place. The so-called Ericksen–Leslie system is a macroscopic continuum description of the time evolution of the materials under the influence of both the flow velocity field u and the macroscopic description of the microscopic orientation configurations d of rod-like liquid crystals.

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When the fluid is an incompressible, viscous fluid, Lin [20] first derived a simplified Ericksen–Leslie equation modeling liquid crystal flows in 1989. Subsequently, Lin and Liu [21,22] made some important analytic studies, such as the existence of weak and strong solutions and the partial regularity of suitable solutions, of the simplified Ericksen–Leslie system, under the assumption that the liquid crystal director field is of varying length by Leslie's terminology or variable degree of orientation by Ericksen's terminology.

When the fluid is allowed to be compressible, the Ericksen–Leslie system becomes more complicate and there seems very few analytic works available yet. We would like to mention that very recently, there have been both modeling study, see Morro [29], and numerical study, see Zakharov and Vakulenko [39], on the hydrodynamics of compressible nematic liquid crystals under the influence of temperature gradient or electromagnetic forces.

This paper, and the companion paper [18], aims to study the strong solutions of the flow of compressible nematic liquid crystals and the blow-up criterions.

Let $\Omega \subset \mathbb{R}^3$ be a domain. We will consider the simplified version of Ericksen–Leslie system modeling the flow of compressible nematic liquid crystals in Ω[1]:

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1.1}$$

$$\rho u_t + \rho u \cdot \nabla u + \nabla\big(P(\rho)\big) = \mathcal{L}u - \nabla d \cdot \Delta d, \tag{1.2}$$

$$d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \tag{1.3}$$

where $\rho : \Omega \times [0, +\infty) \to \mathbb{R}^1$ is the density function of the fluid, $u : \Omega \times [0, +\infty) \to \mathbb{R}^3$ represents velocity field of the fluid, $P = P(\rho)$ represents the pressure function, $d : \Omega \times [0, +\infty) \to S^2$ represents the macroscopic average of the nematic liquid crystal orientation field, $\nabla\cdot$ is the divergence operator in $\mathbb{R}^3$, and $\mathcal{L}$ denotes the Lamé operator:

$$\mathcal{L}u = \mu \Delta u + (\mu + \lambda)\nabla \operatorname{div} u,$$

where μ and λ are shear viscosity and the bulk viscosity coefficients of the fluid respectively that satisfy the physical condition:

$$\mu > 0, \qquad 2\mu + 3\lambda \geqslant 0. \tag{1.4}$$

We refer to the readers to consult the recent preprint [7] by Ding, Huang, Wen and Zi for the derivation for the system (1.1)–(1.3) based on the energetic-variational approach. Throughout this paper, we assume that

$$P : [0, +\infty) \to \mathbb{R} \text{ is a locally Lipschitz continuous function.} \tag{1.5}$$

Notice that (1.1) is the equation of conservation of mass, (1.2) is the equation of linear momentum, and (1.3) is the equation of angular momentum. We would like to point out that the system (1.1)–(1.3) includes several important equations as special cases:

(i) When ρ is constant, Eq. (1.1) reduces to the incompressibility condition of the fluid ($\nabla \cdot u = 0$), and the system (1.1)–(1.3) becomes the equation of incompressible flow of nematic liquid crystals provided that P is an unknown pressure function. This was previously proposed by Lin [20] as a simplified Ericksen–Leslie equation modeling incompressible liquid crystal flows.

[1] Through the energy variational approach presented by [7], we know that the induced stress force by the director field d in the right-hand side of (1.2) should be $-\nabla \cdot (\nabla d \otimes \nabla d - \frac{1}{2}|\nabla d|^2 \mathbb{I}_3)$, where $(\nabla d \otimes \nabla d)_{ij} = \frac{\partial d}{\partial x_i} \cdot \frac{\partial d}{\partial x_j}$ for $1 \leqslant i, j \leqslant 3$ and $\mathbb{I}_3$ is the identity matrix of order 3. However, it is not hard to check that $-\nabla \cdot (\nabla d \otimes \nabla d - \frac{1}{2}|\nabla d|^2 \mathbb{I}_3) = -\Delta d \cdot \nabla d$. For the incompressible nematic liquid crystal flow, since the body force term $\nabla \cdot (\frac{1}{2}|\nabla d|^2 \mathbb{I}_3) = \nabla(\frac{1}{2}|\nabla d|^2)$ can be absorbed into the term ∇P of a unknown pressure function P, in the literature that the induced stress force by d is frequently written by $\nabla \cdot (\nabla d \otimes \nabla d)$.

(ii) When d is a constant vector field, the system (1.1)–(1.2) becomes a compressible Navier–Stokes equation, which is an extremely important equation to describe compressible fluids (e.g., gas dynamics). It has attracted great interests among many analysts and there have been many important developments (see, for example, Lions [27], Feireisl [11] and references therein).
(iii) When both ρ and d are constants, the system (1.1)–(1.2) becomes the incompressible Navier–Stokes equation provided that P is a unknown pressure function, the fundamental equation to describe Newtonian fluids (see, for example, Lions [26] and Temam [35] for survey of important developments).
(iv) When ρ is constant and $u = 0$, the system (1.1)–(1.3) reduces to the equation for heat flow of harmonic maps into S^2. There have been extensive studies on the heat flow of harmonic maps in the past few decades (see, for example, the monograph by Lin and Wang [24] and references therein).

From the viewpoint of partial differential equations, the system (1.1)–(1.3) is a highly nonlinear system coupling between hyperbolic equations and parabolic equations. It is very challenging to understand and analyze such a system, especially when the density function ρ may vanish or the fluid takes vacuum states.

In this paper, we will consider the following initial condition:

$$(\rho, u, d)|_{t=0} = (\rho_0, u_0, d_0), \tag{1.6}$$

and one of the three types of boundary conditions:

(1) Cauchy problem:

$$\Omega = \mathbb{R}^3, \text{ and } \rho,\ u \text{ vanish at infinity and } d \text{ is constant at infinity (in some weak sense).} \tag{1.7}$$

(2) Dirichlet and Neumann boundary condition for (u, d): $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, and

$$\left(u, \frac{\partial d}{\partial \nu}\right)\bigg|_{\partial\Omega} = 0, \tag{1.8}$$

where ν is the unit outer normal vector of $\partial\Omega$.

(3) Navier-slip and Neumann boundary condition for (u, d): $\Omega \subset \mathbb{R}^3$ is bounded, simply connected, smooth domain, and

$$\left(u \cdot \nu, (\nabla \times u) \times \nu,\ \frac{\partial d}{\partial \nu}\right)\bigg|_{\partial\Omega} = 0, \tag{1.9}$$

where $\nabla \times u$ denotes the vorticity field of the fluid.

To state the definition of strong solutions to the initial and boundary value problem (1.1)–(1.3), (1.6) together with (1.7) or (1.8) or (1.9), we introduce some notations.

We denote

$$\int f\,dx = \int_{\Omega} f\,dx.$$

For $1 \leqslant r \leqslant \infty$, denote the L^r spaces and the standard Sobolev spaces as follows:

$$L^r = L^r(\Omega), \qquad D^{k,r} = \left\{u \in L^1_{\mathrm{loc}}(\Omega)\colon \left\|\nabla^k u\right\|_{L^r} < \infty\right\},$$
$$W^{k,r} = L^r \cap D^{k,r}, \qquad H^k = W^{k,2}, \qquad D^k = D^{k,2},$$
$$D^1_0 = \left\{u \in L^6\colon \|\nabla u\|_{L^2} < \infty, \text{ and satisfies (1.7) or (1.8) or (1.9) for the part of } u\right\},$$
$$H^1_0 = L^2 \cap D^1_0, \qquad \|u\|_{D^{k,r}} = \left\|\nabla^k u\right\|_{L^r}.$$

Denote

$$Q_T = \Omega \times [0,T] \quad (T > 0),$$

and let

$$\mathcal{D}(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right)$$

denote the deformation tensor, which is the symmetric part of the velocity gradient.

Definition 1.1. For $T > 0$, (ρ, u, d) is called a strong solution to the compressible nematic liquid crystal flow (1.1)–(1.3) in $\Omega \times (0, T]$, if for some $q \in (3, 6]$,

$$0 \leqslant \rho \in C\left([0,T]; W^{1,q} \cap H^1\right), \quad \rho_t \in C\left([0,T]; L^2 \cap L^q\right);$$
$$u \in C\left([0,T]; D^2 \cap D^1_0\right) \cap L^2\left(0,T; D^{2,q}\right), \quad u_t \in L^2\left(0,T; D^1_0\right), \quad \sqrt{\rho} u_t \in L^\infty\left(0,T; L^2\right);$$
$$\nabla d \in C\left([0,T]; H^2\right) \cap L^2\left(0,T; H^3\right), \quad d_t \in C\left([0,T]; H^1\right) \cap L^2\left(0,T; H^2\right), \quad |d| = 1 \quad \text{in } \overline{Q}_T;$$

and (ρ, u, d) satisfies (1.1)–(1.3) a.e. in $\Omega \times (0, T]$.

The first main result is concerned with local existence of strong solutions.

Theorem 1.2. *Assume that P satisfies* (1.5), $\rho_0 \geqslant 0$, $\rho_0 \in W^{1,q} \cap H^1 \cap L^1$ *for some* $q \in (3, 6]$, $u_0 \in D^2 \cap D^1_0$, $\nabla d_0 \in H^2$ *and* $|d_0| = 1$ *in* $\overline{\Omega}$. *If, in additions, the following compatibility condition*

$$\mathcal{L}u_0 - \nabla\left(P(\rho_0)\right) - \Delta d_0 \cdot \nabla d_0 = \sqrt{\rho_0}\, g \quad \text{for some } g \in L^2\left(\Omega, \mathbb{R}^3\right) \tag{1.10}$$

holds, then there exist a positive time $T_0 > 0$ *and a unique strong solution* (ρ, u, d) *of* (1.1)–(1.3), (1.6) *together with* (1.7) *or* (1.8) *or* (1.9) *in* $\Omega \times (0, T_0]$.

We would like to point out that an analogous existence theorem of local strong solutions to the isentropic[2] compressible Navier–Stokes equation, under the first two boundary conditions (1.7) and (1.8), has been previously established by Choe and Kim [5] and Cho et al. [4]. A byproduct of Theorem 1.2 yields the existence of local strong solutions to a larger class of compressible Navier–Stokes equations under the Navier-slip boundary condition (1.9), which seems not available in the literature. We would also mention that, after completing this work, we receive a preprint by Chen et al. [3] in which they proved the existence of local strong solution to (1.1)–(1.3) under the Dirichlet boundary condition on (u, d), when $\mathcal{L} = \mu\Delta$ and $P = aP^\gamma$, by fixed point arguments.

We would like to comment that it is a standard fact that the local existence of a unique strong solution to the incompressible nematic liquid crystal flow (i.e. $\rho \equiv 1$ and $\nabla \cdot u = 0$) holds for any initial data $(u_0, d_0) \in (D^2 \cap D^1_0) \times H^3(\Omega, S^2)$ with $\nabla \cdot u_0 = 0$. It is readily seen that this local existence

[2] I.e. $P = a\rho^\gamma$ for some $a > 0$ and $\gamma > 1$.

of strong solutions to the incompressible nematic liquid crystal flow is closely related to Theorem 1.2 when we consider the slightly compressible nematic liquid crystal flow, i.e. $\|\rho_0 - 1\|_{W^{1,q}\cap H^1\cap L^1} \ll 1$ is sufficiently small. In fact, the compatibility condition (1.10) clearly holds in this case.

In dimension one, Ding et al. [8] have proven that the local strong solution to (1.1)–(1.3) under (1.6) and (1.8) is global. For dimensions at least two, it is reasonable to believe that the local strong solution to (1.1)–(1.3) may cease to exist globally. In fact, there exist finite time singularities of the (transported) heat flow of harmonic maps (1.3) in dimensions two or higher (we refer the interested readers to [24] for the exact references). An important question to ask would be what is the main mechanism of possible break down of local strong (or smooth) solutions.

Such a question has been studied for the incompressible Euler equation or the Navier–Stokes equation by Beale, Kato and Majda (BKM) in their pioneering work [1], which showed that the L^∞-bound of vorticity $\nabla \times u$ must blow up. Later, Ponce [31] rephrased the BKM-criterion in terms of the deformation tensor $\mathcal{D}(u)$.

When dealing with the isentropic compressible Navier–Stokes equation, there have recently been several very interesting works on the blow-up criterion. For example, if $0 < T_* < +\infty$ is the maximum time for strong solution, then (i) Huang et al. [15] established a Serrin type criterion: $\lim_{T\uparrow T_*}(\|\operatorname{div} u\|_{L^1(0,T;L^\infty)} + \|\sqrt{\rho}u\|_{L^s(0,T;L^r)}) = \infty$ for $\frac{2}{s} + \frac{3}{r} \leqslant 1$, $3 < r \leqslant \infty$; (ii) Sun et al. [34], and independently [15], showed that if $7\mu > \lambda$, then $\lim_{T\uparrow T_*}\|\rho\|_{L^\infty(0,T;L^\infty)} = \infty$; and (iii) Huang et al. [16] showed $\lim_{T\uparrow T_*}\|\mathcal{D}(u)\|_{L^1(0,T;L^\infty)} = \infty$.

When dealing the heat flow of harmonic maps (1.3) (with $u = 0$), Wang [37] obtained a Serrin type regularity theorem, which implies that if $0 < T_* < +\infty$ is the first singular time for local smooth solutions, then $\lim_{T\uparrow T_*}\|\nabla d\|_{L^2(0,T;L^\infty)} = \infty$.

When dealing with the incompressible nematic liquid crystal flow, Lin et al. [25] and Lin and Wang [23] have established the global existence of a unique "almost strong" solution[3] for the initial–boundary value problem in bounded domains in dimension two, see also Hong [14] and Xu and Zhang [38] for some related works. In dimension three, for the incompressible nematic liquid crystal flow Huang and Wang [17] have obtained a BKM type blow-up criterion very recently, while the existence of global weak solutions still remains to be a largely open question.

Motivated by these works on the blow-up criterion of local strong solutions to the Navier–Stokes equation and the incompressible nematic liquid crystal flow, we will establish in this paper the following blow-up criterion of breakdown of local strong solutions under the boundary condition (1.1) or (1.2).[4]

Theorem 1.3. *Let (ρ, u, d) be a strong solution of the initial boundary problem* (1.1)–(1.3), (1.6) *together with* (1.7) *or* (1.8)*. Assume that P satisfies* (1.5)*, and the initial data (ρ_0, u_0, d_0) satisfies* (1.10)*. If $0 < T_* < +\infty$ is the maximum time of existence and $7\mu > 9\lambda$, then*

$$\lim_{T\uparrow T_*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\nabla d\|_{L^3(0,T;L^\infty)}\right) = \infty. \tag{1.11}$$

We would like to make a few comments of Theorem 1.3.

Remark 1.4. (a) Since we can't yet prove Lemma 4.2 for the Navier-slip and Neumann boundary condition (1.9), it is unclear whether Theorem 1.3 remains to be true under the boundary condition (1.9).

(b) Motivated by the Beale–Kato–Majda criterion on Navier–Stokes equations (see [1,31,16]), it is also a natural question to seek other blow-up criterions of (1.1)–(1.3) involving the vorticity filed of fluids. In [18], we obtained such a blow-up criterion of (1.1)–(1.3) under the initial condition (1.6) and the boundary condition (1.7) or (1.8) or (1.9) in terms of u and ∇d that is valid for all P satisfying (1.5)

[3] That has at most finitely many possible singular time.

[4] It is unclear to the authors whether there exists connection between the blow-up criterion on the incompressible nematic liquid crystal flow obtained by Huang and Wang [17] and the blow-up criterion stated in Theorem 1.3 for the compressible nematic liquid crystal flow, even for slightly compressible cases.

and all μ, λ satisfying (1.4): if $0 < T_* < +\infty$ is the maximum time of existence of strong solutions, then

$$\lim_{T \uparrow T_*} \left(\left\| \mathcal{D}(u) \right\|_{L^1(0,T;L^\infty)} + \|\nabla d\|_{L^2(0,T;L^\infty)} \right) = +\infty.$$

However, the techniques involved in [18] are much different from Theorem 1.3, due to the estimates of both $\|\rho\|_{L_t^\infty L_x^\infty}$ and arbitrarily high integrability of ∇d in terms of $\|\mathcal{D}(u)\|_{L_t^1 L_x^\infty}$ and $\|\nabla d\|_{L_t^2 L_x^\infty}$.

(b) For compressible liquid crystal flows without the nematicity constraint ($|d| = 1$),[5] Liu and Liu [28] have recently obtained a Serrin type criterion on the blow-up of strong solutions under Dirichlet conditions on (u, d), when $\mathcal{L} = \mu \Delta$ and $P = a\rho^\gamma$.

(c) It is a very interesting question to ask whether there exists a global weak solution to the initial–boundary value problem of (1.1)–(1.3) in dimensions at least two. In dimension one, such an existence has been obtained by Ding et al. [6].

Now we briefly outline the main ideas of the proof, some of which are inspired by earlier works on the isentropic compressible Navier–Stokes equations by [4,34,16]. To obtain the existence of a unique local strong solution to (1.1)–(1.3), under (1.6) and (1.7) or (1.8) or (1.9), we employ Galerkin's method that requires us to establish a priori estimate of the quantity

$$\left\| \rho(t) \right\|_{H^1 \cap W^{1,q}} + \left\| \nabla u(t) \right\|_{L^2} + \left\| \sqrt{\rho} u_t(t) \right\|_{L^2} + \left\| \nabla^2 d(t) \right\|_{L^2}, \quad 3 < q \leqslant 6$$

for strong solutions (ρ, u, d) in the form of a Gronwall type inequality. See Theorem 2.1. It may be of independent interest that we establish $W^{2,q}$-estimate for the Lamé equation under the Navier-slip boundary condition, see Lemma 3.1.

Notice that (1.1)–(1.3) are much more complicate that compressible Navier–Stokes equations, due to the super critical nonlinearity $|\nabla d|^2 d$ in the transported heat flow of harmonic map equation (1.3) and the strong coupling nonlinear term $\Delta d \cdot \nabla d$ in the momentum equation (1.2). To prove the blow-up criterion (1.11) of Theorem 1.3 in terms of ρ and ∇d, a critical step is to establish the $L_t^\infty L_x^q$-estimate of $\nabla\rho$. From the continuity equation (1.1), this requires that the Lipschitz norm of velocity field u, or $\|\nabla^2 u(t)\|_{L^q}$ is bounded in L_t^1, which in turns requires. This is done in several steps:

(1) We show that under the condition $7\mu > 9\lambda$, the bound of $(\|\rho\|_{L_t^\infty L_x^\infty} + \|\nabla d\|_{L_t^3 L_x^\infty})$ and Eqs. (1.2) and (1.3) can yield both a high integrability and a high order estimate of u and ∇d, i.e. both $(\|\rho^{\frac{1}{5}} u\|_{L_t^\infty L_x^5} + \|\nabla d\|_{L_t^\infty L_x^5})$ and $(\|\nabla u\|_{L_t^\infty L_x^2} + \|\nabla^2 d\|_{L_t^\infty L_x^2})$ are bounded. See Lemma 4.2.
(2) Utilizing the $L_t^\infty L_x^5$-bound of $\rho^{\frac{1}{5}} u$ and ∇d, we manage to establish that $\nabla^3 d$ is bounded in $L_t^\infty L_x^2$ and ∇u is bounded in $L_t^2 W_x^{1,q} + L_t^\infty(\mathrm{BMO}_x)$. To achieve it, we adapt the approach, due to Sun et al. [34], by decomposing $u = w + v$, where $v \in H_0^1(\Omega)$ solves the Lamé equation $\mathcal{L}v = \nabla(P(\rho))$. One can prove that $\nabla v \in L_t^\infty(\mathrm{BMO}_x)$ by the elliptic regularity theory. The difficult part is to show that $\nabla^2 w \in L_t^2 L_x^q$ for $3 < q \leqslant 6$. In order to obtain this estimate, we first establish that $(\|\sqrt{\rho}\dot{u}\|_{L_t^\infty L_x^2} + \|\nabla d_t\|_{L_t^\infty L_x^2})$ and $(\|\nabla \dot{u}\|_{L_t^2 L_x^2} + \|d_{tt}\|_{L_t^2 L_x^2})$ are bounded by viewing (1.2) as an evolution equation of the material derivative $\dot{u} \equiv u_t + u \cdot \nabla u$ and performing second order energy estimates of both Eqs. (1.2) and (1.3). Then we employ $W^{2,q}$-estimate of the Lamé equation to control $\|\nabla^2 w\|_{L^q}$. The details are illustrated by Lemma 4.4 and Corollary 4.5.
(3) We show that $\|\nabla\rho\|_{L^2 \cap L^q}$ is bounded by an argument similar to [34, §5]. Then we apply $W^{2,q}$-estimate of the Lamé equation again to control $\|\nabla^2 u\|_{L_t^\infty L_x^2}$ and $\|u\|_{L_t^\infty D_x^{2,q}}$. See Lemma 4.6, Corollaries 4.7, and 4.8.

[5] The right-hand side of Eq. (1.3) is replaced by $\Delta d + f(d)$ for some smooth function $f : \mathbb{R}^3 \to \mathbb{R}^3$, e.g. $f(d) = (|d|^2 - 1)d$.

It is interesting to notice that during the proof of both the existence of a unique local strong solutions and the blow-up criterion for strong solutions, specific forms of the pressure function $P(\rho)$ play no roles and it is the local Lipschitz regularity of P that matters.

The paper is written as follows. In Section 2, we derive some a priori estimates for strong solutions or approximate solutions via Galerkin's method. In Section 3, we prove both the local existence by Galerkin's method and uniqueness of strong solutions. In Section 4, we discuss the blow-up criterion of strong solutions and prove Theorem 1.3.

2. A priori estimates

In the section, we will derive some a priori estimates for strong or smooth solutions (ρ, u, d) to (1.1)–(1.3) on a bounded domain, associated with the initial condition (1.6) and the boundary condition (1.8) or (1.9), provided that the initial density function has a positive lower bound, $\rho_0 \geqslant \delta > 0$. All these a priori estimates we will obtain are independent of $\delta > 0$ and the size of the domain when $\Omega = B_R$ ($R \geqslant 1$) is a ball in $\mathbb{R}^3$, which are the crucial ingredients to prove the local existence of strong solutions to (1.1)–(1.3) when we allow the initial data $\rho_0 \geqslant 0$ and unbounded domain $\Omega = \mathbb{R}^3$. Although these estimates may have their own interests, we mainly apply them to the approximate solutions to (1.1)–(1.3) that are constructed by Galerkin's method.

Throughout the paper, we denote by C generic constants that depend on $\|\rho_0\|_{W^{1,q}\cap H^1\cap L^1}$, $\|u_0\|_{D^2\cap D_0^1}$, $\|\nabla d_0\|_{H^2}$, and P, but are independent of $\delta > 0$, the solutions (ρ, u, d) and the size of domain when $\Omega = B_R$ ($R \geqslant 1$) is a ball in $\mathbb{R}^3$. We will also use the obvious notation

$$\|\cdot\|_{X_1\cap\cdots\cap X_k} = \sum_{i=1}^{k} \|\cdot\|_{X_i}$$

for Banach spaces X_i, $1 \leqslant i \leqslant k$ and $k = 2, 3$. We will use $A \lesssim B$ to denote $A \leqslant CB$ for some constant generic $C > 0$.

Let (ρ, u, d) be a strong solution of (1.1)–(1.3) in $\Omega \times (0, T]$ (or the approximate solutions (ρ^m, u^m, d^m) of (1.1)–(1.3) constructed by Galerkin's method in Section 3.2 below). For simplicity, we assume $0 < T \leqslant 1$. For $0 < t < T$, set

$$\Phi(t) := \sup_{0\leqslant s\leqslant t} \left(\left\|\rho(s)\right\|_{H^1\cap W^{1,q}} + \left\|\nabla u(s)\right\|_{L^2} + \left\|\sqrt{\rho} u_t(s)\right\|_{L^2} + \left\|\nabla^2 d(s)\right\|_{H^1} + 1 \right). \tag{2.1}$$

The main aim of this section is to estimate each term of Φ in terms of some integrals of Φ. In Section 3 below, we will apply arguments of Gronwall's type to prove that Φ is locally bounded.

Throughout this section and Section 3, we will let $\mathcal{F}$ to denote the set that consists of monotonic increasing, locally bounded functions M from $[0, +\infty)$ to $[0, +\infty)$ with $M(0) = 0$, which are independent of δ and the size of Ω. The reader will see that the exact form of $M \in \mathcal{F}$ is not important and may vary from lines to lines during the proof of the lemmas.

Now we state the main theorem of this section.

Theorem 2.1. *There exists $M \in \mathcal{F}$ such that for any $0 < t < T$, it holds*

$$\Phi(t) \leqslant \exp\left[C\mathcal{M}(\rho_0, u_0, d_0) + C \int_0^t M\big(\Phi(s)\big)\, ds \right], \tag{2.2}$$

where

$$\mathcal{M}(\rho_0, u_0, d_0) = 1 + \left\| \frac{\mathcal{L}u_0 - \nabla(P(\rho_0)) - \Delta d_0 \cdot \nabla d_0}{\sqrt{\rho_0}} \right\|_{L^2}. \tag{2.3}$$

The proof of Theorem 2.1 is based on several lemmas. We may assume $P(0)=0$. Observe that (1.5) implies that the Lipschitz norm

$$B_P(R) := \|P'\|_{L^\infty([0,R])} : [0,+\infty) \to [0,+\infty) \text{ is monotonic increasing and locally bounded.} \tag{2.4}$$

Lemma 2.2 *(Energy inequality). There exists $M \in \mathcal{F}$ such that for any $0<t<T$, it holds*

$$\int_\Omega \left(\rho|u|^2 + |\nabla d|^2\right) dx + \int_0^t \int_\Omega \left[|\nabla u|^2 + \left|\Delta d + |\nabla d|^2 d\right|^2\right] dx \leqslant C + \int_0^t M\big(\Phi(s)\big)\, ds. \tag{2.5}$$

Proof. Here we only sketch the proof for the boundary condition (1.9). Multiplying (1.2) by u and integrating over Ω, using $\Delta u = \nabla \operatorname{div} u - \nabla \times (\nabla \times u)$ and (1.1), and applying integration by parts several times, we obtain

$$\frac{1}{2}\frac{d}{dt}\int \rho|u|^2\, dx + \int \left(\mu|\nabla \times u|^2 + (2\mu+\lambda)|\operatorname{div} u|^2\right) dx = \int P(\rho) \operatorname{div} u\, dx - \int u \cdot \nabla d \cdot \Delta d\, dx. \tag{2.6}$$

Since Ω is assumed to be simply connected for the boundary condition (1.9), we have (see [36]):

$$\|\nabla u\|_{L^2} \lesssim \|\nabla \times u\|_{L^2} + \|\operatorname{div} u\|_{L^2}, \quad \forall u \in H^1(\Omega) \text{ with } u \cdot \nu = 0 \text{ on } \partial\Omega. \tag{2.7}$$

This and (1.4) imply

$$\int \left(\mu|\nabla \times u|^2 + (2\mu+\lambda)|\operatorname{div} u|^2\right) dx \geqslant \frac{\mu}{3}\int \left(|\nabla \times u|^2 + |\operatorname{div} u|^2\right) dx \geqslant \frac{1}{C}\int |\nabla u|^2\, dx. \tag{2.8}$$

By Cauchy inequality, we have

$$\left|\int P(\rho) \operatorname{div} u\, dx\right| \leqslant \frac{1}{2C}\int |\nabla u|^2\, dx + C\int \left|P(\rho)\right|^2 dx. \tag{2.9}$$

Multiplying (1.3) by $\Delta d + |\nabla d|^2 d$ and integrating over Ω, using integration by parts and the fact that $|d|=1$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int |\nabla d|^2\, dx + \int \left|\Delta d + |\nabla d|^2 d\right|^2 dx = \int u \cdot \nabla d \cdot \Delta d\, dx. \tag{2.10}$$

Combining (2.6), (2.8), (2.9), and (2.10) together, we obtain

$$\frac{d}{dt}\int \left(\rho|u|^2 + |\nabla d|^2\right) dx + \int \left(\frac{1}{C}|\nabla u|^2 + \left|\Delta d + |\nabla d|^2 d\right|^2\right) dx \leqslant C\int \left|P(\rho)\right|^2 dx. \tag{2.11}$$

To estimate the right-hand side of (2.11), first observe that by (2.4) we have[6]

$$\|\rho\|_{L^\infty} + \big\|P(\rho)\big\|_{L^\infty} + \big\|P(\rho)\big\|_{H^1\cap W^{1,q}} \leqslant C\Phi + CB_P\big(\|\rho\|_{L^\infty}\big)\Phi \leqslant M(\Phi) \tag{2.12}$$

for some $M \in \mathcal{F}$. It follows from (1.1) and Sobolev's inequality that

$$\begin{aligned}\int \big|P(\rho)\big|^2\,dx &= \int \big|P(\rho_0)\big|^2\,dx + 2\int_0^t\!\!\int P(\rho)P'(\rho)(-\rho\operatorname{div}u - \nabla\rho\cdot u)\,dx\,dt\\ &\leqslant C + C\int_0^t B_P\big(\|\rho\|_{L^\infty}\big)\big(\big\|P(\rho)\big\|_{L^3}\|\nabla\rho\|_{L^2} + \big\|P(\rho)\big\|_{L^2}\|\rho\|_{L^\infty}\big)\|\nabla u\|_{L^2}\,ds\\ &\leqslant C + \int_0^t M\big(\Phi(s)\big)\,ds \leqslant C + M\big(\Phi(t)\big)\end{aligned} \tag{2.13}$$

as $M(\Phi(s))$ is increasing and $t \leqslant 1$. Substituting (2.13) into (2.11) and integrating over $[0,t]$ yields (2.5). □

Now we want to estimate $\|\nabla u(t)\|_{H^1}^2$ in terms of $\Phi(t)$.

Lemma 2.3. *There exists $M \in \mathcal{F}$ such that for $0 < t < T$, it holds*

$$\big\|\nabla u(t)\big\|_{H^1} \leqslant M\big(\Phi(t)\big). \tag{2.14}$$

Proof. By the standard H^2-estimate of the Lamé equation with respect to the boundary condition (1.7) or (1.8) or (1.9), (2.12), and Hölder's inequality, we have

$$\begin{aligned}\|\nabla u\|_{H^1}^2 &\lesssim \|\mathcal{L}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\\ &\lesssim \|\rho u_t\|_{L^2}^2 + \|\rho u\cdot\nabla u\|_{L^2}^2 + \big\|\nabla\big(P(\rho)\big)\big\|_{L^2}^2 + \|\Delta d\cdot\nabla d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\\ &\lesssim \|\rho\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\rho\|_{L^\infty}^2\|u\|_{L^6}^2\|\nabla u\|_{L^3}^2 + B_P^2\big(\|\rho\|_{L^\infty}\big)\|\nabla\rho\|_{L^2}^2\\ &\quad + \|\Delta d\|_{L^3}^2\|\nabla d\|_{L^6}^2 + \|\nabla u\|_{L^2}^2\\ &\leqslant M(\Phi)\big(1 + \|u\|_{L^6}^2\|\nabla u\|_{L^3}^2\big) + C\|\Delta d\|_{L^3}^2\|\nabla d\|_{L^6}^2\end{aligned} \tag{2.15}$$

for some $M \in \mathcal{F}$. By the interpolation inequality, Sobolev's inequality,[7] we obtain

$$\|u\|_{L^6}^2\|\nabla u\|_{L^3}^2 \leqslant C\|\nabla u\|_{L^2}^3\|\nabla u\|_{H^1}. \tag{2.16}$$

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[6] When $\Omega = B_R$ for $R \geqslant 1$, one can the independence of C with respect to R as follows:

$$\|\rho\|_{L^\infty(B_R)} \leqslant \max_{x\in B_R}\|\rho\|_{L^\infty(B_1(x))} \leqslant C\max_{x\in B_R}\|\rho\|_{W^{1,q}(B_1(x))} \leqslant C\|\rho\|_{W^{1,q}(B_R)}.$$

[7] When $\Omega = B_R$ for $R \geqslant 1$, by simple scalings, one has

$$\|f\|_{L^6(B_R)} \leqslant C\big(R^{-1}\|f\|_{L^2(B_R)} + \|\nabla f\|_{L^2(B_R)}\big) \leqslant C\|f\|_{H^1(B_R)}.$$

Similar to (2.16), by (2.5), we obtain

$$\begin{aligned}\|\Delta d\|_{L^3}^2\|\nabla d\|_{L^6}^2 &\lesssim \|\Delta d\|_{L^2}\|\Delta d\|_{L^6}\|\nabla d\|_{H^1}^2\\ &\lesssim \|\Delta d\|_{H^1}^2\|\nabla d\|_{L^2}^2+\|\Delta d\|_{H^1}^2\left\|\nabla^2 d\right\|_{L^2}^2\lesssim M(\Phi)\end{aligned}\tag{2.17}$$

for some $M\in\mathcal{F}$. Substituting (2.16), (2.17) into (2.15), and using (2.5) and Cauchy's inequality, we have

$$\|\nabla u\|_{H^1}^2\leqslant\frac{1}{2}\|\nabla u\|_{H^1}^2+M\big(\Phi(t)\big)$$

for some $M\in\mathcal{F}$. This gives (2.14) and completes the proof. □

Now we want to estimate $\|\sqrt{\rho}u_t\|_{L^2}$. More precisely, we have

Lemma 2.4. *There exists $M\in\mathcal{F}$ such that for any $0<t<T$, it holds*

$$\int_\Omega\rho|u_t|^2\,dx+\int_0^t\int_\Omega|\nabla u_t|^2\,dx\,ds\leqslant C\mathcal{M}(\rho_0,u_0,d_0)+\int_0^t M\big(\Phi(s)\big)\,ds.\tag{2.18}$$

Proof. Differentiating (1.2) with respect to t, we have[8]

$$\begin{aligned}&\rho u_{tt}+\rho u\cdot\nabla u_t+\rho_t u_t+\rho_t u\cdot\nabla u+\rho u_t\cdot\nabla u+\nabla\big(P(\rho)\big)_t\\ &\quad=(2\mu+\lambda)\nabla\operatorname{div}u_t-\mu\nabla\times(\nabla\times u_t)-\nabla\cdot(\nabla d_t\otimes\nabla d+\nabla d\otimes\nabla d_t-\nabla d\cdot\nabla d_t\mathbb{I}_3).\end{aligned}\tag{2.19}$$

Multiplying (2.19) by u_t, integrating the resulting equations over Ω, and using (1.1) and integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|u_t|^2\,dx+\int\big((2\mu+\lambda)|\operatorname{div}u_t|^2+\mu|\nabla\times u_t|^2\big)\,dx\\ &\quad=-2\int\rho u u_t\cdot\nabla u_t\,dx-\int\rho_t u\cdot\nabla u\cdot u_t\,dx-\int\rho u_t\cdot\nabla u\cdot u_t\,dx+\int P'(\rho)\rho_t\operatorname{div}u_t\,dx\\ &\qquad+\int(\nabla d_t\otimes\nabla d+\nabla d\otimes\nabla d_t-\nabla d\cdot\nabla d_t\,\mathbb{I}_3):\nabla u_t\,dx=\sum_{i=1}^5 II_i.\end{aligned}\tag{2.20}$$

By Hölder's inequality, Sobolev's inequality, (2.12), and (2.14), we have

$$\begin{aligned}|II_1|&\lesssim\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}u\|_{L^\infty}\\ &\lesssim\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|\nabla u\|_{H^1}\lesssim M(\Phi)\|\nabla u_t\|_{L^2}\end{aligned}\tag{2.21}$$

for some $M\in\mathcal{F}$.

[8] Here we have used the fact that $\Delta d\cdot\nabla d=\nabla\cdot(\nabla d\otimes\nabla d-\frac{1}{2}|\nabla d|^2\mathbb{I}_3)$, where $\nabla d\otimes\nabla d=(d_{x_i}\cdot d_{x_j})_{1\leqslant i,j\leqslant 3}$ and $\mathbb{I}_3$ is the identity matrix of order 3.

By (1.1), Hölder's inequality, Sobolev's inequality, (2.12), and (2.14), we have

$$\begin{aligned}
|II_2| &= \left| \int \rho u \cdot \nabla(u \cdot \nabla u \cdot u_t)\, dx \right| \\
&= \left| \int \rho u \cdot (\nabla u \cdot \nabla u \cdot u_t + u \cdot \nabla\nabla u \cdot u_t + u \cdot \nabla u \cdot \nabla u_t)\, dx \right| \\
&\lesssim \|\rho\|_{L^\infty} \|u\|_{L^6}^2 \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2} + \|\rho\|_{L^\infty} \|u\|_{L^6}^2 \left\|\nabla^2 u\right\|_{L^2} \|u_t\|_{L^6} \\
&\quad + \|\sqrt{\rho}\|_{L^\infty} \|u\|_{L^6} \|\nabla u\|_{L^6}^2 \|\sqrt{\rho} u_t\|_{L^2} \\
&\lesssim M(\Phi)\big(1 + \|\nabla u_t\|_{L^2}\big)
\end{aligned} \tag{2.22}$$

for some $M \in \mathcal{F}$. For II_3, by (2.14) we have

$$\begin{aligned}
|II_3| &\lesssim \|\sqrt{\rho}\|_{L^\infty} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u\|_{L^3} \|u_t\|_{L^6} \\
&\lesssim \|\sqrt{\rho}\|_{L^\infty} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{H^1}^{\frac{1}{2}} \|\nabla u_t\|_{L^2} \lesssim M(\Phi) \|\nabla u_t\|_{L^2}
\end{aligned} \tag{2.23}$$

for some $M \in \mathcal{F}$. For II_4, by (1.1), (2.12), and (2.14) we have

$$\begin{aligned}
|II_4| &\lesssim B_P\big(\|\rho\|_{L^\infty}\big) \|\rho_t\|_{L^2} \|\operatorname{div} u_t\|_{L^2} \\
&\lesssim B_P\big(\|\rho\|_{L^\infty}\big)\big(\|\nabla\rho\|_{L^2} \|u\|_{L^\infty} + \|\rho\|_{L^\infty} \|\operatorname{div} u\|_{L^2}\big) \|\operatorname{div} u_t\|_{L^2} \\
&\lesssim M(\Phi) \|\nabla u_t\|_{L^2}
\end{aligned} \tag{2.24}$$

for some $M \in \mathcal{F}$. For II_5, by (2.5) we have

$$\begin{aligned}
|II_5| &\lesssim \int_\Omega |\nabla d||\nabla d_t||\nabla u_t|\, dx \lesssim \|\nabla u_t\|_{L^2} \|\nabla d\|_{L^\infty} \|\nabla d_t\|_{L^2} \\
&\lesssim \|\nabla u_t\|_{L^2} \|\nabla d\|_{H^2} \|\nabla d_t\|_{L^2} \\
&\lesssim \|\nabla u_t\|_{L^2} \big(\|\nabla d\|_{L^2} + \left\|\nabla^2 d\right\|_{H^1}\big) \|\nabla d_t\|_{L^2} \leqslant \big(C + M(\Phi)\big) \|\nabla u_t\|_{L^2} \|\nabla d_t\|_{L^2}
\end{aligned} \tag{2.25}$$

for some $M \in \mathcal{F}$. Substituting (2.21)–(2.25) into (2.20), and using Cauchy's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\int \rho|u_t|^2\, dx + \frac{1}{C}\int |\nabla u_t|^2\, dx \leqslant \frac{1}{2C}\int |\nabla u_t|^2\, dx + M(\Phi) + \big(C + M(\Phi)\big)\|\nabla d_t\|_{L^2}^2 \tag{2.26}$$

for some $M \in \mathcal{F}$, where we have used the following inequality due to [36]: if (i) either Ω is simply connected and $u \cdot \nu = 0$ on $\partial\Omega$ or (ii) $u = 0$ on $\partial\Omega$,[9] then

$$\|\nabla u_t\|_{L^2} \lesssim \|\operatorname{div} u_t\|_{L^2} + \|\nabla \times u_t\|_{L^2}. \tag{2.27}$$

By (2.26), we have

$$\frac{d}{dt}\int \rho|u_t|^2\, dx + \frac{1}{C}\int |\nabla u_t|^2\, dx \lesssim M(\Phi) + \big(C + M(\Phi)\big)\|\nabla d_t\|_{L^2}^2. \tag{2.28}$$

[9] In fact, in this case, the inequality (2.27) is an equality.

Differentiating (1.3) with respect to x, we have

$$\nabla d_t - \nabla \Delta d = \nabla\big(|\nabla d|^2 d\big) - \nabla(u \cdot \nabla d). \tag{2.29}$$

From (2.29), we have[10]

$$\begin{aligned}
\|\nabla d_t\|_{L^2} &\lesssim \|\nabla u \cdot \nabla d\|_{L^2} + \big\|u \cdot \nabla^2 d\big\|_{L^2} + \|\nabla \Delta d\|_{L^2} + \|\nabla d\|_{L^6}^3 + \big\|\nabla d \cdot \nabla^2 d\big\|_{L^2} \\
&\lesssim \|\nabla d\|_{L^\infty}\|\nabla u\|_{L^2} + \|u\|_{L^6}\big\|\nabla^2 d\big\|_{L^3} + \|\nabla \Delta d\|_{L^2} + \big(1 + \big\|\nabla^2 d\big\|_{L^2}\big)^3 + \|\nabla d\|_{L^\infty}\big\|\nabla^2 d\big\|_{L^2} \\
&\lesssim \|\nabla d\|_{H^2}\|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}\big\|\nabla^2 d\big\|_{H^1} + \|\nabla \Delta d\|_{L^2} + \big(1 + \big\|\nabla^2 d\big\|_{L^2}\big)^3 + \|\nabla d\|_{H^2}\big\|\nabla^2 d\big\|_{L^2} \\
&\lesssim M(\Phi) + 1
\end{aligned} \tag{2.30}$$

for some $M \in \mathcal{F}$.

Substituting (2.30) into (2.28), and using Cauchy's inequality, we have

$$\frac{d}{dt}\int \rho|u_t|^2\,dx + \frac{1}{C}\int |\nabla u_t|^2\,dx \leqslant M(\Phi) + C \tag{2.31}$$

for some $M \in \mathcal{F}$. Integrating (2.31) over $(0,t)$, and using (1.2), and (1.10), we have

$$\begin{aligned}
\int \rho|u_t|^2\,dx + \int_0^t\int_\Omega |\nabla u_t|^2\,dx\,ds &\leqslant C\int \rho|u_t|^2\,dx|_{t=0} + \int_0^t M\big(\Phi(s)\big)\,ds + C \\
&\leqslant C\mathcal{M}(\rho_0, u_0, d_0) + \int_0^t M\big(\Phi(s)\big)\,ds
\end{aligned}$$

for some $M \in \mathcal{F}$. This completes the proof. □

As an immediate consequence of Lemma 2.4, we obtain an estimate of $\|\nabla u\|_{L^2}$.

Lemma 2.5. *There exists $M \in \mathcal{F}$ such that for $0 < t < T$, it holds*

$$\int \big|\nabla u(t)\big|^2\,dx \leqslant C\mathcal{M}(\rho_0, u_0, d_0) + \int_0^t M\big(\Phi(s)\big)\,ds. \tag{2.32}$$

Proof. By Cauchy's inequality, Lemma 2.2, Lemma 2.3, and Lemma 2.4, we have

$$\int |\nabla u|^2(t)\,dx = \int |\nabla u_0|^2\,dx + 2\int_0^t\int_\Omega \nabla u \cdot \nabla u_t\,dx\,ds$$

[10] Here we also use the Sobolev's inequality: $\|\nabla d\|_{L^\infty(\Omega)} \leqslant C\|\nabla d\|_{H^2(\Omega)}$ and the fact that C can be chosen independent of R when $\Omega = B_R$ for $R \geqslant 1$.

$$\leqslant C + \int_0^t \int_\Omega |\nabla u|^2 \, dx \, ds + \int_0^t \int_\Omega |\nabla u_t|^2 \, dx \, ds$$

$$\leqslant C\mathcal{M}(\rho_0, u_0, d_0) + \int_0^t M\big(\Phi(s)\big) \, ds$$

for some $M \in \mathcal{F}$. This completes the proof. □

Lemma 2.6. *There exists $M \in \mathcal{F}$ such that for $0 < t < T$, it holds*

$$\big\|\rho(t)\big\|_{H^1 \cap W^{1,q}} \leqslant \exp\left\{C\mathcal{M}(\rho_0, u_0, d_0) + C \int_0^t M\big(\Phi(s)\big) \, ds\right\}. \tag{2.33}$$

Proof. It follows from [4, p. 249, (2.11)] that

$$\big\|\rho(t)\big\|_{H^1 \cap W^{1,q}} \leqslant \|\rho_0\|_{H^1 \cap W^{1,q}} \exp\left\{C \int_0^t \|\nabla u\|_{H^1 \cap D^{1,q}} \, ds\right\}. \tag{2.34}$$

By $W^{2,q}$-estimate of the Lamé equation under either Dirichlet boundary condition (1.8) or the Navier-slip boundary condition (1.9) (see Lemma 3.1 below), (1.2), and Sobolev's inequality, we have

$$\begin{aligned} \big\|\nabla^2 u\big\|_{L^q} &\lesssim \|\rho u_t\|_{L^q} + \|\rho u \cdot \nabla u\|_{L^q} + \big\|\nabla\big(P(\rho)\big)\big\|_{L^q} + \|\nabla d \cdot \Delta d\|_{L^q} \\ &= \sum_{i=1}^4 III_i. \end{aligned} \tag{2.35}$$

If $q = 6$, then by Sobolev's inequality we have

$$III_1 \lesssim \|\rho\|_{L^\infty} \|u_t\|_{L^6} \lesssim \Phi \|\nabla u_t\|_{L^2}. \tag{2.36}$$

If $q \in (3, 6)$, then by Hölder's inequality and Sobolev's inequality, we have

$$III_1 \lesssim \|\rho\|_{L^{\frac{6q}{6-q}}} \|u_t\|_{L^6} \lesssim \|\rho\|_{L^1}^{\frac{6-q}{6q}} \|\rho\|_{L^\infty}^{1-\frac{6-q}{6q}} \|\nabla u_t\|_{L^2} \lesssim \Phi \|\nabla u_t\|_{L^2}, \tag{2.37}$$

where we have used the fact that $\int \rho \, dx = \int \rho_0 \, dx$. From (2.36) and (2.37), we have that for $q \in (3, 6]$,

$$III_1 \lesssim \Phi \|\nabla u_t\|_{L^2}. \tag{2.38}$$

For III_2, if $q \in (3, 6]$, then by similar arguments, Lemmas 2.2 and 2.3, we have

$$III_2 \lesssim \Phi \|\nabla u\|_{H^1}^2 \leqslant M(\Phi) \tag{2.39}$$

for some $M \in \mathcal{F}$. For III_3 and III_4, if $q \in (3, 6]$, then we have

$$III_3 + III_4 \leqslant CB_P\big(\|\rho\|_{L^\infty}\big)\|\nabla \rho\|_{L^q} + \|\nabla d\|_{H^2}^2 \leqslant M(\Phi) \tag{2.40}$$

for some $M \in \mathcal{F}$. Substituting (2.38), (2.39) and (2.40) into (2.35), we have

$$\|\nabla^2 u\|_{L^q} \lesssim \Phi \|\nabla u_t\|_{L^2} + M(\Phi) \leqslant \|\nabla u_t\|_{L^2}^2 + M(\Phi) \tag{2.41}$$

for some $M \in \mathcal{F}$. Integrating (2.41) over $(0,t)$, and using Cauchy's inequality and (2.18), we have

$$\int_0^t \|\nabla^2 u\|_{L^q} \leqslant C\mathcal{M}(\rho_0, u_0, d_0) + \int_0^t M\big(\Phi(s)\big)\, ds. \tag{2.42}$$

Substituting (2.14) and (2.42) into (2.34), we have

$$\|\rho(t)\|_{H^1 \cap W^{1,q}} \lesssim \exp\left\{C\mathcal{M}(\rho_0, u_0, d_0) + C\int_0^t M\big(\Phi(s)\big)\, ds\right\}$$

for some $M \in \mathcal{F}$. This completes the proof. □

Lemma 2.7. *There exists $M \in \mathcal{F}$ such that for any $0 < t < T$, it holds*

$$\|\nabla^2 d\|_{L^2}^2 + \int_0^t \|\nabla d_t\|_{L^2}^2\, ds \leqslant C + \int_0^t M\big(\Phi(s)\big)\, ds. \tag{2.43}$$

Proof. Multiplying (2.29) by ∇d_t and integrating over Ω, using integration by parts and $\frac{\partial d_t}{\partial \nu} = 0$ on $\partial\Omega$, we obtain

$$\begin{aligned}\|\nabla d_t\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\Delta d\|_{L^2}^2 &= \int \big[\nabla\big(|\nabla d|^2 d\big) - \nabla(u\cdot\nabla d)\big]\nabla d_t\, dx \\ &\leqslant \frac{1}{2}\|\nabla d_t\|_{L^2}^2 + C\int \big|\nabla\big(|\nabla d|^2 d\big)\big|^2\, dx + C\int \big|\nabla(u\cdot\nabla d)\big|^2\, dx.\end{aligned}$$

Thus we have

$$\|\nabla d_t\|_{L^2}^2 + \frac{d}{dt}\|\Delta d\|_{L^2}^2 \lesssim \int \big|\nabla\big(|\nabla d|^2 d\big)\big|^2\, dx + \int \big|\nabla(u\cdot\nabla d)\big|^2\, dx. \tag{2.44}$$

Similar to the proof of (2.30), we obtain

$$\|\nabla d_t\|_{L^2}^2 + \frac{d}{dt}\|\Delta d\|_{L^2}^2 \leqslant M(\Phi) \tag{2.45}$$

for some $M \in \mathcal{F}$. Integrating (2.45) over $(0,t)$ and applying $W^{2,2}$-estimate of Eq. (1.3), we have

$$\|\nabla^2 d\|_{L^2}^2 + \int_0^t \|\nabla d_t\|_{L^2}^2\, ds \leqslant \|\nabla^2 d_0\|_{L^2}^2 + \int_0^t M\big(\Phi(s)\big)\, ds \leqslant C + \int_0^t M\big(\Phi(s)\big)\, ds.$$

This completes the proof. □

Lemma 2.8. *There exists $M \in \mathcal{F}$ such that for $0 < t < T$, it holds*

$$\left\|\nabla^3 d\right\|_{L^2}^2 + \int_0^t \left\|\nabla^2 d_t\right\|_{L^2}^2 ds \leqslant \left(C\mathcal{M}(\rho_0, u_0, d_0) + \int_0^t M\big(\Phi(s)\big)\, ds\right)^4. \tag{2.46}$$

Proof. Multiplying (2.29) by $\nabla\Delta d_t$, integrating over Ω, using $\frac{\partial d_t}{\partial \nu} = 0$ on $\partial\Omega$ and integration by parts, we obtain

$$\begin{aligned}
\|\Delta d_t\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\Delta d\|_{L^2}^2 &= \int \big[\nabla(u\cdot\nabla d) - \nabla\big(|\nabla d|^2 d\big)\big]\cdot\nabla\Delta d_t\, dx \\
&= \frac{d}{dt}\int \big[\nabla(u\cdot\nabla d) - \nabla\big(|\nabla d|^2 d\big)\big]\cdot\nabla\Delta d\, dx \\
&\quad - \int \frac{\partial}{\partial t}\big[\nabla(u\cdot\nabla d) - \nabla\big(|\nabla d|^2 d\big)\big]\cdot\nabla\Delta d\, dx.
\end{aligned} \tag{2.47}$$

Now we need to estimate the second term of right side as follows:

$$\begin{aligned}
-\int \frac{\partial}{\partial t}\big[\nabla(u\cdot\nabla d)\big]\cdot\nabla\Delta d\, dx &= -\int \big[\nabla u_t\cdot\nabla d + \nabla u\cdot\nabla d_t + u_t\cdot\nabla^2 d + u\cdot\nabla^2 d_t\big]\cdot\nabla\Delta d\, dx \\
&= \sum_{i=1}^4 IV_i.
\end{aligned} \tag{2.48}$$

By Hölder's inequality and Sobolev's inequality, we have

$$|IV_1| \lesssim \|\nabla u_t\|_{L^2}\|\nabla d\|_{L^\infty}\|\nabla\Delta d\|_{L^2} \lesssim \|\nabla u_t\|_{L^2}\|\nabla d\|_{H^2}^2 \lesssim M(\Phi) + \|\nabla u_t\|_{L^2}^2 \tag{2.49}$$

for some $M \in \mathcal{F}$.

By Hölder's inequality, Sobolev's inequality, (2.14), (2.30) and Young's inequality, we obtain

$$\begin{aligned}
|IV_2| &\lesssim \|\nabla u\|_{L^6}\|\nabla d_t\|_{L^3}\|\nabla\Delta d\|_{L^2} \\
&\lesssim \|\nabla u\|_{H^1}\|\nabla d_t\|_{H^1}\|\nabla\Delta d\|_{L^2} \\
&\lesssim \|\nabla u\|_{H^1}\left\|\nabla^2 d_t\right\|_{L^2}\|\nabla\Delta d\|_{L^2} + \|\nabla u\|_{H^1}\|\nabla d_t\|_{L^2}\|\nabla\Delta d\|_{L^2} \\
&\leqslant \varepsilon\left\|\nabla^2 d_t\right\|_{L^2}^2 + M(\Phi)
\end{aligned} \tag{2.50}$$

for some $M \in \mathcal{F}$.

By Hölder's inequality, Sobolev's inequality and Cauchy's inequality, we obtain

$$\begin{aligned}
|IV_3| &\lesssim \|u_t\|_{L^6}\left\|\nabla^2 d\right\|_{L^3}\|\nabla\Delta d\|_{L^2} \\
&\lesssim \|\nabla u_t\|_{L^2}\left\|\nabla^2 d\right\|_{H^1}\|\nabla\Delta d\|_{L^2} \lesssim M(\Phi) + \|\nabla u_t\|_{L^2}^2
\end{aligned} \tag{2.51}$$

for some $M \in \mathcal{F}$.

By Hölder's inequality, Sobolev's inequality, (2.14) and Cauchy's inequality, we obtain

$$
\begin{aligned}
|IV_4| &\lesssim \|u\|_{L^\infty} \left\|\nabla^2 d_t\right\|_{L^2} \|\nabla\Delta d\|_{L^2} \lesssim \|\nabla u\|_{H^1} \left\|\nabla^2 d_t\right\|_{L^2} \|\nabla\Delta d\|_{L^2} \\
&\leqslant \varepsilon \left\|\nabla^2 d_t\right\|_{L^2}^2 + M(\Phi)
\end{aligned} \tag{2.52}
$$

for some $M \in \mathcal{F}$.

Combining (2.48), (2.49), (2.50), (2.51) and (2.52), we obtain

$$
-\int \frac{\partial}{\partial t}\big[\nabla(u \cdot \nabla d)\big] \cdot \nabla\Delta d\, dx \leqslant 2\varepsilon \left\|\nabla^2 d_t\right\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2 + M(\Phi) \tag{2.53}
$$

for some $M \in \mathcal{F}$.

By Leibniz's rule and the fact $|d| = 1$, we have

$$
\begin{aligned}
&\int \frac{\partial}{\partial t}\big[\nabla\big(|\nabla d|^2 d\big)\big] \cdot \nabla\Delta d\, dx \\
&\lesssim \int \big[|\nabla d|^2|\nabla d_t| + |\nabla d_t|\left|\nabla^2 d\right| + |\nabla d|\left|\nabla^2 d_t\right| + |\nabla d|\left|\nabla^2 d\right||d_t|\big]|\nabla\Delta d|\, dx \\
&= \sum_{i=1}^4 V_i.
\end{aligned} \tag{2.54}
$$

By Hölder's inequality, Sobolev's inequality and (2.30), Cauchy inequality, and Young inequality, we obtain

$$
|V_1| \lesssim \|\nabla d\|_{L^\infty}^2 \|\nabla d_t\|_{L^2} \|\nabla\Delta d\|_{L^2} \lesssim \|\nabla d\|_{H^2}^2 \|\nabla d_t\|_{L^2} \|\nabla\Delta d\|_{L^2} \leqslant M(\Phi), \tag{2.55}
$$

$$
\begin{aligned}
|V_2| &\lesssim \|\nabla d_t\|_{L^6} \left\|\nabla^2 d\right\|_{L^3} \|\nabla\Delta d\|_{L^2} \lesssim \|\nabla d_t\|_{H^1} \left\|\nabla^2 d\right\|_{H^1}^2 \\
&\lesssim \Phi\big(\left\|\nabla^2 d_t\right\|_{L^2} + \|\nabla d_t\|_{L^2}\big) \leqslant \varepsilon \left\|\nabla^2 d_t\right\|_{L^2}^2 + M(\Phi),
\end{aligned} \tag{2.56}
$$

$$
\begin{aligned}
|V_3| &\lesssim \|\nabla d\|_{L^\infty} \left\|\nabla^2 d_t\right\|_{L^2} \|\nabla\Delta d\|_{L^2} \\
&\lesssim \|\nabla d\|_{H^2} \left\|\nabla^2 d_t\right\|_{L^2} \|\nabla\Delta d\|_{L^2} \leqslant \varepsilon \left\|\nabla^2 d_t\right\|_{L^2}^2 + M(\Phi),
\end{aligned} \tag{2.57}
$$

$$
\begin{aligned}
|V_4| &\lesssim \|d_t\|_{L^6} \|\nabla d\|_{L^\infty} \left\|\nabla^2 d\right\|_{L^3} \|\nabla\Delta d\|_{L^2} \\
&\lesssim \|d_t\|_{H^1} \|\nabla d\|_{H^2} \left\|\nabla^2 d\right\|_{H^1} \|\nabla\Delta d\|_{L^2} \lesssim \|d_t\|_{H^1} M(\Phi)
\end{aligned} \tag{2.58}
$$

for some $M \in \mathcal{F}$. Notice that

$$
\begin{aligned}
\|d_t\|_{L^2} &\lesssim \|\Delta d\|_{L^2} + \|\nabla d\|_{L^4}^2 + \|u \cdot \nabla d\|_{L^2} \lesssim \|\nabla d\|_{H^1}^2 + \|u\|_{L^6} \|\nabla d\|_{L^3} + 1 \\
&\lesssim \|\nabla d\|_{H^1}^2 + \|\nabla u\|_{L^2} \|\nabla d\|_{H^1} + 1 \lesssim \Phi.
\end{aligned} \tag{2.59}
$$

Thus by (2.30), (2.58) and (2.59), we have

$$
|V_4| \leqslant M(\Phi) \tag{2.60}
$$

for some $M \in \mathcal{F}$. Combining (2.54), (2.55), (2.56), (2.57) and (2.60), we have

$$\int \frac{\partial}{\partial t}\big[\nabla\big(|\nabla d|^2 d\big)\big]\cdot\nabla\Delta d\,dx \leqslant 2\varepsilon\big\|\nabla^2 d_t\big\|_{L^2}^2 + M(\Phi) \tag{2.61}$$

for some $M \in \mathcal{F}$.

Putting (2.53) and (2.61) into (2.47), we obtain

$$\begin{aligned}\|\Delta d_t\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\Delta d\|_{L^2}^2 \leqslant{}& \frac{d}{dt}\int\big[\nabla(u\cdot\nabla d) - \nabla\big(|\nabla d|^2 d\big)\big]\cdot\nabla\Delta d\,dx\\ &+ 4\varepsilon\big\|\nabla^2 d_t\big\|_{L^2}^2 + C\|\nabla u_t\|_{L^2}^2 + M(\Phi)\end{aligned} \tag{2.62}$$

for some $M \in \mathcal{F}$. Integrating (2.62) over $(0,t)$, using H^k $(k=2,3)$ estimate of the elliptic equations, and choosing ε small enough, we have

$$\begin{aligned}&\big\|\nabla^3 d\big\|_{L^2}^2 + \int_0^t \big\|\nabla^2 d_t\big\|_{L^2}^2\,ds\\ &\lesssim \int\big|\nabla(u\cdot\nabla d) - \nabla\big(|\nabla d|^2 d\big)\big||\nabla\Delta d|\,dx + \int\big|\nabla(u_0\cdot\nabla d_0) - \nabla\big(|\nabla d_0|^2 d_0\big)\big||\nabla\Delta d_0|\,dx\\ &\quad+ \big\|\nabla^3 d_0\big\|_{L^2}^2 + \int_0^t\|\nabla u_t\|_{L^2}^2\,ds + \int_0^t M\big(\Phi(s)\big)\,ds.\end{aligned} \tag{2.63}$$

For the first term of right side of (2.63), we have

$$\begin{aligned}&\int\big|\nabla(u\cdot\nabla d) - \nabla\big(|\nabla d|^2 d\big)\big||\nabla\Delta d|\,dx\\ &\quad\lesssim \int\big(|\nabla u||\nabla d| + |u|\big|\nabla^2 d\big| + |\nabla d|^3 + |\nabla d|\big|\nabla^2 d\big|\big)|\nabla\Delta d|\,dx\\ &\quad= \sum_{i=1}^4 VI_i.\end{aligned} \tag{2.64}$$

By Hölder's inequality, Nirenberg's interpolation inequality, (2.5), and Young's inequality, we obtain

$$\begin{aligned}|VI_1| &\lesssim \|\nabla d\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2} \lesssim \|\nabla d\|_{L^2}^{\frac14}\|\nabla d\|_{H^2}^{\frac34}\|\nabla u\|_{L^2}\|\nabla\Delta d\|_{L^2}\\ &\lesssim \|\nabla d\|_{H^1}^{\frac34}\|\nabla u\|_{L^2}\big\|\nabla^3 d\big\|_{L^2} + \big\|\nabla^3 d\big\|_{L^2}^{\frac74}\|\nabla u\|_{L^2}\\ &\leqslant \varepsilon\big\|\nabla^3 d\big\|_{L^2}^2 + C\big(\|\nabla d\|_{H^1}^{\frac32}\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^8\big),\end{aligned} \tag{2.65}$$

$$\begin{aligned}|VI_2| &\lesssim \|u\|_{L^6}\big\|\nabla^2 d\big\|_{L^3}\|\nabla\Delta d\|_{L^2} \lesssim \|\nabla u\|_{L^2}\big\|\nabla^2 d\big\|_{L^2}^{\frac12}\big\|\nabla^2 d\big\|_{H^1}^{\frac12}\big\|\nabla^3 d\big\|_{L^2}\\ &\lesssim \|\nabla u\|_{L^2}\big\|\nabla^2 d\big\|_{L^2}\big\|\nabla^3 d\big\|_{L^2} + \|\nabla u\|_{L^2}\big\|\nabla^2 d\big\|_{L^2}^{\frac12}\big\|\nabla^3 d\big\|_{L^2}^{\frac32}\\ &\leqslant \varepsilon\big\|\nabla^3 d\big\|_{L^2}^2 + C\big\|\nabla^2 d\big\|_{L^2}^2\big(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4\big),\end{aligned} \tag{2.66}$$

$$|VI_3| \lesssim \|\nabla d\|_{L^6}^3\|\nabla\Delta d\|_{L^2} \lesssim \|\nabla d\|_{H^1}^3\big\|\nabla^3 d\big\|_{L^2} \leqslant \varepsilon\big\|\nabla^3 d\big\|_{L^2}^2 + C\|\nabla d\|_{H^1}^6, \tag{2.67}$$

and

$$\begin{aligned}|VI_4| &\lesssim \|\nabla d\|_{L^\infty}\|\nabla^2 d\|_{L^2}\|\nabla\Delta d\|_{L^2} \lesssim \|\nabla d\|_{L^2}^{\frac14}\|\nabla d\|_{H^2}^{\frac34}\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2}\\ &\lesssim \|\nabla d\|_{H^1}^{\frac34}\|\nabla^2 d\|_{L^2}\|\nabla^3 d\|_{L^2} + \|\nabla^3 d\|_{L^2}^{\frac74}\|\nabla^2 d\|_{L^2}\\ &\leqslant \varepsilon\|\nabla^3 d\|_{L^2}^2 + C\big(\|\nabla d\|_{H^1}^{\frac32}\|\nabla^2 d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^8\big). \end{aligned} \tag{2.68}$$

Combining (2.64), (2.65), (2.66), (2.67) and (2.68), we obtain

$$\begin{aligned}&\int \big|\nabla(u\cdot\nabla d) - \nabla\big(|\nabla d|^2 d\big)\big||\nabla\Delta d|\,dx\\ &\leqslant 4\varepsilon\|\nabla^3 d\|_{L^2}^2 + C\|\nabla d\|_{H^1}^{\frac32}\big(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2\big) + C\|\nabla^2 d\|_{L^2}^2\big(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4\big)\\ &\quad + C\big(\|\nabla d\|_{H^1}^6 + \|\nabla^2 d\|_{L^2}^8 + \|\nabla u\|_{L^2}^8\big)\\ &\leqslant 4\varepsilon\|\nabla^3 d\|_{L^2}^2 + \left(C\mathcal{M}(\rho_0,u_0,d_0) + \int_0^t M\big(\Phi(s)\big)\,ds\right)^4 \end{aligned} \tag{2.69}$$

for some $M \in \mathcal{F}$, where we have used Lemma 2.2, Lemma 2.5, and Lemma 2.7 in the last step.

Substituting (2.69) into (2.63), choosing ε small enough, and using (2.18), Cauchy's inequality, Lemma 2.5 and (2.43), we have

$$\|\nabla^3 d\|_{L^2}^2 + \int_0^t \|\nabla^2 d_t\|_{L^2}^2\,ds \leqslant \left(C\mathcal{M}(\rho_0,u_0,d_0) + \int_0^t M\big(\Phi(s)\big)\,ds\right)^4$$

for some $M \in \mathcal{F}$. This completes the proof. □

Proof of Theorem 2.1. It is readily seen that the conclusion follows from (2.18), (2.32), (2.33), (2.43) and (2.46). □

3. Proof of Theorem 1.2

3.1. $W^{2,p}$-estimate

In this subsection, we give a proof of $W^{2,p}$-estimate of the Lamé equation on a simply connected, bounded, smooth domain with the Navier-slip boundary condition, which is needed in our proof of Theorem 1.2. We believe that such an estimate may have its own interest.

Lemma 3.1. *For any simply connected, smooth bounded domain $\Omega \subset \mathbb{R}^3$, $1 < p < +\infty$, and $f \in L^p(\Omega, \mathbb{R}^3)$, If $u \in H^1 \cap H^2(\Omega, \mathbb{R}^3)$ is a weak solution of*

$$\begin{aligned}\mathcal{L}u &= f \quad \text{in } \Omega,\\ u\cdot\nu &= (\nabla\times u)\times\nu = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Then $u \in W^{2,p}(\Omega)$, and there exists $C > 0$ depending on p, Ω, and $\mathcal{L}$ such that

$$\|\nabla^2 u\|_{L^p} \leqslant C\big[\|f\|_{L^p} + \|\nabla u\|_{L^2}\big]. \tag{3.2}$$

Proof. By the duality argument, we may assume $1 < p \leqslant 2$. Since $u \cdot \nu = 0$ on $\partial\Omega$, it follows from Bourguignon and Brezis [2] that

$$\|\nabla^2 u\|_{L^p} \lesssim \|\nabla(\operatorname{div} u)\|_{L^p} + \|\nabla(\operatorname{curl} u)\|_{L^p} + \|\nabla u\|_{L^p}. \tag{3.3}$$

Also, since Ω is simply connected and $(\nabla \times u) \times \nu = 0$ on $\partial\Omega$, it follows from Von Wahl [36] that

$$\begin{aligned} \|\nabla(\operatorname{curl} u)\|_{L^p} &\leqslant C\|\nabla \times \operatorname{curl} u\|_{L^p} + \|\nabla \cdot (\operatorname{curl} u)\|_{L^p} = C\|\nabla \times (\operatorname{curl} u)\|_{L^p} \\ &\lesssim \frac{1}{\mu}\big[\|\mathcal{L}u\|_{L^p} + (2\mu+\lambda)\|\nabla(\operatorname{div} u)\|_{L^p}\big] \\ &\lesssim \|\nabla(\operatorname{div} u)\|_{L^p} + \|f\|_{L^p}. \end{aligned} \tag{3.4}$$

Now we estimate $\|\nabla(\operatorname{div} u)\|_{L^p}$ by the duality argument: for $p' = \frac{p}{p-1}$,

$$\|\nabla(\operatorname{div} u)\|_{L^p} \leqslant C \sup\left\{ \int \nabla(\operatorname{div} u) \cdot g\, dx\colon\ g \in C^\infty(\overline{\Omega}, \mathbb{R}^3),\ \|g\|_{L^{p'}} = 1 \right\}.$$

For any $g \in C^\infty(\overline{\Omega}, \mathbb{R}^3)$, with $\|g\|_{L^{p'}} = 1$, by Helmholtz's decomposition theorem (see Fujiwara and Morimoto [12] and Solonnikov [33]), there exist $G \in C^\infty(\overline{\Omega}) \cap W^{1,p'}(\Omega)$ and $H \in C^\infty(\overline{\Omega}) \cap L^{p'}(\Omega, \mathbb{R}^3)$ such that

$$\begin{gathered} g = \nabla G + H, \qquad \operatorname{div} H = 0 \quad \text{in } \Omega, \\ \frac{\partial G}{\partial \nu} = g \cdot \nu \quad \text{on } \partial\Omega, \\ \|G\|_{W^{1,p'}} + \|H\|_{L^{p'}} \leqslant C\|g\|_{L^{p'}} = C. \end{gathered}$$

Thus we have

$$\int \nabla(\operatorname{div} u) \cdot H\, dx = 0$$

so that

$$\begin{aligned} \int \nabla(\operatorname{div} u) \cdot g\, dx &= \int \nabla(\operatorname{div} u) \cdot (\nabla G + H)\, dx = \int \nabla(\operatorname{div} u) \cdot \nabla G\, dx \\ &= \int \left(\nabla(\operatorname{div} u) - \frac{1}{2\mu+\lambda} f\right) \cdot \nabla G\, dx + \frac{1}{2\mu+\lambda} \int f \cdot \nabla G\, dx \\ &= \frac{\mu}{2\mu+\lambda} \int \nabla \times (\operatorname{curl} u) \cdot \nabla G\, dx + \frac{1}{2\mu+\lambda} \int f \cdot \nabla G\, dx \\ &= \frac{1}{2\mu+\lambda} \int f \cdot \nabla G\, dx, \end{aligned}$$

where we have used

$$\int \nabla \times (\operatorname{curl} u) \cdot \nabla G = 0,$$

since $\operatorname{div}(\nabla \times (\operatorname{curl} u)) = 0$ in Ω and $(\operatorname{curl} u) \times \nu = 0$ on $\partial\Omega$. The above inequality implies

$$\left| \int \nabla(\operatorname{div} u) \cdot g\, dx \right| \lesssim \|f\|_{L^p} \|\nabla G\|_{L^{p'}} \leqslant C\|f\|_{L^p}.$$

Taking supremum over all such g's, we obtain

$$\left\| \nabla(\operatorname{div} u) \right\|_{L^p} \leqslant C\|f\|_{L^p}.$$

It is clear that this, with the help of (3.3) and (3.4), implies (3.2). □

3.2. Existence

In this subsection, we will first consider that $\Omega \subset \mathbb{R}^3$ is a bounded domain, and then employ Galerkin's method to obtain a sequence of approximate solutions to (1.1)–(1.3) under (1.6) and (1.8) or (1.9) that enjoy *a priori* estimates obtained in Section 2, which will converge to a strong solution to (1.1)–(1.3). The existence of strong solutions for the Cauchy problem on $\mathbb{R}^3$ follows in a standard way from a priori estimates by the *domain exhaustion* technique, which will be sketched at the end of this subsection.

To implement Galerkin's method, we take the function space X to be either

(i) for the Dirichlet boundary condition (1.8), $X := H_0^1 \cap H^2(\Omega, \mathbb{R}^3)$ and its finite dimensional subspaces as

$$X^m := \operatorname{span}\{\phi^1, \ldots, \phi^m\}, \quad m \geqslant 1,$$

where $\{\phi^m\} \subset X$ is an orthonormal base of $H^1(\Omega)$, formed by the set of eigenfunction of the Lamé operator under the boundary condition $u = 0$ on $\partial\Omega$; or

(ii) for the Navier-slip boundary condition (1.9),

$$X := \left\{ u \in H^2(\Omega, \mathbb{R}^3):\ u \cdot \nu = (\nabla \times u) \times \nu = 0 \text{ on } \partial\Omega \right\},$$

and its finite dimensional subspaces as

$$X^m := \operatorname{span}\{\phi^1, \ldots, \phi^m\}, \quad m \geqslant 1,$$

where $\{\phi^m\} \subset X$ is an orthonormal base of $H^1(\Omega)$, formed by the set of eigenfunction of the Lamé operator under the Navier-slip boundary condition $u \cdot \nu = (\nabla \times u) \times \nu = 0$ on $\partial\Omega$. By the $W^{2,p}$-estimate of Lamé equation under (1.8) or (1.9) (see Lemma 3.1), we see that $\{\phi^m\} \subset W^{2,p}(\Omega)$ for any $1 < p < +\infty$.

Now we outline Galerkin's scheme into several steps:

Step 1 (Modification of initial data). For $\delta > 0$, let $\rho_0^\delta = \rho_0 + \delta$, $d_0^\delta = d_0$, and $u_0^\delta \in X$ be the unique solution of

$$\mathcal{L}u_0^\delta - \nabla\left(P\left(\rho_0^\delta\right)\right) - \Delta d_0 \cdot \nabla d_0 = \sqrt{\rho_0^\delta}\, g \quad \text{in } \Omega, \tag{3.5}$$

$$u_0^\delta = 0; \quad \text{or} \quad u_0^\delta \cdot \nu = \left(\nabla \times u_0^\delta\right) \times \nu = 0 \quad \text{on } \partial\Omega. \tag{3.6}$$

By the $W^{2,2}$-estimate of Lamé equation, it is not hard to show that

$$\lim_{\delta \downarrow 0^+} \left\| u_0^\delta - u_0 \right\|_X = 0.$$

Step 2 (mth approximate solutions). Fix $\delta > 0$ and $3 < q \leqslant 6$. For $m \geqslant 1$ and some $0 < T = T(m) < +\infty$ to be determined below, we let

$$u_0^m = \sum_{k=1}^m \big(u_0^\delta, \phi_k\big)\phi_k$$

and look for the triple

$$\begin{cases} \rho^m \in C\big([0,T]; W^{1,q} \cap H^1\big), \\ u^m(x,t) = \displaystyle\sum_{k=1}^m u_k^m(t)\phi_k(x) \in C\big([0,T]; W^{2,q} \cap H^2\big), \\ d^m \in C\big([0,T]; H^3\big(\Omega, S^2\big)\big) \end{cases}$$

solution of the following problem

$$\begin{cases} \rho_t^m + \nabla\cdot\big(\rho^m u^m\big) = 0, \\ \big(\rho^m u_t^m, \phi_k\big) + \mu\big(\nabla\times u^m, \nabla\phi_k\big) + (2\mu+\lambda)\big(\nabla\cdot u^m, \nabla\phi_k\big) \\ \quad = -\big(\rho^m u^m\cdot\nabla u^m, \phi_k\big) - \big(\nabla\big(P\big(\rho^m\big)\big), \phi_k\big) - \big(\Delta d^m\cdot\nabla d^m, \phi_k\big) \quad (1 \leqslant k \leqslant m), \\ d_t^m + u^m\cdot\nabla d^m = \Delta d^m + \big|\nabla d^m\big|^2 d^m, \\ \big(\rho^m, u^m, d^m\big)\big|_{t=0} = \big(\rho_0^\delta, u_0^m,\ d_0\big), \\ \left(u^m, \dfrac{\partial d^m}{\partial\nu}\right)\bigg|_{\partial\Omega\times[0,T]} = 0, \quad \text{or } \left(u^m\cdot\nu, \big(\nabla\times u^m\big)\times\nu, \dfrac{\partial d^m}{\partial\nu}\right)\bigg|_{\partial\Omega\times[0,T]} = 0. \end{cases} \tag{3.7}$$

The existence of a solution (ρ^m, u^m, d^m) to (3.7) over $\Omega\times[0,T(m)]$ for some $T(m) > 0$ can be obtained by the fixed point theorem, similar to that on the compressible Navier–Stokes equation by Padula [30] (see also [5]). Here we only sketch the argument. First, observe that for any given $0 < T < +\infty$ and $u^m \in C([0,T]; W^{2,q}\cap H^2)$, it is standard to show that there exist

(1) a solution $\rho^m \in C([0,T]; W^{1,q}\cap H^1)$ of $(3.7)_1$ along with $\rho^m|_{t=0} = \rho_0^\delta$.

(2) $0 < t_m \leqslant T$, depending on u^m and $\|d_0\|_{H^3}$, and a solution $d^m \in C([0,t_m], H^3(\Omega, S^2))$ of $(3.7)_3$ along with $d^m|_{t=0} = d_0$ and $\frac{\partial d^m}{\partial\nu}|_{\partial\Omega\times[0,t_m]} = 0$.

It is well known (cf. [30,5] or Lemma 2.5 in Section 2) that

$$\rho^m(x,t) \geqslant \delta\exp\left(-\int_0^t \big\|\nabla u^m\big\|_{L^\infty}\, ds\right) > 0, \quad (x,t)\in Q_T. \tag{3.8}$$

The coefficients $u_k^m(t)$ can be determined by the following system of m first order ordinary differential equations: $1 \leqslant k \leqslant m$,

$$\sum_{i=1}^m \big(\rho^m\phi_i, \phi_k\big)\dot{u}_i^m = F_k\left(u_l^m(t), \int_0^t u_l^m\, ds, t\right); \quad u_k^m(0) = \big(u_0^\delta, \phi_k\big), \tag{3.9}$$

where F_k denotes the right-hand side of $(3.7)_2$. Since ρ^m is strictly positive, the determinant of the $m\times m$ matrix $(\rho^m\phi_i, \phi_k)_{1\leqslant i,k\leqslant m}$ is positive. Hence we can reduce (3.9) into

$$\dot{u}_k^m = G_k\big(u_l^m, b_l^m, t\big), \qquad \dot{b}_k^m = u_k^m; \qquad u_k^m(0) = \big(u_0^\delta, \phi_k\big), \qquad b_k^m(0) = 0, \tag{3.10}$$

where G_k is a regular function of u_l^m, b_l^m. Therefore, by the standard existence theory of ordinary differential equations, we conclude that there exists a $0 < T_m \leqslant t_m$ and a solution $u_k^m(t)$ to (3.9), which in turn implies the existence of solutions ρ^m, d^m of $(3.7)_1$ and $(3.7)_3$ on the same time interval.

Step 3 (A priori estimates). We will show that there exist $0 < T_0 < +\infty$ and $C > 0$, depending only on the norms given by the regularity conditions on P and the initial data ρ_0, u_0, and d_0, but independent of the parameters δ, m, and the size of the domain Ω, such that there exists $M \in \mathcal{F}$ so that for any $m \geqslant 1$, (ϕ^m, u^m, d^m) satisfies:

$$\Phi^m(t) \leqslant \exp\left[C\mathcal{M}\left(\rho_0^\delta, u_0^\delta, d_0^\delta\right) + C\int_0^t M\left(\Phi^m(s)\right) ds\right], \quad 0 < t \leqslant T_0, \tag{3.11}$$

where $\Phi^m(t)$ is defined by (2.1) with (ρ, u, d) replaced by (ρ^m, u^m, d^m) and $\mathcal{M}(\rho_0^\delta, u_0^\delta, d_0^\delta)$ is defined by (2.3) with (ρ_0, u_0, d_0) replaced by $(\rho_0^\delta, u_0^\delta, d_0^\delta)$.

Since the argument to obtain (3.11) is almost identical to proof of Theorem 2.1, we only briefly outline it here:

First, it is easy to see $(3.7)_2$ holds with ϕ_k replaced by u^m. By multiplying $(3.7)_3$ by $(\Delta d^m + |\nabla d^m|^2 d^m)$ and integrating over Ω and adding these two resulting equations, we can show that there is an $M \in \mathcal{F}$ such that the energy inequality (2.5) holds with (ρ, u, d), M, and Φ replaced by (ρ^m, u^m, d^m), M, and Φ^m.

Second, since $(3.7)_2$ implies

$$\mathcal{L}u^m = \mathbb{P}_m\left(\rho^m \dot{u}^m + \nabla\left(P\left(\rho^m\right)\right) + \nabla d^m \cdot \Delta d^m\right), \tag{3.12}$$

where $\mathbb{P}_m(u) = \sum_{i=1}^m (u, \phi_k)\phi_k : X \to X^m$ is the orthogonal projection map, we can check that the same argument as Lemma 2.3 yields that exists $M \in \mathcal{F}$ so that

$$\left\|\nabla u^m\right\|_{H^1}^2 \leqslant M\left(\Phi^m(t)\right), \quad 0 \leqslant t \leqslant T_m. \tag{3.13}$$

Third, by differentiating (3.12) w.r.t. t, multiplying the resulting equation with u_t^m, integrating over Ω, and repeating the proof of Lemma 2.4, we obtain that there exists $M \in \mathcal{F}$ such that for any $m \geqslant 1$,

$$\int \rho^m \left|u_t^m\right|^2 + \int_0^t\int_\Omega \left|\nabla u_t^m\right|^2 \leqslant C\left[\mathcal{M}\left(\rho_0^\delta, u_0^m, d_0^\delta\right) + \int_0^t M\left(\Phi^m(s)\right) ds\right]. \tag{3.14}$$

Fourth, similar to the proof of Lemma 2.5 and Lemma 2.6, we have that there exists $M \in \mathcal{F}$ such that for all $m \geqslant 1$,

$$\left\|\nabla u^m\right\|_{L^2}^2 \leqslant C\left[\mathcal{M}\left(\rho_0^\delta, u_0^m, d_0^\delta\right) + \int_0^t M\left(\Phi^m(s)\right) ds\right], \tag{3.15}$$

and

$$\left\|\rho^m\right\|_{H^1\cap W^{1,q}} \leqslant C\exp\left\{C\left[\mathcal{M}\left(\rho_0^\delta, u_0^m, d_0^\delta\right) + \int_0^t M\left(\Phi^m(s)\right) ds\right]\right\}. \tag{3.16}$$

Fifth, by differentiating $(3.7)_3$ w.r.t. x and multiplying by ∇d_t^m (and $\nabla\Delta d_t^m$ respectively) and integrating over Ω, we can use the same argument as Lemma 2.7 and Lemma 2.8 to show that there exists $M \in \mathcal{F}$ such that for all $m \geqslant 1$,

$$\left\|\nabla^2 d^m\right\|_{L^2}^2 + \int_0^t \left\|\nabla d_t^m\right\|_{L^2}^2 ds \leqslant C\left[1 + \int_0^t M\big(\Phi^m(s)\big)\, ds\right], \tag{3.17}$$

$$\left\|\nabla^3 d^m\right\|_{L^2}^2 + \int_0^t \left\|\nabla^2 d_t^m\right\|_{L^2}^2 ds \leqslant \left(C\mathcal{M}\big(\rho_0^\delta, u_0^m, d_0^\delta\big) + \int_0^t M\big(\Phi^m(s)\big)\, ds\right)^4. \tag{3.18}$$

It is readily seen that combining all these estimates together yields (3.11) with T_0 replaced by T_m and u_0^δ replaced by u_0^m.

Step 4 (Convergence and solution). By the definition of u_0^δ, $\mathcal{M}$ given by (2.1), and the condition (1.10), we have

$$\mathcal{M}\big(\rho_0^\delta, u_0^\delta, d_0^\delta\big) = 1 + \|g\|_{L^2},$$

and

$$\left|\mathcal{M}\big(\rho_0^\delta, u_0^m, d_0^\delta\big) - \mathcal{M}\big(\rho_0^\delta, u_0^\delta, d_0^\delta\big)\right| \leqslant \frac{C}{\delta}\left\|u_0^m - u_0^\delta\right\|_{H^2} \to 0, \quad \text{as } m \to \infty.$$

Thus there exists $N = N(\delta) > 0$ such that

$$\mathcal{M}\big(\rho_0^\delta, u_0^m, d_0^\delta\big) \leqslant 2 + \|g\|_{L^2}, \quad \forall m \geqslant N. \tag{3.19}$$

It follows from (3.19), (3.11), and Gronwall's inequality (see, for example, [4, p. 263] or [32, Lemma 6]) that there exists a small $T_0 > 0$, independent of δ and m, such that

$$\sup_{0\leqslant t\leqslant T_0} \Phi^m(t) \leqslant C\exp\big(C\|g\|_{L^2}\big), \quad \forall m \geqslant M. \tag{3.20}$$

By virtue of (3.20), we obtain that for any $m \geqslant M$,

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T_0}\left(\left\|\sqrt{\rho^m}u_t^m\right\|_{L^2}^2 + \left\|\rho^m\right\|_{W^{1,q}\cap H^1}^2 + \left\|\nabla u^m\right\|_{H^1}^2 + \left\|d_t^m\right\|_{H^1}^2 + \left\|\nabla d^m\right\|_{H^2}^2\right) \\
&\quad + \int_0^{T_0}\left(\left\|u^m\right\|_{D^{2,q}}^2 + \left\|\nabla u_t^m\right\|_{L^2}^2 + \left\|\nabla^4 d^m\right\|_{L^2}^2 + \left\|\nabla^2 d_t^m\right\|_{L^2}^2\right) \leqslant C\exp\big(C\|g\|_{L^2}^2\big).
\end{aligned} \tag{3.21}$$

Based on the estimate (3.21), we can deduce that after taking subsequences, there exists $(\rho^\delta, u^\delta, d^\delta)$ such that

$$\begin{gathered}
\rho^m \rightharpoonup \rho^\delta \quad \text{weak* in } L^\infty\big(0, T_0; W^{1,q}\cap H^1\big), \qquad u^m \rightharpoonup u^\delta \quad \text{weak* in } L^\infty\big(0, T_0; D^1\cap D^2\big), \\
u^m \rightharpoonup u^\delta \quad \text{weak in } L^2\big(0, T_0; D^{2,q}\big), \qquad u_t^m \rightharpoonup u_t^\delta \quad \text{weak in } L^2\big(0, T_0; D^1\big), \\
\sqrt{\rho^m}u_t^m \rightharpoonup \sqrt{\rho^\delta}u_t^\delta \quad \text{weak* in } L^\infty\big(0, T; L^2\big), \\
d^m \rightharpoonup d^\delta \quad \text{weak* in } L^\infty\big(0, T_0; D^1\cap D^3\big) \text{ and } L^2\big(0, T_0; D^4\big), \\
d_t^m \rightharpoonup d_t^\delta \quad \text{in } L^2\big(0, T; H^2\big) \quad \text{and} \quad \text{weak* in } L^\infty\big(0, T; H^1\big).
\end{gathered}$$

By the lower semicontinuity, (3.21) implies that for $0 \leqslant t \leqslant T_0$, $(\rho^\delta, u^\delta, d^\delta)$ satisfies

$$\sup_{0\leqslant t\leqslant T_0} \left(\left\|\sqrt{\rho^\delta} u_t^\delta\right\|_{L^2}^2 + \left\|\rho^\delta\right\|_{W^{1,q}\cap H^1}^2 + \left\|\nabla u^\delta\right\|_{H^1}^2 + \left\|d_t^\delta\right\|_{H^1}^2 + \left\|\nabla d^\delta\right\|_{H^2}^2\right)$$
$$+ \int_0^{T_0} \left(\left\|u^\delta\right\|_{D^{2,q}}^2 + \left\|\nabla u_t^\delta\right\|_{L^2}^2 + \left\|\nabla^4 d^\delta\right\|_{L^2}^2 + \left\|\nabla^2 d_t^\delta\right\|_{L^2}^2\right) \leqslant C \exp\left(C\|g\|_{L^2}^2\right). \tag{3.22}$$

Furthermore, it is straightforward to check that $(\rho^\delta, u^\delta, d^\delta)$ is a strong solution in $[0, T_0]$ of (1.1)–(1.3) under the initial condition $(\rho^\delta, u^\delta, d^\delta)|_{t=0} = (\rho_0^\delta, u_0^\delta, d_0^\delta)$ and the boundary condition (1.8) or (1.9). Since $T_0 > 0$ is independent of δ, $(\rho^\delta, u^\delta, d^\delta)$ satisfies (3.22), $\rho_0^\delta \to \rho_0$ in $W^{1,q} \cap H^1$, $u_0^\delta \to u_0$ in $D^1 \cap D^2$, and $d_0^\delta = d_0$, the same limiting process as above would imply that after taking a subsequence $\delta \downarrow 0$, $(\rho^\delta, u^\delta, d^\delta)$ converges (weakly in the corresponding spaces) to a strong solution (ρ, u, d) of (1.1)–(1.3) on $\Omega \times [0, T_0]$ along with (1.6) and (1.8) or (1.9).

For the Cauchy problem on $\mathbb{R}^3$, we proceed as follows. For $R \uparrow \infty$, it is standard (cf. [24]) that there exists $d_0^R \in H^3(\mathbb{R}^3, S^2)$ such that $d_0^R \equiv n_0$ outside $B_{\frac{R}{2}}$ for some constant $n_0 \in S^2$ and

$$\lim_{R\uparrow\infty} \left\|\nabla d_0^R - \nabla d_0\right\|_{H^2(\mathbb{R}^3)} = 0. \tag{3.23}$$

Now we let $u_0^R \in H_0^1(B_R) \cap H^2(B_R)$ be the unique solution of

$$\mathcal{L} u_0^R - \nabla\left(P(\rho_0)\right) - \Delta d_0^R \cdot \nabla d_0^R = \sqrt{\rho_0} g \quad \text{on } B_R, u_0^R\big|_{\partial B_R} = 0, \tag{3.24}$$

where $g \in L^2(\mathbb{R}^3)$ is given by (1.10). Extending u_0^R to $\mathbb{R}^3$ by letting it be zero outside B_R. Then it is not hard to show that for any compact subset $K \subset \mathbb{R}^3$,

$$\lim_{R\uparrow\infty} \left\|\nabla u_0^R - \nabla u_0\right\|_{H^1(K)} = 0. \tag{3.25}$$

By the above existence, we know that there exists $T_0 > 0$, independent of R, and a strong solution $(\rho^R,\ u^R,\ d^R)$ of (1.1)–(1.3) on $B_R \times [0, T_0]$ of (1.1)–(1.3), under the initial and boundary condition:

$$\left(\rho^R,\ u^R,\ d^R\right)\big|_{B_R\times\{t=0\}} = \left(\rho_0,\ u_0^R, d_0^R\right); \qquad \left(u^R, \frac{\partial d^R}{\partial R}\right)\bigg|_{\partial B_R\times[0,T_0]} = 0. \tag{3.26}$$

Furthermore, (ρ^R, u^R, d^R) satisfies the estimate:

$$\sup_{0\leqslant t\leqslant T_0} \left(\left\|\sqrt{\rho^R} u_t^R\right\|_{L^2}^2 + \left\|\rho^R\right\|_{W^{1,q}\cap H^1}^2 + \left\|\nabla u^R\right\|_{H^1}^2 + \left\|d_t^R\right\|_{H^1}^2 + \left\|\nabla d^R\right\|_{H^2}^2\right)$$
$$+ \int_0^{T_0} \left(\left\|u^R\right\|_{D^{2,q}}^2 + \left\|\nabla u_t^R\right\|_{L^2}^2 + \left\|\nabla^4 d^R\right\|_{L^2}^2 + \left\|\nabla^2 d_t^R\right\|_{L^2}^2\right) \leqslant C \exp\left(C\|g\|_{L^2}^2\right), \tag{3.27}$$

with $C > 0$ independent of R. It is readily seen that (3.27), (3.23), and (3.25) imply that after taking a subsequence, we may assume that (ρ^R, u^R, d^R) locally converges (weakly in the corresponding spaces) to a strong solution (ρ, u, d) of (1.1)–(1.3) on $\mathbb{R}^3 \times [0, T_0]$ under the initial condition (1.6) and the boundary condition (1.7). This completes the proof of Theorem 1.2.

3.3. Uniqueness

In this subsection, we will show the uniqueness of the local strong solutions obtained in Theorem 1.2.

Let (ρ_i, u_i, d_i) $(i = 1, 2)$ be two strong solutions on $\Omega \times (0, T]$ of (1.1)–(1.3) with (1.6) and either (1.7), or (1.8), or (1.9). Set $\bar\rho = \rho_2 - \rho_1$, $\bar u = u_2 - u_1$, $\bar d = d_2 - d_1$. Then we have

$$\begin{cases} \bar\rho_t + (u_1 \cdot \nabla)\bar\rho + \bar u \cdot \nabla\rho_2 + \bar\rho \operatorname{div} u_2 + \rho_1 \operatorname{div} \bar u = 0, \\ \rho_1 \bar u_t + \rho_1 u_1 \cdot \nabla \bar u + \nabla\big(P(\rho_2) - P(\rho_1)\big) \\ \quad = \mathcal{L}\bar u - \bar\rho(u_{2t} + u_2 \cdot \nabla u_2) - \rho_1 \bar u \cdot \nabla u_2 - \Delta \bar d \cdot \nabla d_2 - \Delta d_1 \cdot \nabla \bar d, \\ \bar d_t - \Delta \bar d = \nabla \bar d \cdot (\nabla d_2 + \nabla d_1) d_1 + |\nabla d_2|^2 \bar d - \bar u \cdot \nabla d_2 - u_1 \cdot \nabla \bar d, \end{cases} \tag{3.28}$$

with the initial condition:

$$(\bar\rho, \bar u, \bar d)|_{t=0} = 0, \quad x \in \overline{\Omega},$$

and the boundary condition:

$$\left(\bar u, \frac{\partial \bar d}{\partial \nu}\right)\bigg|_{\partial\Omega} = 0, \quad \text{or} \left(\bar u \cdot \nu, (\nabla \times \bar u) \times \nu, \frac{\partial \bar d}{\partial \nu}\right)\bigg|_{\partial\Omega} = 0.$$

Multiplying $(3.28)_2$ by $\bar u$, integrating over Ω, and using integration by parts, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho_1 |\bar u|^2\,dx + \int \big((2\mu + \lambda)|\operatorname{div} \bar u|^2 + \mu|\nabla \times \bar u|^2\big)\,dx \\ &\quad = -\int \bar\rho(u_{2t} + u_2 \cdot \nabla u_2) \cdot \bar u\,dx - \int \rho_1 \bar u \cdot \nabla u_2 \cdot \bar u\,dx + \int \big(P(\rho_2) - P(\rho_1)\big) \operatorname{div} \bar u\,dx \\ &\qquad + \int (\nabla \bar d \cdot \nabla\nabla d_2 \cdot \bar u + \nabla \bar d \cdot \nabla d_2 \cdot \nabla \bar u)\,dx - \int \Delta d_1 \cdot \nabla \bar d \cdot \bar u\,dx. \end{aligned}$$

Observe that

$$\big|P(\rho_2) - P(\rho_1)\big| \leqslant B_P\big(\|\rho_1\|_{L^\infty} + \|\rho_2\|_{L^\infty}\big)|\bar\rho| \leqslant C|\bar\rho|.$$

Hence, by Hölder's inequality and Cauchy's inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho_1 |\bar u|^2\,dx + \int \big((2\mu + \lambda)|\operatorname{div} \bar u|^2 + \mu|\nabla \times \bar u|^2\big)\,dx \\ &\quad \lesssim \|\bar\rho\|_{L^{\frac{3}{2}}} \|u_{2t} + u_2 \cdot \nabla u_2\|_{L^6} \|\bar u\|_{L^6} + \|\nabla u_2\|_{L^\infty} \int_\Omega \rho_1 |\bar u|^2\,dx + \|\bar\rho\|_{L^2} \|\operatorname{div} \bar u\|_{L^2} \\ &\qquad + \|\nabla \bar d\|_{L^2} \big\|\nabla^2 d_2\big\|_{L^3} \|\bar u\|_{L^6} + \|\nabla \bar d\|_{L^2} \|\nabla d_2\|_{L^\infty} \|\nabla \bar u\|_{L^2} + \|\Delta d_1\|_{L^3} \|\nabla \bar d\|_{L^2} \|\bar u\|_{L^6} \\ &\quad \lesssim \|\bar\rho\|_{L^{\frac{3}{2}}} \|u_{2t} + u_2 \cdot \nabla u_2\|_{L^6} \|\nabla \bar u\|_{L^2} + \|\nabla u_2\|_{W^{1,q}} \int \rho_1 |\bar u|^2\,dx \\ &\qquad + \|\bar\rho\|_{L^2} \|\operatorname{div} \bar u\|_{L^2} + \|\nabla \bar d\|_{L^2} \|\nabla \bar u\|_{L^2} \end{aligned}$$

$$\leqslant \epsilon \int |\nabla \bar{u}|^2\,dx + C\bigg[\|\bar{\rho}\|^2_{L^{\frac{3}{2}}}\|u_{2t} + u_2 \cdot \nabla u_2\|^2_{L^6} + \|\nabla u_2\|_{W^{1,q}} \int \rho_1 |\bar{u}|^2\,dx + \|\bar{\rho}\|^2_{L^2} + \|\nabla \bar{d}\|^2_{L^2}\bigg].$$

Thus, by choosing ϵ sufficiently small, we have

$$\frac{d}{dt}\int \rho_1 |\bar{u}|^2\,dx + \int |\nabla \bar{u}|^2\,dx \leqslant C\bigg[\|\bar{\rho}\|^2_{L^{\frac{3}{2}}}\|u_{2t} + u_2 \cdot \nabla u_2\|^2_{L^6} + \|\nabla u_2\|_{W^{1,q}} \int \rho_1 |\bar{u}|^2\,dx + \|\bar{\rho}\|^2_{L^2} + \|\nabla \bar{d}\|^2_{L^2}\bigg]. \tag{3.29}$$

Multiplying $(3.28)_1$ by $2\bar{\rho}$, integrating over Ω, and using integration by parts, we have

$$\begin{aligned}
\frac{d}{dt}\int |\bar{\rho}|^2\,dx &\lesssim \int |\bar{\rho}\,\bar{u} \cdot \nabla \rho_2|\,dx + \int |\bar{\rho}|^2\big(|\operatorname{div} u_1| + |\operatorname{div} u_2|\big)\,dx + \int |\bar{\rho}\rho_1 \operatorname{div} \bar{u}|\,dx \\
&\lesssim \|\bar{\rho}\|_{L^2}\|\nabla \rho_2\|_{L^3}\|\bar{u}\|_{L^6} + \big(\|\operatorname{div} u_1\|_{L^\infty} + \|\operatorname{div} u_2\|_{L^\infty}\big)\int |\bar{\rho}|^2\,dx + \|\bar{\rho}\|_{L^2}\|\operatorname{div} \bar{u}\|_{L^2} \\
&\lesssim \|\bar{\rho}\|_{L^2}\|\nabla \bar{u}\|_{L^2} + \big(\|\operatorname{div} u_1\|_{W^{1,q}} + \|\operatorname{div} u_2\|_{W^{1,q}}\big)\int |\bar{\rho}|^2\,dx \\
&\lesssim \|\bar{\rho}\|_{L^2}\big(\|\operatorname{div} \bar{u}\|_{L^2} + \|\nabla \times \bar{u}\|_{L^2}\big) + \big(\|\operatorname{div} u_1\|_{W^{1,q}} + \|\operatorname{div} u_2\|_{W^{1,q}}\big)\int |\bar{\rho}|^2\,dx \\
&\leqslant \epsilon \int |\nabla \bar{u}|^2\,dx + C_\epsilon \|\bar{\rho}\|^2_{L^2} + C\big(\|\operatorname{div} u_1\|_{W^{1,q}} + \|\operatorname{div} u_2\|_{W^{1,q}}\big)\int |\bar{\rho}|^2\,dx,
\end{aligned} \tag{3.30}$$

for any $\epsilon > 0$. Similarly, we have

$$\begin{aligned}
\frac{d}{dt}\int |\bar{\rho}|^{\frac{3}{2}}\,dx &\lesssim \int \big|\bar{\rho}^{\frac{1}{2}}\bar{u} \cdot \nabla \rho_2\big|\,dx + \int |\bar{\rho}|^{\frac{3}{2}}\big(|\operatorname{div} u_1| + |\operatorname{div} u_2|\big)\,dx + \int \big|\bar{\rho}^{\frac{1}{2}}\rho_1 \operatorname{div} \bar{u}\big|\,dx \\
&\lesssim \|\bar{\rho}\|^{\frac{1}{2}}_{L^{\frac{3}{2}}}\|\nabla \rho_2\|_{L^2}\|\bar{u}\|_{L^6} \\
&\quad + \big(\|\operatorname{div} u_1\|_{L^\infty} + \|\operatorname{div} u_2\|_{L^\infty}\big)\int |\bar{\rho}|^{\frac{3}{2}}\,dx + \|\bar{\rho}\|^{\frac{1}{2}}_{L^{\frac{3}{2}}}\|\operatorname{div} \bar{u}\|_{L^2}\|\rho_1\|_{L^6} \\
&\lesssim \|\bar{\rho}\|^{\frac{1}{2}}_{L^{\frac{3}{2}}}\|\nabla \bar{u}\|_{L^2} + \big(\|\operatorname{div} u_1\|_{W^{1,q}} + \|\operatorname{div} u_2\|_{W^{1,q}}\big)\int |\bar{\rho}|^{\frac{3}{2}}\,dx \\
&\lesssim \|\bar{\rho}\|^{\frac{1}{2}}_{L^{\frac{3}{2}}}\|\nabla \bar{u}\|_{L^2} + \big(\|\operatorname{div} u_1\|_{W^{1,q}} + \|\operatorname{div} u_2\|_{W^{1,q}}\big)\int |\bar{\rho}|^{\frac{3}{2}}\,dx.
\end{aligned} \tag{3.31}$$

Multiplying (3.31) by $\|\bar{\rho}\|^{\frac{1}{2}}_{L^{\frac{3}{2}}}$, and using Cauchy's inequality, we have

$$\begin{aligned}
\frac{d}{dt}\|\bar{\rho}\|^2_{L^{\frac{3}{2}}} &\lesssim \|\bar{\rho}\|_{L^{\frac{3}{2}}}\|\nabla \bar{u}\|_{L^2} + \big(\|\operatorname{div} u_1\|_{W^{1,q}} + \|\operatorname{div} u_2\|_{W^{1,q}}\big)\|\bar{\rho}\|^2_{L^{\frac{3}{2}}} \\
&\leqslant \epsilon \int |\nabla \bar{u}|^2\,dx + C_\epsilon \|\bar{\rho}\|^2_{L^{\frac{3}{2}}} + C\big(\|\operatorname{div} u_1\|_{W^{1,q}} + \|\operatorname{div} u_2\|_{W^{1,q}}\big)\|\bar{\rho}\|^2_{L^{\frac{3}{2}}}.
\end{aligned} \tag{3.32}$$

Multiplying $(3.28)_3$ by $-\Delta\bar{d}$, integrating over Ω, and using integration by parts and Cauchy's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\nabla\bar{d}|^2\,dx+\int |\Delta\bar{d}|^2\,dx \\
&\quad\lesssim \|\nabla\bar{d}\|_{L^2}\|\Delta\bar{d}\|_{L^2}\|\nabla d_2+\nabla d_1\|_{L^\infty}+\|\Delta\bar{d}\|_{L^2}\|\nabla d_2\|_{L^6}^2\|\bar{d}\|_{L^6} \\
&\qquad+\|\Delta\bar{d}\|_{L^2}\|\bar{u}\|_{L^6}\|\nabla d_2\|_{L^3}+\|\Delta\bar{d}\|_{L^2}\|u_1\|_{L^\infty}\|\nabla\bar{d}\|_{L^2} \\
&\quad\lesssim \|\nabla\bar{d}\|_{L^2}\|\Delta\bar{d}\|_{L^2}\|\nabla d_2+\nabla d_1\|_{H^2}+\|\Delta\bar{d}\|_{L^2}\|\nabla d_2\|_{H^1}^2\|\nabla\bar{d}\|_{L^2} \\
&\qquad+\|\Delta\bar{d}\|_{L^2}\|\nabla\bar{u}\|_{L^2}\|\nabla d_2\|_{L^2}^{\frac{1}{2}}\|\nabla d_2\|_{L^6}^{\frac{1}{2}}+\|\Delta\bar{d}\|_{L^2}\|\nabla u_1\|_{H^1}\|\nabla\bar{d}\|_{L^2} \\
&\quad\lesssim \|\Delta\bar{d}\|_{L^2}\|\nabla\bar{d}\|_{L^2}+\|\Delta\bar{d}\|_{L^2}\|\nabla\bar{u}\|_{L^2} \\
&\quad\leqslant \frac{1}{2}\|\Delta\bar{d}\|_{L^2}^2+C\|\nabla\bar{d}\|_{L^2}^2+C\|\nabla\bar{u}\|_{L^2}^2.
\end{aligned}$$

This gives

$$\frac{d}{dt}\int_\Omega |\nabla\bar{d}|^2\,dx+\int_\Omega |\Delta\bar{d}|^2\,dx\leqslant C\|\nabla\bar{d}\|_{L^2}^2+C\|\nabla\bar{u}\|_{L^2}^2. \tag{3.33}$$

Multiplying (3.29) by $3C$, putting the resulting inequality, (3.30) and (3.32) to (3.33), and taking $\epsilon>0$ small enough, we have

$$\begin{aligned}
&\frac{d}{dt}\big(3C\|\sqrt{\rho_1}\bar{u}\|_{L^2}^2+\|\bar{\rho}\|_{L^{\frac{3}{2}}}^2+\|\bar{\rho}\|_{L^2}^2+\|\nabla\bar{d}\|_{L^2}^2\big)+C\int |\nabla\bar{u}|^2\,dx \\
&\quad\lesssim \|\bar{\rho}\|_{L^{\frac{3}{2}}}^2\|u_{2t}+u_2\cdot\nabla u_2\|_{L^6}^2+\|\nabla u_2\|_{W^{1,q}}\int \rho_1|\bar{u}|^2\,dx+\|\bar{\rho}\|_{L^2}^2+\|\nabla\bar{d}\|_{L^2}^2 \\
&\qquad+\big(\|\operatorname{div}u_1\|_{W^{1,q}}+\|\operatorname{div}u_2\|_{W^{1,q}}\big)\int |\bar{\rho}|^2\,dx+\|\bar{\rho}\|_{L^{\frac{3}{2}}}^2+\big(\|\operatorname{div}u_1\|_{W^{1,q}}+\|\operatorname{div}u_2\|_{W^{1,q}}\big)\|\bar{\rho}\|_{L^{\frac{3}{2}}}^2 \\
&\quad\lesssim \big(\|u_{2t}+u_2\cdot\nabla u_2\|_{L^6}^2+\|\nabla u_1\|_{W^{1,q}}+\|\nabla u_2\|_{W^{1,q}}+1\big) \\
&\qquad\cdot\big(3C\|\sqrt{\rho_1}\bar{u}\|_{L^2}^2+\|\bar{\rho}\|_{L^{\frac{3}{2}}}^2+\|\bar{\rho}\|_{L^2}^2+\|\nabla\bar{d}\|_{L^2}^2\big).
\end{aligned} \tag{3.34}$$

By (3.34), Gronwall's inequality, and $(\overline{\rho_0},\ \overline{u_0},\ \overline{d_0})=0$, we have

$$\|\sqrt{\rho_1}\bar{u}\|_{L^2}^2+\|\bar{\rho}\|_{L^{\frac{3}{2}}}^2+\|\bar{\rho}\|_{L^2}^2+\|\nabla\bar{d}\|_{L^2}^2+\int_0^t\int_\Omega |\nabla\bar{u}|^2\,dx\,ds=0. \tag{3.35}$$

This yields

$$(\bar{\rho},\bar{u},\nabla\bar{d})=0. \tag{3.36}$$

To see $\bar{d}=0$, observe that after substituting (3.36) into $(3.28)_3$, we have

$$\bar{d}_t=|\nabla d_2|^2\bar{d},\qquad \bar{d}|_{t=0}=0.$$

This implies $\bar{d}=0$. This completes the proof. □

4. Proof of Theorem 1.3

Let $0 < T_* < \infty$ be the maximum time for the existence of strong solution (ρ, u, d) to (1.1)–(1.3). Namely, (ρ, u, d) is a strong solution to (1.1)–(1.3) in $\Omega \times (0, T]$ for any $0 < T < T_*$, but not a strong solution in $\Omega \times (0, T_*]$. Suppose that (1.11) were false, i.e.

$$\limsup_{T \nearrow T_*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \int_0^T \|\nabla d(t)\|_{L^\infty}^3 \, dt \right) = M_0 < \infty. \tag{4.1}$$

The goal is to show that under the assumption (4.1), there is a bound $C > 0$ depending only on M_0, ρ_0, u_0, d_0, and T_* such that

$$\sup_{0 \leqslant t < T_*} \Big[\max_{r=2,q} \big(\|\rho\|_{W^{1,r}} + \|\rho_t\|_{L^r} \big) + \big(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{H^1} \big) + \big(\|d_t\|_{H^1} + \|\nabla d\|_{H^2} \big) \Big] \leqslant C, \tag{4.2}$$

and

$$\int_0^{T_*} \big(\|u_t\|_{D^1}^2 + \|u\|_{D^{2,q}}^2 + \|d_t\|_{H^2}^2 + \|\nabla d\|_{H^3}^2 \big) \, dt \leqslant C. \tag{4.3}$$

With (4.2) and (4.3), we can then show without much difficulty that T_* is not the maximum time, which is the desired contradiction.

The proof is based on several lemmas.

Lemma 4.1. *Assume* (4.1)*, we have*

$$\int_0^{T_*} \int_\Omega \big| \nabla^2 d \big|^2 \, dx \, dt \leqslant C. \tag{4.4}$$

Proof. To see (4.4), observe that (4.1) implies $\int_0^{T_*} \|\nabla d\|_{L^\infty}^2 \, dt \leqslant M_0$ so that

$$\begin{aligned}
\int_0^{T_*} \int |\nabla d|^4 \, dx \, dt &\leqslant M_0 \cdot \left(\sup_{0 \leqslant t < T_*} \int |\nabla d|^2 \, dx \right) \\
&\leqslant M_0 \left[\int_0^{T_*} \int \big| P(\rho) \big|^2 \, dx \, dt + \int \big(\rho_0 |u_0|^2 + |\nabla d_0|^2 \big) \, dx \right]
\end{aligned}$$

where we have used (2.11) in the last step. Applying (2.11) again, this then implies

$$\begin{aligned}
\int_0^{T_*} \int |\Delta d|^2 \, dx \, dt &= \int_0^{T_*} \int \big| \Delta d + |\nabla d|^2 d \big|^2 \, dx \, dt + \int_0^{T_*} \int |\nabla d|^4 \, dx \, dt \\
&\leqslant (1 + M_0) \left[\int_0^{T_*} \int \big| P(\rho) \big|^2 \, dx \, dt + \int \big(\rho_0 |u_0|^2 + |\nabla d_0|^2 \big) \, dx \right].
\end{aligned}$$

Since

$$\big|P(\rho)\big| \leqslant B_P\big(\|\rho\|_{L^\infty}\big)|\rho| \leqslant C|\rho|,$$

we have, by the conservation of mass and (4.1),

$$\int_0^{T_*}\int \big|P(\rho)\big|^2\,dx\,dt \leqslant CT_* \sup_{0\leqslant t<T_*} \|\rho\|_{L^1}\|\rho\|_{L^\infty} \leqslant C.$$

Thus the standard L^2-estimate yields (4.4). □

Following the argument by [34], we let $v = \mathcal{L}^{-1}\nabla(P(\rho))$ be the solution of the Lamé system:

$$\begin{cases} \mathcal{L}v = \nabla\big(P(\rho)\big), \\ v|_{\partial\Omega} = 0, \quad \text{or } v \to 0 \quad \text{as } |x| \to \infty \ \big(\text{when } \Omega = \mathbb{R}^3\big). \end{cases} \tag{4.5}$$

Then it follows from [34] Proposition 2.1 that

$$\|\nabla v\|_{L^q} \leqslant C\big\|P(\rho)\big\|_{L^q} \leqslant CB_P\big(\|\rho\|_{L^\infty}\big)\|\rho\|_{L^q} \leqslant C, \quad 1 < q \leqslant 6, \tag{4.6}$$

where we have used (4.1) and the conservation of mass in the last step.

Denote $w = u - v$, then w satisfies

$$\begin{cases} \rho w_t - \mathcal{L}w = \rho F - \nabla d \cdot \Delta d, \\ w|_{t=0} = w_0 = u_0 - v_0, \\ w|_{\partial\Omega} = 0 \quad \text{or} \quad w \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{4.7}$$

where

$$F = -u\cdot\nabla u - \mathcal{L}^{-1}\nabla\big(\partial_t\big(P(\rho)\big)\big) = -u\cdot\nabla u + \mathcal{L}^{-1}\nabla\operatorname{div}\big(P(\rho)u\big) - \mathcal{L}^{-1}\nabla\big(\big(P - P'(\rho)\rho\big)\operatorname{div} u\big).$$

Then we have the following estimate.

Lemma 4.2. *Under the assumptions of Theorem* 1.3*, if* $\lambda < \frac{7\mu}{9}$*, then* (ρ, u, d) *satisfies that for any* $0 \leqslant t < T_*$,

$$\int_\Omega \big(\rho|u|^5 + |\nabla w|^2 + |\nabla d|^5 + \big|\nabla^2 d\big|^2\big)\,dx + \int_0^t\int_\Omega \big(\big|\nabla^3 d\big|^2 + \big|\nabla^2 w\big|^2 + |\nabla d_t|^2\big)\,dx\,ds \leqslant C. \tag{4.8}$$

Proof. The proof of this lemma is divided into five steps:

Step 1. Estimates of $\int |\nabla w|^2\,dx$. Multiplying $(4.7)_1$ by w_t, integrating over Ω, and using integration by parts and Cauchy's inequality, we have

$$\begin{aligned} &\frac{d}{dt}\int \big(\mu|\nabla w|^2 + (\mu+\lambda)|\operatorname{div} w|^2\big)\,dx + \int \rho|w_t|^2\,dx \\ &\quad \leqslant \|\sqrt{\rho}F\|_{L^2}^2 + 2\frac{d}{dt}\int\left(\nabla d\otimes\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right) : \nabla w\,dx + C\int |\nabla d||\nabla d_t||\nabla w|\,dx = \sum_{i=1}^3 I_i. \end{aligned} \tag{4.9}$$

For I_1, we have

$$I_1 \lesssim \|\sqrt{\rho}u\cdot\nabla u\|_{L^2}^2 + \big\|\sqrt{\rho}\mathcal{L}^{-1}\nabla\operatorname{div}\big(P(\rho)u\big)\big\|_{L^2}^2 + \big\|\sqrt{\rho}\mathcal{L}^{-1}\nabla\big(\big(P(\rho)-P'(\rho)\rho\big)\operatorname{div}u\big)\big\|_{L^2}^2 = \sum_{j=1}^{3} I_{1j}. \tag{4.10}$$

For I_{11}, by Hölder's inequality, (4.1), Sobolev inequality, interpolation inequality, and (4.6), we have

$$\begin{aligned} I_{11} &\lesssim \big\|\rho^{\frac{1}{5}}u\big\|_{L^5}^2\|\nabla u\|_{L^{\frac{10}{3}}}^2 \lesssim \big\|\rho^{\frac{1}{5}}u\big\|_{L^5}^2\|\nabla u\|_{L^2}^{\frac{4}{5}}\|\nabla u\|_{L^6}^{\frac{6}{5}} \\ &\lesssim \big\|\rho^{\frac{1}{5}}u\big\|_{L^5}^2\|\nabla u\|_{L^2}^{\frac{4}{5}}\|\nabla w\|_{L^6}^{\frac{6}{5}} + \big\|\rho^{\frac{1}{5}}u\big\|_{L^5}^2\|\nabla u\|_{L^2}^{\frac{4}{5}}\|\nabla v\|_{L^6}^{\frac{6}{5}} \\ &\lesssim \big\|\rho^{\frac{1}{5}}u\big\|_{L^5}^2\|\nabla u\|_{L^2}^{\frac{4}{5}}\big(\big\|\nabla^2 w\big\|_{L^2}^{\frac{6}{5}} + \|\nabla w\|_{L^2}^{\frac{6}{5}} + 1\big). \end{aligned} \tag{4.11}$$

Again by [34], Proposition 2.1 and (4.7), we have

$$\big\|\nabla^2 w\big\|_{L^2} \lesssim \|\sqrt{\rho}w_t\|_{L^2} + \|\sqrt{\rho}F\|_{L^2} + \|\nabla d\cdot\Delta d\|_{L^2}. \tag{4.12}$$

Substituting (4.12) into (4.11), and using Young's inequality, we obtain for any $\varepsilon > 0$

$$\begin{aligned} I_{11} &\leqslant \varepsilon\big(\|\sqrt{\rho}w_t\|_{L^2}^2 + \|\sqrt{\rho}F\|_{L^2}^2\big) \\ &\quad + C\big(\big\|\rho^{\frac{1}{5}}u\big\|_{L^5}^5\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\|\Delta d\|_{L^2}^2 + 1\big). \end{aligned} \tag{4.13}$$

For I_{12} and I_{13}, by [34] Proposition 2.1, (4.1), (2.5), and (1.5), and Sobolev's inequality, we have

$$I_{12} \lesssim \big\|P(\rho)u\big\|_{L^2}^2 \lesssim \|\rho u\|_{L^2}^2 \lesssim \|\sqrt{\rho}u\|_{L^2}^2 \leqslant C, \tag{4.14}$$

$$\begin{aligned} I_{13} &\lesssim \|\sqrt{\rho}\|_{L^3}^2\big\|\mathcal{L}^{-1}\nabla\big(\big(P(\rho)-P'(\rho)\rho\big)\operatorname{div}u\big)\big\|_{L^6}^2 \\ &\lesssim \big\|\nabla\mathcal{L}^{-1}\nabla\big(\big(P(\rho)-P'(\rho)\rho\big)\operatorname{div}u\big)\big\|_{L^2}^2 \lesssim \big\|\big(P(\rho)-P'(\rho)\rho\big)\nabla u\big\|_{L^2}^2 \\ &\leqslant CB_P\big(\|\rho\|_{L^\infty}\big)\|\rho\|_{L^\infty}\|\nabla u\|_{L^2}^2 \leqslant C\|\nabla u\|_{L^2}^2, \end{aligned} \tag{4.15}$$

where we have used the Sobolev inequality when $\Omega = \mathbb{R}^3$, and both Sobolev and Poincaré inequalities when Ω is a bounded domain.

Putting (4.13), (4.14) and (4.15) into (4.10), and choosing ε sufficiently small, we obtain

$$\begin{aligned} I_1 &\leqslant \frac{1}{2}\|\sqrt{\rho}w_t\|_{L^2}^2 + C\big(\big\|\rho^{\frac{1}{5}}u\big\|_{L^5}^5\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\|\Delta d\|_{L^2}^2 + 1\big) \\ &\leqslant \frac{1}{2}\|\sqrt{\rho}w_t\|_{L^2}^2 + C\big(\big\|\rho^{\frac{1}{5}}u\big\|_{L^5}^5\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\|\Delta d\|_{L^2}^2 + 1\big) \\ &\leqslant \frac{1}{2}\|\sqrt{\rho}w_t\|_{L^2}^2 + C\big(\big\|\rho^{\frac{1}{5}}u\big\|_{L^5}^5\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\|\Delta d\|_{L^2}^2 + 1\big), \end{aligned} \tag{4.16}$$

where we have used (4.6) with $q = 2$. For I_3, using Cauchy's inequality, we have

$$I_3 \leqslant \frac{1}{2}\int |\nabla d_t|^2\,dx + C\int |\nabla d|^2|\nabla w|^2\,dx \leqslant \frac{1}{2}\int |\nabla d_t|^2\,dx + C\|\nabla d\|_{L^\infty}^2 \int |\nabla w|^2\,dx. \quad (4.17)$$

Substituting (4.16) and (4.17) into (4.9), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int \big(\mu|\nabla w|^2 + (\mu+\lambda)|\operatorname{div} w|^2\big)\,dx + \frac{1}{2}\int \rho|w_t|^2\,dx \\
&\quad\leqslant 2\frac{d}{dt}\int \left(\nabla d\otimes\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right):\nabla w\,dx + \frac{1}{2}\|\nabla d_t\|_{L^2}^2 \\
&\qquad + C\big(\|\nabla d\|_{L^\infty}^2\big(\|\nabla w\|_{L^2}^2 + \|\Delta d\|_{L^2}^2\big) + \big\|\rho^{\frac{1}{5}}u\big\|_{L^5}^5\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + 1\big). \qquad (4.18)
\end{aligned}$$

Step 2. Estimates of $\int \rho|u|^5\,dx$. Multiplying (1.2) by $5|u|^3u$, integrating over Ω, and using integration by parts and Cauchy's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\int \rho|u|^5\,dx + \int 5|u|^3\big(\mu|\nabla u|^2 + (\mu+\lambda)|\operatorname{div} u|^2 + 3\mu\big|\nabla|u|\big|^2\big)\,dx \\
&= \int 5P(\rho)\operatorname{div}\big(|u|^3u\big)\,dx + \int 5\left(\nabla d\otimes\nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3\right)\operatorname{div}\big(|u|^3u\big) \\
&\quad - \int 15(\mu+\lambda)(\operatorname{div} u)|u|^2u\cdot\nabla|u| \\
&\leqslant C\left(\int \rho|u|^3|\nabla u| + \int |\nabla d|^2|u|^3|\nabla u|\right) + \int 5(\mu+\lambda)|u|^3|\operatorname{div} u|^2 + \int \frac{45}{4}(\mu+\lambda)|u|^3\big|\nabla|u|\big|^2.
\end{aligned}$$

By Kato's inequality $|\nabla u|^2 \geqslant |\nabla|u||^2$, we have

$$\begin{cases}
\left(15\mu - \dfrac{45(\mu+\lambda)}{4}\right)\displaystyle\int |u|^3\big|\nabla|u|\big|^2 \\
\qquad \geqslant \left(15\mu - \dfrac{45(\mu+\lambda)}{4}\right)\displaystyle\int |u|^3|\nabla u|^2, \quad \text{if } \mu - \dfrac{3(\mu+\lambda)}{4} \leqslant 0, \\
\left(15\mu - \dfrac{45(\mu+\lambda)}{4}\right)\displaystyle\int |u|^3\big|\nabla|u|\big|^2 \geqslant 0, \quad \text{if } \mu - \dfrac{3(\mu+\lambda)}{4} > 0.
\end{cases}$$

Hence we obtain

$$\begin{aligned}
&\frac{d}{dt}\int \rho|u|^5\,dx + 5\min\left\{\mu, \left(4\mu - \frac{9(\mu+\lambda)}{4}\right)\right\}\int |u|^3|\nabla u|^2\,dx \\
&\quad\leqslant C\left(\int \rho|u|^3|\nabla u|\,dx + \int |\nabla d|^2|u|^3|\nabla u|\,dx\right). \qquad (4.19)
\end{aligned}$$

Since $\lambda < \frac{7\mu}{9}$, we have

$$c_0 := 5\min\left\{\mu, \left(4\mu - \frac{9(\mu+\lambda)}{4}\right)\right\} > 0. \qquad (4.20)$$

Thus by Cauchy's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\int \rho|u|^5\,dx + c_0\int |u|^3|\nabla u|^2\,dx \\
&\quad\leqslant C\left(\int \rho|u|^3|\nabla u|\,dx + \int |\nabla d|^2|u|^3|\nabla u|\,dx\right) \\
&\quad\leqslant \frac{c_0}{2}\int |u|^3|\nabla u|^2\,dx + C\left[\int \rho^2|u|^3\,dx + \int |\nabla d|^4|u|^3\,dx\right].
\end{aligned}$$

Hence by Hölder's inequality, Sobolev's inequality, the conservation of mass, (4.1) and Young's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\int \rho|u|^5\,dx + \frac{c_0}{2}\int |u|^3|\nabla u|^2\,dx \\
&\quad\lesssim \int \rho^2|u|^3\,dx + \int |\nabla d|^4|u|^3\,dx \\
&\quad\lesssim \left(\int \rho|u|^5\,dx\right)^{\frac{3}{5}}\left(\int \rho^{\frac{7}{2}}\,dx\right)^{\frac{2}{5}} + \left\|\nabla\left(|u|^{\frac{5}{2}}\right)\right\|_{L^2}^{\frac{6}{5}}\left(\int |\nabla d|^5\,dx\right)^{\frac{4}{5}} \\
&\quad\leqslant \frac{c_0}{4}\int |u|^3|\nabla u|^2\,dx + C\left[1 + \int \rho|u|^5\,dx + \left(\int |\nabla d|^5\,dx\right)^2\right].
\end{aligned}$$

Thus by (2.5) we have

$$\begin{aligned}
\frac{d}{dt}\int \rho|u|^5\,dx + \frac{c_0}{4}\int |u|^3|\nabla u|^2\,dx &\lesssim \int \rho|u|^5\,dx + \left(\int |\nabla d|^5\,dx\right)^2 + 1 \\
&\lesssim \int \rho|u|^5\,dx + \|\nabla d\|_{L^\infty}^3\|\nabla d\|_{L^5}^5 + 1.
\end{aligned} \tag{4.21}$$

Step 3. Estimates of $\int |\nabla d|^5\,dx$. Differentiating (1.3) with respect to x, we obtain

$$\nabla d_t - \nabla\Delta d + \nabla(u\cdot\nabla d) = \nabla\left(|\nabla d|^2 d\right). \tag{4.22}$$

Multiplying (4.22) by $5|\nabla d|^3\nabla d$ and integrating by parts over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\int |\nabla d|^5\,dx + 5\int |\Delta d|^2|\nabla d|^3\,dx \\
&\quad= 5\int \left[\nabla\left(|\nabla d|^2 d\right) - \nabla(u\cdot\nabla d)\right]\cdot|\nabla d|^3\nabla d\,dx - 5\int \Delta d\cdot\nabla\left(|\nabla d|^3\right)\cdot\nabla d\,dx \\
&\quad\leqslant \int \left(|\nabla d|^5\left|\nabla^2 d\right| + |\nabla d|^7 + |\nabla u||\nabla d|^5 + |\nabla d|^3\left|\nabla^2 d\right|^2\right)dx.
\end{aligned}$$

This, combined with Cauchy's inequality and the fact

$$|\nabla d|^2 = -d\cdot\Delta d \quad \left(\text{since } |d| = 1\right), \tag{4.23}$$

gives

$$\begin{aligned}
&\frac{d}{dt}\int |\nabla d|^5\,dx + 5\int |\Delta d|^2|\nabla d|^3\,dx \\
&\quad\lesssim \int \left(|\nabla d|^3\left|\nabla^2 d\right|^2 + |\nabla d|^7 + |\nabla u||\nabla d|^3\left|\nabla^2 d\right|\right)dx \\
&\quad\lesssim \|\nabla d\|_{L^\infty}^3\left\|\nabla^2 d\right\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\|\nabla d\|_{L^5}^5 + \|\nabla d\|_{L^\infty}^3\|\nabla u\|_{L^2}\left\|\nabla^2 d\right\|_{L^2} \\
&\quad\lesssim \|\nabla d\|_{L^\infty}^3\left(\left\|\nabla^2 d\right\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) + \|\nabla d\|_{L^\infty}^2\|\nabla d\|_{L^5}^5. \qquad (4.24)
\end{aligned}$$

By (4.6) and (4.24), we have

$$\begin{aligned}
&\frac{d}{dt}\int |\nabla d|^5\,dx + 5\int |\Delta d|^2|\nabla d|^3\,dx \\
&\quad\lesssim \|\nabla d\|_{L^\infty}^3\left(\left\|\nabla^2 d\right\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) + \|\nabla d\|_{L^\infty}^3 + \|\nabla d\|_{L^\infty}^2\|\nabla d\|_{L^5}^5. \qquad (4.25)
\end{aligned}$$

Step 4. Estimates of $\int |\nabla^2 d|^2\,dx$. Multiplying (4.22) by ∇d_t, integrating by parts over Ω, and using Cauchy's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta d|^2\,dx + \int |\nabla d_t|^2\,dx \\
&\quad= \int \left(\nabla\left(|\nabla d|^2 d\right) - \nabla(u\cdot\nabla d)\right)\cdot\nabla d_t\,dx \\
&\quad\leqslant \varepsilon\|\nabla d_t\|_{L^2}^2 + C\int \left(|\nabla d|^6 + |\nabla d|^2\left|\nabla^2 d\right|^2 + |\nabla u|^2|\nabla d|^2 + |u|^2\left|\nabla^2 d\right|^2\right)dx \\
&\quad\leqslant \varepsilon\|\nabla d_t\|_{L^2}^2 + C\left[\|\nabla d\|_{L^\infty}^2\left(\left\|\nabla^2 d\right\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) + \int |u|^2\left|\nabla^2 d\right|^2 dx\right], \qquad (4.26)
\end{aligned}$$

where we have used (4.23) to estimate

$$\int |\nabla d|^6\,dx \lesssim \|\nabla d\|_{L^\infty}^2\int \left|\nabla^2 d\right|^2 dx. \qquad (4.27)$$

For the last term on the right-hand side of (4.26), using Nirenberg's interpolation inequality and Cauchy's inequality, we have

$$\begin{aligned}
\int |u|^2\left|\nabla^2 d\right|^2 dx &\lesssim \left\||u|^{\frac{5}{2}}\right\|_{L^6}^{\frac{12}{15}}\left\|\nabla^2 d\right\|_{L^{\frac{30}{13}}}^2 \leqslant \varepsilon\left\|\nabla|u|^{\frac{5}{2}}\right\|_{L^2}^2 + C\left\|\nabla^2 d\right\|_{L^{\frac{30}{13}}}^{\frac{10}{3}} \\
&\leqslant \varepsilon\left\|\nabla|u|^{\frac{5}{2}}\right\|_{L^2}^2 + C\|\nabla d\|_{L^6}^{\frac{8}{3}}\|\nabla d\|_{H^2}^{\frac{2}{3}} \\
&\leqslant \varepsilon\left\|\nabla|u|^{\frac{5}{2}}\right\|_{L^2}^2 + \varepsilon\left\|\nabla^3 d\right\|_{L^2}^2 + C\left(\|\nabla d\|_{L^6}^4 + \left\|\nabla^2 d\right\|_{L^2}^2 + 1\right) \\
&\leqslant 5\varepsilon\int |u|^3|\nabla u|^2\,dx + \varepsilon\left\|\nabla^3 d\right\|_{L^2}^2 + C\left(\left\|\nabla^2 d\right\|_{L^2}^4 + 1\right). \qquad (4.28)
\end{aligned}$$

By (1.3), H^3-estimate for elliptic equations, and (4.27), we have

$$\begin{aligned}\|\nabla^3 d\|_{L^2}^2 &\lesssim \|\nabla d_t\|_{L^2}^2 + \|\nabla(u\cdot\nabla d)\|_{L^2}^2 + \|\nabla(|\nabla d|^2 d)\|_{L^2}^2 \\ &\lesssim \|\nabla d_t\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\big(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2\big) + \|\nabla d\|_{L^6}^6 + \int |u|^2|\nabla^2 d|^2\,dx \\ &\leqslant C\bigg[\|\nabla d_t\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\big(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2\big) + \int |u|^2|\nabla^2 d|^2\,dx\bigg]. \end{aligned} \tag{4.29}$$

Substituting (4.29) into (4.28), and choosing ε sufficiently small, we have

$$\begin{aligned}&\int |u|^2|\nabla^2 d|^2\,dx \\ &\quad\leqslant C\bigg[\int |u|^3|\nabla u|^2\,dx + \|\nabla^2 d\|_{L^2}^4 + \varepsilon\|\nabla d_t\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^2\big(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2\big) + 1\bigg]. \end{aligned} \tag{4.30}$$

Substituting (4.30) into (4.26), using (4.6), and choosing ε sufficiently small, we obtain

$$\begin{aligned}&\frac{d}{dt}\int |\Delta d|^2\,dx + \int |\nabla d_t|^2\,dx \\ &\quad\leqslant C\bigg[\|\nabla d\|_{L^\infty}^2\big(\|\nabla^2 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\big) + \int |u|^3|\nabla u|^2\,dx + \|\nabla^2 d\|_{L^2}^4 + 1\bigg] \\ &\quad\leqslant C\bigg[\|\nabla d\|_{L^\infty}^2\big(\|\nabla^2 d\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\big) + \|\nabla d\|_{L^\infty}^2 + \int |u|^3|\nabla u|^2\,dx + \|\nabla^2 d\|_{L^2}^4 + 1\bigg]. \end{aligned} \tag{4.31}$$

Step 5. Completion of proof of Lemma 4.2. Adding (4.21), (4.18), (4.25) and (4.31) together, and choosing ε sufficiently small, we obtain

$$\begin{aligned}&\frac{d}{dt}\int \big(\rho|u|^5 + \mu|\nabla w|^2 + (\mu+\lambda)|\operatorname{div} w|^2 + |\nabla d|^5 + |\Delta d|^2\big)\,dx + \frac12\int \rho|w_t|^2\,dx + \frac12\int |\nabla d_t|^2\,dx \\ &\leqslant 2\frac{d}{dt}\int\Big(\nabla d\otimes\nabla d - \frac12|\nabla d|^2\mathbb{I}_3\Big) : \nabla w\,dx + C\bigg[\big(\|\nabla u\|_{L^2}^2 + 1\big)\int \rho|u|^5\,dx\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^3 \\ &\quad + \|\nabla d\|_{L^\infty}^3\big(\|\nabla d\|_{L^5}^5 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\big) + \|\nabla d\|_{L^\infty}^2\big(\|\nabla d\|_{L^5}^5 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\big) \\ &\quad + \|\nabla d\|_{L^\infty}^2 + \|\nabla^2 d\|_{L^2}^4 + 1\bigg].\end{aligned}$$

This, combined with Cauchy's inequality, implies

$$\begin{aligned}&\frac{d}{dt}\int \big(\rho|u|^5 + \mu|\nabla w|^2 + (\mu+\lambda)|\operatorname{div} w|^2 + |\nabla d|^5 + |\Delta d|^2\big)\,dx + \frac12\bigg(\int \rho|w_t|^2\,dx + \int |\nabla d_t|^2\,dx\bigg) \\ &\leqslant 2\frac{d}{dt}\int\Big(\nabla d\otimes\nabla d - \frac12|\nabla d|^2\mathbb{I}_3\Big) : \nabla w\,dx + C\big[\|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^3 \\ &\quad + \big(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^\infty}^3 + 1\big)\big(\|\rho^{\frac15}u\|_{L^5}^5 + \|\nabla d\|_{L^5}^5 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\big) + 1\big].\end{aligned}$$

Integrating over $(0,t)$, and using (4.1), (2.5), we have

$$\int \left(\rho|u|^5+|\nabla w|^2+|\nabla d|^5+\left|\nabla^2 d\right|^2\right)dx+\int_0^t\int \left(\rho|w_t|^2+|\nabla d_t|^2\right)dx\,ds$$
$$\leqslant C\left[\int |\nabla d|^2|\nabla w|\,dx+\int_0^t K(s)\left(\left\|\rho^{\frac{1}{5}}u\right\|_{L^5}^5+\|\nabla w\|_{L^2}^2+\|\nabla d\|_{L^5}^5+\left\|\nabla^2 d\right\|_{L^2}^2\right)ds+1\right], \quad (4.32)$$

where

$$K(s)=\left\|\nabla u(s)\right\|_{L^2}^2+\left\|\nabla^2 d(s)\right\|_{L^2}^2+\left\|\nabla d(s)\right\|_{L^\infty}^3+1.$$

By (4.32) and Young's inequality, we have

$$\int \left(\rho|u|^5+|\nabla w|^2+|\nabla d|^5+\left|\nabla^2 d\right|^2\right)dx+\int_0^t\int \left(\rho|w_t|^2+|\nabla d_t|^2\right)dx\,ds$$
$$\leqslant \frac{1}{2}\int |\nabla w|^2\,dx+C\left[\int |\nabla d|^4\,dx+\int_0^t K(s)\left(\left\|\rho^{\frac{1}{5}}u\right\|_{L^5}^5+\|\nabla w\|_{L^2}^2+\|\nabla d\|_{L^5}^5+\left\|\nabla^2 d\right\|_{L^2}^2\right)ds+1\right]$$
$$\leqslant \frac{1}{2}\left(\int |\nabla w|^2\,dx+\int |\nabla d|^5\,dx\right)$$
$$+C\left[\int_0^t K(s)\left(\left\|\rho^{\frac{1}{5}}u\right\|_{L^5}^5+\|\nabla w\|_{L^2}^2+\|\nabla d\|_{L^5}^5+\left\|\nabla^2 d\right\|_{L^2}^2\right)ds+1\right].$$

Thus we obtain

$$\int \left(\rho|u|^5+|\nabla w|^2+|\nabla d|^5+\left|\nabla^2 d\right|^2\right)dx+\int_0^t\int \left(\rho|w_t|^2+|\nabla d_t|^2\right)dx\,ds$$
$$\leqslant C\left[1+\int_0^t K(s)\left(\left\|\rho^{\frac{1}{5}}u\right\|_{L^5}^5+\|\nabla w\|_{L^2}^2+\|\nabla d\|_{L^5}^5+\left\|\nabla^2 d\right\|_{L^2}^2\right)ds\right]. \quad (4.33)$$

By (4.1), (2.5) and (4.4), we know

$$\int_0^t K(s)\,ds\leqslant C. \quad (4.34)$$

By (4.33), (4.34) and Gronwall's inequality, we obtain that for any $0\leqslant t<T_*$,

$$\int \left(\rho|u|^5+|\nabla w|^2+|\nabla d|^5+\left|\nabla^2 d\right|^2\right)dx+\int_0^t\int \left(\rho|w_t|^2+|\nabla d_t|^2\right)dx\,ds\leqslant C.$$

This completes the proof of Lemma 4.2. □

Corollary 4.3. *Under the same assumptions of Lemma* 4.2*, we have that for any* $2 \leqslant q \leqslant 6$*,*

$$\sup_{0\leqslant t<T_*} \left(\|u\|_{L^6} + \|\nabla u\|_{L^2} + \|\nabla d\|_{L^q} + \|d_t\|_{L^2}\right) + \|\nabla u\|_{L^2(0,T;L^6)} \leqslant C. \tag{4.35}$$

Proof. Combining (4.6) with (4.8), we get

$$\left\|\nabla u(t)\right\|_{L^2} \lesssim \left\|\nabla w(t)\right\|_{L^2} + \left\|\nabla v(t)\right\|_{L^2} \leqslant C. \tag{4.36}$$

The upper bound of $\sup_{0\leqslant t<T_*}\|u\|_{L^6}$ follows from (4.36) and Sobolev's inequality. The bound of $\sup_{0\leqslant t<T_*}\|\nabla d\|_{L^q}$ follows from (4.8) and interpolation inequality. For the last term of (4.35), by Sobolev's inequality, (4.6) and (4.8), we have

$$\begin{aligned}\|\nabla u\|_{L^2(0,T;L^6)} &\lesssim \|\nabla w\|_{L^2(0,T;L^6)} + \|\nabla v\|_{L^2(0,T;L^6)} \\ &\lesssim \left\|\nabla^2 w\right\|_{L^2(0,T;L^2)} + \|\nabla w\|_{L^2(0,T;L^2)} + 1 \leqslant C.\end{aligned}$$

By Eqs. (1.3), (4.8) and Hölder's inequality, we have

$$\begin{aligned}\sup_{0\leqslant t<T_*}\|d_t\|_{L^2} &\lesssim \sup_{0\leqslant t<T_*}\left(\|\Delta d\|_{L^2} + \|\nabla d\|_{L^4}^2 + \|u\cdot\nabla d\|_{L^2}\right) \\ &\lesssim \sup_{0\leqslant t<T_*}\left(\|u\|_{L^6}\|\nabla d\|_{L^3}\right) + 1 \leqslant C.\end{aligned}$$

This completes the proof. □

Lemma 4.4. *Under the same assumptions of Lemma* 4.2*,* (ρ, u, d) *satisfies that for any* $0 \leqslant t < T_*$*,*

$$\int_\Omega \left(\rho\left|\dot u(t)\right|^2 + |\nabla d_t|^2\right)(t)\,dx + \int_0^t\int_\Omega \left(|\nabla \dot u|^2 + |d_{tt}|^2\right)dx\,ds \leqslant C, \tag{4.37}$$

where $\dot f$ *is the material derivative*:

$$\dot f := f_t + u\cdot\nabla f.$$

Proof. *Step 1. Estimates of* $\int \rho|\dot u(t)|^2\,dx$*.* By the definition of material derivative, we can write (1.2) as follows,

$$\rho\dot u + \nabla\left(P(\rho)\right) = \mathcal{L}u - \nabla d\cdot\Delta d. \tag{4.38}$$

Differentiating (4.38) with respect to t and using (1.1), we have

$$\begin{aligned}&\rho\dot u_t + \rho u\cdot\nabla\dot u + \nabla\left(P(\rho)_t\right) + (\nabla d\cdot\Delta d)_t \\ &\quad = \mathcal{L}\dot u - \mathcal{L}(u\cdot\nabla u) + \operatorname{div}\left[\mathcal{L}u\otimes u - \nabla\left(P(\rho)\right)\otimes u - (\nabla d\cdot\Delta d)\otimes u\right].\end{aligned} \tag{4.39}$$

Multiplying (4.39) by $\dot u$, integrating by parts over Ω and using the fact $\dot u = 0$ on $\partial\Omega$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|\dot u|^2\,dx + \int \big(\mu|\nabla \dot u|^2 + (\mu+\lambda)|\operatorname{div}\dot u|^2\big)\,dx \\
&\quad = \int \big(\big(P(\rho)\big)_t \operatorname{div}\dot u + u\otimes\nabla\big(P(\rho)\big):\nabla\dot u\big)\,dx + \mu\int\big(\operatorname{div}(\Delta u\otimes u) - \Delta(u\cdot\nabla u)\big)\cdot\dot u\,dx \\
&\qquad + (\mu+\lambda)\int\big(\operatorname{div}(\nabla\operatorname{div}u\otimes u) - \nabla\operatorname{div}(u\cdot\nabla u)\big)\cdot\dot u\,dx + \int\big(u\otimes(\Delta d\cdot\nabla d)\big):\nabla\dot u\,dx \\
&\qquad + \int(\nabla d_t\otimes\nabla d + \nabla d\otimes\nabla d_t - \nabla d\cdot\nabla d_t\mathbb{I}_3):\nabla\dot u\,dx = \sum_{i=1}^{5} J_i. \qquad (4.40)
\end{aligned}$$

By Eqs. (1.1) and (4.1), we have

$$\begin{aligned}
J_1 &= \int\big(-\operatorname{div}\big(P(\rho)u\big)\operatorname{div}\dot u - \big(P'(\rho)\rho - P(\rho)\big)\operatorname{div}u\operatorname{div}\dot u + u\otimes\nabla\big(P(\rho)\big):\nabla\dot u\big)\,dx \\
&= \int\big(P(\rho)u\cdot\nabla\operatorname{div}\dot u + \big(P(\rho)-P'(\rho)\rho\big)\operatorname{div}u\operatorname{div}\dot u + P(\rho)(\nabla u)^t:\nabla\dot u - P(\rho)u\cdot\nabla\operatorname{div}\dot u\big)\,dx \\
&= \int\big(\big(P(\rho)-P'(\rho)\rho\big)\operatorname{div}u\operatorname{div}\dot u\,dx + P(\rho)(\nabla u)^t:\nabla\dot u\big)\,dx \lesssim \|\nabla u\|_{L^2}\|\nabla\dot u\|_{L^2}.
\end{aligned}$$

By the product rule, we can see

$$\operatorname{div}(\Delta u\otimes u) - \Delta(u\cdot\nabla u) = \nabla_k(\operatorname{div}u\nabla_k u) - \nabla_k\big(\nabla_k u^j\nabla_j u\big) - \nabla_j\big(\nabla_k u^j\nabla_k u\big),$$

so that by integration by parts, we have

$$J_2 = \mu\int\big(\nabla_k(\operatorname{div}u\nabla_k u) - \nabla_k\big(\nabla_k u^j\nabla_j u\big) - \nabla_j\big(\nabla_k u^j\nabla_k u\big)\big)\cdot\dot u\,dx \lesssim \|\nabla\dot u\|_{L^2}\|\nabla u\|_{L^4}^2.$$

Similarly, since

$$\operatorname{div}(\nabla\operatorname{div}u\otimes u) - \nabla\operatorname{div}(u\cdot\nabla u) = \nabla_k\big(\nabla_j u^j\nabla_i u^i\big) - \nabla_k\big(\nabla_j u^i\nabla_i u^j\big) - \nabla_i\big(\nabla_k u^i\nabla_j u^j\big),$$

we have

$$J_3 = (\mu+\lambda)\int\big(\nabla_k\big(\nabla_j u^j\nabla_i u^i\big) - \nabla_k\big(\nabla_j u^i\nabla_i u^j\big) - \nabla_i\big(\nabla_k u^i\nabla_j u^j\big)\big)\dot u^k\,dx \lesssim \|\nabla\dot u\|_{L^2}\|\nabla u\|_{L^4}^2.$$

By Hölder's inequality, and Corollary 4.3, we have

$$J_4 \lesssim \|\nabla\dot u\|_{L^2}\|\Delta d\|_{L^6}\|\nabla d\|_{L^6}\|u\|_{L^6} \lesssim \|\nabla\dot u\|_{L^2}\|\Delta d\|_{L^6},$$

$$J_5 \lesssim \int|\nabla\dot u||\nabla d_t||\nabla d|\,dx \lesssim \|\nabla\dot u\|_{L^2}\|\nabla d_t\|_{L^2}\|\nabla d\|_{L^\infty}.$$

Putting all these estimates into (4.40), using Young's inequality and Sobolev's inequality, and Lemma 4.2 and Corollary 4.3, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|\dot{u}|^2\,dx+\int \big(\mu|\nabla\dot{u}|^2+(\mu+\lambda)|\operatorname{div}\dot{u}|^2\big)\,dx\\
&\quad\lesssim \|\nabla u\|_{L^2}\|\nabla\dot{u}\|_{L^2}+\|\nabla u\|_{L^4}^2\|\nabla\dot{u}\|_{L^2}+\|\nabla\dot{u}\|_{L^2}\|\Delta d\|_{L^6}+\|\nabla\dot{u}\|_{L^2}\|\nabla d_t\|_{L^2}\|\nabla d\|_{L^\infty}\\
&\quad\leqslant \frac{\mu}{2}\|\nabla\dot{u}\|_{L^2}^2+C\big(\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^4}^4+\|\Delta d\|_{H^1}^2+\|\nabla d_t\|_{L^2}^2\|\nabla d\|_{L^\infty}^2\big)\\
&\quad\leqslant \frac{\mu}{2}\|\nabla\dot{u}\|_{L^2}^2+C\big(\|\nabla u\|_{L^4}^4+\big\|\nabla^3 d\big\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\|\nabla d\|_{L^\infty}^2+1\big).
\end{aligned}$$

Thus we obtain

$$\frac{d}{dt}\int \rho|\dot{u}|^2\,dx+\mu\int|\nabla\dot{u}|^2\,dx\lesssim \|\nabla u\|_{L^4}^4+\big\|\nabla^3 d\big\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\|\nabla d\|_{L^\infty}^2+1. \tag{4.41}$$

By H^3-estimate of elliptic equations, Lemma 4.2, Corollary 4.3, and Nirenberg's interpolation inequality, we have

$$\begin{aligned}
\big\|\nabla^3 d\big\|_{L^2}&\lesssim \|\nabla d_t\|_{L^2}+\|u\cdot\nabla d\|_{L^2}+\big\||\nabla u||\nabla d|\big\|_{L^2}+\big\||\nabla d|\big|\nabla^2 d\big|\big\|_{L^2}+\big\||\nabla d|^3\big\|_{L^2}\\
&\lesssim \|\nabla d_t\|_{L^2}+\|u\|_{L^6}\|\nabla d\|_{L^3}+\|\nabla u\|_{L^3}\|\nabla d\|_{L^6}+\|\nabla d\|_{L^6}\big\|\nabla^2 d\big\|_{L^3}+1\\
&\lesssim \|\nabla d_t\|_{L^2}+\|\nabla u\|_{L^2}^{\frac12}\|\nabla u\|_{H^1}^{\frac12}+\big\|\nabla^2 d\big\|_{L^2}^{\frac12}\big\|\nabla^2 d\big\|_{H^1}^{\frac12}+1\\
&\leqslant \frac12\big\|\nabla^3 d\big\|_{L^2}+C\big(\|\nabla d_t\|_{L^2}+\big\|\nabla^2 u\big\|_{L^2}+1\big).
\end{aligned}$$

Thus we obtain

$$\big\|\nabla^3 d\big\|_{L^2}\lesssim \|\nabla d_t\|_{L^2}+\big\|\nabla^2 u\big\|_{L^2}+1. \tag{4.42}$$

By the definition of w, we have

$$\mathcal{L}w=\rho\dot{u}+\Delta d\cdot\nabla d. \tag{4.43}$$

By H^2-estimate of Eqs. (4.43), (4.1), Corollary 4.3, Nirenberg's interpolation inequality, and (4.42), we obtain

$$\begin{aligned}
\big\|\nabla^2 w\big\|_{L^2}^2&\lesssim \|\rho\dot{u}\|_{L^2}^2+\|\Delta d\cdot\nabla d\|_{L^2}^2\lesssim \big\|\rho^{\frac12}\dot{u}\big\|_{L^2}^2+\|\Delta d\|_{L^3}^2\|\nabla d\|_{L^6}^2\\
&\lesssim \big\|\rho^{\frac12}\dot{u}\big\|_{L^2}^2+\|\Delta d\|_{L^2}\|\Delta d\|_{H^1}\lesssim \big\|\rho^{\frac12}\dot{u}\big\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}+1\\
&\lesssim \big\|\rho^{\frac12}\dot{u}\big\|_{L^2}^2+\|\nabla d_t\|_{L^2}+\big\|\nabla^2 u\big\|_{L^2}+1.
\end{aligned}\tag{4.44}$$

By interpolation inequality, Corollary 4.3, (4.6) (for $q=6$), (4.44), and Cauchy's inequality, we obtain

$$\begin{aligned}
\|\nabla u\|_{L^4}^4&\lesssim \|\nabla u\|_{L^2}\|\nabla u\|_{L^6}^3\lesssim \|\nabla u\|_{L^6}\|\nabla u\|_{L^6}^2\\
&\lesssim \|\nabla u\|_{L^6}\big(\|\nabla w\|_{L^6}^2+\|\nabla v\|_{L^6}^2\big)\lesssim \|\nabla u\|_{L^6}\big(\big\|\nabla^2 w\big\|_{L^2}^2+1\big)\\
&\lesssim \|\nabla u\|_{L^6}\big(\big\|\rho^{\frac12}\dot{u}\big\|_{L^2}^2+\|\nabla d_t\|_{L^2}+\big\|\nabla^2 u\big\|_{L^2}+1\big)\\
&\lesssim \|\nabla u\|_{L^6}\big\|\rho^{\frac12}\dot{u}\big\|_{L^2}^2+\|\nabla u\|_{L^6}^2+\|\nabla d_t\|_{L^2}^2+\big\|\nabla^2 u\big\|_{L^2}^2+1\\
&\lesssim \|\nabla u\|_{L^6}\big\|\rho^{\frac12}\dot{u}\big\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2+\big\|\nabla^2 u\big\|_{L^2}^2+1.
\end{aligned}\tag{4.45}$$

Putting (4.45) and (4.42) into (4.41), we have

$$\frac{d}{dt}\int \rho|\dot{u}|^2\,dx+\mu\int|\nabla\dot{u}|^2\,dx \lesssim \|\nabla u\|_{L^6}\big\|\rho^{\frac{1}{2}}\dot{u}\big\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\big(\|\nabla d\|_{L^\infty}^2+1\big)+\big\|\nabla^2 u\big\|_{L^2}^2+1. \tag{4.46}$$

Step 2. Estimates of $\int|\nabla d_t|^2\,dx$. Differentiating (1.3) with respect to t, we have

$$d_{tt}-\Delta d_t=\partial_t\big(|\nabla d|^2 d-u\cdot\nabla d\big). \tag{4.47}$$

Multiplying (4.47) by d_{tt}, integrating by parts over Ω and using $\frac{\partial d_t}{\partial \nu}|_{\partial\Omega}=0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla d_t|^2\,dx+\int|d_{tt}|^2\,dx\\
&\quad=\int\partial_t\big(|\nabla d|^2 d-u\cdot\nabla d\big)d_{tt}\,dx\\
&\quad\lesssim\int\big(|\nabla d|^2|d_t|+|\nabla d||\nabla d_t|\big)|d_{tt}|\,dx+\int\big(|u_t||\nabla d|+|u||\nabla d_t|\big)|d_{tt}|\,dx\\
&\quad=K_1+K_2.
\end{aligned} \tag{4.48}$$

By Hölder's inequality, Sobolev's inequality, Corollary 4.3, and Young's inequality, we have

$$\begin{aligned}
|K_1|&\lesssim\|d_{tt}\|_{L^2}\|d_t\|_{L^6}\|\nabla d\|_{L^6}^2+\|d_{tt}\|_{L^2}\|\nabla d_t\|_{L^2}\|\nabla d\|_{L^\infty}\\
&\lesssim\|d_{tt}\|_{L^2}\big(\|\nabla d_t\|_{L^2}+1\big)+\|d_{tt}\|_{L^2}\|\nabla d_t\|_{L^2}\|\nabla d\|_{L^\infty}\\
&\leqslant\frac{1}{8}\|d_{tt}\|_{L^2}^2+C\big(\|\nabla d_t\|_{L^2}^2+\|\nabla d\|_{L^\infty}^2\|\nabla d_t\|_{L^2}^2+1\big).
\end{aligned}$$

By the definition of $\dot{u}$, Hölder's inequality, Sobolev's inequality, Corollary 4.3, and Young's inequality, we have

$$\begin{aligned}
|K_2|&\lesssim\int\big[\big(|\dot{u}|+|u||\nabla u|\big)|\nabla d|+|u||\nabla d_t|\big]|d_{tt}|\,dx\\
&\lesssim\|d_{tt}\|_{L^2}\|\dot{u}\|_{L^6}\|\nabla d\|_{L^3}+\|d_{tt}\|_{L^2}\|u\|_{L^6}\|\nabla u\|_{L^6}\|\nabla d\|_{L^6}+\|d_{tt}\|_{L^2}\|u\|_{L^6}\|\nabla d_t\|_{L^3}\\
&\lesssim\|d_{tt}\|_{L^2}\|\nabla\dot{u}\|_{L^2}+\|d_{tt}\|_{L^2}\big(\big\|\nabla^2 u\big\|_{L^2}+1\big)+\|d_{tt}\|_{L^2}\|\nabla d_t\|_{L^3}\\
&\leqslant\frac{1}{8}\|d_{tt}\|_{L^2}^2+C\big(\|\nabla\dot{u}\|_{L^2}^2+\big\|\nabla^2 u\big\|_{L^2}^2+\|\nabla d\|_{L^3}^2+1\big).
\end{aligned}$$

Putting these two estimates into (4.48), using Nirenberg's interpolation inequality, and Young's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla d_t|^2\,dx+\frac{3}{4}\int|d_{tt}|^2\,dx\\
&\quad\lesssim\|\nabla\dot{u}\|_{L^2}^2+\big\|\nabla^2 u\big\|_{L^2}^2+\big(1+\|\nabla d\|_{L^\infty}^2\big)\|\nabla d_t\|_{L^2}^2+\|\nabla d\|_{L^3}^2+1\\
&\quad\lesssim\|\nabla\dot{u}\|_{L^2}^2+\big\|\nabla^2 u\big\|_{L^2}^2+\big(1+\|\nabla d\|_{L^\infty}^2\big)\|\nabla d_t\|_{L^2}^2+\|\nabla d_t\|_{L^2}\big\|\nabla^2 d_t\big\|_{L^2}+1\\
&\quad\leqslant\frac{1}{8}\big\|\nabla^2 d_t\big\|_{L^2}^2+C\big(\|\nabla\dot{u}\|_{L^2}^2+\big\|\nabla^2 u\big\|_{L^2}^2+\big(1+\|\nabla d\|_{L^\infty}^2\big)\|\nabla d_t\|_{L^2}^2+1\big).
\end{aligned} \tag{4.49}$$

By H^2-estimate of Eq. (4.47) and estimates similar to K_1 and K_2, we obtain

$$\begin{aligned}\|\nabla^2 d_t\|_{L^2} &\lesssim \|d_{tt}\|_{L^2} + \|\partial_t(u\cdot\nabla d)\|_{L^2} + \|\partial_t(|\nabla d|^2 d)\|_{L^2}\\ &\lesssim \|d_{tt}\|_{L^2} + \|\dot u\cdot\nabla d\|_{L^2} + \|(u\cdot\nabla u)\cdot\nabla d\|_{L^2} + \|u\|_{L^6}\|\nabla d_t\|_{L^3}\\ &\quad + \|d_t\|_{L^6}\|\nabla d\|_{L^6}^2 + \|\nabla d_t\|_{L^3}\|\nabla d\|_{L^6}\\ &\lesssim \|d_{tt}\|_{L^2} + \|\dot u\|_{L^6}\|\nabla d\|_{L^3} + \|u\|_{L^6}\|\nabla u\|_{L^6}\|\nabla d\|_{L^6} + \|\nabla d_t\|_{L^2}^{\frac12}\|\nabla d_t\|_{L^6}^{\frac12} + \|\nabla d_t\|_{L^2} + 1\\ &\leqslant \frac12\|\nabla^2 d_t\|_{L^2} + C\big(\|d_{tt}\|_{L^2} + \|\nabla\dot u\|_{L^2} + \|\nabla^2 u\|_{L^2} + \|\nabla d_t\|_{L^2} + 1\big).\end{aligned}$$

Thus

$$\|\nabla^2 d_t\|_{L^2} \lesssim \|d_{tt}\|_{L^2} + \|\nabla\dot u\|_{L^2} + \|\nabla^2 u\|_{L^2} + \|\nabla d_t\|_{L^2} + 1. \tag{4.50}$$

Substituting this inequality into (4.49), we obtain

$$\frac{d}{dt}\int|\nabla d_t|^2\,dx + \int|d_{tt}|^2\,dx \lesssim \|\nabla\dot u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \big(1+\|\nabla d\|_{L^\infty}^2\big)\|\nabla d_t\|_{L^2}^2 + 1. \tag{4.51}$$

Combining (4.46) and (4.51), and applying Gronwall's inequality, we establish the conclusions of Lemma 4.4. □

By Eq. (4.43) and Lemma 4.4, we obtain the following corollary:

Corollary 4.5. *Under the same assumptions of Lemma* 4.2*, we have that for* $q \in (3,6]$,

$$\sup_{0\leqslant t<T_*}\big(\|\nabla^3 d\|_{L^2} + \|\nabla d\|_{L^\infty}\big) + \|\nabla w\|_{L^2(0,T_*;L^\infty)} + \|\nabla^2 w\|_{L^2(0,T_*;L^q)} \leqslant C. \tag{4.52}$$

Proof. By H^3-estimate of elliptic equations, (1.3), Lemma 4.4, Corollary 4.3, and Nirenberg's interpolation inequality, we have

$$\begin{aligned}\|\nabla^3 d\|_{L^2} &\lesssim \|\nabla d_t\|_{L^2} + \|u\cdot\nabla d\|_{L^2} + \big\||\nabla u||\nabla d|\big\|_{L^2} + \big\||\nabla d||\nabla^2 d|\big\|_{L^2} + \big\||\nabla d|^3\big\|_{L^2}\\ &\lesssim \|u\|_{L^6}\|\nabla d\|_{L^3} + \|\nabla u\|_{L^2}\|\nabla d\|_{L^\infty} + \|\nabla d\|_{L^6}\|\nabla^2 d\|_{L^3} + 1\\ &\lesssim \|\nabla d\|_{L^2}^{\frac14}\|\nabla d\|_{H^2}^{\frac34} + \|\nabla^2 d\|_{L^2}^{\frac12}\|\nabla^2 d\|_{H^1}^{\frac12} + 1 \leqslant \frac12\|\nabla^3 d\|_{L^2} + C.\end{aligned}$$

Hence

$$\sup_{0\leqslant t<T_*}\|\nabla^3 d\|_{L^2} \leqslant C.$$

By Sobolev's inequality, this yields

$$\sup_{0\leqslant t<T_*}\|\nabla d\|_{L^\infty} \leqslant C.$$

For simplicity, we only consider the case $q=6$. By $W^{2,q}$-estimate of Eqs. (4.43), (4.1), and Sobolev's inequality, we obtain

$$\left\|\nabla^2 w\right\|_{L^6} \lesssim \|\rho\dot{u}\|_{L^6} + \|\Delta d \cdot \nabla d\|_{L^6} \lesssim \|\dot{u}\|_{L^6} + \|\Delta d\|_{H^1}\|\nabla d\|_{L^\infty} \lesssim \|\nabla\dot{u}\|_{L^2} + 1.$$

Therefore, by (4.37), we have

$$\left\|\nabla^2 w\right\|_{L^2(0,T_*;L^6)} \lesssim \int_0^{T_*} \left(\|\nabla\dot{u}\|_{L^2}^2 + 1\right) ds \leqslant C. \qquad \square$$

Following the same argument of [34, Section 5], we have

Lemma 4.6. *Under the same assumptions of Lemma* 4.2*, we have that for* $q \in (3, 6]$*,*

$$\sup_{0\leqslant t<T_*} \|\nabla\rho\|_{L^q\cap L^2} \leqslant C. \tag{4.53}$$

Corollary 4.7. *Under the same assumptions of Lemma* 4.2*, we have for* $q \in (3, 6]$*,*

$$\sup_{0\leqslant t<T_*} \left\|\nabla^2 u\right\|_{L^2} + \|u\|_{L^2(0,T_*;D^{2,q})} \leqslant C. \tag{4.54}$$

Proof. By [34, Proposition 2.1], (4.38), (4.1) and Lemma 4.6, we obtain that for $r_1 = 2$ or q,

$$\begin{aligned}\left\|\nabla^2 u\right\|_{L^{r_1}} &\lesssim \|\rho\dot{u}\|_{L^{r_1}} + \left\|\nabla\big(P(\rho)\big)\right\|_{L^{r_1}} + \|\nabla d \cdot \Delta d\|_{L^{r_1}} \\ &\lesssim \|\rho\dot{u}\|_{L^{r_1}} + \|\nabla d \cdot \Delta d\|_{L^{r_1}} + \|\nabla\rho\|_{L^{r_1}}.\end{aligned} \tag{4.55}$$

When $r_1 = 2$, (4.1), (4.55), Lemmas 4.2, 4.4, and Corollary 4.5 imply

$$\left\|\nabla^2 u\right\|_{L^2} \lesssim \left\|\rho^{\frac{1}{2}}\dot{u}\right\|_{L^2} + \|\nabla d\|_{L^\infty}\|\Delta d\|_{L^2} + 1 \leqslant C.$$

When $r_1 = q$, for simplicity, we only consider the case $q = 6$. By (4.1), (4.55), Lemma 4.2, Lemma 4.4, Corollary 4.5, and Sobolev's inequality, we have

$$\begin{aligned}\left\|\nabla^2 u\right\|_{L^2(0,T_*;L^6)} &\lesssim \|\rho\|_{L^\infty(0,T_*;L^\infty)}\|\dot{u}\|_{L^2(0,T_*;L^6)} + \sup_{0\leqslant t<T_*}\|\nabla d\|_{L^\infty}\|\Delta d\|_{L^2(0,T_*;L^6)} + 1 \\ &\lesssim \|\nabla\dot{u}\|_{L^2(0,T_*;L^2)} + \|\Delta d\|_{L^2(0,T_*;H^1)} + 1 \leqslant C.\end{aligned}$$

This completes the proof. $\square$

Corollary 4.8. *Under the same assumptions of Lemma* 4.2*, we have that for* $r_1 = 2$ *or* q*,*

$$\sup_{0\leqslant t<T_*} \int_\Omega \left(\rho|u_t|^2 + |\rho_t|^{r_1}\right) dx + \int_0^{T_*}\int_\Omega \left(|\nabla u_t|^2 + \left|\nabla^2 d_t\right|^2 + \left|\nabla^4 d\right|^2\right) dx\, ds \leqslant C. \tag{4.56}$$

Proof. It follows from (4.1), Lemma 4.4, Sobolev's inequality, (4.35), and Corollary 4.7 that

$$\begin{aligned}\int \rho|u_t|^2\, dx &\lesssim \int \rho|\dot{u}|^2\, dx + \int \rho|u\cdot\nabla u|^2\, dx \\ &\lesssim \|\rho\|_{L^\infty}\|u\|_{L^\infty}\int |\nabla u|^2\, dx + 1 \lesssim \|\nabla u\|_{H^1} + 1 \leqslant C.\end{aligned}$$

By (1.1), (4.1), Sobolev's inequality, (4.35), Lemma 4.6 and Corollary 4.7, we get

$$\begin{aligned}\|\rho_t\|_{L^{r_1}} &\lesssim \|\rho \operatorname{div} u\|_{L^{r_1}} + \|u\cdot\nabla\rho\|_{L^{r_1}} \lesssim \|\rho\|_{L^\infty}\|\operatorname{div} u\|_{L^{r_1}} + \|u\|_{L^\infty}\|\nabla\rho\|_{L^{r_1}} \\ &\lesssim \|\rho\|_{L^\infty}\|\operatorname{div} u\|_{H^1} + \|\nabla u\|_{H^1}\|\nabla\rho\|_{L^{r_1}} \leqslant C.\end{aligned}$$

By Lemma 4.4, interpolation inequality, Sobolev's inequality, (4.35), and Corollary 4.7, we have

$$\begin{aligned}\int_0^{T_*}\int_\Omega |\nabla u_t|^2\,dx\,ds &\lesssim \int_0^{T_*}\int_\Omega |\nabla\dot u|^2\,dx\,ds + \int_0^{T_*}\int_\Omega \left|\nabla(u\cdot\nabla u)\right|^2\,dx\,ds \\ &\lesssim \int_0^{T_*}\int_\Omega |\nabla u|^4\,dx\,ds + \int_0^{T_*}\int_\Omega \left|u\cdot\nabla^2 u\right|^2\,dx\,ds + 1 \\ &\lesssim \int_0^{T_*}\|\nabla u\|_{L^2}\|\nabla u\|_{H^1}^3\,ds + \int_0^{T_*}\|u\|_{L^\infty}^2\int_\Omega \left|\nabla^2 u\right|^2\,dx\,ds + 1 \\ &\lesssim \int_0^{T_*}\|\nabla u\|_{H^1}^2\int_\Omega \left|\nabla^2 u\right|^2\,dx\,ds + 1 \leqslant C.\end{aligned}$$

By (4.50), Lemma 4.4, and Corollary 4.7, we get

$$\int_0^{T_*}\int_\Omega \left|\nabla^2 d_t\right|^2\,dx\,ds \leqslant C. \tag{4.57}$$

By H^4-estimate of Eq. (1.3), we have

$$\left\|\nabla^4 d\right\|_{L^2}^2 \lesssim \|d_t\|_{H^2}^2 + \|u\cdot\nabla d\|_{H^2}^2 + \left\||\nabla d|^2 d\right\|_{H^2}^2 = \sum_{i=1}^3 L_i. \tag{4.58}$$

For L_1, (4.35) and Lemma 4.4 imply

$$L_1 \lesssim \left\|\nabla^2 d_t\right\|_{L^2}^2 + 1. \tag{4.59}$$

For L_2, Hölder's inequality, Sobolev's inequality, (2.5), (4.8), (4.35), Corollaries 4.5 and 4.7, we have

$$\begin{aligned}L_2 &\lesssim \left\||u|\left(|\nabla d| + \left|\nabla^2 d\right| + \left|\nabla^3 d\right|\right)\right\|_{L^2}^2 + \left\||\nabla u|\left(|\nabla d| + \left|\nabla^2 d\right|\right)\right\|_{L^2}^2 + \left\|\left|\nabla^2 u\right||\nabla d|\right\|_{L^2}^2 \\ &\lesssim \|u\|_{L^\infty}^2\left(\|\nabla d\|_{L^2}^2 + \left\|\nabla^2 d\right\|_{L^2}^2 + \left\|\nabla^3 d\right\|_{L^2}^2\right) + \|\nabla d\|_{L^\infty}^2\left(\|\nabla u\|_{L^2}^2 + \left\|\nabla^2 u\right\|_{L^2}^2\right) \\ &\quad + \|\nabla u\|_{H^1}^2\left\|\nabla^2 d\right\|_{H^1}^2 \leqslant C.\end{aligned} \tag{4.60}$$

Similarly, for L_3, we have

$$
\begin{aligned}
L_3 &\lesssim \big\| |\nabla d|^2 \big\|_{L^2}^2 + \big\| |\nabla d| \big|\nabla^2 d\big| \big\|_{L^2}^2 + \big\| \big|\nabla^2 d\big|^2 \big\|_{L^2}^2 + \big\| |\nabla d| \big|\nabla^3 d\big| \big\|_{L^2}^2 \\
&\lesssim \|\nabla d\|_{L^\infty}^2 \|\nabla d\|_{H^2}^2 + \big\|\nabla^2 d\big\|_{H^1}^4 \leqslant C.
\end{aligned} \tag{4.61}
$$

Substituting (4.59)–(4.61) into (4.58), we have

$$
\big\|\nabla^4 d\big\|_{L^2}^2 \lesssim \big\|\nabla^2 d_t\big\|_{L^2}^2 + 1. \tag{4.62}
$$

Integrating (4.62) over $(0, t)$, and using (4.57), we establish Corollary 4.8. □

Proof of Theorem 1.3. By the above estimates, we know that both (4.2) and (4.3) are valid. Hence T_* is not the maximum time for the strong solution (ρ, u, d). This contradicts the definition of T_*. The proof of Theorem 1.3 is complete. □

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