Linear Algebra and its Applications 437 (2012) 2523–2537

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

An elementary, illustrative proof of the Rado–Horn theorem ☆

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ARTICLE INFO

Article history:
Received 8 December 2011
Accepted 12 June 2012
Available online 23 July 2012

Submitted by R.A. Brualdi

AMS classification:
15A03
05A17

Keywords:
Rado–Horn
Vector partitions

ABSTRACT

The Rado–Horn theorem provides necessary and sufficient conditions for when a family of vectors can be partitioned into a fixed number of linearly independent sets. Such partitions exist if and only if every subfamily of the vectors satisfies the so-called Rado–Horn inequality. In this paper we provide an elementary proof of the Rado–Horn theorem as well as results for the redundant case. Previous proofs give no information about how to actually partition the vectors; we use ideas present in our proof to find subfamilies of vectors which may be used to construct a kind of "optimal" partition.

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1. Introduction

The terminology *Rado–Horn theorem* was first introduced in [3]. This theorem [12,15] provides necessary and sufficient conditions for a family of vectors to be partitioned into k linearly independent sets:

Theorem 1.1 (Rado–Horn). *Consider a family of non-zero vectors $\Phi = \{\varphi_i\}_{i=1}^M$ in a vector space. Then the following are equivalent:*

(i) The set $\{1, \dots, M\}$ can be partitioned into sets $\{A_j\}_{j=1}^k$ such that $\{\varphi_i\}_{i\in A_j}$ is a linearly independent set for all $j = 1, 2, \dots, k$.
(ii) For any non-empty subset $J \subseteq \{1, \dots, M\}$, $|J| \,/\, \dim \operatorname{span}(\{\varphi_i\}_{i\in J}) \leqslant k$.

The Rado–Horn theorem has found application in several areas including progress on the Feichtinger conjecture [5], a characterization of Sidon sets in $\Pi_{k=1}^{\infty}\mathbb{Z}_p$ [13,14], and a notion of redundancy for finite

☆ The first author is supported by NSF DMS 1008183, NSF ATD 1042701 and AFOSR F1ATA00183G003.
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0024-3795/$ - see front matter Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.laa.2012.06.017

frames [1]. A generalized version of the Rado–Horn theorem has also found use in frame theory where redundancy is at the heart of the subject [2].

Unfortunately, proving the Rado–Horn theorem tends to be very intricate. Pisier, when discussing a characterization of Sidon sets in $\Pi_{k=1}^{\infty}\mathbb{Z}_p$ states "... d'un lemme d'alébre dû à Rado–Horn dont la démonstration est relativement délicate" [14]. Today there are at least six proofs of the Rado–Horn theorem [4,5,10–12,15]. The theorem was proved in a more general algebraic setting in [12,15] and then for matroids in [10]; these proofs are all delicate. Harary and Welsh [11] improved upon the matroid version of the Rado–Horn theorem with a short and elegant proof; however, their argument requires a development of certain deep structures within matroid theory. The Rado–Horn theorem was generalized in [4] to include partitions of a family of vectors with subfamilies of specified sizes removed, and the authors also proved results for the redundant case – the case where a family of vectors cannot be partitioned into k linearly independent sets. Unfortunately the proofs for these refinements to the theorem are even more delicate than the original. Finally, the Rado–Horn theorem was rediscovered in [5], where the authors give an induction proof which may be considered elementary. This proof has some limitations, however, as it does not clearly generalize nor does it describe the redundant case; it does not reveal the origin of the Rado–Horn inequality.

In this paper, we present an elementary proof which is at the core of the Rado–Horn theorem. With slight modification, these simple arguments prove a generalization of the Rado–Horn theorem and provide results for the redundant case similar to those in [4]. Most appealing, the arguments we present may be thought of visually and provide insight into the specific conditions which give rise to the inequality in the Rado–Horn theorem. These ideas can then be used to construct partitions which contain the fewest possible number of linearly independent sets and which are optimal with regard to certain spanning properties. We will make this clear in the definition of a fundamental partition.

This paper is organized into three sections. The first develops constructions and main arguments used throughout the paper. The second section uses these tools to prove the Rado–Horn theorem, the original and the redundant case. The final section describes which subfamilies maximize the Rado–Horn inequality and how similar subfamilies may be used to construct a so-called fundamental partition.

2. Preliminaries

We will always consider $\Phi = \{\varphi_i\}_{i=1}^M$ to be a finite family of non-zero vectors in a real or complex vector space. Note the vectors in this family are not necessarily unique. Our proof of the Rado–Horn theorem relies on a special partition of this family. In this section we define fundamental partitions and demonstrate several of their remarkable properties.

Definition 2.1. Given a family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$, let $\{A_j\}_{j=1}^k$ be a partition of the index set $\{1, \ldots, M\}$. We call $\{\{\varphi_i\}_{i\in A_j}\}_{j=1}^k$ an **ordered partition** of Φ if $|A_j| \geqslant |A_{j+1}|$ for all $j = 1, \ldots, k-1$.

Definition 2.2. Given a family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$, let $\{P_k\}_{k=1}^m$ be all possible ordered partitions of Φ into linearly independent sets. Let $P_k = \{\{\varphi_i\}_{i\in F_{kj}}\}_{j=1}^{r_k}$ so that $\{\varphi_i\}_{i\in F_{kj}}$ denotes the jth set in the kth partition. Now define

$$a_1 = \max_{k=1,\ldots,m} |F_{k1}|.$$

Consider only the partitions $\{P_k : |F_{k1}| = a_1\}$, and define

$$a_2 = \max_{\{k:|F_{k1}|=a_1\}} |F_{k2}|.$$

We continue so that given $a_1, \ldots, a_n$,

$$a_{n+1} = \max_{\{k:|F_{k1}|=a_1,\ldots,|F_{kn}|=a_n\}} \left|F_{k(n+1)}\right|.$$

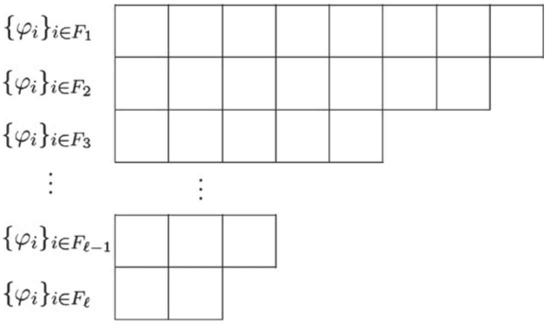

Fig. 1. Example of a fundamental partition $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$.

When $\sum_{i=1}^{\ell} a_i = M$, any remaining partition is in the set $\{P_k : |F_{k1}| = a_1, \ldots, |F_{k\ell}| = a_\ell\}$. We call any such ordered partition of Φ a **fundamental partition** which we write as $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$. Also, we will use the notation $\varphi_{(j)}$ when denoting some vector φ_i in $\{\varphi_i\}_{i\in F_j}$.

We introduce a fundamental partition as in Definition 2.2 because existence is clear. However, a fundamental partition is a specific example of a basis for a sum of matroids [6,7]. The following theorem gives a useful alternative definition and is Theorem 1 from [7].

Theorem 2.3. *Let $\Phi = \{\varphi_i\}_{i=1}^{M}$ be a family of vectors. Then $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ is a fundamental partition if and only if for any other ordered partition $\{\{\varphi_i\}_{i\in A_j}\}_{j=1}^{k}$ of Φ into linearly independent sets,*

(i) $\ell \leqslant k$.
(ii) $\sum_{j=1}^{n} |A_j| \leqslant \sum_{j=1}^{n} |F_j|$, $n = 1, 2, \ldots, \ell$.

It is helpful to visualize a fundamental partition as a Young diagram where each square represents a vector, and the rows correspond to the sets $\{\varphi_i\}_{i\in F_j}$; see Fig. 1. Intuitively, if Young diagrams represent ordered partitions of vectors into linearly independent sets, a fundamental partition is a diagram which is as top-heavy as possible.

Next we will examine spanning properties of a fundamental partition. We will often use the following well known result.

Proposition 2.4. *Let $\Phi = \{\varphi_i\}_{i=1}^{M}$ be a set of linearly independent vectors. Suppose $\psi \in \mathrm{span}(\Phi)$ so that $\psi = \sum_{i=1}^{M} c_i\varphi_i$. Then for any $j \in \{1, \ldots, M\}$ such that $c_j \neq 0$, $\Psi_j = (\Phi \setminus \{\varphi_j\}) \cup \{\psi\}$ is linearly independent and* $\mathrm{span}(\Psi_j) = \mathrm{span}(\Phi)$.

The following lemma is trivial but does provides some information concerning spanning properties of a fundamental partition.

Lemma 2.5. *Let $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ be a fundamental partition of $\Phi = \{\varphi_i\}_{i=1}^{M}$. Then $\mathrm{span}(\{\varphi_i\}_{i\in F_j}) \subseteq \mathrm{span}(\{\varphi_i\}_{i\in F_r})$ for $r \leqslant j$.*

Proof. Suppose there existed some $\varphi_{(j)} \in \{\varphi_i\}_{i\in F_j}$, such that $\varphi_{(j)} \notin \mathrm{span}(\{\varphi_i\}_{i\in F_r})$. Then

$$\{\varphi_i\}_{i\in F_r'} = \{\varphi_i\}_{i\in F_r} \cup \{\varphi_{(j)}\}$$

is linearly independent with $|F_r'| > |F_r|$ contradicting our assumption that $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ is a fundamental partition. $\square$

This shows that in a fundamental partition, any vector is contained in the spans of the sets before it. Next we show some vectors must be contained in the spans of almost every set.

Lemma 2.6. *Let $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ be a fundamental partition of $\Phi = \{\varphi_i\}_{i=1}^{M}$. Pick any $\varphi_{(\ell)} \in \{\varphi_i\}_{i\in F_\ell}$ and fix any $k \leqslant \ell - 1$. Let $S_k \subseteq F_k$ be the smallest set such that $\varphi_{(\ell)} \in \text{span}(\{\varphi_i\}_{i\in S_k})$. Then $\text{span}(\{\varphi_i\}_{i\in S_k}) \subseteq \text{span}(\{\varphi_i\}_{i\in F_j})$, $j = 1, \ldots, \ell - 1$.*

Proof. Clearly the set S_k exist by Lemma 2.5. We will prove the statement for $j = \ell - 1$. The result will then follow for all $j = 1, \ldots, \ell - 1$ since $\text{span}(\{\varphi_i\}_{i\in F_{\ell-1}}) \subseteq \text{span}(\{\varphi_i\}_{i\in F_j})$ for $j \leqslant \ell - 1$.

We will assume the result fails and get a contradiction. Suppose there exists some $\varphi_{(k)} \in \{\varphi_i\}_{i\in S_k}$ such that $\varphi_{(k)} \notin \text{span}(\{\varphi_i\}_{i\in F_{\ell-1}})$. By Proposition 2.4,

$$(\{\varphi_i\}_{i\in S_k} \setminus \{\varphi_{(k)}\}) \cup \{\varphi_{(\ell)}\}$$

is linearly independent with the same span as $\{\varphi_i\}_{i\in S_k}$. Thus we can partition $\Phi \setminus \{\varphi_{(k)}\}$ into ℓ linearly independent sets, say $\{\{\varphi_i\}_{i\in G_j}\}_{j=1}^{\ell}$ given by

$$\{\varphi_i\}_{i\in G_j} = \begin{cases} (\{\varphi_i\}_{i\in F_k} \setminus \{\varphi_{(k)}\}) \cup \{\varphi_{(\ell)}\} & \text{for } j = k \\ \{\varphi_i\}_{i\in F_\ell} \setminus \{\varphi_{(\ell)}\} & \text{for } j = \ell \\ \{\varphi_i\}_{i\in F_j} & \text{for } j \neq k, \ell. \end{cases}$$

Notice $|G_j| = |F_j|$ and $\text{span}(\{\varphi_i\}_{i\in G_j}) = \text{span}(\{\varphi_i\}_{i\in F_j})$ for $j = 1, \ldots, \ell - 1$, but then

$$\{\varphi_i\}_{i\in G_{\ell-1}} \cup \{\varphi_{(k)}\}$$

is also linearly independent with $\left|\{\varphi_i\}_{i\in G_{\ell-1}} \cup \{\varphi_{(k)}\}\right| > \left|\{\varphi_i\}_{i\in F_{\ell-1}}\right|$. This contradicts the fact that $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ was a fundamental partition. □

We can extend Lemma 2.6 to obtain a larger set of vectors which must be contained in the spans of each $\{\varphi_i\}_{i\in F_j}, j = 1, \ldots, \ell - 1$; this is done by iterating the argument.

Theorem 2.7. *Let $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ be a fundamental partition of $\Phi = \{\varphi_i\}_{i=1}^{M}$. Pick any $\varphi_{(\ell)} \in \{\varphi_i\}_{i\in F_\ell}$ and for $j = 1, \ldots, \ell - 1$, let $S_j^{(1)} \subseteq F_j$ be the smallest set such that $\varphi_{(\ell)} \in \text{span}(\{\varphi_i\}_{i\in S_j^{(1)}})$. Pick a k_1 so*

$$\left|S_{k_1}^{(1)}\right| = \max_{j=1,\ldots,\ell-1} \left|S_j^{(1)}\right|,$$

and set $S_{k_1}^{(1)} = S_{k_1}^{(2)}$. Now define $S_j^{(1)} \subseteq S_j^{(2)} \subseteq F_j, j = 1, \ldots, \ell - 1$ as the smallest subset such that $\text{span}(\{\varphi_i\}_{i\in S_{k_1}^{(2)}}) \subseteq \text{span}(\{\varphi_i\}_{i\in S_j^{(2)}})$ and choose k_2 so

$$\left|S_{k_2}^{(2)}\right| = \max_{j=1,\ldots,\ell-1} \left|S_j^{(2)}\right|.$$

Continue this process so given $S_j^{(n-1)}$ and k_{n-1}, we set $S_{k_{n-1}}^{(n-1)} = S_{k_{n-1}}^{(n)}$ and define $S_j^{(n-1)} \subseteq S_j^{(n)} \subseteq F_j$, $j = 1, \ldots \ell - 1$ as the smallest subset such that $\text{span}(\{\varphi_i\}_{i\in S_{k_{n-1}}^{(n)}}) \subseteq \text{span}(\{\varphi_i\}_{i\in S_j^{(n)}})$. Choosing k_n so

$$\left|S_{k_n}^{(n)}\right| = \max_{j=1,\ldots,\ell-1} \left|S_j^{(n)}\right|,$$

then $\text{span}(\{\varphi_i\}_{i\in S_{k_n}^{(n)}}) \subseteq \text{span}(\{\varphi_i\}_{i\in F_j})$, for all $j = 1, \ldots, \ell - 1$.

Proof. For $n = 1$, this is Lemma 2.6. Notice this guarantees the sets $S_j^{(2)}, j = 1, \ldots, \ell - 1$ are well defined. It now suffices to show $\text{span}(\{\varphi_i\}_{i \in S_{k_n}^{(n)}}) \subseteq \text{span}(\{\varphi_i\}_{i \in F_{\ell-1}})$.

We introduce the notation $\varphi_{(j)}^{(n)}$ to represent a vector in $\{\varphi_i\}_{i \in F_j}$ present in the nth iteration. That is $\varphi_{(j)}^{(n)} \in S_j^{(n)}$.

We proceed by contradiction. Suppose instead there existed some $\varphi_{(k_n)}^{(n)} \in \{\varphi_i\}_{i \in S_{k_n}^{(n)}}$ such that $\varphi_{(k_n)}^{(n)} \notin \text{span}(\{\varphi_i\}_{i \in F_{\ell-1}})$. By Proposition 2.4, there exists some $\varphi_{(k_{n-1})}^{(m_1)} \in \{\varphi_i\}_{i \in S_{k_{n-1}}^{(m_1)}}$, $m_1 < n$, such that

$$\left(\{\varphi_i\}_{i \in S_{k_n}^{(n)}} \setminus \{\varphi_{(k_n)}^{(n)}\}\right) \cup \left\{\varphi_{(k_{n-1})}^{(m_1)}\right\}$$

is linearly independent and has the same span as $\{\varphi_i\}_{i \in S_{k_n}^{(n)}}$. Note there may be several such m_i for which there is an appropriate $\varphi_{(k_{n-1})}^{(m_i)} \in \{\varphi_i\}_{i \in S_{k_{n-1}}^{(m_i)}}$, but we may choose $S_{k_{n-1}}^{(m_1)}$ so that m_1 is minimal. Indeed simply note if $m_1 < m$ and $k_{m_1} = k_m$ then $S_{k_{m_1}}^{(m_1)} \subseteq S_{k_m}^{(m)}$.

Then we consider $\{\varphi_i\}_{i \in S_{k_{n-1}}^{(m_1)}}$ and again apply Proposition 2.4. There exists some $\varphi_{(k_{m_1-1})}^{(m_2)} \in \{\varphi_i\}_{i \in S_{k_{m_1-1}}^{(m_2)}}$, $m_2 < m_1$ such that

$$\left(\{\varphi_i\}_{i \in S_{k_{n-1}}^{(m_1)}} \setminus \{\varphi_{(k_{n-1})}^{(m_1)}\}\right) \cup \left\{\varphi_{(k_{m_1-1})}^{(m_2)}\right\}$$

is linearly independent and has the same span as $\{\varphi_i\}_{i \in S_{k_{m_1}}^{(m_1)}}$. Choose the smallest such m_2.

By continuing this process $\{m_i\}_{i=1}^r$ is a decreasing sequence which terminates with $m_r = 1$. By one final application of Proposition 2.4,

$$\left(\{\varphi_i\}_{i \in S_{k_1}}^{(1)} \setminus \{\varphi_{(k_1)}^{(1)}\}\right) \cup \{\varphi_{(\ell)}\}$$

is linearly independent and has the same span as $\{\varphi_i\}_{i \in S_{k_1}^{(1)}}$.

Thus we can partition $\Phi \setminus \{\varphi_{(k_n)}^{(n)}\}$ into ℓ sets of linear independent vectors, say $\{\{\varphi_i\}_{i \in G_j}\}_{j=1}^{\ell}$ where $|G_j| = |F_j|$ and $\text{span}(\{\varphi_i\}_{i \in G_j}) = \text{span}(\{\varphi_i\}_{i \in F_j})$ for $j = 1, \ldots, \ell - 1$. However, recalling $\varphi_{(k_n)}^{(n)} \notin \text{span}(\{\varphi_i\}_{i \in F_{\ell-1}})$,

$$\{\varphi_i\}_{i \in G_{\ell-1}} \cup \left\{\varphi_{(k_n)}^{(n)}\right\}$$

is also linearly independent contradicting that $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$ was a fundamental partition. □

The argument in Theorem 2.7 can be easily visualized; see Fig. 2 where we consider two iterations.

We have shown spans of specific subsets of $\{\varphi_i\}_{i \in F_j}, j = 1, \ldots, \ell - 1$ are contained in a common subspace. As a corollary, the next step will be to show specific subsets span exactly the same subspace. This will lead to so-called transversals.

Definition 2.8. Given a fundamental partition $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$ of $\Phi = \{\varphi_i\}_{i=1}^M$, let $t \leqslant \ell$ and $T \subseteq \{1, \ldots, M\}$. We call $\{\varphi_i\}_{i \in T}$ a **t-transversal** of $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$ if $T = \cup_{j=1}^t S_j$ where $S_j \subseteq F_j$ and $\text{span}(\{\varphi_i\}_{i \in S_j}) = \text{span}(\{\varphi_i\}_{i \in S_k})$ for all $j, k \in \{1, \ldots, t\}$.

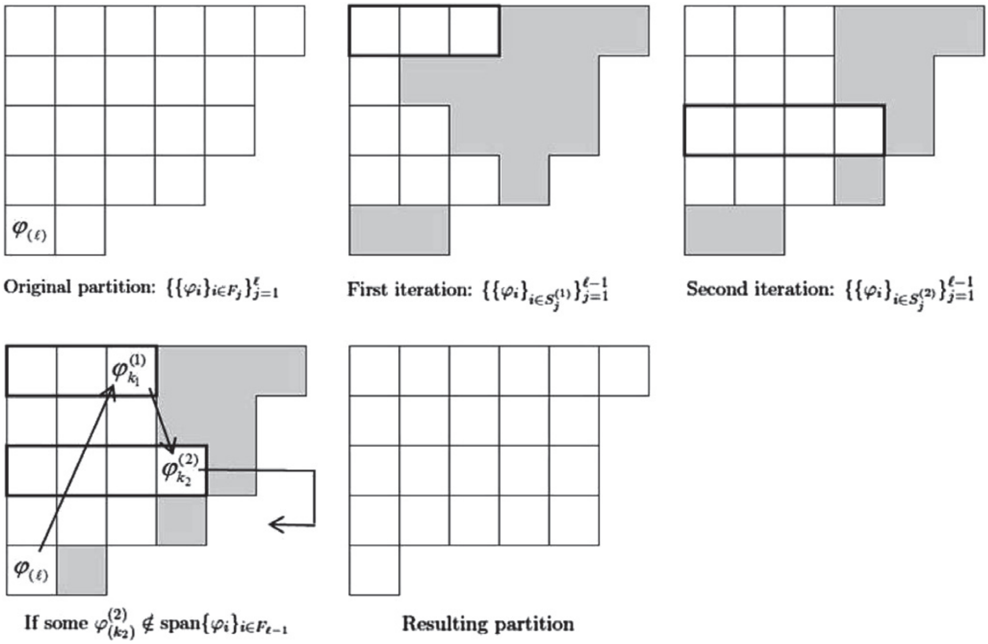

Fig. 2. Partitions after performing argument in Theorem 2.7.

We use the term transversal since this definition is (almost) a special case for the concept of the same name for a sum of matroids, the difference being transversals for sums of matroids are independent of a basis (independent of a fundamental partition) [6]. Our use of the term clearly depends on a given fundamental partition.

Proving the existence of transversals in a sum of matroids, while well known (see Lemma 2.3 in [6] for example), is not elementary. However, we may show existence in our case by using Theorem 2.7 to construct a transversal.

Corollary 2.9. *Consider the family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ with a fundamental partition $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$. Fix $t < r \leqslant \ell$ and choose any $\varphi_{(r)} \in \{\varphi_i\}_{i\in F_r}$. Then $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ contains a t-transversal $\{\varphi_i\}_{i\in T}$, $T = \cup_{j=1}^t S_j$, with $\varphi_{(r)} \in \text{span}(\{\varphi_i\}_{i\in S_j})$ for all $j = 1, \ldots, t$.*

Proof. Notice if $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ is a fundamental partition and we remove sets $\{\varphi_i\}_{i\in F_j}, j = t+1, \ldots, r-1, r+1, \ldots, \ell$, then

$$\{\{\varphi_i\}_{i\in F_1}, \ldots, \{\varphi_i\}_{i\in F_t}, \{\varphi_i\}_{i\in F_r}\}$$

remains a fundamental partition for the remaining vectors

$$\Phi \setminus \{\varphi_i\}_{i\in \cup_{j=t+1, j\neq r}^{\ell} F_j}.$$

It therefore suffices to prove the statement for $t = \ell - 1$, and $r = \ell$.

Consider the sets $S_j^{(n)}, j = 1, \ldots, \ell-1, n = 1, 2, \ldots$ as given in Theorem 2.7 where again $S_{k_n}^{(n)}$ is a largest such set for each n. Notice $\text{span}(\{\varphi_i\}_{i\in S_{k_n}^{(n)}}) \subseteq \text{span}(\{\varphi_i\}_{i\in S_{k_{n+1}}^{(n+1)}})$. Since we have only finitely many vectors, there exits an n_0 such that

$$\left|S_{k_{n_0-1}}^{(n_0-1)}\right| = \left|S_{k_{n_0}}^{(n_0)}\right|.$$

Then

$$\left|S_{k_{n_0-1}}^{(n_0)}\right| = \left|S_j^{(n_0)}\right|, \quad j = 1, \ldots, \ell - 1.$$

Since $\text{span}(\{\varphi_i\}_{i \in S_{k_{n_0-1}}^{(n_0)}}) \subseteq \text{span}(\{\varphi_i\}_{i \in S_j^{(n_0)}})$ for all $j = 1, \ldots, \ell - 1$, we conclude

$$\text{span}\left(\{\varphi_i\}_{i \in S_{k_{n_0-1}}^{(n_0)}}\right) = \text{span}(\{\varphi_i\}_{i \in S_j^{(n_0)}}).$$

Clearly $\varphi_{(\ell)} \in \text{span}(\{\varphi_i\}_{i \in S_j^{(n_0)}})$ and $S_j^{(n_0)} \subseteq F_j$ for all $j = 1, \ldots, \ell - 1$ by construction. Set $T = \cup_{j=1}^{\ell-1} S_j = \cup_{j=1}^{\ell-1} S_j^{(n_0)}$, and we have the desired $\ell - 1$ transversal $\{\varphi_i\}_{i \in T}$. $\square$

It is simple to see that given multiple t-transversals in a fundamental partition, their union is a t-transversal in the same fundamental partition; we omit the proof. In its matroid version, this is Proposition 2.4 in [6].

Lemma 2.10. *Let $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$ be a fundamental partition of $\Phi = \{\varphi_i\}_{i=1}^{M}$. Suppose $\{\varphi_i\}_{i \in T_1}$ and $\{\varphi_i\}_{i \in T_2}$ are t-transversals of $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$ where $T_1 = \cup_{j=1}^{t} U_j$ and $T_2 = \cup_{j=1}^{t} V_j$. Setting $T = \cup_{j=1}^{t}(U_j \cup V_j)$, $\{\varphi_i\}_{i \in T}$ is a t-transversal of $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$.*

We are now ready to prove the Rado–Horn theorem.

3. Proof of Rado–Horn and its generalizations

We begin with the original.

Theorem 3.1 (Rado–Horn). *Consider the family of vectors $\Phi = \{\varphi_i\}_{i=1}^{M}$. Then the following are equivalent:*

(i) The set $\{1, \ldots, M\}$ can be partitioned into sets $\{A_j\}_{j=1}^{k}$ such that $\{\varphi_i\}_{i \in A_j}$ is a linearly independent set for all $j = 1, 2, \ldots, k$.
(ii) For any non-empty subset $J \subseteq \{1, \ldots, M\}$, $|J| / \dim \text{span}(\{\varphi_i\}_{i \in J}) \leqslant k$.

Proof. ($i \Rightarrow ii$). Suppose $\{A_j\}_{j=1}^{k}$ is a partition of $\{1, \ldots, M\}$ such that $\{\varphi_i\}_{i \in A_j}$ is a linearly independent set for all $j = 1, 2, \ldots, k$. For any $J \subseteq \{1, \ldots, M\}$, let $J_j = J \cap A_j$. Then

$$|J| = \sum_{j=1}^{k} |J_j| = \sum_{j=1}^{k} \dim \text{span}(\{\varphi_i\}_{i \in J_j}) \leqslant k \dim \text{span}(\{\varphi_i\}_{i \in J})$$

giving the result.

($ii \Rightarrow i$). We prove the contrapositive. Suppose Φ cannot be partitioned into k linearly independent sets. Then for any fundamental partition $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$, we must have $\ell > k$. By Corollary 2.9, for any $\varphi_{(\ell)} \in \{\varphi_i\}_{i \in F_\ell}$, $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$ contains a k-transversal, T with $\varphi_{(\ell)} \in \text{span}(\{\varphi_i\}_{i \in T})$. Then we have

$$\frac{|T \cup \{(\ell)\}|}{\dim \text{span}(\{\varphi_i\}_{i \in T \cup \{(\ell)\}})} = k + \frac{1}{\dim \text{span}(\{\varphi_i\}_{i \in T})} > k. \quad \square \tag{1}$$

One of the benefits of this proof is that the ideas generalize to many other versions of the Rado–Horn theorem. It is a simple matter to adapt the ideas of this proof to show the following generalized version of the Rado–Horn theorem which originally appeared in [4]. We omit the details as the ideas are similar to the previous proof.

Theorem 3.2 (Generalized Rado–Horn)**.** *Consider the family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$. Then the following are equivalent.*

(i) There exists a subset $H \subseteq \{1, \ldots, M\}$ such that $\{\varphi_i\}_{i \notin H}$ can be partitioned into k linearly independent sets.
(ii) For any non-empty subset $J \subseteq \{1, \ldots, M\}$, we have $(|J| - |H|)/\dim \operatorname{span}(\{\varphi_i\}_{i \in J}) \leqslant k$.

Transversals in a fundamental partition also explain why the Rado–Horn inequality can fail when Φ cannot be partitioned into k linearly independent sets. The following redundant version of Rado–Horn was originally proven in [4].

Theorem 3.3 (Redundant Rado–Horn)**.** *Consider the family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ in a vector space V. If this set cannot be partitioned into k linearly independent sets, then there exists a partition $\{A_j\}_{j=1}^k$ of $\{1, \ldots, M\}$ and a subspace S of V such that the following hold:*

(i) For all $1 \leqslant j \leqslant k$, there exists a subset $S_j \subseteq A_j$ such that $S = \operatorname{span}(\{\varphi_i\}_{i \in S_j})$.
(ii) For $J = \{i : \varphi_i \in S\}$, $|J| / \dim \operatorname{span}(\{\varphi_i\}_{i \in J}) > k$.
(iii) For all $1 \leqslant j \leqslant k$, $\{\varphi_i\}_{i \in A_j \setminus S_j}$ is linearly independent.

Proof. Take a fundamental partition $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$ of Φ, and consider the partition of indices $\{A_j\}_{j=1}^k = \{F_1, \ldots, F_{k-1}, \cup_{r=k}^{\ell} F_r\}$. We will show there exists a subspace S which satisfies (i), (ii), and (iii) for $\{\{\varphi_i\}_{i \in A_j}\}_{j=1}^k$.

By Corollary 2.9, for each $r \in F_j$, $j = k+1, \ldots, \ell$, there exits a k-transversal, say $\{\varphi_i\}_{i \in T_r}$, of $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$ containing φ_r in $\operatorname{span}(\{\varphi_i\}_{i \in T_r})$. By Lemma 2.10, we take the union

$$T = \cup_{\{r : \varphi_r \in F_j, j=k+1,\ldots,l\}} T_r$$

so $\{\varphi_i\}_{i \in T}$ is a k-transversal of $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$ which satisfies $\varphi_r \in \operatorname{span}(\{\varphi_i\}_{i \in T})$ for all $\varphi_r \in F_j$, $j = k+1, \ldots, \ell$. Thus

$$\operatorname{span}(\{\varphi_i\}_{i \in F_j}) \subseteq \operatorname{span}(\{\varphi_i\}_{i \in T})$$

for all $j = k+1, \ldots, \ell$.

Finally, set $S = \operatorname{span}(\{\varphi_i\}_{i \in T})$ and $S_i = T \cap F_i$ for $i = 1, \ldots, k-1$ with $S_k = T \cap (\cup_{j=k}^{\ell} F_j)$. Then (i) and (ii) follow since $\{\varphi_i\}_{i \in T}$ is a k-transversal which contains in its span at least one $\varphi \in \{\varphi_i\}_{i \in F_j}, j > k$ (in this case all of them). Clearly for $j = 1, \ldots, k-1$, $\{\varphi_i\}_{i \in A_j \setminus S_j} \subseteq \{\varphi_i\}_{i \in F_j}$ is linearly independent. Lastly by the way we constructed our transversal,

$$\{\varphi_i\}_{i \in A_k \setminus S_k} \subseteq \{\varphi_i\}_{i \in (\cup_{j=k}^{\ell} F_j) \setminus (\cup_{j=k+1}^{\ell} F_j)} \subseteq \{\varphi_i\}_{i \in F_k},$$

which is also linearly independent. □

4. Constructing a fundamental partition

The previous sections rely only on the existence of a fundamental partition. Interestingly, we can build a fundamental partition where we use Rado–Horn as a tool in the construction. This process is much like a finding the so-called flag transversal for a sum of matroids [9]. It will be helpful to define the concept of a quasi-transversal which, like the transversal, is inspired from a matroid version [8].

Definition 4.1. Given a fundamental partition $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ of $\Phi = \{\varphi_i\}_{i=1}^{M}$, let $t \leqslant \ell$ and $T \subseteq \{1, \ldots, M\}$. We call $\{\varphi_i\}_{i\in T}$ a **t-quasi-transversal** of $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ if $T = \cup_{j=1}^{t} S_j$ where $S_j \subseteq F_j$ and

$$\begin{aligned}&\text{span}(\{\varphi_i\}_{i\in S_j}) = \text{span}(\{\varphi_i\}_{i\in S_k}) \quad j, k \in \{1, \ldots, t-1\}\\ &\text{span}(\{\varphi_i\}_{i\in S_t}) \subseteq \text{span}(\{\varphi_i\}_{i\in S_j}) \quad j \in \{1, \ldots, t-1\}.\end{aligned}$$

Quasi-transversals will form building blocks for our construction; initially this is a problem since quasi-transversals are defined in terms of existing fundamental partitions. In order to proceed, we must find vectors which necessarily form a quasi-transversal in some yet unknown fundamental partition. Choosing vectors which maximize the Rado–Horn inequality is a reasonable starting point.

Proposition 4.2. *Given a family of vectors $\Phi = \{\varphi_i\}_{i=1}^{M}$, suppose $J \subseteq \{1, \ldots, M\}$ maximizes $|J|/\dim \text{span}(\{\varphi_i\}_{i\in J})$. Then in a fundamental partition of $\{\varphi_i\}_{i\in J}$, say $\{\{\varphi_i\}_{i\in F'_j}\}_{j=1}^{\ell}$, $\{\varphi_i\}_{i\in J}$ is an ℓ-quasi-transversal.*

Proof. By Corollary 2.9 and Lemma 2.10, $\{\{\varphi_i\}_{i\in F'_j}\}_{j=1}^{\ell}$ contains a maximal $(\ell-1)$-transversal, $\{\varphi_i\}_{i\in T}$, where $\text{span}(\{\varphi_i\}_{i\in F'_\ell}) \subseteq \text{span}(\{\varphi_i\}_{i\in T})$. Define the set $T' = T \cup F'_\ell$ so that $\{\varphi_i\}_{i\in T'}$ is an ℓ-quasi-transversal of $\{\{\varphi_i\}_{i\in F'_j}\}_{j=1}^{\ell}$. For contradiction, suppose $\{\varphi_i\}_{i\in J}$ was not an ℓ-quasi-transversal of $\{\{\varphi_i\}_{i\in F'_j}\}_{j=1}^{\ell}$. Then $\{\varphi_i\}_{i\in J\setminus T'}$ cannot be an $(\ell-1)$-transversal in $\{\{\varphi_i\}_{i\in F'_j}\}_{j=1}^{\ell}$. Specifically,

$$\left|F'_{\ell-1} \setminus T'\right| < \left|F'_1 \setminus T'\right| = \dim \text{span}(\{\varphi_i\}_{i\in J}) - \dim \text{span}(\{\varphi_i\}_{i\in T'}),$$

which then implies

$$|J| - \left|T'\right| = \left|J \setminus T'\right| < \sum_{j=1}^{\ell-1} \left|F'_j \setminus T'\right| < (\ell-1)[\dim \text{span}(\{\varphi_i\}_{i\in J}) - \dim \text{span}(\{\varphi_i\}_{i\in T'})]. \qquad (2)$$

Using (2) and that $\{\varphi_i\}_{i\in T'}$ is an ℓ-quasi-transversal of $\{\{\varphi_i\}_{i\in F'_j}\}_{j=1}^{\ell}$, we have

$$\frac{|J| - |T'|}{\dim \text{span}(\{\varphi_i\}_{i\in J}) - \dim \text{span}(\{\varphi_i\}_{i\in T'})} < \ell - 1 < \frac{|T'|}{\dim \text{span}(\{\varphi_i\}_{i\in T'})}.$$

It now follows that

$$\begin{aligned}&\left|T'\right| \dim \text{span}(\{\varphi_i\}_{i\in J})\\ &\quad = \left|T'\right| [\dim \text{span}(\{\varphi_i\}_{i\in T'}) + (\dim \text{span}(\{\varphi_i\}_{i\in J}) - \dim \text{span}(\{\varphi_i\}_{i\in T'}))]\\ &\quad > \left|T'\right| \dim \text{span}(\{\varphi_i\}_{i\in T'}) + (|J| - \left|T'\right|) \dim \text{span}(\{\varphi_i\}_{i\in T'})\\ &\quad = |J| \dim \text{span}(\{\varphi_i\}_{i\in T'}),\end{aligned}$$

giving $|T'| / \dim \text{span}(\{\varphi_i\}_{i\in T'}) > |J| / \dim \text{span}(\{\varphi_i\}_{i\in J})$, a contradiction. □

Proposition 4.2 is not adequate since it does not consider the entire family Φ. By picking a slightly different $J \subseteq \{1, \ldots, M\}$, we can find the needed quasi-transversals.

Lemma 4.3. *Suppose $\Phi = \{\varphi_i\}_{i=1}^{M}$ can be partitioned into at fewest ℓ linearly independent sets. Let $K \subseteq \{1, \ldots, M\}$ be such that*

$$\frac{|K|}{\dim \text{span}(\{\varphi_i\}_{i\in K})} = \ell - 1,$$

and for any other set L satisfying this equality,

$$|\{i : \varphi_i \in \operatorname{span}(\{\varphi_i\}_{i\in K})\} \setminus K| \geqslant |\{i : \varphi_i \in \operatorname{span}(\{\varphi_i\}_{i\in L})\} \setminus L| \,.$$

Let $J = \{i : \varphi_i \in \operatorname{span}(\{\varphi_i\}_{i\in K})\}$. *Then for any fundamental partition* $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ *of* Φ, $\{\varphi_i\}_{i\in J}$ *is an* ℓ*-quasi-transversal.*

Proof. First note such a set $K \neq \emptyset$ since an $(\ell-1)$-transversal in a fundamental partition, which we have by Corollary 2.9, satisfies the equality.

With J now chosen, let $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ be any fundamental partition of Φ, and consider the sets $J \cap F_j$. We must have

$$|J \cap F_\ell| \leqslant |J \setminus K| \tag{3}$$

for otherwise we could find a maximal $(\ell-1)$-transversal $\{\varphi_i\}_{i\in L}$ as a consequence of Corollary 2.9 and Lemma 2.10. This would imply

$$\frac{|L|}{\dim \operatorname{span}(\{\varphi_i\}_{i\in L})} = \ell - 1,$$

and

$$\begin{aligned} |\{i : \varphi_i \in \operatorname{span}(\{\varphi_i\}_{i\in L})\} \setminus L| &\geqslant |J \cap F_\ell| \\ &> |J \setminus K| \\ &= |\{i : \varphi_i \in \operatorname{span}(\{\varphi_i\}_{i\in K})\} \setminus K| \end{aligned}$$

a contradiction.

With (3) in mind, notice

$$|J \cap F_j| \leqslant \dim \operatorname{span}(\{\varphi_i\}_{i\in J}) = \dim \operatorname{span}(\{\varphi_i\}_{i\in K}),$$

but suppose the inequality was strict for some $j \in \{1, \ldots, \ell-1\}$. Then we have

$$\begin{aligned} |K| + |J \setminus K| &= |J| \\ &= \sum_{j=1}^{\ell} |J \cap F_j| \\ &= \sum_{j=1}^{\ell-1} |J \cap F_j| + |J \cap F_\ell| \\ &< (\ell-1) \dim \operatorname{span}(\{\varphi_i\}_{i\in K}) + |J \cap F_\ell| \\ &= |K| + |J \cap F_\ell| \,, \end{aligned}$$

and we see $|J \cap F_\ell| > |J \setminus K|$, a contradiction.

We conclude

$$|J \cap F_j| = \dim \operatorname{span}(\{\varphi_i\}_{i\in J})$$

for all $j \in \{1, \ldots, \ell-1\}$ and

$$|J \cap F_\ell| = |F_\ell| \,.$$

It follows that $\{\varphi_i\}_{i\in J}$ is an ℓ-quasi-transversal of $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$. □

Definition 4.4. Given a family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ which can be partitioned into at fewest ℓ linearly independent sets, let $J \subseteq \{1, \dots, M\}$ be given as in Lemma 4.3. Then we will say $\{\varphi_i\}_{i \in J}$ is a **universal quasi-transversal** of Φ.

Notice that for any fundamental partition $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$ of $\{\varphi_i\}_{i=1}^M$, a universal quasi-transversal $\{\varphi_i\}_{i \in J}$ must be an ℓ-quasi-transversal with $F_\ell \subseteq J$.

Now that we have a quasi-transversal for some fundamental partition, albeit unknown, the next two results show projecting onto the orthogonal complements of the spans of such transversals maintains some structure of the partition. Theorem 4.6 is the main result needed for our construction.

Lemma 4.5. *Consider family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ with a fundamental partition $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell}$. Suppose $\{\varphi_i\}_{i \in T}$ is an ℓ-quasi-transversal of this partition which satisfies $F_\ell \subseteq T$. Let P_T be the orthogonal projection onto $\operatorname{span}(\{\varphi\}_{i \in T})$ and suppose $F_j' = \{i : i \in F_j \setminus T\}$. Then $\{\{(I - P_T)\varphi_i\}_{i \in F_j'}\}_{j=1}^{\ell'}$ is a fundamental partition of $\{(I - P_T)\varphi_i\}_{i \notin T}$ where*

$$\ell' = \max\left\{j : F_j \neq F_j'\right\}. \tag{4}$$

Proof. Note the family $\{(I - P_T)\varphi_i\}_{i \notin T}$ is precisely the elements of Φ under the projection $I - P_T$ which are non-zero.

We first show $\{(I - P_T)\varphi_i\}_{i \in F_j'}$ is linearly independent for $j = 1, \dots, \ell'$, each set being non-empty due to (4). Indeed suppose there exists scalars a_i such that $\sum_{i \in F_j'} a_i (I - P_T)\varphi_i = 0$. Then $\sum_{i \in F_j'} a_i \varphi_i \in \operatorname{span}(\{\varphi_i\}_{i \in F_j \setminus F_j'})$. Since $\{\varphi_i\}_{i \in F_j}$ is linearly independent, $a_i = 0$ for all $i \in F_j'$.

Now suppose these independent sets do not form a fundamental partition. Then there exists some other partition of $\{1, \dots, M\} \setminus T$, say $\{A_j\}_{j=1}^s$ such that $\{(I - P_T)\varphi_i\}_{i \in A_j}$ is linearly independent for all $j = 1, \dots, s$ and there is some $k < \ell'$ such that $|A_k| > |F_k'|$ but $|A_j| = \left|F_j'\right|$ for all $j < k$. It now suffices to show $\{\varphi_i\}_{i \in (F_j \setminus F_j') \cup A_j}$ is linearly independent for $j = 1, \dots, k$, for this would contradict that $\{\{\varphi_i\}_{i \in F_j}\}_{j=1}^{\ell'}$ was a fundamental partition.

For scalars a_i, consider $\sum_{i \in (F_j \setminus F_j') \cup A_j} a_i \varphi_i = 0$. Under the projection $I - P_T$, this becomes

$$\sum_{i \in (F_j \setminus F_j') \cup A_j} a_i (I - P_T)\varphi_i = \sum_{i \in A_j} a_i (I - P_T)\varphi_i = 0,$$

and $a_i = 0$ for $i \in A_j$. But then

$$\sum_{i \in (F_j \setminus F_j') \cup A_j} a_i \varphi_i = \sum_{i \in F_j \setminus F_j'} a_i \varphi_i = 0,$$

and $a_i = 0$ for all $i \in (F_j \setminus F_j') \cup A_j$. $\square$

Theorem 4.6. *Suppose the family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ can be partitioned into at fewest ℓ linearly independent sets. Let $\{\varphi_i\}_{i \in J}$ be a universal quasi-transversal of Φ, and let P_J be the orthogonal projection onto $\operatorname{span}(\{\varphi_i\}_{i \in J})$. Assuming $J \neq \{1, \dots, M\}$, let $\{\{\varphi_i\}_{i \in G_j}\}_{j=1}^{\ell}$ and $\{\{(I - P_J)\varphi_i\}_{i \in G_j'}\}_{j=1}^{\ell'}$ be fundamental partitions of $\{\varphi_i\}_{i \in J}$ and $\{(I - P_J)\varphi_i\}_{i \notin J}$ respectively. Then $\{\{\varphi_i\}_{i \in G_j \cup G_j'}\}_{j=1}^{\ell}$ is a fundamental partition of Φ where we set $G_j' = \emptyset$ for $\ell' < j \leqslant \ell$.*

Proof. First note $\{\varphi_i\}_{i \in G_j'}$ are not all empty since $J \neq \{1, \dots, M\}$.

We will show $\{\varphi_i\}_{i\in G_j\cup G_j'}$ are linearly independent for $j\in\{1,\ldots,\ell\}$. For $j>\ell'$, $\{\varphi_i\}_{i\in G_j\cup G_j'}=\{\varphi_i\}_{i\in G_j}$ is clearly linearly independent. Thus let $j\leqslant\ell'$, and suppose

$$\sum_{i\in G_j\cup G_j'} a_i\varphi_i=\sum_{i\in G_j} a_i\varphi_i+\sum_{i\in G_j'} a_i\varphi_i=0.$$

Under the projection $(I-P_J)$, this becomes

$$\sum_{i\in G_j'} a_i(I-P_J)\varphi_i=0,$$

and $a_i=0$ for $i\in G_j'$ since $\{\{(I-P_J)\varphi_i\}_{i\in G_j'}\}_{j=1}^{r}$ is a fundamental partition. Then

$$\sum_{i\in G_j\cup G_j'} a_i\varphi_i=\sum_{i\in G_j} a_i\varphi_i=0,$$

but $\{\{\varphi_i\}_{i\in G_j}\}_{j=1}^{\ell}$ is also a fundamental partition. We conclude $\{\varphi_i\}_{i\in G_j\cup G_j'}$ is a linearly independent set for $j\in\{1,\ldots,\ell\}$.

Now that we have linear independence, we will show $\{\{\varphi_i\}_{i\in G_j\cup G_j'}\}_{j=1}^{\ell}$ forms a fundamental partition of Φ. For contradiction, suppose this was not the case. Then there exists a fundamental partition $\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ such that for some $1\leqslant t<\ell$, $|F_j|=\left|G_j\cup G_j'\right|$ for $j<t$ but $|F_t|>\left|G_t\cup G_t'\right|$. We define

$$F_j'=F_j\setminus J$$

and compare $\{(I-P_J)\varphi_i\}_{i\in F_j'}$ with $\{(I-P_J)\varphi_i\}_{i\in G_j'}$.

Since $\{\varphi_i\}_{i\in J}$ is a universal quasi-transversal, Lemma 4.5 implies $\{\{(I-P_J)\varphi_i\}_{i\in F_j'}\}_{j=1}^{\ell'}$ is a fundamental partition of $\{(I-P_J)\varphi_i\}_{i\notin J}$, and by hypothesis, so is $\{\{(I-P_J)\varphi_i\}_{i\in G_j'}\}_{j=1}^{\ell'}$. Hence

$$\left|F_j'\right|=\left|G_j'\right|$$

for $j\in\{1,\ldots,\ell'\}$. Then for $j<t\leqslant\ell'$,

$$|G_j|+\left|G_j'\right|=\left|G_j\cup G_j'\right|=|F_j|=\left|F_j\setminus F_j'\right|+\left|F_j'\right|=\left|F_j\setminus F_j'\right|+\left|F_j'\right|$$

yielding $|G_j|=\left|F_j\setminus F_j'\right|$. The same argument for $j=t$ shows $|G_t|<\left|F_t\setminus F_t'\right|$ contradicting that $\{\{\varphi_i\}_{i\in G_j}\}_{j=1}^{\ell}$ was a fundamental partition. □

We can now construct a fundamental partition by repeated application of Theorem 4.6.

4.1. Construction of a fundamental partition

Let $\Phi=\{\varphi_i\}_{i=1}^{M}=\{\varphi_{1i}\}_{i=1}^{M}$ be a family of vectors where we have added the extra index in order to track an iterative process of projections. Suppose

$$\max_{J\subseteq\{1,\ldots,M\}}\left\lceil\frac{|J|}{\dim\operatorname{span}(\{\varphi_i\}_{i\in J})}\right\rceil=k_1.$$

Then a fundamental partition of Φ contains k_1 linearly independent sets, and we may find a universal quasi-transversal of Φ by searching through $K \subseteq \{1, \ldots, M\}$ such that $|K| / \dim \operatorname{span}(\{\varphi_i\}_{i \in K}) = k_1 - 1$. Choose $T_1 \subseteq \{1, \ldots, M\}$ so that $\{\varphi_i\}_{i \in T_1}$ comprises such a universal quasi-transversal. Let

$$t_1 = \dim \operatorname{span}(\{\varphi_{1i}\}_{i \in T_1}),$$
$$s_1 = |T_1| - (k_1 - 1)t_1.$$

Then we know exactly how this quasi-transversal appears in a fundamental partition. It is not difficult to see that we may partition T_1 as $\{T_{1j}\}_{j=1}^{k_1}$ where

(i) $|T_{1j}| = t_1, j = 1, \ldots, k_1 - 1$.
(ii) $|T_{1j}| = s_1, j = k_1$.
(iii) $\operatorname{span}(\{\varphi_i\}_{i \in T_{1n}}) = \operatorname{span}(\{\varphi_i\}_{i \in T_{1m}}), n, m \neq k_1$.
(iv) $\operatorname{span}(\{\varphi_i\}_{i \in T_{1k_1}}) \subseteq \operatorname{span}(\{\varphi_i\}_{i \in T_{1j}}), j = 1, \ldots, k_1 - 1$.

Let P_{T_1} be the orthogonal projection of Φ onto $\operatorname{span}(\{\varphi_{1i}\}_{i \in T_1})$. Define $\Phi_2 = \{(I - P_{T_1})\varphi_{1i}\}_{i \notin T_1} = \{\varphi_{2i}\}_{i \notin T_1}$. Finding a fundamental partition of Φ_2 will give us a fundamental partition of Φ via Theorem 4.6.

Examine subsets of the indices $\{1, \ldots, M\} \setminus T_1$ so that

$$\max_{J \subseteq \{1,\ldots,M\} \setminus T_1} \left\lceil \frac{|J|}{\dim \operatorname{span}(\{\varphi_{2i}\}_{i \in J})} \right\rceil = k_2.$$

We now know a fundamental partition of Φ_2 contains k_2 linearly independent sets, and we may again find a universal quasi-transversal. Choose $T_2 \subseteq \{1, \ldots, M\} \setminus T_1$ so that $\{\varphi_{2i}\}_{i \in T_2}$ comprises a universal quasi-transversal, and let

$$t_2 = \dim \operatorname{span}(\{\varphi_{2i}\}_{i \in T_2}),$$
$$s_2 = |T_2| - (k_2 - 1)t_2.$$

We may partition T_2 as $\{T_{2j}\}_{j=1}^{k_2}$ where

(i) $|T_{2j}| = t_2, j = 1, \ldots, k_2 - 1$.
(ii) $|T_{2j}| = s_2, j = k_2$.
(iii) $\operatorname{span}(\{\varphi_i\}_{i \in T_{2n}}) = \operatorname{span}(\{\varphi_i\}_{i \in T_{2m}}), n, m \neq k_2$.
(iv) $\operatorname{span}(\{\varphi_i\}_{i \in T_{2k_2}}) \subseteq \operatorname{span}(\{\varphi_i\}_{i \in T_{2j}}), j = 1, \ldots, k_2 - 1$.

We continue so that P_{T_r} is the orthogonal projection of Φ_r onto $\operatorname{span}(\{\varphi_{ri}\}_{i \in T_r})$. Define $\Phi_{r+1} = \{(I - P_{T_r})\varphi_{ri}\}_{i \notin T_1 \cup \ldots \cup T_r} = \{\varphi_{(r+1)i}\}_{i \notin T_1 \cup \ldots \cup T_r}$. Examine subsets of the indices $\{1, \ldots, M\} \setminus (\cup_{j=1}^r T_j)$ so that

$$\max_{J \subseteq \{1,\ldots,M\} \setminus (\cup_{j=1}^r T_j)} \left\lceil \frac{|J|}{\dim \operatorname{span}(\{\varphi_{(r+1)i}\}_{i \in J})} \right\rceil = k_{r+1}.$$

Now choose $T_{r+1} \subseteq \{1, \ldots, M\} \setminus (\cup_{j=1}^r T_j)$ so that $\{\varphi_{(r+1)i}\}_{i \in T_{r+1}}$ is a universal quasi-transversal in Φ_{r+1}. Letting

$$t_{r+1} = \dim \operatorname{span}(\{\varphi_{(r+1)i}\}_{i \in T_{r+1}}),$$
$$s_{r+1} = |T_{r+1}| - (k_{r+1} - 1)t_{r+1},$$

we may partition T_{r+1} as $\{T_{(r+1)j}\}_{j=1}^{k_{r+1}}$

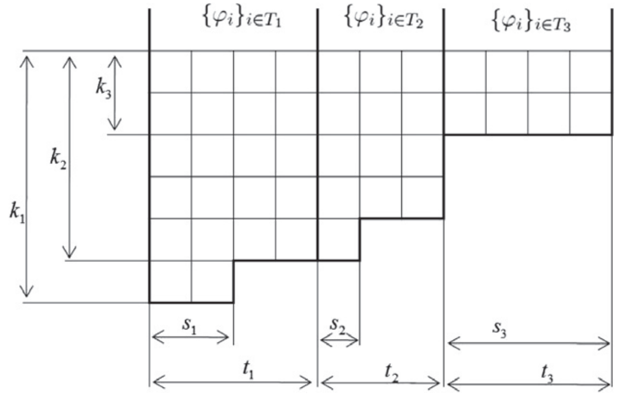

Fig. 3. Fundamental partition constructed from quasi-transversals of appropriate projections.

(i) $|T_{(r+1)j}| = t_{r+1}, j = 1, \ldots, k_{r+1} - 1.$
(ii) $|T_{(r+1)j}| = s_{r+1}, j = k_{r+1}.$
(iii) $\mathrm{span}(\{\varphi_i\}_{i\in T_{(r+1)n}}) = \mathrm{span}(\{\varphi_i\}_{i\in T_{(r+1)m}}), n, m \neq k_{r+1}.$
(iv) $\mathrm{span}(\{\varphi_i\}_{i\in T_{(r+1)k_{r+1}}}) \subseteq \mathrm{span}(\{\varphi_i\}_{i\in T_{(r+1)j}}), j = 1, \ldots, k_{r+1} - 1.$

Notice $k_r > k_{r+1}$. At some point, we will have used up all our indices. To be precise, this occurs after z iterations where $k_z \neq 0$ but $k_{z+1} = 0$. Finally, for $j > k_r$ adopt the convention $T_{rj} = \emptyset$. Then letting

$$F_j = \cup_{r=1,\ldots,z} T_{rj}, \quad j = 1, \ldots, k_1,$$

$\{\{\varphi_i\}_{i\in F_j}\}_{j=1}^{\ell}$ is a fundamental partition of Φ.

We have constructed a fundamental partition by repeatedly finding universal quasi-transversals and applying Theorem 4.6. Fig. 3 provides an example of a constructed fundamental partition showing values $t_i, k_i, s_i, i = 1, \ldots, z$ where $z = 3$.

Remark 4.7. We have essentially used Rado–Horn and transversals to describe many of the spanning properties of the vectors. For example, using the notation from the above construction, a family of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ spans a $(\sum_{i=1}^z t_i)$-dimensional space and can be partitioned into at most k_z spanning sets when $t_z = s_z$ and at most $k_z - 1$ spanning sets when $t_z > s_z$.

Acknowledgment

The authors thank the referee for the helpful suggestions and bringing our attention to the paper [9].

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