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Bifurcations of links of periodic orbits in non-singular Morse–Smale systems with a rotational symmetry on S^3

B. Campos [a,*], J. Martínez Alfaro [b], P. Vindel [a,1]

[a] Dpt. de Matemàtiques, Universitat Jaume I, 12071 Castelló, Spain
[b] Dpt. Matemàtica Aplicada, Facultat de Matemàticas, Universitat de València, Burjassot, València, Spain

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Abstract

In this paper we consider a rotational symmetry on a non-singular Morse–Smale (NMS) system analyzing the restrictions this symmetry imposes on the links defined by the set of its periodic orbits and to the appearance of local generic codimension one bifurcations in the set of NMS flows on S^3. The topological characterization is obtained by writing the involved links in terms of Wada operations.

It is also obtained that symmetry implies that in general bifurcations have to be multiple. On the other hand, we also see that there exists a set of links that cannot be related to any other by sequences of this kind of bifurcation.

Keywords: NMS systems; Symmetric links; Bifurcations

AMS classification: 58F09; 58F14; 58F22

1. Introduction

In the study of a Dynamical System we can focus our attention to the set of its periodic orbits. The structure of this set can be very complicated; nevertheless, if the phase space is a three-dimensional manifold and there are a finite number of periodic orbits we can characterize the set of all periodic orbits as a link. There exists an extensive literature on the relation between periodic orbits and knots, that began with Birman and Williams [1,2], Franks [10], and Holmes and Williams [13]. For a review, see Ghrist, Holmes and Sullivan [11].

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Let $\chi^r(M)$ denote the set of C^r-vector fields on M with the usual C^r-topology.

When M is a two-dimensional manifold the set of structurally stable vector fields is dense in $\chi^r(M)$ and it can be used to describe the dynamics of any system. But, if $\dim M > 2$ this set is not dense and it is usually replaced by one of its subsets: the Morse–Smale vector fields $MS(M)$ [16]. Another motivation for studying *MS* flows is that they characterize entropy zero flows. On the other hand it is shown that Bott integrable Hamiltonian fields on a non-singular compact constant-energy surface Q are in the boundary of the *NMS* vector fields [7,9]. Bifurcations take place in its closure $\overline{MS(M)}$ and the vector fields where bifurcations are produced are in the boundary $\partial MS(M)$. The characterization of the set of periodic orbits of a non-singular Morse–Smale $NMS(M)$ flow on a three-dimensional manifold has been studied by other authors, e.g., [14,15,18–20]. So links and bifurcations of Hamiltonian systems can be characterized as in [5].

An arc in $\chi^r(M)$ jointing two vector fields can avoid any codimension two bifurcation through small deformations, but not codimension one bifurcations. So, the study of generic codimension one families of vector fields is associated with the study of codimension one bifurcations. In this paper bifurcations are considered in the direction in which new orbits appear.

Let $NMS(S^3)$ denote the set of NMS vector fields on S^3. Let $f(0) \in NMS(S^3)$ be the initial smooth field of a uniparametric family $f(\mu)$, $\mu \in [0, \varepsilon]$. Let μ_0 be the first point in which $f(\mu)$ intersects the boundary $\partial(NMS(S^3))$. Therefore, the link type of the periodic orbits of the flow can change. The generic codimension one bifurcations of links of periodic orbits in NMS systems on S^3 are studied in [6] following Wada's result [19]. Let $\mathcal{L}(S^3)$ denote the set of links of periodic orbits of NMS systems.

In this paper we continue this study assuming some symmetry properties of the flow. We consider that the flow is an NMS flow before and after a codimension one bifurcation, preserving a rotational symmetry, and we study the restrictions that have to be imposed on links corresponding to vector fields $f(\mu)$ to satisfy this condition of symmetry (see Section 2). While Wada's results are restated below, we shall assume that the reader is familiar with the results and techniques in our earlier paper [6].

When a link $l \in \mathcal{L}(S^3)$ suffers any kind of bifurcation to a new configuration of periodic orbits $bif[l]$, we cannot assume that $bif[l] \in \mathcal{L}(S^3)$ because there are some restrictions imposed by the kind of the system and manifold. To assure that $bif[l]$ is in $\mathcal{L}(S^3)$ we will normally find a "simpler" link (or links) $l_0 \in \mathcal{L}(S^3)$ and a sequence of Wada operations taking l_0 to l. To handle this characterization, it will be necessary to impose some conditions on l. The corresponding statements are given in Section 3. From the point of view of the dynamics of a system, we obtain conditions for a link to undergo a generic codimension one bifurcation and we characterize this type of bifurcation. Let us remark that this interpretation gives a topological description of the links obtained after the bifurcation. We also see that every bifurcation can be associated with different Wada operations so, the different topological descriptions obtained point out the way the new orbits generated by the bifurcation appear.

Some consequences are obtained from these characterizations, in particular, we find some links that cannot be related to any other by these kind of bifurcations (Theorem 3) and some links can be related by a sequence of generic codimension one bifurcations (Theorem 2).

Finally, the study of the change of the link type when the parameter varies permits us to look for the characterization of the critical link (Section 4).

1.1. Wada's theorem

Wada's theorem [19] characterizes the set of indexed links which arise as the closed orbits of a non-singular Morse–Smale flow on S^3 in terms of a generator, the Hopf link h with indices 0 and 2 attached to the components, and six operations (the index i of a periodic orbit is the dimension of its unstable manifold minus one).

Every indexed link which consists of all the closed orbits of a non-singular Morse–Smale flow on S^3 is obtained from Hopf links by applying the following six operations. Conversely, every indexed link obtained from Hopf links by applying the operations is the set of all the closed orbits of a non-singular Morse–Smale flow on S^3.

Operations. For given indexed links l_1 and l_2, the six operations are defined as follows. Let $l_1 \cdot l_2$ denote the split sum of l_1 and l_2 and $N(k, M)$ the regular neighborhood of k in M.

- $I(l_1, l_2) = l_1 \cdot l_2 \cdot u$, where u is an unknot with index 1.
- $II(l_1, l_2) = l_1 \cdot (l_2 - k_2) \cdot u$, where k_2 is a component of l_2 of index 0 or 2.
- $III(l_1, l_2) = (l_1 - k_1) \cdot (l_2 - k_2) \cdot u$, where k_1 is a component of l_1 of index 0 and k_2 is a component of l_2 of index 2.
- $IV(l_1, l_2) = (l_1 \# l_2) \cup m$. The connected sum $(l_1 \# l_2)$ is obtained by composing a component k_1 of l_1 and a component k_2 of l_2, each of which has index 0 or 2. The index of the composed component $k_1 \# k_2$ is equal to either $i(k_1)$ or $i(k_2)$. Finally, m is a meridian of $k_1 \# k_2$ with $i = 1$.
- $V(l_1)$: Choose a component k_1 of l_1 of index 0 or 2, and replace $N(k_1, S^3)$ by $D^2 \times S^1$ with three indexed circles in it; $\{0\} \times S^1$, k_2 and k_3. Here, k_2 and k_3 are parallel (p, q)-cables on $\partial N(\{0\} \times S^1, D^2 \times S^1)$, where p is the number of longitudinal turns and q the number of the transverse ones. The indices of $\{0\} \times S^1$ and k_2 are either 0 or 2, and one of them is equal to $i(k_1)$. The index of k_3 is 1.
- $VI(l_1)$: Choose a component k_1 of l_1 of index 0 or 2. Replace $N(k_1, S^3)$ by $D^2 \times S^1$ with two indexed circles in it: $\{0\} \times S^1$ and the $(2, q)$-cable k_2 of $\{0\} \times S^1$. The index of $\{0\} \times S^1$ is 1, and $i(k_2) = i(k_1)$.

We call the first three operations (I–III) type A operations, and the other three (IV–VI), that produce unsplittable links, type B operations.

1.2. Symmetries of S^3

Symmetries frequently appear in Dynamical Systems. The presence of symmetry forces others kinds of behaviour that are not generic in general but they are generic in symmetric systems. We are interested in the study of the symmetries that can appear in an NMS system on S^3.

The group of symmetries of S^3 is $O(4)$, that contains rotations $SO(4)$, and reflections. A reflection implies the existence of an invariant surface that produces, by means of the Poincaré–Bendixon theorem, the existence of fixed points so reflections cannot be considered in NMS systems. Therefore, we restrict our study to the rotational group $SO(4)$.

In this paper, we consider as the symmetry group G, the subgroup generated by a finite rotation around one axis.

Because of the symmetry we can split S^3 into n equal strata around the symmetry axis and, each of these stratum, S^3_n, can be considered as S^3 identifying the points of the boundary corresponding to the same G-orbit, where a G-orbit of a point x is defined by $G(x) = \{gx, g \in G\}$. An invariant periodic orbit γ is associated to this symmetry axis.

Therefore, an NMS system without symmetry is considered in each stratum. If periodic orbits are not linked to the invariant one, to reproduce the symmetry implies to repeat them n times.

When a periodic orbit is linked to γ we can consider the cycle that represents the points where the periodic orbit crosses the boundary of a stratum. Symmetric links are obtained composing the associated cycles n times to reproduce the symmetry of the system. So, we recall some results of its algebra that are very useful in our reasonings.

Let σ be a p-cycle, if order$(\sigma^k) = m$ then $km = \dot{p}$, as m has to be the least positive integer that satisfies this relation, km has to be the *least common multiple of* k *and* p. So:

$$\mathrm{o}(\sigma^k) = m = \frac{\mathrm{lcm}(k, p)}{k}. \tag{1}$$

Every permutation can be written as a product of disjoint cycles, and its order is the least common multiple of the orders of its cycles.

The permutation σ^k can be decomposed as a product of disjoint cycles where each of them represents an equivalence class of order m, then, if the order of σ is p we obtain p/m cycles.

2. Symmetric links

We consider an NMS system with rotational symmetry around one axis associated with the invariant periodic orbit γ. Let $\varphi = 2\pi/n$ be the minimum rotation angle that leaves the flow invariant. The symmetry group G we consider is the cyclic group generated by φ; then S^3 can be split into n equal strata, S^3_n. Therefore, an NMS system without symmetry is considered in each stratum where we can use the characterization of links made in [6], then repeating the stratum n times we obtain the characterization of symmetric links in the complete system. Let us notice that to build symmetric links in this way is analogous to the construction of periodic knots (see [4]).

When a periodic orbit is not linked to γ the symmetric link is obtained repeating the corresponding link in a stratum n times. The characterization of links in each stratum is studied in [6] and we do not refer to them in the following.

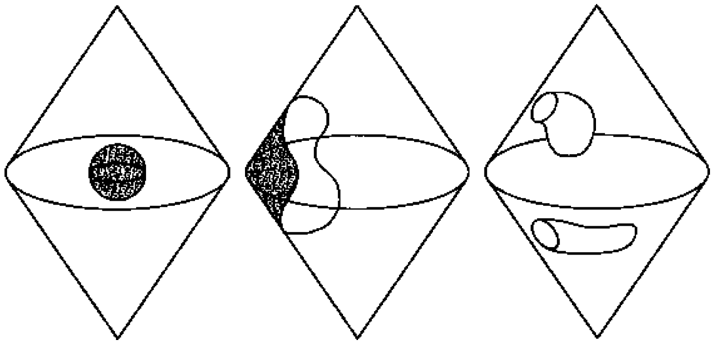

Fig. 1. A three-dimensional ball in S^3.

When a periodic orbit is linked to γ the symmetric link is obtained taking into account the following results.

Lemma 1. *Let k be a periodic orbit linked to γ. Then there exists a meridional disk D in the solid torus $S^3 - N(\gamma)$ such that at each point where k intersects D, the orientation given to k by the NMS flow enters D transversely from the same side.*

Proof. When a periodic orbit k is linked to γ and it is an iterated toral knot of k_0 it follows a given direction around γ, inherited by k_0, and this cable never turns back in a toroidal neighborhood of γ, $N(\gamma)$, so a braid can be defined from the p points where the periodic orbit crosses the boundary of the stratum D.

If k is not an iterated toral knot but it is linked to γ (γ has to be a trivial knot to maintain the symmetry), operations *V* or *VI* of Wada over an initial component k_0 linked to the invariant one (or over γ) are used, obtaining an iterated toral knot k' in its corresponding neighborhood and, after this, operation *IV* over a component of k' has been applied to obtain this kind of orbit. So, k will be the result of connecting a factor k' linked to γ with another factor k''. Both factors are necessarily in disjoint three-balls and, as it can be seen in figures, k'' is not linked to γ (see Figs. 1 and 2).

So, the projection of these links can be done in such a way that these non-toroidal pieces do not intersect the boundary of the stratum and they appear repeated n times to maintain the symmetry (see Fig. 3). Then we can consider the boundary of the strata avoiding these non-toroidal pieces.

Hence, in any case, the orientation given to k by the *NMS* flow crosses D transversely from the same side. □

Therefore, a symmetric link in S^3 can be built from its different components in each stratum: if a component in S^3_n is not linked to the invariant orbit it will be repeated n times and if a periodic orbit is linked to γ it crosses the boundary of S^3_n in p points and a cycle can be associated to it. In the last case, we use composition of cycles to see what happens in S^3.

We refer to p as the order of the periodic orbit.

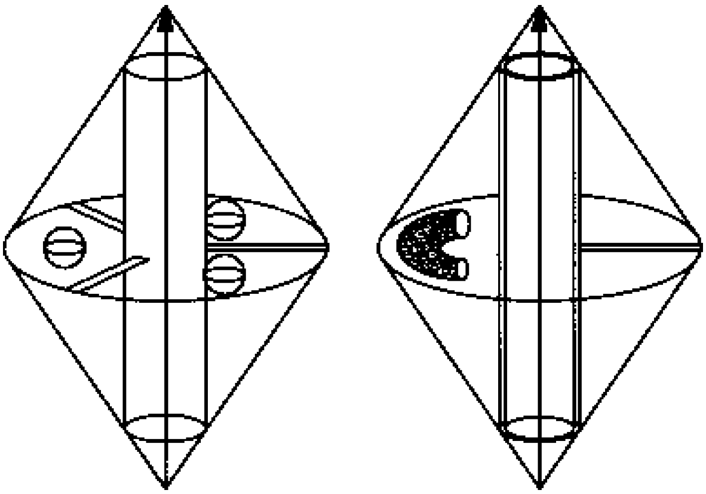

Fig. 2. Three-dimensional balls in S^3 without and with a rotational symmetry around γ. A minimal ball is covering a toroidal neighborhood of γ.

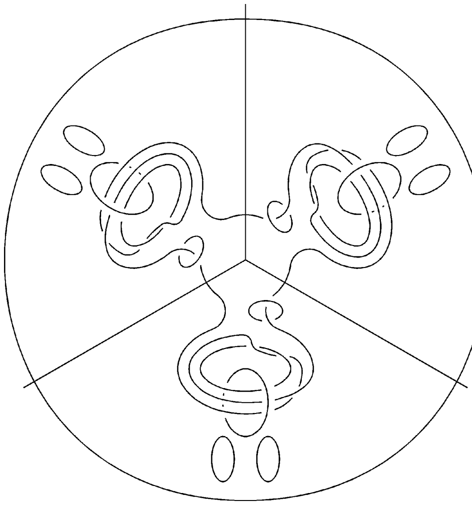

Fig. 3. Repeating non-toroidal pieces.

Corollary 1. *If a p-cycle is associated to a component linked to γ in a stratum, in the symmetric system there exist r components linked to γ and a m-cycle is associated to each of them, where*:

$$r = \frac{p}{m} = \gcd(n, p), \qquad m = \frac{\operatorname{lcm}(n, p)}{n}. \tag{2}$$

When a component k of a link has cables around it, they are in a neighborhood $N(k)$ that does not intersect another periodic orbit. So, when we use cycle theory to represent

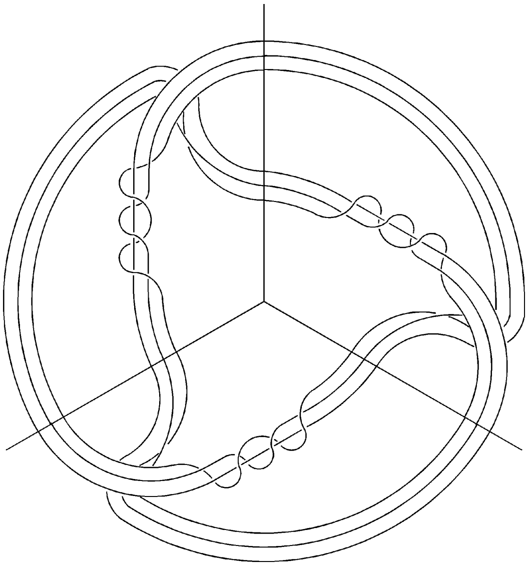

Fig. 4. A symmetric link obtained for $n = 3$.

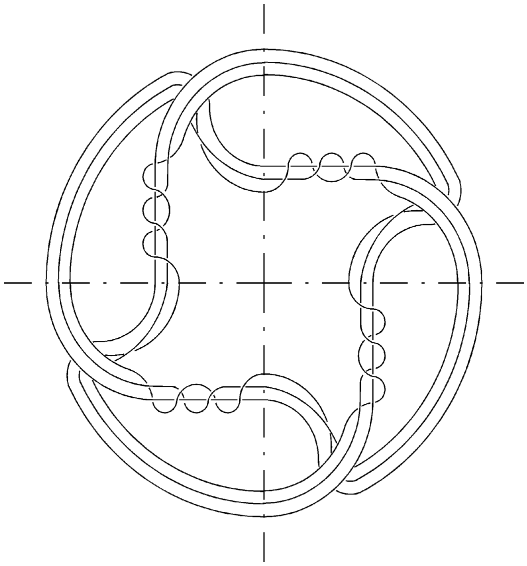

Fig. 5. For $n = 4$ the symmetry forces the splitting of the initial orbits.

what happens in a stratum, these cables are in the corresponding toral neighborhood of the orbit in this stratum, therefore, when the original system is restored the cables follow the orbit k (see Figs. 4 and 5). Moreover, if k splits into m orbits, their cables also split in such a way that every orbit obtained has its corresponding cables in its toral neighborhood (see Fig. 5).

Corollary 2. *When there are two orbits in a stratum, one with order t and the other one is a $(2,q)$-cable of it, in the symmetric system there exist r orbits of order a with $s(m/a, nq/k)$-cables of the same index, where $s = \gcd(n, 2t)/\gcd(n,t)$.*

For $s = 1$ the order of the cables is $2a$ and for $s = 2$ the order of the cables is a, the same as the base component.

Proof. When there are two orbits in a stratum, one with order t and the other one is in a toral neighborhood and it is a $(2,q)$-cable, they can be written as a product of two cycles: the t-cycle b_1 and a $2t$ cycle corresponding to the $(2,q)$-cable, b_2. Then, the set of periodic orbits in the symmetric system can be written as:

$$b^n = b_1^n \cdot b_2^n.$$

As we have seen before, b_2 splits into j orbits of order m, where $j = 2t/m = \gcd(n, 2t)$ and $m = \mathrm{lcm}(n, 2t)/n$, and b_1 splits into $r = t/a = \gcd(n,t)$ orbits of order $a = \mathrm{lcm}(n,t)/n$.

Therefore, there are r orbits of order a with $s(m/a, nq/k)$-cables of the same index, where $s = \gcd(n, 2t)/\gcd(n,t)$ is 1 or 2 in the complete system. For $s = 1$ the order of the cable is $2a$ and for $s = 2$ the order of the cables is a, the same as the base component. □

Therefore, in terms of Wada operations it is deduced that when the system has a rotational symmetry these operations have to be applied in such a way that the link obtained has this rotational symmetry. We can conclude that:

(1) We have also to take into account the symmetry of the system when we apply type B operations. If they are applied over components that are not linked to γ, they have also to be applied a number of times that has to be a multiple of n in order to obtain a symmetric link. In other case, as we can see in the following, the sum of all transversal turns of the (p_i, q_i)-cables that appear must be a multiple of n.

(2) Type A operations have to be applied a number of times multiple of n in order to obtain a symmetric link because each A operation produces an unknot.

(3) Moreover, the symmetry of the system leads to the non-admissibility of some NMS links:

Lemma 2. *For $n > 2$, a link obtained using only operation III over Hopf links is not a symmetric link.*

Proof. The link obtained applying only operation *III* over Hopf links consists in the split sum of two trivial orbits with indices 0 or 2 and unknots and it is not a symmetric link except for $n = 2$. □

In the following, let $\mathcal{L}_n(S^3)$ denote the set of NMS links in a symmetric system and $\mathcal{L}(S^3_n)$ be the set of NMS links in a stratum. Capital letters denote links and bifurcations in the symmetric system, and small letters links and bifurcations in a non-symmetric stratum.

Notice that Wada's operations have to be applied in order to obtain a symmetric link. It will depend on the type of operation. When $L \in \mathcal{L}_n(S^3)$ involves type A operations, we can observe that these operations have been applied $\dot{n}$ times because they generate unknotted orbits. Likewise for operation *IV*, because non toroidal pieces, that must be repeated $\dot{n}$ times, appear. If L only involves operations V and *VI* over periodic orbits linked to γ, as we have seen before, the general form for L is $L = B^r(L_0)$ where B only represents V and *VI* and L_0 is a symmetric link.

For instance, if we consider the link in Fig. 3 we can observe that operations *II* and *IV* have been applied three times. On the other hand, operation *VI* has been applied three times because the cabling is made over orbits that are not linked to γ. So, an expression for this link is $L = \boldsymbol{W}(L_0, l, \overset{3)}{\ldots}, l)$ where $\boldsymbol{W}$ represents operations *IV* and *II* applied three times, L_0 is the symmetric Hopf link (γ is one of its components) and $l = (l_1, l_2)$ with $l_1 = VI(h)$ and $l_2 = h$. We can also represent this link by $L = \boldsymbol{W}(L_0, l, \overset{3)}{\ldots}, l)$ where $\boldsymbol{W}$ represents operation *IV* applied three times and $l = II(VI(h), h)$.

3. Generic codimension one bifurcations

We consider an NMS flow depending on a parameter on S^3_n. For a given value of the parameter the system bifurcates to a new NMS flow on S^3_n. The generic local codimension one bifurcations that can be obtained are saddle-node, period-doubling and Hopf bifurcations and they are characterized in [6] in terms of links using Wada's operations.

Following these results, in this paper we analyze the admissible bifurcations in the complete symmetric dynamical system from the bifurcations obtained in each stratum. In addition a pitchfork type bifurcation is obtained as a consequence of the symmetry of the system.

3.1. Multiple saddle-node bifurcation

Consider a vector field $f(\mu) \in NMS(S^3)$. Let k be a periodic orbit for a critical value $\mu = \mu_0$ that bifurcates by means of a saddle-node to two new periodic orbits, k_1 and k_2. Then, for μ near μ_0 one has:

(1) There exists a toral neighborhood $N(k)$ which does not intersect another periodic orbit. By continuity, k_1 and k_2 are in $N(k)$, so they are linked in a similar way to the rest of the periodic orbits.

(2) As $f \in C^1$ the solutions are also C^1, therefore the knot type of the periodic orbits is tame and there are a finite number of crosses in a regular projection; k_1 and k_2 are in $N(k)$ with the same number and type of crosses as k, so they are of the same knot type.

This occurs in an NMS system, so we have the different cases obtained in [6] in each of the strata S^3_n into which S^3 has been split. Now, we analyze the bifurcations that are obtained in S^3 building the symmetric link from the links in the stratum following the same G-orbit, i.e., $L = G(l)$ and $SN[L] = G(sn[l])$.

Let us say that $SN[L]$ is a link obtained from a symmetric link L adding r-pairs of parallel periodic orbits of the same knot type. Each pair in the same toral neighborhood similarly linked to the rest of orbits in agreement with symmetry, one orbit with index 1 and the other with index 0 or 2.

Proposition 1. *Let* $L = \boldsymbol{W}(L_0, l, \overset{n)}{\ldots}, l) \in \mathcal{L}_n(S^3)$, *where* $L_0 \in \mathcal{L}_n(S^3)$, $l = (l_1, \ldots, l_\nu)$, $l_i \in \mathcal{L}(S^3_n)$ *and* $\boldsymbol{W}$ *represents a set of* ν *Wada operations.*

A saddle bifurcation in a symmetric system when the new orbits are linked to the invariant orbit is characterized by:

$$SN[L] = \boldsymbol{W}\big(V^r(L_0), l, \overset{n)}{\ldots}, l\big), \tag{3}$$

where $r = \gcd(n, p')$ *and* p' *is the order of the new orbits in each stratum.*

Proof. In first place, we can consider that the new orbits that appear in a stratum are linked to γ but not linked between each other. In this case they have to be $(0,1)$-cables, so they are trivial knots and can be represented by a product of 1-cycles and the same bifurcation is obtained in the symmetric system. In terms of Wada operations it can be written as:

$$SN[L] = IV(L, h) = V(L), \tag{4}$$

where the connected sum is made over the invariant periodic orbit when operation IV is used; operation V is equivalent to it when $(0,1)$-cables are used. This bifurcation is a saddle-node bifurcation in the symmetric system.

If they are linked between each other it is necessary to use operation V. Let us begin with the case in which they are also linked to an orbit k_0 with index 0 or 2 and we can consider them as cables of k_0. In this case to make operation V in a stratum consists in adding two (p,q)-cables (see [19]), where p is the number of longitudinal turns around a given component of order t linked to the invariant periodic orbit and q is the number of transversal turns. The order of the cables is $t' = p \cdot t$. These three orbits can be associated to a permutation of $(2p+1) \cdot t$ points, and its matrix b can be written as a product of three cycles, two of them b_1 and b_2, corresponding to the cables, are t'-cycles and the other, b_3, is a t-cycle. Then in the original system the associated element of the group can be written as:

$$b^n = b_1^n \cdot b_2^n \cdot b_3^n.$$

Using the results of composing a cycle n times, we see that b_1 and b_2 split into r m-cycles, where $r = t'/m = \gcd(n, t')$ and $m = o(b_i^n) = \operatorname{lcm}(n, t')/n$, and the orbit associated to b_3 splits into $s = t/a = \gcd(n, t)$ orbits corresponding to a-cycles, with $a = \operatorname{lcm}(n, t)/n$.

That means that there were s orbits with period a in the original system and after the bifurcation $2z$ $(m/a, nq/r)$-cables appear around each of them, where $z = r/s = \gcd(n, t')/\gcd(n, t)$, z cables with index 1 and z cables with index 0 or 2, so the link obtained after the bifurcation is the result of applying r-times operation V of Wada over the previous link L:

$$SN[L] = V^r(L). \tag{5}$$

In general, if the new orbits appear in the neighborhood of k_0 containing a set of orbits coming from different Wada operations $\boldsymbol{W}$, following the reasonings of [6] this bifurcation can be characterized as:

$$SN[L] = SN\big[\boldsymbol{W}\big(L_0, l, \overset{n)}{\ldots}, l\big)\big] = \boldsymbol{W}\big(V^r(L_0), l, \overset{n)}{\ldots}, l\big). \tag{6}$$

In the particular case that the new orbits that appear linked to a component k_0 of order t which gives l longitudinal turns to γ, we can also associate a product of cycles, as in Theorem 1. If they correspond to a (p,q)-cable of k_0 in a stratum, they will give pl longitudinal turns and pt transversal turns to γ and their characterization will be as the previous one. □

Corollary 3. *A saddle-node bifurcation in a symmetric system can be produced on every link $L \in \mathcal{L}_n(S^3)$, except when the bifurcation pair does not appear as a cable of a component of the previous link. In this case L has to be*

$$L = \boldsymbol{A}\big(L_0, l, \overset{n)}{\ldots}, l\big). \tag{7}$$

where $L, L_0 \in \mathcal{L}_n(S^3)$, $l = (l_1, \ldots, l_\nu)$ with $l_i \in \mathcal{L}(S^3_n)$ and $\boldsymbol{A}$ represents a set of ν Wada operations of type A.

3.2. Multiple period-doubling bifurcation

When a period doubling bifurcation occurs a periodic orbit k changes its stability and sheds two periodic orbits, one of them constitutes a period-two orbit for k. This period-doubled orbit forms the boundary of a Möbius band having the original orbit as a spine. We can distinguish two cases of period doubling bifurcation depending on the change of index of k, from 0 or 2 to 1, or vice versa.

Following the method used before for obtaining the bifurcations in the symmetric system, for $L = G(l)$ we denote $FLIP_i[L] = G(flip_i(l))$, $i = 1, 2$, when s (defined in Corollary 2) is 1.

As we will see, $FLIP_1[L]$ is the link obtained from a link L where a set $k_1, \ldots, k_r$ of G-invariant orbits have changed their indices (from 0 or 2 to 1) and a $(2,q)$-cable, with the previous index of the orbit, is around each of them, $FLIP_2[L]$ the link obtained from a link L where some orbits have changed their indices (from 1 to 0 or 2) and a $(2,q)$-cable with index 1, is around each of them. So, $FLIP_1$ and $FLIP_2$ represent a multiple period double bifurcations in a symmetric system.

Let O represent operation *II* or *III* of Wada.

Proposition 2. *Let $L \in \mathcal{L}_n(S^3)$, when the orbits that bifurcate are linked to γ*

(a) *$FLIP_1$ is characterized by*:

$$FLIP_1[L] = VI^r(L); \tag{8}$$

(b) *$FLIP_2$ is characterized by one of the following statements*:

$$FLIP_2[L] = FLIP_2\Big[A^{n-r}\big(O^r\big(VI^r(L_0), l, \overset{n)}{\ldots}, l\big)\big)\Big]$$
$$= A^{n-r}\big(O^r\big(V^r(L_0), l, \overset{n)}{\ldots}, l\big)\big), \tag{9}$$
$$FLIP_2[L] = FLIP_2\big[VI^r(L_0)\big] = V^r(L_0), \tag{10}$$

where $L_0 \in \mathcal{L}_n(S^3)$, $l \in \mathcal{L}(S^3_n)$, $r = \gcd(n,t)$ *corresponds to the number of orbits that bifurcate and* t *is the order of these orbits in each stratum.*

Proof. From Corollary 2 there are two orbits in a stratum, one with order t and the other one is in a toral neighborhood and it is a $(2,q)$-cable, they can be written as a product of two cycles: the t-cycle b_1 and a $2t$ cycle corresponding to the $(2,q)$-cable, b_2. So, in the symmetric system, there exist r orbits of order a with s cables around each of them.

If $s=1$, in the symmetric system these r orbits suffer a period double bifurcation simultaneously. Following results obtained in [6], these bifurcations can be characterized as formulas (8) and (9). Recall that in order to obtain symmetric links type A operations must be applied n times.

If the new orbits correspond to $(2,1)$-cables of the r orbits that bifurcate and appear parallel to some previous one, this bifurcation can be characterized by formula (10).

If the orbit that bifurcates is the invariant one, it admits the flip and the pitchfork bifurcation only in the case $n=2$. When $n \neq 2$, the bifurcation of γ keeping the symmetry of the system, since γ has index 1 before or after these bifurcations, leads to the fact that asymptotic sets are not manifolds, so the system is not an NMS system and we will not consider it.

When the orbit that bifurcates is γ, it changes its index and a cable of order 2 appears around it. If the index of the invariant orbit before the bifurcation is 0 or 2, then: $FLIP_1[L] = VI[L]$ doing operation VI over the invariant orbit. If its index is 1, L has to be one of the types of (9) with $r=1$. □

Corollary 4. *A* $FLIP_1$ *bifurcation in a symmetric system can be produced on any link* $L \in \mathcal{L}_n(S^3)$. *A* $FLIP_2$ *bifurcation can be produced when the link* L *can be written as*:

$$L = \overline{L}_0 \cdot l \cdot u \cdot \overset{n)}{\cdots} \cdot l \cdot u,$$
$$L = \overline{L}_0 \cdot (l-k) \cdot u \cdot \overset{n)}{\cdots} \cdot (l-k) \cdot u, \tag{11}$$
$$L = \big[VI(L_0)\big]^k,$$

where $L_0 \in \mathcal{L}_n(S^3)$, $l \in \mathcal{L}(S^3_n)$, r *is the number of orbits that bifurcate and* $\overline{L}_0$ *means that* r *orbits of a symmetric link* L_0 *have changed their index to* 1.

3.3. Multiple pitchfork bifurcation

When a pitchfork bifurcation occurs a periodic orbit k changes its index and two new periodic orbits, $k_1$and k_2, appear. We can distinguish two cases of pitchfork bifurcation depending on the change of index of k, from 0 (or 2) to 1, or vice versa and the corresponding indices of the new orbits will be 1 or 0 (or 2). For values of the parameter near the bifurcation value one has the same conditions that for saddle-node bifurcation (Section 3.1). Following the method used before for obtaining the bifurcations in the symmetric system we denote:

$$PITCH_i[L] = G\big(flip_i(l)\big), \quad i = 1, 2,$$

when $s = 2$.

When the orbits that bifurcate are trivial the link characterization after the bifurcation is topologically equivalent to the obtained for a saddle-node bifurcation (see formula (4)). So, in the following we only study the case when the orbits that bifurcate are not trivial or the cables are linked between each other.

We obtain that $PITCH_1[L]$ the link obtained from a link L where a set $k_1, \ldots, k_r$ of orbits corresponding to the same G-orbit have changed their indices from 0 or 2 to 1 and two linked $(1, q)$-cables with the same index the orbits had before, appear around each of them. Similarly, $PITCH_2[L]$ denotes the link obtained from a link L in the same way but the change of the indices of the orbits $k_1, \ldots, k_r$ is from 1 to 0 or 2.

Proposition 3. *Let $L \in \mathcal{L}_n(S^3)$, a pitchfork bifurcation is characterized by*:

$$\begin{aligned} PITCH_1[L] &= PITCH_1\Big[A^{n-r}\big(O^r\big(VI^r\big(V^r(L_0)\big), l, \overset{n)}{\ldots}, l\big)\big)\Big] \\ &= A^{n-r}\Big(O^r\big(VI^r\big(V^{2r}(L_0)\big)l, \overset{n)}{\ldots}, l\big)\Big), \end{aligned} \tag{12}$$

$$\begin{aligned} PITCH_2[L] &= PITCH_2\Big[A^{n-r}\big(O^r\big(VI^r(L_0), l, \overset{n)}{\ldots}, l\big)\big)\Big] \\ &= A^{n-r}\Big(O^r\big(VI^r\big(V^r(L_0)\big), l, \overset{n)}{\ldots}, l\big)\Big), \end{aligned} \tag{13}$$

where $L_0 \in \mathcal{L}_n(S^3)$, $l \in \mathcal{L}(S^3_n)$ and r is the number of orbits that bifurcate.

Proof. Following the results of Corollary 2, if $s = 2$, we obtain, in the symmetric system, a bifurcation consisting in the appearance of two cables of order a with the same index, around each of the initial r orbits that have changed their indices. Depending on the index of the orbits that bifurcate we have the following cases:

If the initial orbits, of order a, have index different from 1, after the bifurcation they have index 1 and two $(1, q')$-cables of index 0 or 2 appear around each of them. Then, the link must be written in terms of operation V applied once over each of the r orbits and once over the cables with index different from 1, adding in this last case $(1, 0)$-cables in order to obtain parallel cables. But it is not possible to throw out the cables with index 1 generated, so they have to be in the link before the bifurcation, then L has to be of the form:

$$L = A^{n-r}\Big(O^r\big(VI^r\big(V^r(L_0)\big), l, \overset{n)}{\ldots}, l\big)\Big),$$

because applying r times operation V and then operation VI, the $2r$ cables with index 1 are obtained and applying r times type O operations the double period cables generated are eliminated.

If $r \neq n$ it is necessary to make type A operations $(n - r)$ times for getting L to be symmetric.

After the bifurcation, as the orbits have index 1 we need to apply r times operation *VI*, operations *II* or *III* for getting rid of the double period cables generated by this operation and type A operations for obtaining a symmetric link, so:

$$PITCH_1[L] = A^{n-r}\Big(O^r\big(VI^r\big(V^{2r}(L_0)\big), l, \overset{n)}{\ldots}, l\big)\Big).$$

If the initial orbits, of order a, have index 1, they have index 0 or 2 after the bifurcation and two cables of index 1 appear around each of them. The orbits that appear after the bifurcation are $(1, q' = nq/k)$-cables with index 1 and the only way to get them is using operation V.

As the r components p_j that bifurcate have index 1, the link before the bifurcation must be obtained using operation *VI* to change the index of p_j and operations *II* or *III* to throw out the double period cables, so:

$$L = A^{n-r}\big(O^r\big(VI^r(L_0), l, \overset{n)}{\ldots}, l\big)\big).$$

The link $PITCH_2[L]$ comes from operation V applied r times in order to get the r pairs of parallel $(1, q)$-cables and from operation *VI* applied r times for changing the indices to 1 and then, double period cables generated by operations *VI* will be eliminate with r operations *II* or *III*; $n - r$ type A operations are also necessary to build a symmetric link, then:

$$PITCH_2[L] = A^{n-r}\Big(O^r\big(VI^r\big(V^r(L_0)\big), l, \overset{n)}{\ldots}, l\big)\Big).$$

When the orbit that bifurcates is γ, we have seen that the rotational symmetry has to be $n = 2$. Then, we obtain the previous results where $n = 2$. □

Corollary 5. *A $PITCH_1$ bifurcation in a symmetric system can be produced on a link L if L can be written as*:

$$L = \overline{V^r(L_0)} \cdot l \cdot u \cdot \overset{n)}{\cdots} \cdot l \cdot u, \tag{14}$$
$$L = \overline{V^r(L_0)} \cdot (l - k) \cdot u \cdot \overset{n)}{\cdots} \cdot (l - k) \cdot u.$$

A $PITCH_2$ bifurcation can be produced when the link L can be written as:

$$L = \overline{L}_0 \cdot l \cdot u \cdot \overset{n)}{\cdots} \cdot l \cdot u \tag{15}$$
$$L = \overline{L}_0 \cdot (l - k) \cdot u \cdot \overset{n)}{\cdots} \cdot (l - k) \cdot u.$$

where $L_0 \in \mathcal{L}_n(S^3)$, $l \in \mathcal{L}(S^3_n)$, r is the number of orbits that bifurcate and $\overline{L}_0$ means that r orbits of a symmetric link L_0 have changed their index to be 1.

3.4. Multiple Hopf bifurcation

When a Hopf bifurcation occurs an invariant torus appears around a (attractive or repulsive) periodic orbit that changes its index. If the rotation number that the flow induces on this invariant torus is rational, we have, generically, a finite and even number of hyperbolic periodic orbits, $2r$, and the flow on S^3 is still NMS.

As before, we obtain the bifurcation of a symmetric link L from the bifurcation in a stratum denoting:

$$HOPF[L] = G\big(hopf(l)\big).$$

In the following proposition it is obtained that $HOPF[L]$ is the link obtained by changing the index of a set of symmetric orbits of L (with index 0 or 2) and adding an even number of hyperbolic periodic orbits (r with index 0 or 2 and r with index 1).

Similarly to the previous propositions it can be shown that:

Proposition 4. *Let $L \in \mathcal{L}_n(S^3)$, a Hopf bifurcation of L, when the orbits that bifurcates are linked to γ, is characterized by*:

$$HOPF[L] = V^{rs}(L), \tag{16}$$

where s is the number of pairs of orbits that appear in each stratum, $r = \gcd(n, p')$ and p' is the order of these orbits.

Then, it can be deduced that:

Corollary 6. *A HOPF bifurcation in a symmetric system can be produced on every link $L \in \mathcal{L}_n(S^3)$.*

4. Final remarks

(1) The first conclusion we would like to underline is the existence of new symmetric bifurcations for a given critical value of the parameter, in contrast to a trivial repetition of local generic codimension one bifurcations that will occur after passing several times through different bifurcation points.

We also obtain some important results about the kind of links that can suffer a given bifurcation, maintaining the symmetry of the system, we state it as:

Theorem 1. *In NMS systems on S^3 with rotational symmetry around one axis, the generic codimension one bifurcations are multiple except when the invariant orbit γ bifurcates or when the order of the new orbits that appear after the bifurcation and the symmetry of the system, n, are prime.*

The proof follows from the propositions developed in Section 3.

(2) Similar results to those obtained in NMS systems are obtained when symmetry is present, that is, we also have the possibility of connecting certain links by a sequence of these kind of bifurcation.

Theorem 2. *Given two symmetric unsplittable links differing only in orbits that correspond to cables, one can be obtained from the other by a sequence of symmetric generic codimension one bifurcations.*

Proof. Given two unsplittable links, we can write them in terms of type B Wada operations. If they only differ in orbits that are cables (in number, type and/or index), as we have seen that operations of cabling are associated with certain bifurcations and no orbit have been eliminated (type A operations are not involved), we can obtain the sequence of generic codimension one bifurcations that carries a link onto the other. □

It is directly deduced from characterizations obtained the existence of links that can not be obtained one from any other by means of generic local codimension one bifurcations of symmetric NMS systems over the three-sphere:

Theorem 3. *A symmetric link composed by the split sum of Hopf links and unknots with index* 1 *cannot be obtained from the generic codimension one bifurcation of any link.*

(3) *Critical links.* As a consequence of the previous propositions it is possible to obtain all the possible configurations of links of periodic orbits for the critical value of the parameter $\mu = \mu_0$, for each type of bifurcation.

If the orbits that bifurcates are not linked to the invariant one, the symmetry forces the repetition of the links obtained in [6]. When the orbit that bifurcates is linked to the invariant, new intermediate configurations are obtained from the previous propositions.

Let v be the nonhyperbolic orbit representing the orbit that bifurcates which, then the intermediate configurations for $\mu = \mu_0$ is:

– L with r linked components, v, that are nonhyperbolic and correspond to the same G-orbit.

Let us notice that the set of limit configurations of NMS systems is different from the set of limit configurations of NMS systems with symmetry because in this last case it can appear several critical orbits in the link. Both configurations will coincide when one critical orbit appears in the bifurcation point, that is when the order of the orbit is prime with respect to the symmetry n.

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