On the directional dynamics of additive cellular automata ☆

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ABSTRACT

We continue the study of cellular automata (CA) directional dynamics, *i.e.* , the behavior of the joint action of CA and shift maps. This notion has been investigated for general CA in the case of expansive dynamics by Boyle and Lind; and by Sablik for sensitivity and equicontinuity. In this paper we give a detailed classification for the class of additive CA providing non-trivial examples for some classes of Sablik's classification. Moreover, we extend the directional dynamics studies by considering also factor languages and attractors.

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1. Introduction

Cellular automata (CA) are simple formal models for complex systems. They have been widely studied in a number of disciplines (Computer Science, Physics, Mathematics, Biology, Chemistry, *etc.*) with different purposes (simulation of natural phenomena, pseudo-random number generation, image processing, analysis of universal model of computations, quasi-crystals, *etc.*). For recent results and an extensive and up-to-date bibliography, for example, see [10,19,5,9,24,15,26,16,35, 18,14,28,6,21].

The huge variety of distinct dynamical behaviors is one of the main features which determined the success of CA in applications. Paradoxically, the formal (decidable) classification of such behaviors is still a major open problem in CA theory. Indeed, many classifications have been introduced over the years but none of them is decidable [20,11,4,25,23,17,27].

Inspired by [33,3], M. Sablik proposed to refine Kůrka's equicontinuity classification along "directions" different from the standard time arrow [36]. The idea is to see how "robust" a given dynamical behavior is when changing the way by which time samples are taken into account. In other words, Sablik studies the space-time structure of CA evolutions by classifying the dynamics of $\sigma^k \circ F^h$, where σ is the shift map and F is the global rule of a CA ($k \in \mathbb{Z}$, $h \in \mathbb{N}^+$, see Section 2 for the definitions). Sablik's work is concerned particularly with directions of equicontinuity and (left/right) expansivity: he provides a directional dynamics classification of CA according to such properties. Despite his classification sheds new light about the complexity of CA behavior, most of his classes are still not well understood. Moreover, it is actually unknown whether his classification is (at least partially) decidable or not.

Additive CA (ACA) are the subclass of CA whose local rule is defined by an additive function. Despite their simplicity that makes it possible a detailed algebraic analysis, ACA exhibit many of the complex features of general CA. Several important properties of ACA have been studied during the last twenty years and in some cases exact characterizations have been obtained [22,38,39,32,31,8,7].

In this paper we use ACA to further illustrate the work of Sablik and we extend the directional dynamics picture by further introducing attractors and factor languages directions. We provide a very detailed directional dynamics classification of ACA and we compare our classes with Sablik's ones. Moreover we show that our classification is completely decidable.

The paper is organized as follows. Sections 2–4 are devoted to the basic background on the subject of CA and ACA. In Section 5, we consider factor languages directions, in particular we show that all ACA are regular. In Section 6 we consider attractor directions. In Section 7 we provide a directional dynamics classification of ACA and compare our classes with Sablik's ones. In Section 8, we draw some conclusions about this work.

2. Cellular automata

A CA consists in an infinite set of finite automata distributed over a regular lattice $\mathcal{L}$. All finite automata are identical. Each automaton assumes a *state*, chosen from a finite set A, called the *set of states* or the *alphabet*. A *configuration* is a snapshot of all the states of the automata, *i.e.*, a function from $\mathcal{L}$ to A. Denote $A^{\mathcal{L}}$ the set of all configurations. In the present paper, we consider one-dimensional CA in which $\mathcal{L} = \mathbb{Z}$. A *local rule* updates the state of each automaton on the basis of its current state and the ones of a fixed set of neighboring automata individuated by the neighborhood frame $N = \{m, m + 1, \ldots, a\}$, where $m, a \in \mathbb{Z}$, with $m \leq a$. The integers m, a and $r = \max\{|m|, |a|\}$ are called the *memory*, the *anticipation* and the *radius* of the CA, respectively. Formally, the local rule is a function $f : A^{a-m+1} \to A$. All automata in the lattice are updated synchronously. In other words, the local rule f induces a *global* rule $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ describing the evolution of the whole system from time t to $t + 1$:

$$\forall c \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad F(c)_i = f(c_{i+m}, \ldots, c_{i+a}) .$$

We say that a CA is *one-sided* if either $m \geq 0$ or $a \leq 0$. The *shift map* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, defined as $\forall c \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \sigma(c)_i = c_{i+1}$ is one of the simplest examples of CA.

In this work we restrict our attention to the class of additive CA, *i.e.*, CA based on an additive local rule defined over the ring $\mathbf{Z}_s = \{0, 1, \ldots, s - 1\}$. A function $f : \mathbf{Z}_s^{a-m+1} \to \mathbf{Z}_s$ is said to be additive if there exist coefficients $\lambda_m, \ldots, \lambda_a \in \mathbf{Z}_s$ such that it can be expressed as:

$$\forall (x_m, \ldots, x_a) \in \mathbf{Z}_s^{a-m+1}, \quad f(x_m, \ldots, x_a) = \left[\sum_{j=m}^{a} \lambda_j x_j \right]_s$$

where $[x]_s$ is the integer x taken modulo s. A CA is *additive* if its local rule is additive.

A rule $f : A^{a-m+1} \to A$ is *permutive in the position i* if $\forall b_m, b_{m+1}, \ldots, b_{i-1}, b_{i+1}, \ldots, b_a \in A, \forall b \in A, \exists! b_i \in A, f(b_m, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_a) = b$. The local rule of an ACA is permutive in the position i iff $\gcd(s, \lambda_i) = 1$.

3. Dynamical properties of DTDS and CA

A discrete time dynamical system (DTDS) is a pair (X, g) where X is a compact metric space and g is a continuous mapping from X to itself. Denote $d(\cdot, \cdot)$, the metric on X. When A is equipped with the discrete topology and $A^{\mathbb{Z}}$ with the induced product topology, for any CA F, the structure $(A^{\mathbb{Z}}, F)$ is a DTDS. From now on, for the sake of simplicity we identify a CA with the dynamical system induced by itself or even with its global rule F.

Dynamical and set theoretical properties for DTDS. A DTDS (X, g) is *injective* (resp., *surjective*, *open*) iff g is injective (resp., surjective, open).

A DTDS (X, g) is *sensitive to the initial conditions* (or simply *sensitive*) if there exists a constant $\varepsilon > 0$ such that for any configuration $x \in X$ and any $\delta > 0$ there is a configuration $y \in X$ such that $d(y, x) < \delta$ and $d(g^n(y), g^n(x)) > \varepsilon$ for some $n \in \mathbb{N}$. A DTDS (X, g) is *positively expansive* if there exists a constant $\varepsilon > 0$ such that for any pair of distinct elements x, y we have $d(g^n(y), g^n(x)) \geq \varepsilon$ for some $n \in \mathbb{N}$. When g is a homeomorphism, the notion of *expansivity* can be considered too. It is obtained by replacing $n \in \mathbb{N}$ with $n \in \mathbb{Z}$ in the definition of positive expansivity.

An element $x \in X$ is an *equicontinuity point* for g if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in X$, $d(y, x) < \delta$ implies that $\forall n \in \mathbb{N}$, $d(g^n(y), g^n(x)) < \varepsilon$. A DTDS is said to be *equicontinuous* if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$, $d(y, x) < \delta$ implies that $\forall n \in \mathbb{N}$, $d(g^n(y), g^n(x)) < \varepsilon$. A DTDS (X, g) is equicontinuous iff the set E of its equicontinuity points is the whole X. A DTDS is *almost equicontinuous* if E is residual (*i.e.*, E can be obtained by an infinite intersection of dense open subsets).

A DTDS (X, g) is *(topologically) transitive* if for any pair of non-empty open sets $U, V \subseteq X$ there exists an integer $n \in \mathbb{N}$ such that $g^n(U) \cap V \neq \emptyset$. A DTDS (X, g) is *(topologically) mixing* if for any pair of non-empty open sets $U, V \subseteq A^{\mathbb{Z}}$ there exists an integer $n \in \mathbb{N}$ such that for any $t \geq n$ we have $g^t(U) \cap V \neq \emptyset$. Trivially, any mixing DTDS is also transitive.

A morphism between two DTDS (X, g) and (Y, h) is a continuous mapping $\phi : X \to Y$ such that $h \circ \phi = \phi \circ g$. If ϕ is surjective then (Y, h) is a *factor* of (X, g). If ϕ is a homeomorphism, the two systems are said to be *(topologically) conjugated*. The conjugacy preserves most of the properties seen so far.

Limit sets and attractors for DTDS. For a given DTDS (X, g), a subset $V \subseteq X$ is said to be *invariant* if $g(V) \subseteq V$. The *omega limit* of a closed invariant subset $V \subseteq X$ is defined as

$$\omega(V) = \bigcap_{n>0} \overline{\bigcup_{m>n} g^m(V)}.$$

The *limit set* of the DTDS (X, g) is $\omega(X)$. A dynamical system is called *stable* if it reaches its limit set in a finite amount of time, *i.e.*, if there exists some $n > 0$ such that $\forall m > n, g^m(X) = g^n(X)$. A set $Y \subseteq X$ is an attractor if there exists a non-empty open set V such that $F(\overline{V}) \subseteq V$ and $Y = \omega(V)$. In totally disconnected spaces, attractors are omega limit sets of clopen invariant sets. A set $Y \subseteq X$ is a *minimal* attractor if it is an attractor and no proper subset of Y is an attractor. A *quasi-attractor* is a countable intersection of attractors which is not an attractor. DTDS with a unique attractor which is a singleton are called *nilpotent*.

Topology on CA configurations and related properties. In order to study the dynamical properties of CA, $A^{\mathbb{Z}}$ is usually equipped with the Cantor metric d defined as

$$\forall c, c' \in A^{\mathbb{Z}}, \ d(c, c') = 2^{-n}, \ \text{where } n = \min\left\{i \geq 0 : c_i \neq c'_i \text{ or } c_{-i} \neq c'_{-i}\right\}.$$

The topology induced by d coincides with the product topology defined above. In this case, $A^{\mathbb{Z}}$ is a Cantor space, *i.e.*, it is compact, perfect and totally disconnected.

For any configuration $c \in A^{\mathbb{Z}}$ and any pair $i, j \in \mathbb{Z}$, with $i \leq j$, denote by $c_{[i,j]}$ the word $c_i \cdots c_j \in A^{j-i+1}$, *i.e.*, the portion of the configuration $c \in A^{\mathbb{Z}}$ inside the interval $[i, j] = \{k \in \mathbb{Z} : i \leq k \leq j\}$. A *cylinder* of block $u \in A^k$ and position $i \in \mathbb{Z}$ is the set $C_i(u) = \{c \in A^{\mathbb{Z}} : c_{[i,i+k-1]} = u\}$. Cylinders are clopen (*i.e.*, closed and open) sets *w.r.t.* the Cantor metric.

In the case of CA, it is possible to study other forms of expansivity. For any $n \in \mathbb{Z}$, let $c_{[n,\infty)}$ (resp., $c_{(-\infty,n]}$) denote the portion of a configuration c inside the infinite integer interval $[n, \infty)$ (resp., $(-\infty, n]$). A CA $(A^{\mathbb{Z}}, F)$ is *right* (resp., *left*) *expansive* if there exists a constant $\varepsilon > 0$ such that for any pair of configurations $c, c' \in A^{\mathbb{Z}}$ with $c_{[0,\infty)} \neq c'_{[0,\infty)}$ (resp., $c_{(-\infty,0]} \neq c'_{(-\infty,0]}$) we have $d(F^n(c), F^n(c')) \geq \varepsilon$ for some $n \in \mathbb{N}$.

Subshifts and column subshifts. Denote $A^{\mathbb{N}}$ the set of all one-sided configurations *i.e.* the set of functions from A to $\mathbb{N}$. A *subshift* on the alphabet A is a DTDS (S, σ) where S is a closed σ-invariant subset of $A^{\mathbb{N}}$ (or $A^{\mathbb{Z}}$). From now on, we identify a subshift (S, σ) with the set S. For $w = w_1 \cdots w_n \in A^n$ and $y \in A^{\mathbb{N}}$, $w \prec y$ means that w is a proper factor of $y \in A^{\mathbb{N}}$, *i.e.*, there exists $i \in \mathbb{N}$ such that $y_{[i,i+n-1]} = w$. Let $\mathcal{F} \subseteq A^*$ and $S_{\mathcal{F}} = \{y \in A^{\mathbb{N}} : \forall w \prec y, \ w \notin \mathcal{F}\}$, where A^* is the set of (finite) words over the alphabet A. $S_{\mathcal{F}}$ is a subshift, and $\mathcal{F}$ is its set of *forbidden patterns*. A subshift S is said to be a *subshift of finite type* (SFT) if $S = S_{\mathcal{F}}$ for some finite set $\mathcal{F}$. The language of a subshift S is $L_S = \{w \in A^* : \exists y \in A^{\mathbb{N}}, w \prec y\}$. A subshift is *sofic* if it is a factor of some SFT. Refer to [29] for an introduction on subshifts.

Let S_1 and S_2 be two subshifts. A function $\varphi : S_1 \to S_2$ is said to be a *block map* if it is continuous and σ-commuting, i.e. $\varphi \circ \sigma = \sigma \circ \varphi$. In particular, CA are block maps from the subshift $A^{\mathbb{Z}}$ to itself.

The *column subshift* of width $k > 0$ of a given CA $(A^{\mathbb{Z}}, F)$, is the subshift $(\Sigma_k(F), \sigma)$ on the alphabet $B = A^k$ where

$$\Sigma_k(F) = \left\{y \in B^{\mathbb{N}} : \exists c \in A^{\mathbb{Z}}, \forall i \in \mathbb{N}, y_i = F^i(c)_{[1,k]}\right\}.$$

A language $L \subseteq A^*$ is *bounded periodic* if there exist two integers $l \geq 0$ and $n > 0$ such that for every $u \in L$ and $i \geq l$ we have $u_i = u_{i+n}$. A CA is said to be *bounded periodic* (resp., *regular*) if for any $k > 0$ the the language of the column subshift $(\Sigma_k(F), \sigma)$ is bounded periodic (resp., regular).

Directional dynamics of CA. The directional dynamics of CA concerns the study of the joint action of CA with the shift map. More precisely, for a given CA F and for any rational k/h ($k \in \mathbb{Z}, h \in \mathbb{N}^+$), the focus is the dynamical behavior of the CA $\sigma^k F^h$. A CA F is said to be equicontinuous (resp., almost equicontinuous, resp. left expansive, resp., right expansive, resp., positively expansive, resp., expansive) along the direction $k/h, k \in \mathbb{Z}, h \in \mathbb{N}^+$, if the CA $\sigma^k F^h$ is equicontinuous (resp., almost equicontinuous, resp. left expansive, resp., right expansive, resp., positively expansive, resp., expansive).

3.1. Classifications of CA

We now recall three important classifications of CA based on the complexity of their column subshift languages, the degree of stability/instability of their behavior, and the existence of attractors, respectively. All these classifications have been defined and compared in [25].

Theorem 1 (*[25]*). *Every CA $(A^{\mathbb{Z}}, F)$ falls exactly in one of the following classes:*

L1. $(A^{\mathbb{Z}}, F)$ *is bounded periodic.*
L2. $(A^{\mathbb{Z}}, F)$ *is regular not bounded periodic.*
L3. $(A^{\mathbb{Z}}, F)$ *is not regular.*

Theorem 2 (*[25]*). *Every CA ($A^{\mathbb{Z}}$, F) falls exactly in one of the following classes:*

E1. *($A^{\mathbb{Z}}$, F) is equicontinuous;*
E2. *($A^{\mathbb{Z}}$, F) is almost equicontinuous but not equicontinuous;*
E3. *($A^{\mathbb{Z}}$, F) is sensitive but not positively expansive;*
E4. *($A^{\mathbb{Z}}$, F) is positively expansive.*

Theorem 3 (*[25]*). *Every CA ($A^{\mathbb{Z}}$, F) falls exactly in one of the following classes.*

A1. *There exist two disjoint attractors.*
A2. *There exists a unique minimal quasi-attractor.*
A3. *There exists a unique minimal attractor different from $\omega(A^{\mathbb{Z}})$.*
A4. *There exists a unique attractor $\omega(A^{\mathbb{Z}}) \neq A^{\mathbb{Z}}$.*
A5. *There exists a unique attractor $\omega(A^{\mathbb{Z}}) = A^{\mathbb{Z}}$.*

A recent classification concerns the directional dynamics of a CA *F*. In order to illustrate it, we introduce the following notation.

Definition 1. The *equicontinuous, almost equicontinuous, positively expansive* and *left-or-right expansive* direction sets of a CA ($A^{\mathbb{Z}}$, F) are defined as follows

- $\mathfrak{E}_F = \{k/h \mid k \in \mathbb{Z}, h \in \mathbb{N}^+ : \sigma^k F^h$ is equicontinuous$\}$.
- $\mathfrak{A}_F = \{k/h \mid k \in \mathbb{Z}, h \in \mathbb{N}^+ : \sigma^k F^h$ is almost equicontinuous$\}$.
- $\mathfrak{X}_F^- = \{k/h \mid k \in \mathbb{Z}, h \in \mathbb{N}^+ : \sigma^k F^h$ is left expansive$\}$.
- $\mathfrak{X}_F^+ = \{k/h \mid k \in \mathbb{Z}, h \in \mathbb{N}^+ : \sigma^k F^h$ is right expansive$\}$.
- $\mathfrak{X}_F = \{k/h \mid k \in \mathbb{Z}, h \in \mathbb{N}^+ : \sigma^k F^h$ is expansive$\}$.

The sets $\mathfrak{E}_F, \mathfrak{A}_F, \mathfrak{X}_F^-$ and $\mathfrak{X}_F^+$ are convex (in $\mathbb{Q}$ or in $\mathbb{R}$). Moreover, note that the set of positively expansive directions is $\mathfrak{X}_F^+ \cap \mathfrak{X}_F^-$.

Theorem 4 (*[36]*). *Let ($A^{\mathbb{Z}}$, F) be a CA with memory m and anticipation a.*

- *If $|\mathfrak{E}_F| > 1$ then $\mathfrak{E}_F = \mathbb{Q}$ and ($A^{\mathbb{Z}}$, F) is nilpotent.*
- *If $\mathfrak{E}_F \neq \emptyset$ and $\mathfrak{E}_F \neq \mathbb{Q}$ then $\exists ! \alpha \in [-a, -m], \mathfrak{E}_F = \{\alpha\}$. Moreover, if ($A^{\mathbb{Z}}$, F) is surjective then $\mathfrak{X}_F^- = (-\infty, \alpha), \mathfrak{X}_F^+ = (\alpha, \infty)$ and, in particular, ($A^{\mathbb{Z}}$, F) is injective.*

Theorem 5 (*[36]*). *Every ($A^{\mathbb{Z}}$, F) CA with memory m and anticipation a falls exactly in one of the following classes:*

C1. $\mathfrak{E}_F = \mathfrak{A}_F = \mathbb{Q}$ and $\mathfrak{X}_F^- = \mathfrak{X}_F^+ = \emptyset$. *This happens iff ($A^{\mathbb{Z}}$, F) is nilpotent.*
C2. *There exists $\alpha \in [-a, -m], \mathfrak{E}_F = \mathfrak{A}_F = \{\alpha\}$. Moreover, if ($A^{\mathbb{Z}}$, F) is surjective, $\mathfrak{X}_F^- = (-\infty, \alpha)$ and $\mathfrak{X}_F^+ = (\alpha, \infty)$.*
C3. *There exists $\alpha \in [-a, -m], \mathfrak{E}_F = \emptyset, \mathfrak{A}_F = \{\alpha\}$.*
C4. *There exist $\alpha_1 < \alpha_2$ such that $(\alpha_1, \alpha_2) \subseteq \mathfrak{A}_F \subseteq [\alpha_1, \alpha_2] \subseteq [-a, -m]$ and $\mathfrak{E}_F = \mathfrak{X}_F^- = \mathfrak{X}_F^+ = \emptyset$.*
C5. $\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ \neq \emptyset$. *This implies $\mathfrak{E}_F = \mathfrak{A}_F = \emptyset$.*
C6. $\mathfrak{E}_F = \mathfrak{A}_F = \emptyset$ and $\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ = \emptyset$.

3.2. Main properties of ACA

The dynamical behavior of ACA has been extensively studied. We briefly report the main results which characterize the most important dynamical and set theoretical properties for ACA.

Theorem 6 (*[22,30,32,8,7]*). *Let ($\mathbb{Z}_s^{\mathbb{Z}}$, F) be an ACA with coefficients $\lambda_m, \ldots, \lambda_a$ and $s = p_1^{n_1} \cdot p_2^{n_2} \cdots p_l^{n_l}$ where $p_1, \ldots, p_l$ are primes. Then,*

- *($\mathbb{Z}_s^{\mathbb{Z}}$, F) is surjective iff $\gcd(s, \lambda_m, \ldots, \lambda_a) = 1$*
- *($\mathbb{Z}_s^{\mathbb{Z}}$, F) is injective iff $\forall p_i, \exists ! \lambda_j, p_i \nmid \lambda_j$*
- *($\mathbb{Z}_s^{\mathbb{Z}}$, F) is equicontinuous iff $\forall p_i, p_i \mid \gcd(\lambda_m, \ldots, \lambda_{-1}, \lambda_1, \ldots, \lambda_a)$*
- *($\mathbb{Z}_s^{\mathbb{Z}}$, F) is sensitive iff $\exists p_i, p_i \nmid \gcd(\lambda_m, \ldots, \lambda_{-1}, \lambda_1, \ldots, \lambda_a)$*
- *($\mathbb{Z}_s^{\mathbb{Z}}$, F) is transitive iff it is mixing iff $\gcd(s, \lambda_m, \ldots \lambda_{-1}, \lambda_1, \ldots, \lambda_a) = 1$*
- *($\mathbb{Z}_s^{\mathbb{Z}}$, F) is pos. expansive iff $\gcd(s, \lambda_m, \ldots, \lambda_{-1}) = \gcd(s, \lambda_1, \ldots, \lambda_a) = 1$*
- *($\mathbb{Z}_s^{\mathbb{Z}}$, F) is expansive iff $\gcd(s, \lambda_m, \ldots, \lambda_{-1}, \lambda_1, \ldots, \lambda_a) = 1$*

Remark that, as immediate consequence of Theorem 6, **E2** $= \emptyset$ for ACA. Moreover, all the characterizations are given in terms of coefficients of the local rule and hence they are decidable.

We now recall two tools which are fundamental in order to study ACA.

Theorem 7 (*[12]*). *Consider an ACA* $(\mathbf{Z}_{pq}^{\mathbb{Z}}, F)$ *with* $\gcd(p, q) = 1$. *Then* $(\mathbf{Z}_{pq}^{\mathbb{Z}}, F)$ *is conjugated to the ACA* $(\mathbf{Z}_p^{\mathbb{Z}} \times \mathbf{Z}_q^{\mathbb{Z}}, [F]_p \times [F]_q)$.

On the basis of this theorem, if $s = p_1^{n_1} \cdots p_l^{n_l}$ is the prime factor decomposition of s, an ACA on $\mathbf{Z}_s$ is conjugated to the product of ACA on $\mathbf{Z}_{p_i^{n_i}}$. So all the properties which are preserved under product and under topological conjugacy are lifted from ACA on $\mathbf{Z}_{p^k}$ to $\mathbf{Z}_s$.

Theorem 8 (*[31]*). *Let* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *be an ACA. Then* $\forall h \geq \lfloor \log_2 s \rfloor$, $F^h(\mathbf{Z}_s^{\mathbb{Z}}) = F^{\lfloor \log_2 s \rfloor}(\mathbf{Z}_s^{\mathbb{Z}}) = \omega(\mathbf{Z}_s^{\mathbb{Z}})$. *Moreover,* $(F^h(\mathbf{Z}_s^{\mathbb{Z}}), F)$ *is conjugated to some surjective ACA* $(\mathbf{Z}_{s*}^{\mathbb{Z}}, F*)$.

Remark that the conjugacy map involved in the proof of Theorem 8 preserves factor languages complexities, i.e. for $k > 0$, the column factor of width k of $(F^h(\mathbf{Z}_s^{\mathbb{Z}}), F)$ is a SFT if and only if the column factor of width k of $(\mathbf{Z}_{s*}^{\mathbb{Z}}, F*)$ is SFT. This property will be useful in the sequel.

4. Surjective ACA

Thanks to Theorems 7 and 8, most of the properties of general ACA can be deduced from *undecomposable ACA*, namely surjective ACA over $\mathbf{Z}_{p^k}$ for some prime number p. In this section we classify the possible dynamics of this class in order to understand the directional dynamics of general ACA.

Lemma 1. *Let* $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *be a surjective ACA with p prime whose local rule has memory m and anticipation a. Then, there exists* $i \in [m, a]$ *such that* $\gcd(\lambda_i, p) = 1$.

Proof. If for all λ_i it happens that $\gcd(p, \lambda_i) = p$ then, by Theorem 6, $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ is not surjective, contradicting the hypothesis. □

Lemma 2 (*[12]*). *Let* $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *be a surjective ACA with p prime. Set*

$$L = \min\{j : \gcd(\lambda_j, p) = 1\} \quad and \quad R = \max\{j : \gcd(\lambda_j, p) = 1\}.$$

Then there exists $h \geq 1$ *such that the rule* f^h *associated to* F^h *has the form*

$$f^h(x_{hm}, \ldots, x_{ha}) = \left[\Sigma_{i=hL}^{hR} \mu_i x_i \right]_{p^k} \quad with \quad \gcd(\mu_{hL}, p) = \gcd(\mu_{hR}, p) = 1.$$

Recall that the condition $\gcd(\mu_{hL}, p) = \gcd(\mu_{hR}, p) = 1$ implies permutivity in hL and hR. The following proposition characterizes the possible dynamics of undecomposable CA.

Proposition 1. *Consider a surjective ACA* $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *with p prime. Then, exactly one of the following cases occurs:*

1. $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *is equicontinuous.*
2. $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *is positively expansive.*
3. $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *is either left or right expansive.*

Proof. Let $L = \min\{j : \gcd(\lambda_j, p) = 1\}$ and $R = \max\{j : \gcd(\lambda_j, p) = 1\}$. By Lemma 1, L and R are not empty. There are three possible cases:

1. $L = R = 0$. Then, by Theorem 6, $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ is equicontinuous.
2. $L < 0 < R$. Then, by Theorem 6, $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ is positively expansive.
3. $L < 0$ and $R \leq 0$ (the case $L \geq 0$ and $R > 0$ is similar). Then, by Lemma 2, there exists some $h > 0$ such that the local rule $f^h(x_{hm}, \ldots, x_{ha}) = \left[\Sigma_{i=hL}^{hR} \mu_i x_i \right]_{p^k}$ of F^h is permutive in $hL < hR \leq 0$. Then f^h is left one-sided and permutive. In particular F^h is left expansive which easily implies that F is also left expansive. □

Remark 1. In a similar way as in the proof of Proposition 1, one can show that right/left expansive ACA on $\mathbf{Z}_{p_i^{n_i}}$ are mixing.

The following theorem classifies the directional dynamics of undecomposable surjective ACA: any undecomposable ACA either contains exactly one equicontinuous direction (and it is injective) or contains a positively expansive direction (and it is not injective).

Theorem 9. *Let* $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *be a surjective ACA with p prime. Then, exactly one of the following cases can occur*

1. $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *is injective. Then,*

$$\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ = \emptyset, |\mathfrak{E}_F| = 1 \quad and \quad \mathfrak{X}_F = \mathfrak{X}_F^- \cup \mathfrak{X}_F^+ = \mathbb{Q} \setminus \mathfrak{E}_F.$$

2. $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ *is not injective. Then,*

$$\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ \neq \emptyset \quad and \quad \mathfrak{E}_F = \mathfrak{X}_F = \emptyset.$$

Proof. By Proposition 1, $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ can be equicontinuous or positively expansive or left/right expansive. In the first two cases the conclusion is immediate. If $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ is left/right expansive then there are two possible cases. Let L, R be defined as in Lemma 2.

1. $L = R \neq 0$. By Theorem 7, $(\mathbf{Z}_{p^k}^{\mathbb{Z}}, F)$ must be expansive and, in particular, it is the power of some shift map. Then the thesis follows from Theorem 4.
2. $L \neq R$. Then, by Lemma 2, there exists some $h > 0$ such that the local rule of F^h is permutive in its leftmost and rightmost positions hL, hR. Without loss of generality, we can assume that $hR - hL > 1$ (this condition can be obtained by simply taking some power of F^h). Then there exists some k such that $\sigma^k F^h$ is permutive in its leftmost and rightmost positions $hL + k < 0 < hR + k$. Then $\sigma^k F^h$ is positively expansive. □

From Theorem 9, one can find an easy proof for the following result. We remark that there are several easy proofs for such result and we add ours here just for completeness.

Theorem 10 (*[39]*). *Any surjective ACA is open.*

Proof. Since the openness property is preserved in every direction and it is preserved also under product, by Theorems 7 and 9 it follows that any surjective ACA is open. □

5. Directional dynamics of factor languages for ACA

In this section we show that all ACA are regular. This fact implies that the dynamics of ACA is regular in all rational directions.

Lemma 3. *Let $\Sigma \subseteq A^{\mathbb{N}}$ be a subshift. Then, the following conditions are equivalent:*

1. (Σ, σ) *is open*
2. $\forall n > 0$, (Σ, σ^n) *is open.*
3. $\exists n > 1$ *such that* (Σ, σ^n) *is open.*

Proof. Trivially, $1 \implies 2 \implies 3$. We show that $3 \implies 1$. Since every open set in Σ is the union of clopen sets and every clopen set is a finite union of cylinders it is sufficient to show that σ is open on every cylinder in Σ. For the moment let $k > n$ and choose some cylinder $C_0(u)$ where $u = u_0 u_1 \ldots u_{k-1} \in A^k$. By hypothesis, $V = \sigma^n(C_0(u))$ is clopen then V is the finite union of cylinder sets. Now let define $W = \bigcup_{C_n(v) \subseteq V} C_1(u_1 u_2 \ldots u_{n-1} v)$. Then W is clopen and moreover $F(C_0(u)) = W$. This proves that the image under σ of every cylinder of width $k > n$ is a clopen set. Now, since every cylinder of width $1 \leq k' \leq n$ can be defined as the finite union of cylinders of width $k > n$, the conclusion follows. □

A proof of Lemma 3 in a more general setting can be found in [2].

Lemma 4. *Let $(\mathbf{Z}_{p^n}^{\mathbb{Z}}, F)$ be a right (left) expansive ACA with p prime. Then, for all sufficiently large k, $(\Sigma_k(F), \sigma)$ is a SFT.*

Proof. Since $(\mathbf{Z}_{p^n}^{\mathbb{Z}}, F)$ is right expansive, as shown in the proof of Proposition 1, there exists some $h > 0$ such that F^h is one-sided and permutive in its rightmost position which in turn implies that for all sufficiently large $k > 0$ the column factor $\Sigma_k(F^h)$ is a SFT. Consider the subshift $X_h = \{(x_1, \ldots, x_h) \in \mathbf{Z}_{p^n}^{\mathbb{Z}} \times \cdots \times \mathbf{Z}_{p^n}^{\mathbb{Z}} \mid F(x_i) = x_{i+1}, 1 \leq i < h\}$. It is not difficult to see that (X_h, σ) is conjugated to $(\mathbf{Z}_{p^n}^{\mathbb{Z}}, \sigma)$ and, in particular, X_h is a mixing SFT. The map F^h induces on X_h a continuous and σ-commuting function $G_h : X_h \to X_h$ defined by $G_h(x) = y$ if and only if $x = (x_1, \ldots, x_h), y = (y_1, \ldots, y_h) \in X_h$ and $F(x_h) = y_1$. Then (X_h, G_h) is conjugated to $(\mathbf{Z}_{p^n}^{\mathbb{Z}}, F^h)$. In particular, the local rule of G_h is one-sided and right expansive. Since X_h is a mixing SFT, it follows that for all sufficiently large $k > 0$ the column factor $\Sigma_k(G_h)$ is a SFT. Now we have that $\forall k > 0$, $(\Sigma_k(G_h), \sigma)$ is conjugated to $(\Sigma_k(F), \sigma^h)$. By Parry's Theorem [34], a one-sided subshift Y is a SFT if and only if $\sigma : Y \to Y$ is open. Then, by Lemma 3 we obtain that for all sufficiently large $k > 0$, $\sigma^h : \Sigma_k(F) \to \Sigma_k(F)$ is open and then $(\Sigma_k(F), \sigma)$ is a SFT. □

Note that the condition that for all sufficiently large $k > 0$, $\Sigma_k(F)$ is a SFT is sufficient to conclude that $(\mathbf{Z}_{p^n}^{\mathbb{Z}}, F)$ is regular.

Theorem 11. *Any ACA is regular.*

Proof. Since, by Theorem 8, ACA are stable, to study the factor language complexity of ACA, we can restrict our attention only to the dynamics on the limit set (a finite prefix does not change the language complexity of column factors). Moreover, by Theorem 8, the limit set of an ACA is conjugated to some ACA by some map which preserves factor languages complexities. This implies that ACA are regular if and only if surjective ACA are regular. Since the product of regular CA is regular and since equicontinuous and positively expansive CA are regular, by Theorem 7, Proposition 1 and Lemma 4, it follows that any surjective ACA is regular. □

Remark 2. Actually a stronger result holds for an ACA F: for any $k > 0$, $\Sigma_k(F)$ is a SFT. Indeed, consider a CA F which is permutive in both its rightmost and leftmost variables. Then, for any $k > 0$, $\Sigma_k(F)$ is a SFT [37]. This fact is sufficient to conclude that all factor subshifts of any ACA are SFT.

6. Directional dynamics of ACA according to attractors

In this section we study the class of attractors of ACA according to rational directions. In [31], Manzini and Margara show that any ACA can have either a unique attractor or a pair of disjoint attractors. Here we show some properties of *disjoint attractor* directions of ACA. We will need the two following results.

Lemma 5. *Let* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *be a surjective ACA and let* $s = p_1^{n_1} \cdot p_2^{n_2} \cdots p_l^{n_l}$ *be the prime factor decomposition of s. Then the following conditions are equivalent:*

1. $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *has two disjoint attractors,*
2. $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is not mixing,*
3. $(\mathbf{Z}_{p_i^{n_i}}^{\mathbb{Z}}, [F]_{p_i^{n_i}})$ *is equicontinuous for some* $p_i^{n_i}$.

Proof. $(1 \Rightarrow 2)$ By [25, Prop. 13], if $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ has two disjoint attractors then it cannot be mixing. $(2 \Rightarrow 3)$ Assume by absurd that for each $p_i^{n_i}$ the undecomposable ACA $(\mathbf{Z}_{p_i^{n_i}}^{\mathbb{Z}}, [F]_{p_i^{n_i}})$ is not equicontinuous. Then, by Lemma 1 and Theorem 7, $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is conjugated to the product of mixing CA and then it is mixing contradicting the hypothesis. $(3 \Rightarrow 1)$ Assume that for some $p_i^{n_i}$, $(\mathbf{Z}_{p_i^{n_i}}^{\mathbb{Z}}, [F]_{p_i^{n_i}})$ is equicontinuous. Let $A_1, A_2 \subset \mathbf{Z}_{p_i^{n_i}}^{\mathbb{Z}}$ be two disjoint attractors of $[F]_{p_i^{n_i}}$. Let $q = s/p_i^{n_i}$ and let $A \subset \mathbf{Z}_q^{\mathbb{Z}}$ be an attractor of $(\mathbf{Z}_q^{\mathbb{Z}}, [F]_q)$. Then $A_1 \times A$ and $A_2 \times A$ are disjoint attractors of $(\mathbf{Z}_{p_i^{n_i}}^{\mathbb{Z}} \times \mathbf{Z}_q^{\mathbb{Z}}, [F]_{p_i^{n_i}} \times [F]_q)$. Then, by Theorem 7, $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ has two disjoint attractors. $\square$

The class of attractors of ACA can be easily characterized using the class of attractors of surjective undecomposable ACA.

Theorem 12 (*[31]*). *Any ACA has either a unique attractor or a pair of disjoint attractors.*

Proof. Let $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ be an ACA. It is easy to see that $A \subset \mathbf{Z}_s^{\mathbb{Z}}$ is an attractor for $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ if and only if it is for $(\omega(\mathbf{Z}_s^{\mathbb{Z}}), F)$. The thesis follows from Theorem 7, Proposition 1, and Lemma 5. $\square$

At this point, we have all the elements to study the set of *disjoint attractor directions* of ACA.

Definition 2. Let $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ be an ACA. The *disjoint attractors* direction set of $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is

$$\mathfrak{D}_F = \{k/h \mid k \in \mathbb{Z}, h \in \mathbb{N}^+ : \sigma^k F^h \text{has two disjoint attractors}\}.$$

The following proposition shows some properties of the set $\mathfrak{D}_F$. In particular, we have that $\mathfrak{D}_F$ is finite and that between two disjoint attractors directions $\alpha_1, \alpha_2 \in \mathfrak{D}_F$ there cannot exist left/right expansive directions.

Proposition 2. *Let* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *be an ACA with memory m and anticipation a. Then the following conditions hold.*

1. *If* $|\mathfrak{E}_F| > 1$ *then* $\mathfrak{D}_F = \emptyset$.
2. *If* $\mathfrak{E}_F = \{\alpha\}$ *then* $\mathfrak{D}_F = \{\alpha\}$.
3. *If* $|\mathfrak{D}_F| > 1$ *then* $\mathfrak{E}_F = \emptyset$.
4. $\mathfrak{D}_F \subset [-a, -m]$ *is finite.*
5. *If* $\mathfrak{D}_F = \{\alpha_1, \ldots, \alpha_n\}$ *then* $\forall \alpha_i \le \alpha_j, [\alpha_i, \alpha_j] \not\subset \mathfrak{X}_F^- \cup \mathfrak{X}_F^+$.

Proof. Properties 1, 2 follow directly from Theorem 4 and Lemma 5. Property 3 follows from 1 and 2.

4. Since $A \subset \mathbf{Z}_s^{\mathbb{Z}}$ is an attractor for $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ if and only if it is for $(\omega(\mathbf{Z}_s^{\mathbb{Z}}), F)$, we can assume, without loss of generality, that $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is surjective. Since any surjective ACA is the (finite) product of surjective undecomposable CA (which, by Theorem 4, can have at most one equicontinuous direction), by Lemma 5 it follows that $\mathfrak{D}_F$ is finite (possibly empty). Moreover, since we are assuming that $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is surjective and $(\mathbf{Z}_s^{\mathbb{Z}}, \sigma^k F)$ is mixing for $k \in \mathbb{Z} \setminus [-a, -m]$ (see [1, Prop. 3]), immediately follows that $\mathfrak{D}_F \subset [-a, -m]$.
5. If $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is not surjective then $\mathfrak{X}_F^+ = \mathfrak{X}_F^- = \emptyset$ and the thesis holds. Assume now that F is surjective and let $\alpha_i \le \alpha_j$. By Lemma 5, $\alpha_i, \alpha_j \notin \mathfrak{X}_F^- \cup \mathfrak{X}_F^+$. By Theorems 7 and 9, it follows that for every surjective ACA there exists some α such that $\mathfrak{X}_F^+ = [\alpha, \infty)$ (the property is symmetric for $\mathfrak{X}_F^-$). Since $\mathfrak{X}_F^-, \mathfrak{X}_F^+$ are convex sets, it follows that $\forall \alpha \in [\alpha_i, \alpha_j], \alpha \notin \mathfrak{X}_F^- \cup \mathfrak{X}_F^+$. $\square$

To conclude we enumerate some classes of ACA for which $\mathfrak{D}_F$ is easy to characterize.

Corollary 1. *Let* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *be an ACA.*

- *If* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is nilpotent then* $\mathfrak{D}_F = \emptyset$.
- *If* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is equicontinuous and not nilpotent then* $\mathfrak{D}_F = \{0\}$.
- *If* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is positively expansive then* $\mathfrak{D}_F = \emptyset$.
- *If* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is expansive then* $\mathfrak{D}_F \ne \emptyset$.

In the case of ACA, the presence of a direction with two disjoint attractors is tightly linked to the presence of some form of equicontinuity. Indeed, such an ACA is either equicontinuous (not nilpotent) or it is the product of an ACA having an equicontinuous direction with some other ACA (see Lemma 5). It is not known if the same holds for general CA.

7. Directional dynamics of ACA

In this section, we classify the directional dynamics of ACA according to equicontinuous, left/right expansive, expansive and disjoint attractor directions. We do not consider explicitly factor languages directions since, by Theorem 11, for ACA all language directions are regular, and, by [25, Th. 4], directions which have bounded periodic languages coincide exactly with equicontinuous directions. To have a more clear picture we introduce explicitly the class of strictly sensitive non-expansive directions.

Definition 3. The *strictly sensitive* direction sets of the ACA $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is defined by $\mathfrak{S}_F = \mathbb{Q} \setminus (\mathfrak{E}_F \cup \mathfrak{X}_F^- \cup \mathfrak{X}_F^+ \cup \mathfrak{X}_F)$.

We consider separately the directional dynamics of non-surjective, strictly surjective and injective ACA. Note that, since there are no almost equicontinuous ACA, classes **C3** and **C4** of Theorem 5 are empty for ACA. By Theorem 9, it follows that surjective ACA always have left and right expansive directions. In particular, since surjective ACA are open, it is not difficult to see that for any surjective ACA of memory m and anticipation a it happens that $(-\infty, -a) \subseteq \mathfrak{X}_F^-$ and $(-m, \infty) \subseteq \mathfrak{X}_F^+$. This implies that surjective ACA can only belong to classes **C2**, **C5**, **C6**. In particular, injective ACA are contained in class **C2**∪**C6** and strictly surjective ACA are contained in **C5**∪**C6**. Obviously, in the strictly surjective case there are not expansive directions which arise uniquely in the injective case. For injective ACA it happens also that $\mathfrak{D}_F \neq \emptyset$ and that expansive directions are always the complement in $\mathbb{Q}$ of $\mathfrak{D}_F$.

Theorem 13. *Let $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ be an injective ACA with memory m and anticipation a. Then $\mathfrak{X}_F = \mathbb{Q} \setminus \mathfrak{D}_F$. Moreover, exactly one of the following cases can occur:*

1. $\mathfrak{E}_F \neq \emptyset$. Then $\mathfrak{D}_F = \mathfrak{E}_F = \{\alpha\} \subset [-a, -m]$, $\mathfrak{X}_F^+ = (\alpha, \infty)$, $\mathfrak{X}_F^- = (-\infty, \alpha)$.
2. $\mathfrak{E}_F = \emptyset$. Then $\mathfrak{D}_F = \{\alpha_1, \ldots, \alpha_n\} \subset [-a, -m]$, with $\alpha_1 < \cdots < \alpha_n$, $n > 1$ and $\mathfrak{X}_F^- = (-\infty, \alpha_1)$, $\mathfrak{X}_F^+ = (\alpha_n, \infty)$.

Proof. Since $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is injective, by Theorems 7 and 9, it is the product of undecomposable CA which have exactly one equicontinuity direction and which are expansive in all other directions. Then, since expansivity is preserved under product, by Lemma 5, it follows that $\mathfrak{X}_F = \mathbb{Q} \setminus \mathfrak{D}_F$.

1. If $\mathfrak{E}_F \neq \emptyset$ then the conclusion follows directly from Theorem 4 and Proposition 2. In particular, we are in class **C2** of Theorem 5.
2. If $\mathfrak{E}_F = \emptyset$ then $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ must be the product of at least two injective undecomposable CA (whose respective equicontinuous directions have different slope). Then F must have a finite set of at least two disjoint attractors directions $\alpha_1 < \cdots < \alpha_n$. By Proposition 2, the interval $[\alpha_1, \alpha_n] \subset [-a, -m]$ cannot contain left-or-right expansive directions and by Theorem 4, it must be $\mathfrak{X}_F^- = (-\infty, \alpha_1)$, $\mathfrak{X}_F^+ = (\alpha_n, \infty)$. In particular, we are in class **C6**. □

Strictly surjective ACA trivially cannot contain equicontinuous directions but they can have disjoint attractors directions.

Theorem 14. *Let $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ be a surjective but non-injective ACA with memory m and anticipation a. Then, $\mathfrak{E}_F = \emptyset$. Moreover, exactly one of the following cases occurs.*

1. $\mathfrak{D}_F = \emptyset$ and $\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ = \emptyset$. Then $\exists \alpha_1, \alpha_2 \in [-a, -m]$, $\alpha_1 < \alpha_2$, $\mathfrak{X}_F^- = (-\infty, \alpha_1)$, $\mathfrak{X}_F^+ = (\alpha_2, \infty)$, $\mathfrak{S}_F = [\alpha_1, \alpha_2]$.
2. $\mathfrak{D}_F = \emptyset$ and $\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ \neq \emptyset$. Then $\exists \alpha_1, \alpha_2 \in [-a, -m]$, $\alpha_2 \leq \alpha_1$, $\mathfrak{X}_F^- = (-\infty, \alpha_1)$, $\mathfrak{X}_F^+ = (\alpha_2, \infty)$, $\mathfrak{S}_F = \emptyset$.
3. $\mathfrak{D}_F \neq \emptyset$. Then $\exists -a \leq \alpha_1 \leq \beta_1 \leq \cdots \leq \beta_n \leq \alpha_2 \leq -m$, $\mathfrak{D}_F = \{\beta_1, \ldots, \beta_n\}$, $\mathfrak{X}_F^- = (-\infty, \alpha_1)$, $\mathfrak{X}_F^+ = (\alpha_2, \infty)$, $\mathfrak{S}_F = [\alpha_1, \alpha_2]$.

Proof. If $\mathfrak{E}_F$ is not empty then, by Theorem 4, $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is either injective or nilpotent contradicting the hypothesis.

1. $\mathfrak{D}_F = \emptyset$ and $\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ = \emptyset$. Then $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ must be the product of undecomposable CA which have at least one positively expansive direction (Theorems 7, 9 and Lemma 5). If $\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ = \emptyset$ then $\mathfrak{S}_F = [\alpha_1, \alpha_2] = \mathbb{Q} \setminus (\mathfrak{X}_F^- \cup \mathfrak{X}_F^+)$. In particular, we are in class **C6** of Theorem 5.
2. $\mathfrak{D}_F = \emptyset$ and $\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ \neq \emptyset$. Then $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ must be the product of undecomposable CA which have at least one positively expansive direction (Theorems 7, 9 and Lemma 5). If $\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ \neq \emptyset$ then $\mathfrak{S}_F = \emptyset = \mathbb{Q} \setminus (\mathfrak{X}_F^- \cup \mathfrak{X}_F^+)$ and $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is positively expansive in $\mathfrak{X}_F^- \cap \mathfrak{X}_F^+$. In particular, this is exactly class **C5** of Theorem 5.
3. $\mathfrak{D}_F \neq \emptyset$. Then there exist p and q with $s = pq$ such that $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is the product of a strictly surjective ACA $(\mathbf{Z}_p^{\mathbb{Z}}, [F]_p)$ with $\mathfrak{D}_{[F]_p} = \emptyset$ (i.e. case 1) with an injective ACA $(\mathbf{Z}_q^{\mathbb{Z}}, [F]_q)$ (Theorems 7, 9 and Lemma 5). Then $\mathfrak{D}_F = \mathfrak{D}_{[F]_q}$ and $\mathfrak{S}_F = \mathfrak{S}_{[F]_p}$. In particular F must have at least one disjoint attractor direction which, by Proposition 2, implies that $\mathfrak{X}_F^- \cap \mathfrak{X}_F^+ = \emptyset$ then we are in class **C6**. □

For any non-surjective CA trivially $\mathfrak{X}_F^- = \mathfrak{X}_F^+ = \mathfrak{X}_F = \emptyset$.

Theorem 15. *Let $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ be a non-surjective ACA. Then, exactly one of the following cases can occur.*

1. $\mathfrak{E}_F = \mathbb{Q}$ and $\mathfrak{D}_F = \mathfrak{S}_F = \emptyset$.
2. $\mathfrak{E}_F = \mathfrak{D}_F = \{\alpha\} \subseteq [-a, -m]$ and $\mathfrak{S}_F = \mathbb{Q} \setminus \{\alpha\}$.
3. $\mathfrak{S}_F = \mathbb{Q}$, $\mathfrak{E}_F = \emptyset$ (with either $\mathfrak{D}_F = \emptyset$ or $\mathfrak{D}_F \neq \emptyset$).

Proof. Since $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is not surjective it cannot be of class **C5** of Theorem 5. Then here class 1, 2 and 3 coincide respectively with classes **C1**, **C2**, **C6**. For classes 1 and 2 the characterization of $\mathfrak{D}_F$ follows from Proposition 2. For what concerns class 3, note that all CA in this class can be built from the product of a finite number of ACA $F_1, \ldots, F_n$ such that at least one F_k is not surjective and such that for some $1 \le i, j \le n$, $\mathfrak{E}_{F_i} \cap \mathfrak{E}_{F_j} = \emptyset$. $\square$

In the next theorem we redefine the above classes in terms of the coefficients of the local rule.

Theorem 16. *Let* $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *be an ACA with coefficients* $\lambda_m, \ldots, \lambda_a$ *and* $s = p_1^{n_1} \cdot p_2^{n_2} \cdots p_l^{n_l}$ *where* $p_1, \ldots, p_l$ *are primes. Then,*

1.1 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 1 *of Theorem* 13 *iff*

$$\exists! \lambda_j, \forall p_i, p_i \nmid \lambda_j$$

1.2 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 2 *of Theorem* 13 *iff*

$$\forall p_i, \exists \lambda_j, p_i \nmid \lambda_j \quad and \quad \nexists! \lambda_j, \forall p_i, p_i \nmid \lambda_j$$

2.1 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 1 *of Theorem* 14 *iff*

$$\forall p_i, \exists \lambda_{j'} \neq \lambda_{j''}, p_i \nmid \lambda_{j'}, p_i \nmid \lambda_{j''} \quad and \quad \nexists k \in [m, a], \forall p_i, \exists \lambda_{j'} < k \le \lambda_{j''}, p_i \nmid \lambda_{j'}, p_i \nmid \lambda_{j''}$$

2.2 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 2 *of Theorem* 14 *iff*

$$\exists k \in [m, a], \forall p_i, \exists \lambda_{j'} < k \le \lambda_{j''}, p_i \nmid \lambda_{j'}, p_i \nmid \lambda_{j''}$$

2.3 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 3 *of Theorem* 14 *iff*

$$\forall p_i, \exists \lambda_j, p_i \nmid \lambda_j \quad and \quad \exists p_i, \exists! \lambda_j, p_i \nmid \lambda_j \quad and \quad \exists p_{i'}, \exists \lambda_{j'} \neq \lambda_{j''}, p_{i'} \nmid \lambda_{j'}, p_{i'} \nmid \lambda_{j''}$$

3.1 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 1 *of Theorem* 15 *(i.e., it is nilpotent) iff*

$$\forall p_i, \forall \lambda_j, p_i \mid \lambda_j$$

3.2 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 2 *of Theorem* 15 *iff*

$$\gcd(s, \lambda_m, \ldots, \lambda_a) \neq 1 \quad and \quad \exists p_i, \exists \lambda_j, p_i \nmid \lambda_j \quad and \quad \exists k \in [m, a], \forall p_i, p_i \mid \gcd(\lambda_m, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_a)$$

3.3 $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ *is in class* 3 *of Theorem* 15 *iff*

$$\gcd(s, \lambda_m, \ldots, \lambda_a) \neq 1 \quad and \quad \exists p_i, \exists \lambda_j, p_i \nmid \lambda_j \quad and \quad \nexists k \in [m, a], \forall p_i, p_i \mid \gcd(\lambda_m, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_a).$$

Proof. We jointly consider the cases 1.1, 1.2 and the cases 2.1, 2.2, 2.3.

1. If $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is either in class 1 or in class 2 of Theorem 13 then it is injective and then, by Theorem 6, $\forall p_i, \exists! \lambda_j, p_i \nmid \lambda_j$. Trivially, every undecomposable ACA $[F]_{p_i^{n_i}}$ in the canonical decomposition of $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ must be injective and, by Theorem 9, it must contain exactly one equicontinuous direction. It is not difficult to see that all these equicontinuous directions coincide iff $\exists! \lambda_j$ such that $\forall p_i, p_i \nmid \lambda_j$ (then we are in class 1 of Theorem 13). On the contrary, all such equicontinuous directions do not coincide iff $\exists p_i \neq p_{i'}, \exists \lambda_j \neq \lambda_{j'}$ such that $p_i \nmid \lambda_j$ and $p_{i'} \nmid \lambda_{j'}$ (then we are in class 2 of Theorem 13). Note that, by Theorem 6, the conditions on the local rule coefficients in points 1.1 and 1.2 imply injectivity.

2. If $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is in some class of Theorem 14 then it must be surjective not injective and, in particular, every undecomposable ACA $[F]_{p_i^{n_i}}$ in its canonical decomposition must be surjective. Then, by Lemma 1, for every p_i there exists some $j \in [m, a]$ such that $p_i \nmid \lambda_j$ and, since $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is strictly surjective, there must exist (at least one) p_i and $\lambda_{j'} \neq \lambda_{j''}$ such that $p_i \nmid \lambda_{j'}$ and $p_i \nmid \lambda_{j''}$ (on the contrary $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ would be injective). We are in class 3 of Theorem 14 iff in the canonical decomposition of $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ there is an injective ACA, that is, iff $\exists p_i, \exists! \lambda_j, p_i \nmid \lambda_j$ (point 2.3). Recall that, by Theorem 9, every surjective non-injective undecomposable ACA contains at least one positively expansive direction. We are in class 2 of Theorem 14 iff there is no injective ACA in the canonical decomposition of $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ and there is one direction in which all $[F]_{p_i^{n_i}}$ are positively expansive. This happens iff $\exists k \in [m, a]$ such that $\forall p_i, \exists \lambda_{j'} < k \le \lambda_{j''}, p_i \nmid \lambda_{j'}, p_i \nmid \lambda_{j''}$ (point 2.3). Finally, we are in class 1 of Theorem 14 iff there is no injective ACA in the canonical decomposition of $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ and there is no k as in point 2.3.

3.1 The automaton $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is in class 1 of Theorem 15 iff it is nilpotent iff every undecomposable ACA $[F]_{p_i^{n_i}}$ in its canonical decomposition is nilpotent. Assume that $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is nilpotent. This implies that there exists some $n > 0$ such that for every $[F]_{p_i^{n_i}}$, all coefficients of the local rule of $([F]_{p_i^{n_i}})^n$ modulo $p_i^{n_i}$ are zero (on the contrary $[F]_{p_i^{n_i}}$ would be not nilpotent because we could find some configuration $x \in [\mathbf{Z}_s^{\mathbb{Z}}]_{p_i^{n_i}}$ for which $([F]_{p_i^{n_i}})^n(x)$ is not equal to the configuration of all zeroes). Since p_i is prime, the only possibility is that p_i divides all coefficients of the local rule of $[F]_{p_i^{n_i}}$ and, in particular, that it divides all λ_j. The same argument holds for all p_i. Then we can conclude that if $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is nilpotent it follows that $\forall p_i, \forall \lambda_j, p_i \mid \lambda_j$. Assume now that $\forall p_i, \forall \lambda_j, p_i \mid \lambda_j$. We have to show that $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is nilpotent. Since, by hypothesis, every prime p_i divides every coefficient λ_j, it is not difficult to see that there must exist some power n such that all coefficients of the local rule of F^n are divisible by $s = p_1^{n_1} \cdot p_2^{n_2} \cdots p_l^{n_l}$ which implies that F is nilpotent.

3.2 As a consequence of equicontinuity and surjectivity characterization given by Theorem 6, an ACA $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is in class 2 of Theorem 15 iff it is not surjective and it has exactly one equicontinuous direction iff $\gcd(s, \lambda_m, \ldots, \lambda_a) \neq 1$ and $\exists p_i, \exists \lambda_j, p_i \nmid \lambda_j$ and $\exists k \in [m, a], \forall p_i, p_i \mid \gcd(\lambda_m, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_a)$.

3.3 The automaton $(\mathbf{Z}_s^{\mathbb{Z}}, F)$ is in class 3 of Theorem 15 iff it is not surjective and it does not belong to class 1 and class 2 of Theorem 15. $\square$
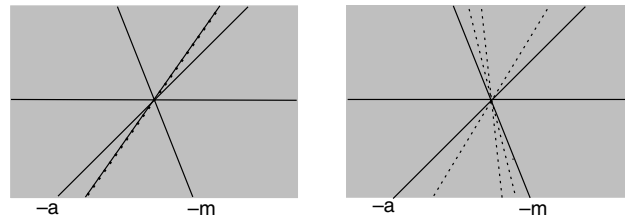
Fig. 1. Directional dynamics for injective ACA. Gray area depicts expansive directions. Trailed line is a direction of equicontinuity with two disjoint attractors. Dotted lines are directions presenting two disjoint attractors.
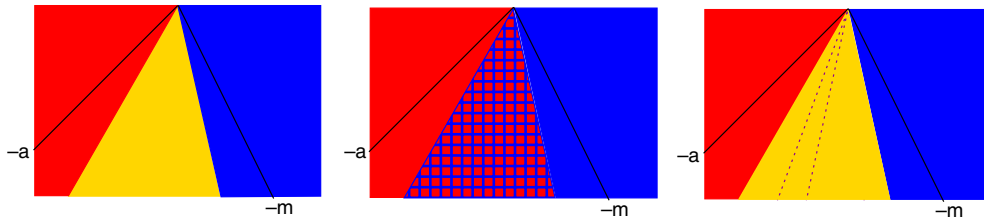

Fig. 2. Directional dynamics for surjective ACA. Light gray (resp., dark gray) areas depict left (resp., right) expansive directions. White area indicates directions presenting sensitivity. Very light gray areas show the positively expansive directions. Dotted lines are directions presenting two disjoint attractors.
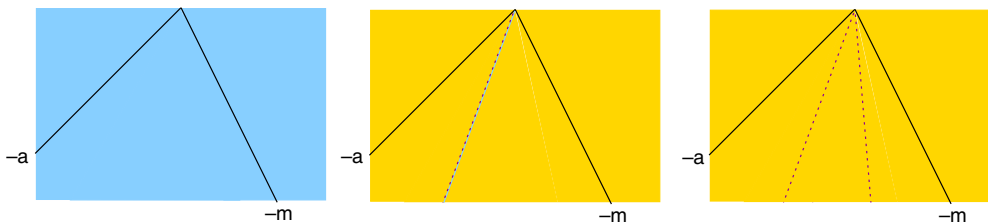
Fig. 3. In the leftmost (resp., central and rightmost) picture, grayed areas indicate equicontinuity (resp., sensitivity) directions. Dotted lines in the central (resp., rightmost) picture represent equicontinuity directions with two disjoint attractors (resp., two disjoint attractors).

8. Conclusions

In this paper we have completely characterized the directional dynamics of ACA, not only *w.r.t.* equicontinuity or expansivity (as in the Sablik's approach) but also *w.r.t.* attractors and factor languages. Figs. 1–3 summarize all the possible scenarios.

Looking at the pictures, one immediately sees that the algebraic nature of ACA has greatly reduced the number and the complexity of the possible dynamics. For example, we have proved that the factor languages of any ACA are regular along any direction. Of course, this is not true for the general case but it would be very interesting to investigate which is the largest class of CA with such a property.

The directional classification proposed by Sablik [36] sheds some light on how the information propagates in space-time diagrams of CA. For instance, there is no exchange of information between zones delimited by two directions of equicontinuity (almost equicontinuity) and the rest of phase space. In this paper, we showed that this is also the case for CA having directions with two disjoint attractors (see Fig. 2 right or Fig. 3). Remark that in the case of ACA, directions with two disjoint attractors are always tightly linked to the presence of equicontinuity (see Lemma 5). We wonder if this happens also in the general case where the situation is much more complicated, since one must take into account also almost equicontinuity and other types of attractors.

To conclude we remark that, by Theorem 16, our classification is completely decidable.

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