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On Nash Equilibrium Strategy of Two-person Zero-sum Games with Trapezoidal Fuzzy Payoffs

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Abstract In this paper, we investigate Nash equilibrium strategy of two-person zero-sum games with fuzzy payoffs. Based on fuzzy max order, Maeda and Cunlin constructed several models in symmetric triangular and asymmetric triangular fuzzy environment, respectively. We extended their models in trapezoidal fuzzy environment and proposed the existence of equilibrium strategies for these models. We also established the relation between Pareto Nash equilibrium strategy and parametric bi-matrix game. In addition, numerical examples are presented to find Pareto Nash equilibrium strategy and weak Pareto Nash equilibrium strategy from bi-matrix game.

Keywords Fuzzy numbers · Two-person zero-sum games · Fuzzy payoffs · Parametric bi-matrix game · Pareto Nash equilibrium strategy

1. Introduction

Game theory is a mathematical tool to describe strategic interactions among multiple decision makers who behave rationale. In 1944, Von Neumann and Morgenstern [1] introduced game theory in their pioneer work "Theory of Games and Economic

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Behavior". Since then many diverse kinds of mathematical games have been defined with their solution concept proposed. Games are broadly classified into two major categories: cooperative and non-cooperative games. In non-cooperative games, one important class of games are two-person zero-sum matrix games in which we assume that payoffs of outcomes are well defined and certain to both players. But in real-life games, such as economics, finance, business competition, voting, auctions, research and development races cartel behavior, e-commerce etc., the certainty assumption of payoffs is not realistic. Basically in real-life games, players are not able to estimate exactly payoffs of outcomes in the game due to lack of adequate information and/or imprecision of the available information on the environments. For example, different advertising strategies of two competing companies leading to different market shares must be estimated by using approximate values [2]. To overcome this kind of uncertainty and imprecision in available information, Zadeh [3] introduced the concept of fuzzy set theory.

Butnariu [4] introduced the concept of fuzzy sets in non-cooperative game theory for the first time to model each player's beliefs about action of the other players as fuzzy sets. Later, Billiot [5] extended work of [4]. Buckley [6] investigated a two party non-cooperative games involving both uncertainty and multiple goals. Campos [7] was the first to study non-cooperative matrix games with fuzzy payoffs. Using Yager's fuzzy numbers ranking index [8], the author in [7] transformed the problem of finding solution to fuzzy matrix game into a pair of fuzzy linear programming problems. In [9, 10], Li proposed a multi-objective programming approach to two-person zero-sum matrix games with triangular fuzzy payoffs. Bector et al. [11] studied matrix game with fuzzy goals. Utilizing fuzzy linear programming duality results, they transformed the finding solution to matrix game into a pair of fuzzy primal-dual linear programming problems. Vijay et al. [12] investigated two-person zero-sum matrix games with fuzzy payoffs and fuzzy goals and by using a suitable defuzzification function, they proved that such a game is equivalent to a primal-dual pair of certain fuzzy linear programming problems in which both goals as well as parameters are fuzzy. Sakawa and Nishizaki [13] studied two-person zero-sum multi-objective matrix games with fuzzy payoffs and fuzzy goals. In [14], Cevikel and Ahlatcioglu presented two models for studying two persons zero-sum games with fuzzy payoffs and fuzzy goals. In [15], Li developed a fast approach for computing fuzzy values of matrix games with payoffs of triangular fuzzy numbers.

The most commonly used solution concept in traditional game theory is that of Nash equilibrium, which has been introduced by Nash [16]. Maeda [17] extended Nash equilibrium solution concept for two-person zero-sum games with fuzzy payoffs. Using α-cut and fuzzy max order, introduced by Ramik [18], Maeda studied three kinds of equilibrium strategies, their existence conditions and relation with parametric bi-matrix games in the setting of symmetric triangular fuzzy number. Han et al. [19] developed Nash equilibrium solution concepts for bi-matrix game with fuzzy payoffs of symmetric triangular fuzzy numbers. Cunlin and Qiang [20] defined the max-order principles for asymmetric fuzzy numbers and extended the Nash equilibrium solution concepts for two person zero sum games with fuzzy payoffs of asymmetric triangular fuzzy numbers. They also introduced two special types of fuzzy games: constant fuzzy game and proportional fuzzy game.

In [17, 19, 20], the uncertainty and imprecision in payoffs have been represented by either symmetric or asymmetric fuzzy numbers. The triangular fuzzy number uses one parameter to represent the most possible value. This is very difficult for the decision makers to determine a single point for representing their most preferred value. Therefore this article models the fuzzy payoffs of decision makers by trapezoidal fuzzy numbers (TrFNs). TrFN permits the decision makers to represent the most possible value by an interval instead of a single point. Therefore, TrFNs can better reflect the uncertainty and ambiguous nature of subjective judgements of decision makers.

In this article, we generalize Maeda [17] and, Cunlin and Qiang [20] Nash equilibrium solution concepts based on symmetric and asymmetric triangular fuzzy numbers by modelling the fuzzy payoffs as TrFNs and investigate all the three equilibrium strategies, their existence conditions and characterization by bi-matrix game with parameters. To do this, the rest of paper is organized as follows. In Section 2, we present some basic definitions and notations on fuzzy sets, such as α-cut, ordering in $\mathbb{R}^n$, and establish fuzzy max order relation using α-cut as well as fuzzy max order relation for trapezoidal fuzzy numbers. In Section 3, we introduce the notion of two-person zero-sum matrix games with trapezoidal fuzzy payoffs, the different types of equilibrium strategies and investigate their existence conditions for the fuzzy games. Then, we establish the relation between equilibrium strategies of fuzzy game and Nash equilibrium strategies of crisp parametric bi-matrix games. In Section 4, we give examples to illustrate the relation between parametric bi-matrix games and Nash equilibrium. Section 5 contains the conclusion.

2. Preliminaries

In this section, we give some basic concepts and results of fuzzy numbers and fuzzy arithmetic operations which are used throughout the paper.

A fuzzy set is defined as a subset $\tilde{a}$ of universal set $X \subseteq \mathbb{R}$ by its membership function $\mu_{\tilde{a}}(\cdot)$ which assigns to each element $x \in \mathbb{R}$, a real number $\mu_{\tilde{a}}(x)$ in the interval $[0, 1]$.

Definition 2.1 [21] *A fuzzy subset $\tilde{a}$ defined on $\mathbb{R}$, is said to be a fuzzy number if its membership function $\mu_{\tilde{a}}(x)$ satisfies the following conditions*:

(i) $\mu_{\tilde{a}}(x) : \mathbb{R} \to [0, 1]$ *is upper semi-continuous*;

(ii) $\mu_{\tilde{a}}(x) = 0$ *outside some interval* $[a, d]$;

(iii) *There exist real numbers b, c such that $a \le b \le c \le d$ and*

  (a) $\mu_{\tilde{a}}(x)$ *is monotonic increasing on* $[a, b]$;
  (b) $\mu_{\tilde{a}}(x)$ *is monotonic decreasing on* $[c, d]$;
  (c) $\mu_{\tilde{a}}(x) = 1$ *in* $[b, c]$.

The α-cut of a fuzzy number $\tilde{a}$ plays an important role in parametric ordering of fuzzy numbers. The α-cut or α-level set of a fuzzy number $\tilde{a}$, denoted by $\tilde{a}_\alpha$, is defined as $\tilde{a}_\alpha = \{x \in \mathbb{R} \mid \mu_{\tilde{a}}(x) \ge \alpha\}$ for all $\alpha \in (0, 1]$. The support or 0-cut $\tilde{a}_0$ is defined as the closure of the set $\tilde{a}_0 = \{x \in \mathbb{R} \mid \mu_{\tilde{a}}(x) > 0\}$. Every α-cut is a closed interval $\tilde{a}_\alpha = [a^L_\alpha, a^U_\alpha] \subset \mathbb{R}$, where $a^L_\alpha = \inf\{x \in \mathbb{R} \mid \mu_{\tilde{a}}(x) \geq \alpha\}$ and $a^U_\alpha = \sup\{x \in \mathbb{R} \mid \mu_{\tilde{a}}(x) \geq \alpha\}$ for any $\alpha \in [0,1]$.

Definition 2.2 *A fuzzy number $\tilde{a} = (a, b, h, k)$ is said to be trapezoidal if its membership function is defined as*

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x-(a-h)}{h}, & \text{for} \quad a-h \leq x \leq a, \\ 1, & \text{for} \quad a \leq x \leq b, \\ \dfrac{(b+k)-x}{k}, & \text{for} \quad b \leq x \leq b+k, \\ 0, & \text{otherwise}, \end{cases}$$

where $[a,b]$ is the core of $\tilde{a}$ and $h > 0$, $k > 0$ are its left and right fuzziness of $\tilde{a}$, respectively. The membership function of the trapezoidal fuzzy number $\tilde{a} = (a, b, h, k)$ is depicted in Fig. 1.

We denote the set of all trapezoidal fuzzy number on $\mathbb{R}$ by $F(\mathbb{R})$. Let $\tilde{a} = (a_1, b_1, h_1, k_1)$ and $\tilde{b} = (a_2, b_2, h_2, k_2)$ be two trapezoidal fuzzy numbers. Then arithmetic operations on $\tilde{a}$ and $\tilde{b}$ are defined as follows:

Addition: $\tilde{a} + \tilde{b} = (a_1 + a_2, b_1 + b_2, h_1 + h_2, k_1 + k_2)$.

Subtraction: $\tilde{a} - \tilde{b} = (a_1 - b_2, b_1 - a_2, h_1 + k_2, h_2 + k_1)$.

Scalar multiplication:

$$x > 0, x \in \mathbb{R}; \quad x\tilde{a} = (xa_1, xb_1, xh_1, xk_1),$$
$$x < 0, x \in \mathbb{R}; \quad x\tilde{a} = (xb_1, xa_1, xk_1, xh_1).$$

Definition 2.3 [22] *Let $x, y \in \mathbb{R}^n$. Then comparison between x and y will be done in accordance with the following understanding*:

(i) *$x \geqq y$ if and only if $x_i \geqq y_i$, for all $i = 1, 2, \cdots, n$,*

(ii) *$x \geq y$ if and only if $x \geqq y$, and $x \neq y$,*

(iii) *$x > y$ if and only if $x_i > y_i$, for all $i = 1, 2, \cdots, n$.*

Definition 2.4 [22] *Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Then treating $\tilde{a}_\alpha = (a^L_\alpha, a^U_\alpha)$ and $\tilde{b}_\alpha = (b^L_\alpha, b^U_\alpha)$ as a vector in $\mathbb{R}^2$ and following the Definition* 2.3, *we define the fuzzy max order '$\gtrapprox$', the strict fuzzy max order '$\succeq$', strong fuzzy max order '$\succ$' as follows*:

(i) *$\tilde{a} \gtrapprox \tilde{b}$ iff $(a^L_\alpha, a^U_\alpha)^T \geqq (b^L_\alpha, b^U_\alpha)^T$, for all $\alpha \in [0,1]$;*

(ii) *$\tilde{a} \succeq \tilde{b}$ iff $(a^L_\alpha, a^U_\alpha)^T \geq (b^L_\alpha, b^U_\alpha)^T$, for all $\alpha \in [0,1]$;*

(iii) *$\tilde{a} \succ \tilde{b}$ iff $(a^L_\alpha, a^U_\alpha)^T > (b^L_\alpha, b^U_\alpha)^T$, for all $\alpha \in [0,1]$.*

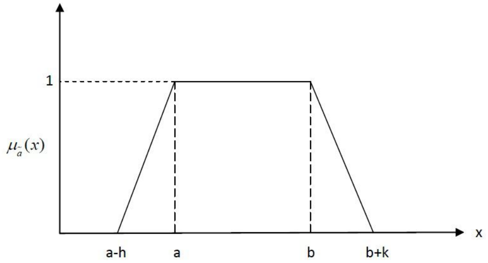

Fig. 1 Trapezoidal fuzzy number $\tilde{a} = (a, b, h, k)$

The following theorem gives a characterization of above orders for the case of trapezoidal fuzzy numbers.

Theorem 2.1 *Let $\tilde{a} = (a_1, b_1, h_1, k_1)$ and $\tilde{b} = (a_2, b_2, h_2, k_2)$ be two trapezoidal fuzzy numbers. Then*

(i) $\tilde{a} \lessapprox \tilde{b}$ iff $\max\{h_2 - h_1, 0\} \leq a_2 - a_1$ and $\max\{k_1 - k_2, 0\} \leq b_2 - b_1$;
(ii) $\tilde{a} \prec \tilde{b}$ iff $\max\{h_2 - h_1, 0\} < a_2 - a_1$ and $\max\{k_1 - k_2, 0\} < b_2 - b_1$.

Proof Let us assume that $\tilde{a} \lessapprox \tilde{b}$. Then, by using Definition 2.4, we have

$$[a^L_\alpha, a^U_\alpha]^T \ \leqq \ [b^L_\alpha, b^U_\alpha]^T \text{ for all } \alpha \in [0, 1].$$

This implies

$$a^L_\alpha \ \leq \ b^L_\alpha \text{ and } a^U_\alpha \ \leq \ b^U_\alpha \text{ for all } \alpha \in [0, 1].$$

Putting the values of α-cut of $\tilde{a}$ and $\tilde{b}$ in above inequalities and rearranging the terms, we get

$$(1 - \alpha)(h_2 - h_1) \leq a_2 - a_1 \text{ for all } \alpha \in [0, 1],$$

and

$$(1 - \alpha)(k_1 - k_2) \leq b_2 - b_1 \text{for all } \alpha \in [0, 1].$$

The result now follows directly.

Similarly, the proof for the case $\tilde{a} \prec \tilde{b}$ can be obtained in an analogous way. Hence the proof ends.

3. Two-person Zero-sum Game with Fuzzy Payoffs and Its Nash Equilibrium Strategy

A matrix game involves two players, a set of strategies for each player, and a payoff that qualitatively describes the outcome of each play of the game in terms of the amount that each player gains or loses. A pure strategy for each player is a plan, determined at the start of the game that describes what a player will do in every possible situation. Let $S^m = \{\eta_1, \eta_2, \cdots, \eta_m\}$ and $S^n = \{\delta_1, \delta_2, \cdots, \delta_n\}$ be sets of pure strategies of player I and player II, respectively. A matrix game is said to be two-person zero-sum game if player I's gain is players II's loss. Let $A = (a_{ij})_{m\times n}$ be the payoff matrix whose entries a_{ij} denote the payoff that player I gains and player II loses, when player I chooses the pure strategy η_i and player II chooses the pure strategy δ_j. A mixed strategy $x = (x_1, x_2, \cdots, x_m)^T$ for player I is a probability distribution on the set S^m of his/her pure strategies η_i $(i = 1, 2, \cdots, m)$, where x^T is the transposition of x. The set of mixed strategies of player I is represented by

$$X = \{(x_1, x_2, \cdots, x_m) \in R^m \mid \sum_{i=1}^{m} x_i = 1, x_i \geq 0 \text{ for all } i = 1, 2, \cdots, m\}.$$

Similarly, the set of mixed strategy for player II is represented by

$$Y = \{(y_1, y_2, \cdots, y_n) \in R^n \mid \sum_{j=1}^{n} y_j = 1, y_j \geq 0 \text{ for all } j = 1, 2, \cdots, n\}.$$

If player I chooses any mixed strategy $x \in X$ and player II chooses any mixed strategy $y \in Y$, then the expected payoff of player I is given by $x^T Ay = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} y_j$. For more details interested readers can see [23, 24].

Let $\tilde{a}_{ij} = (a_{ij}, b_{ij}, h_{ij}, k_{ij}) \in F(\mathbb{R})$ and $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$ be the fuzzy counterparts of a_{ij} and $A = (a_{ij})_{m\times n}$, respectively. Now, we call this game as two-person zero-sum game with fuzzy payoff or fuzzy matrix game and denote it by $\tilde{G} = (X, Y, \tilde{A})$. We denote $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$, $H = (h_{ij})_{m\times n}$, $K = (k_{ij})_{m\times n}$. Expected fuzzy payoff of player I is $x^T\tilde{A}y = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i\tilde{a}_{ij}y_j = (x^T Ay, x^T By, x^T Hy, x^T Ky)$, which is also a trapezoidal fuzzy number.

Definition 3.1 [20] *A pair of mixed strategies $(\bar{x}, \bar{y}) \in X \times Y$ is called a Nash equilibrium strategy of the game $\tilde{G}$ if*

(i) $x^T\tilde{A}\bar{y} \lessapprox \bar{x}^T\tilde{A}\bar{y}$ *for all $x \in X$, and*
(ii) $\bar{x}^T\tilde{A}\bar{y} \lessapprox \bar{x}^T\tilde{A}y$ *for all $y \in Y$.*

Definition 3.2 [20] *A pair of mixed strategies $(\bar{x}, \bar{y}) \in X \times Y$ is called a Pareto Nash equilibrium strategy of the game $\tilde{G}$ if*

(i) *there does not exist any $x \in X$ such that $\bar{x}^T\tilde{A}\bar{y} \preccurlyeq x^T\tilde{A}\bar{y}$, and*
(ii) *there does not exist any $y \in Y$ such that $\bar{x}^T\tilde{A}y \preccurlyeq \bar{x}^T\tilde{A}\bar{y}$.*

Definition 3.3 [20] *A pair of mixed strategies $(\bar{x}, \bar{y}) \in X \times Y$ is called a weak Pareto Nash equilibrium strategy of the game $\tilde{G}$ if*

(i) *there does not exist any $x \in X$ such that $\bar{x}^T\tilde{A}\bar{y} \prec x^T\tilde{A}\bar{y}$, and*

(ii) *there does not exist any* $y \in Y$ *such that* $\bar{x}^T \tilde{A} y \prec \bar{x}^T \tilde{A} \bar{y}$.

Remark 1 From above definitions, it is obvious that following relationship holds among these definitions:

(i) If $(\bar{x}, \bar{y}) \in X \times Y$ is a Nash equilibrium strategy of the fuzzy game $\tilde{G}$, then it will be a Pareto Nash equilibrium strategy.

(ii) If $(\bar{x}, \bar{y}) \in X \times Y$ is a Pareto Nash equilibrium strategy of the fuzzy game $\tilde{G}$, then it will be a weak Pareto Nash equilibrium strategy.

This means

Nash equilibrium strategy ⇒ Pareto Nash equilibrium strategy
⇒ weak Pareto Nash equilibrium strategy.

Remark 2 If payoff matrix $\tilde{A}$ is crisp, the fuzzy game $\tilde{G}$ reduces to usual two-person zero-sum matrix game $G = (X, Y, A)$, then all the above three definitions of Nash equilibrium coincide and become the definition of crisp min-max equilibrium strategy. In the following theorem, we will find the existence condition of Nash equilibrium strategy in trapezoidal fuzzy matrix game.

Theorem 3.1 *A pair of mixed strategies* $(\bar{x}, \bar{y}) \in X \times Y$ *is a Nash equilibrium strategy of the fuzzy two-person zero-sum matrix game* $\tilde{G} = (X, Y, \tilde{A})$ *iff*

(i) $x^T A\bar{y} \le \bar{x}^T A\bar{y} \le \bar{x}^T Ay,$ (1)

(ii) $x^T B\bar{y} \le \bar{x}^T B\bar{y} \le \bar{x}^T By,$ (2)

(iii) $x^T (A - H)\bar{y} \le \bar{x}^T (A - H)\bar{y} \le \bar{x}^T (A - H)y,$ (3)

(iv) $x^T (B + K)\bar{y} \le \bar{x}^T (B + K)\bar{y} \le \bar{x}^T (B + K)y,$ (4)

holds for all $x \in X,\ y \in Y$.

Proof Let $(\bar{x}, \bar{y}) \in X \times Y$ be a Nash equilibrium strategy of the game $\tilde{G}$. Since $(x^T Ay, x^T By, x^T Hy, x^T Ky)$, therefore from Definition 3.1 (i) and Theorem 2.1, we have

$$\max\{\bar{x}^T H\bar{y} - x^T H\bar{y}, 0\} \le \bar{x}^T A\bar{y} - x^T A\bar{y}, \tag{5}$$

and

$$\max\{\bar{x}^T K\bar{y} - x^T K\bar{y}, 0\} \le \bar{x}^T B\bar{y} - x^T B\bar{y}. \tag{6}$$

By expanding and rearranging (5) and (6), we get

$$x^T (A - H)\bar{y} \le \bar{x}^T (A - H)\bar{y}, \quad x^T A\bar{y} \le \bar{x}^T A\bar{y}, \tag{7}$$

and

$$x^T (B + K)\bar{y} \le \bar{x}^T (B + K)\bar{y}, \quad x^T B\bar{y} \le \bar{x}^T B\bar{y}. \tag{8}$$

Similarly from Definition 3.1 (ii), we have

$$\bar{x}^T(A-H)\bar{y} \le \bar{x}^T(A-H)y\,, \quad \bar{x}^T A\bar{y} \le \bar{x}^T Ay, \tag{9}$$

and

$$\bar{x}^T(B+K)\bar{y} \le \bar{x}^T(B+K)y, \quad \bar{x}^T B\bar{y} \le \bar{x}^T By. \tag{10}$$

Finally, arranging the inequalities (7)-(10), we have the required inequalities (1)-(4).

Similarly, the converse part of the theorem can be obtained. Hence the result comes out.

For the rest of paper, we adopt the following notations: $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$, $H = (h_{ij})_{m\times n}$, $K = (k_{ij})_{m\times n}$, $A_0^L = A - H$ and $B_0^U = B + K$.

In the view of Theorem 3.1, if we wish to solve the fuzzy two-person zero-sum game $\tilde{G}$, then we have to consider four crisp two-person zero-sum games, $G_a = (X, Y, A), G_b = (X, Y, B), G_h = (X, Y, A-H)$ and $G_k = (X, Y, B+K)$ and attempt to determine a point $(\bar{x}, \bar{y}) \in X \times Y$ which is simultaneously a saddle point of G_a, G_b, G_h and G_k. By setting $x^T Gy = (x^T G_a y,\, x^T G_b y,\, x^T G_h y,\, x^T G_k y) \in R^4$ for all $x \in X$, $y \in Y$, from (1)-(4), we have

$$x^T G\bar{y} \;\leqq\; \bar{x}^T G\bar{y} \;\leqq\; \bar{x}^T Gy.$$

In Theorem 3.1, we have established the equivalent conditions of Nash equilibrium strategy in two persons zero-sum games with trapezoidal fuzzy payoffs. But in many cases, it has been observed that there does not exist Nash equilibrium solutions to a fuzzy matrix game. However, in the following cases, we can guarantee that there exists at least one Nash equilibrium point.

Definition 3.4 *A two-person zero-sum fuzzy game $\tilde{G} = (X, Y, \tilde{A})$ is said to be a proportional fuzzy game iff there exists $\gamma_1, \gamma_2 \in (0, 1]$ such that*

$$h_{ij} = \gamma_1 a_{ij} \text{ and } k_{ij} = \gamma_2 b_{ij} \text{ for all } i = 1, 2, 3, \cdots, m, \;\; j = 1, 2, 3, \cdots, n.$$

Theorem 3.2 *A pair of mixed strategies $(\bar{x}, \bar{y}) \in X \times Y$ is a Nash equilibrium strategy of the proportional fuzzy matrix game $\tilde{G}$ iff $(\bar{x}, \bar{y}) \in X \times Y$ is the Nash equilibrium of crisp two-person zero-sum games $G_a = (X, Y, A)$ and $G_b = (X, Y, B)$.*

Proof Let $(\bar{x}, \bar{y}) \in X \times Y$ be the Nash equilibrium strategy of the proportional fuzzy two-person zero-sum game $\tilde{G}$. From Theorem 3.1, we have

$$x^T A\bar{y} \;\le\; \bar{x}^T A\bar{y} \;\le\; \bar{x}^T Ay, \tag{11}$$

$$x^T B\bar{y} \;\le\; \bar{x}^T B\bar{y} \;\le\; \bar{x}^T By, \tag{12}$$

$$x^T(A-H)\bar{y} \le \bar{x}^T(A-H)\bar{y} \le \bar{x}^T(A-H)y, \tag{13}$$

$$x^T(B+K)\bar{y} \le \bar{x}^T(B+K)\bar{y} \le \bar{x}^T(B+K)y. \tag{14}$$

Since $\tilde{G}$ is a proportional fuzzy matrix game, it follows that

$$h_{ij} = \gamma_1 a_{ij} \text{ and } k_{ij} = \gamma_2 b_{ij} \text{ for all } i = 1, 2, \cdots, m, j = 1, 2, \cdots, n.$$

This implies that

$$H = \gamma_1 A \text{ and } K = \gamma_2 B.$$

Putting the values of H and K in (13) and (14), we obtain (11) and (12) again. Therefore, $(\bar{x}, \bar{y}) \in X \times Y$ is the Nash equilibrium strategy of the crisp two-person zero-sum games G_a and G_b.

Definition 3.5 *A two-person zero-sum fuzzy game $\tilde{G} = (X, Y, \tilde{A})$ is said to be a constant fuzzy game iff there exist $h, k > 0$ such that*
$h_{ij} = h$ *and* $k_{ij} = k$ *for all* $i = 1, 2, 3, \cdots, m,\ j = 1, 2, 3, \cdots, n.$

Theorem 3.3 *A pair of mixed strategies $(\bar{x}, \bar{y}) \in X \times Y$ is called the Nash equilibrium strategy of the constant fuzzy matrix game $\tilde{G}$ iff $(\bar{x}, \bar{y}) \in X \times Y$ is the Nash equilibrium strategy of crisp two-person zero-sum games $G_a = (X, Y, A)$ and $G_b = (X, Y, B)$.*

Proof From Definition 3.5, H and K are constant matrices with all the entries h and k, respectively. Hence, we have

$$x^T H y = h \text{ and } x^T K y = k \text{ for all } x \in X,\ y \in Y.$$

Now, the proof directly follows from Theorem 3.1.

In Theorem 3.4, we have imposed one more condition on proportional fuzzy game $\tilde{G}$ to show that Nash equilibrium of this special kind of fuzzy game is equivalent to a Nash equilibrium of a crisp two-person zero-sum game.

Theorem 3.4 *Let $\tilde{G}$ be proportional fuzzy game with payoff matrix $\tilde{A} = (\tilde{a}_{ij})$, where $\tilde{a}_{ij} = (a_{ij}, b_{ij}, h_{ij}, k_{ij}) \in F(\mathbb{R})$. Suppose that $b_{ij} = \gamma_3 a_{ij},\ \forall\ i, j$ with $\gamma_3 \geq 1$. Then a pair of mixed strategies $(\bar{x}, \bar{y}) \in X \times Y$ is called the Nash equilibrium strategy of the proportional fuzzy matrix game $\tilde{G}$ iff $(\bar{x}, \bar{y}) \in X \times Y$ is the Nash equilibrium strategy of crisp two-person zero-sum game $G_a = (X, Y, A)$.*

Proof It follows directly from Theorem 3.2.

Remark 3 It may be remarked here that the results obtained in the paper, generalized the previous known results in [17, 20] as

(i) taking $a_{ij} = b_{ij}\ \forall\ i$ and j, i.e., trapezoidal fuzzy payoffs become non-symmetric triangular fuzzy numbers and therefore, we obtained the results of Cunlin and Qiang [20] fuzzy game model.

(ii) taking $a_{ij} = b_{ij}$ and $h_{ij} = k_{ij}\ \forall\ i$ and j, i.e., trapezoidal fuzzy payoffs become symmetric triangular fuzzy payoffs and therefore we obtain the results in Maeda's fuzzy game model [17].

In the next theorem, we will derive the conditions for existence of Pareto Nash equilibrium strategies in fuzzy game $\tilde{G}$.

Theorem 3.5 *A pair of mixed strategies $(\bar{x}, \bar{y}) \in X \times Y$ is said to be Pareto Nash equilibrium strategy of the game $\tilde{G} = (X, Y, \tilde{A})$ iff*

(i) *there exists no $x \in X$ such that*

$$(\bar{x}^T A_0^L \bar{y}, \bar{x}^T B_0^U \bar{y}) \leq (x^T A_0^L \bar{y}, x^T B_0^U \bar{y}),\quad \bar{x}^T A \bar{y} \leq x^T A y \text{ and } \bar{x}^T B \bar{y} \leq x^T B y, \tag{15}$$

(ii) *there exists no* $y \in Y$ *such that*

$$(\bar{x}^T A_0^L y, \bar{x}^T B_0^U y) \leq (\bar{x}^T A_0^L \bar{y}, \bar{x}^T B_0^U \bar{y}), \quad \bar{x}^T A y \leq \bar{x}^T A \bar{y} \ \ and \ \ \bar{x}^T B y \leq \bar{x}^T B \bar{y}. \qquad (16)$$

Proof We will proceed by contradiction. Suppose $(\bar{x}, \bar{y}) \in X \times Y$ be the Pareto Nash equilibrium strategy of $\tilde{G}$. Assume that there exists a strategy $\hat{x} \in X$ such that (15) holds, i.e.,

$$(\bar{x}^T A_0^L \bar{y}, \bar{x}^T B_0^U \bar{y}) \ \leq \ (\hat{x}^T A_0^L \bar{y}, \hat{x}^T B_0^U \bar{y}), \quad \bar{x}^T A \bar{y} \ \leq \hat{x}^T A y \ \ and \ \ \bar{x}^T B \bar{y} \ \leq \ \hat{x}^T B y.$$

It follows that

$$\bar{x}^T A_0^L \bar{y} \ \leq \hat{x}^T A_0^L \bar{y}, \ \ \bar{x}^T B_0^R \bar{y} \ \leq \hat{x}^T B_0^R \bar{y}.$$

Now, by Definition 2.3, at least one of the above inequality should be strict. Therefore, for $\alpha \in [0, 1]$ from the above inequalities, we obtain

$$\begin{aligned}&(\bar{x}^T (\alpha A + (1-\alpha) A_0^L)\bar{y}, \ \bar{x}^T (\alpha B + (1-\alpha) B_0^U)\bar{y})\\ &\quad \leq (\hat{x}^T (\alpha A + (1-\alpha) A_0^L)\bar{y}, \ \hat{x}^T (\alpha B + (1-\alpha) B_0^U)\bar{y}).\end{aligned}$$

Putting $A_0^L = A - H, \ B_0^U = B + K$ in above inequality, we get

$$(\bar{x}^T (A-(1-\alpha)H)\bar{y}, \ \bar{x}^T (B+(1-\alpha)K)\bar{y}) \ \leq \ (\hat{x}^T (A-(1-\alpha)H)\bar{y}, \ \hat{x}^T (B+(1-\alpha)K)\bar{y}),$$

which implies that

$$\bar{x}^T \tilde{A} \bar{y} \ \preccurlyeq \ \hat{x}^T \tilde{A} \bar{y}.$$

This contradicts the fact that $(\bar{x}, \bar{y}) \in X \times Y$ is the Pareto Nash equilibrium strategy.

Conversely, let the pair of mixed strategy $(\bar{x}, \bar{y}) \in X \times Y$ be satisfy (15) and (16). Now, on the contrary, suppose that there exists a strategy $\hat{x}^T \in X$ such that

$$\bar{x}^T \tilde{A} \bar{y} \ \preccurlyeq \ \hat{x}^T \tilde{A} \bar{y}.$$

Then by Definition 2.4, we have

$$(\bar{x}^T A_\alpha^L \bar{y}, \bar{x}^T B_\alpha^U \bar{y}) \ \leq \ (\hat{x}^T A_\alpha^L \bar{y}, \hat{x}^T B_\alpha^U \bar{y}) \ \text{ for all } \ \alpha \in [0, 1], \qquad (17)$$

where, $A_\alpha^L = A - (1-\alpha)H$ and $B_\alpha^U = B + (1-\alpha)K$.

Set $\alpha = 0$ and $\alpha = 1$ in (13), we get

$$(\bar{x}^T A_0^L \bar{y}, \bar{x}^T B_0^U \bar{y}) \ \leq \ (\hat{x}^T A_0^L \bar{y}, \hat{x}^T B_0^U \bar{y}), \qquad (18)$$

and

$$(\bar{x}^T A \bar{y}, \ \bar{x}^T B \bar{y}) \ \leq \ (\hat{x}^T A \bar{y}, \ \hat{x}^T B \bar{y}). \qquad (19)$$

The above inequalities (18) and (19) contradict (15). Similarly, one can show that there does not exist any $y \in Y$ such that

$$\bar{x}^T \tilde{A} y \ \preccurlyeq \ \bar{x}^T \tilde{A} \bar{y}.$$

This completes the proof of the theorem.

In the following, we will establish the relationship of fuzzy game $\tilde{G}$ with crisp parametric bi-matrix game. Before doing this, we give the brief description of bi-matrix game as follows: Let $S^m = \{\eta_1, \eta_2, \cdots, \eta_m\}$ and $S^n = \{\delta_1, \delta_2, \cdots, \delta_n\}$ be sets of pure strategy for player I and player II, respectively. Let c_{ij}, d_{ij} be the payoffs that player I and player II receive when player I plays with pure strategy η_i and player II plays with pure strategy δ_j. Suppose $C = (c_{ij})_{m\times n}$ and $D = (d_{ij})_{m\times n}$ are the payoffs matrices of player I and player II, respectively. Then we call this game a bi-matrix game and denote it by $G = (X, Y, C, D)$.

We introduce the notion of parametric bi-matrix game as follows: Suppose $\lambda,\ \mu \in [0, 1]$ and let $(1-\lambda)(a_{ij} - h_{ij}) + \lambda(a_{ij} + r_{ij})$ be the gains of players I and $(1-\mu)(a_{ij} - h_{ij}) + \mu(a_{ij} + r_{ij})$ be the losses of players II when players I employing pure strategy i and player II employing pure strategy j. Let $A(\lambda) = (1-\lambda)(A - H) + \lambda(B + K)$ and $-A(\mu) = -[(1-\mu)(A - H) + \mu(B + K)]$ be the payoff matrices of player I and II, respectively. Now onwards, we consider the parametric bi-matrix game $G(\lambda, \mu) = (X, Y, A(\lambda), -A(\mu))$, where λ and μ are parameters.

Definition 3.6 [16] *A pair of mixed strategies $(\bar{x}, \bar{y}) \in X \times Y$ is said to be Nash equilibrium strategy of the parametric bi-matrix game $G(\lambda, \mu)$ if*

(i) $x^T A(\lambda)\bar{y} \ \leq\ \bar{x}^T A(\lambda)\bar{y},\ \forall\ x \in X,$

(ii) $\bar{x}^T A(\mu)\bar{y} \ \leq\ \bar{x}^T A(\mu) y,\ \forall\ y\ \in Y.$

Lemma 3.1 [16] *For each $\lambda, \mu \in [0, 1]$, bi-matrix game $G(\lambda, \mu)$ has at least one Nash equilibrium strategy.*

In the following theorems, we will establish the relationship between Nash equilibrium strategies of parametric bi-matrix game $G(\lambda, \mu)$ and fuzzy matrix game $\tilde{G}$.

Theorem 3.6 *Let the pair of mixed strategies $(\bar{x}, \bar{y}) \in X \times Y$ be Nash equilibrium strategy of the parametric bi-matrix game $G(\lambda, \mu)$ with $\lambda,\ \mu \in (0, 1)$. Then $(\bar{x}, \bar{y}) \in X \times Y$ is the Pareto Nash equilibrium strategy of the fuzzy two-person zero-sum game $\tilde{G}$.*

Proof Let $(\bar{x}, \bar{y}) \in X \times Y$ be the Nash equilibrium strategy of the parametric bi-matrix game $G(\lambda, \mu)$, where $\lambda,\ \mu \in (0, 1)$. From Definition 3.6, we have

$$(1-\lambda)x^T A_0^L\bar{y} + \lambda x^T B_0^U\bar{y} \ \leq\ (1-\lambda)\bar{x}^T A_0^L\bar{y} + \lambda\bar{x}^T B_0^U\bar{y}\ \forall\ x \in X, \tag{20}$$

$$(1-\mu)x^T A_0^L\bar{y} + \mu x^T B_0^U\bar{y} \ \leq\ (1-\mu)\bar{x}^T A_0^L\bar{y} + \mu\bar{x}^T B_0^U\bar{y}\ \forall\ x \in X. \tag{21}$$

In order to show that $(\bar{x}, \bar{y}) \in X \times Y$ is a Pareto Nash equilibrium strategy of $\tilde{G}$, we have to prove that

$$\bar{x}^T\tilde{A}\bar{y} \preccurlyeq x^T\tilde{A}\bar{y}.$$

On the contrary, suppose that there exists $\hat{x} \in X$ such that $\bar{x}^T\tilde{A}\bar{y} \preccurlyeq x^T\tilde{A}\bar{y}$ holds. By Definition 2.4, we get

$$(\bar{x}^T A_\alpha^L\bar{y},\ \bar{x}^T B_\alpha^U\bar{y}) \ \leq\ (\hat{x}^T A_\alpha^L\bar{y},\ \hat{x}^T B_\alpha^U\bar{y}) \text{ for all } \alpha \in [0, 1].$$

Setting $\alpha = 0$ in above inequality, we obtain

$$(\bar{x}^T A_0^L\bar{y},\ \bar{x}^T B_0^U\bar{y}) \ \leq\ (\hat{x}^T A_0^L\bar{y},\ \hat{x}^T B_0^U\bar{y}).$$

Since both the inequalities do not hold simultaneously, there exists $\lambda \in (0, 1)$ such that

$$(1-\lambda)x^T A_0^L \bar{y} + \lambda x^T B_0^U \bar{y} \leq (1-\lambda)\hat{x}^T A_0^L \hat{y} + \lambda \bar{x}^T B_0^U \bar{y} \; \forall \; x \in X,$$

which contradicts inequality (21).

Similarly, it can be proved that there does not exist any $y \in Y$ such that $\bar{x}^T \tilde{A} y \preccurlyeq \bar{x}^T \tilde{A} \bar{y}$.

Theorem 3.7 *Let the pair of mixed strategies $(\bar{x}, \bar{y}) \in X \times Y$ be Nash equilibrium strategy of the parametric bi-matrix game $G(\lambda, \mu)$ with $\lambda,\ \mu \in [0,1]$. Then $(\bar{x}, \bar{y}) \in X \times Y$ is the weak Pareto Nash equilibrium strategy of the fuzzy two-person zero sum game $\tilde{G}$.*

Proof It follows on the lines of Theorem 3.6.

The following result is the immediate consequence of Theorem 3.5, Theorem 3.6 and Lemma 3.1.

Theorem 3.8 *A fuzzy two-person zero-sum game $\tilde{G}$ satisfies the following properties:*

(i) *There exists at least one Pareto Nash equilibrium strategy of the fuzzy game $\tilde{G}$;*
(ii) *There exists at least one weak Pareto Nash equilibrium strategy of the fuzzy game $\tilde{G}$.*

4. Numerical Examples

Example 1 Consider the fuzzy two-person zero-sum game $\tilde{G}$ with trapezoidal fuzzy payoff matrix $\tilde{A}$ given by

$$\tilde{A} = \begin{pmatrix} (90, 100, 10, 15) & (75, 80, 5, 10) \\ (80, 90, 10, 20) & (170, 180, 20, 30) \end{pmatrix}.$$

We want to find the Nash equilibrium strategy, pareto Nash equilibrium strategy and weak pareto Nash equilibrium strategy of the game $\tilde{G}$.

Let $x^T = (p,\ 1-p)$ and $y^T = (q,\ 1-q)$ be the mixed strategy of player I and II, respectively. In order to check the existence of Nash equilibrium strategies, we have to check the conditions in Eqs. (1)-(4). It can be done by finding the Nash equilibrium strategies of the four crisp matrix games whose payoffs matrices are A, B, $A-H$ and $B+K$, respectively. Since the four matrix games have different Nash equilibrium strategies, so there exists no $(x, y) \in X \times Y$ such that Eqs. (1)-(4) hold simultaneously, i.e., Nash equilibrium does not exist for the given fuzzy game $\tilde{G}$. Now, we find the Pareto Nash equilibrium strategy of the fuzzy game $\tilde{G}$. From Theorem 3.6, we know that the Nash equilibrium strategies of the parametric bi-matrix game is also the Pareto Nash equilibrium strategies of fuzzy game $\tilde{G}$. So for finding Pareto Nash equilibrium strategies, it is sufficient to find the Nash equilibrium strategies of parametric bi-matrix game $\tilde{G}$. For this purpose, we construct the bi-matrix game $G(\lambda, \mu)$ from fuzzy matrix game $\tilde{G}$ as described before the introduction of parametric bi-matrix game in Section 3:

$$A(\lambda) = \begin{pmatrix} 80 + 35\lambda & 70 + 20\lambda \\ 70 + 40\lambda & 150 + 60\lambda \end{pmatrix}, \qquad A(\mu) = \begin{pmatrix} 80 + 35\mu & 70 + 20\mu \\ 70 + 40\mu & 150 + 60\mu \end{pmatrix},$$

where $\lambda, \mu \in [0,1]$. To check how to find the existence of Nash equilibrium strategies of bi-matrix and how to compute the Nash equilibrium strategies, we refer to [25]. Nash equilibrium strategy of the parametric bi-matrix game $G(\lambda,\mu)$ is given by [25],

$$(1,0)A(\lambda)y \le x^TA(\lambda)y, \qquad (0,1)A(\lambda)y \le x^TA(\lambda)y,$$
$$x^TA(\mu)y \le x^TA(\mu)(0,1)^T, \qquad x^TA(\mu)y \le x^TA(\mu)(1,0)^T.$$

From the above inequalities, we obtain

$$(1-p)q(90+35\lambda)-(1-p)(80+40\lambda)\le 0, \quad pq(90+35\lambda)-p(80+40\lambda)\ge 0,$$
$$p(1-q)(90+35\mu)-(1-q)(80+20\mu)\le 0, \quad pq(90+35\mu)-q(80+20\mu)\ge 0.$$

Nash equilibrium strategy of $G(\lambda,\mu)$ is as follows:

$$\bar{x}^T = (\bar{x}_1,\bar{x}_2) = \left(\frac{80+20\mu}{90+35\mu}, \frac{10+15\mu}{90+35\mu}\right), \ \bar{y}^T = (\bar{y}_1,\bar{y}_2) = \left(\frac{80+40\lambda}{90+35\lambda}, \frac{10-5\lambda}{90+35\lambda}\right)$$

and value of the expected payoffs are given by

$$\begin{aligned}\bar{v}(\lambda,\mu) &= \bar{x}^TA(\lambda)\bar{y}\\ &= \frac{639000+526500\lambda+117000\lambda^2+248500\mu+204750\lambda\mu+45500\lambda^2\mu}{(90+35\lambda)(90+35\mu)},\end{aligned}$$

$$\begin{aligned}\bar{w}(\lambda,\mu) &= \bar{x}^TA(\mu)\bar{y}\\ &= \frac{639000+526500\mu+117000\mu^2+248500\lambda+204750\lambda\mu+45500\lambda\mu^2}{(90+35\lambda)(90+35\mu)}.\end{aligned}$$

From Theorem 3.7 and 3.8, the Pareto Nash equilibrium strategy (PNS) and weak Pareto Nash equilibrium strategies (WPNS) are given by

$$PNS = \left\{(\bar{x}^T,\bar{y}^T) = \left(\left(\frac{80+20\mu}{90+35\mu}, \frac{10+15\mu}{90+35\mu}\right), \left(\frac{80+40\lambda}{90+35\lambda}, \frac{10-5\lambda}{90+35\lambda}\right)\right) \mid \lambda,\ \mu \in [0,1]\right\},$$

$$WPNS = \left\{(\bar{x}^T,\bar{y}^T) = \left(\left(\frac{80+20\mu}{90+35\mu}, \frac{10+15\mu}{90+35\mu}\right), \left(\frac{80+40\lambda}{90+35\lambda}, \frac{10-5\lambda}{90+35\lambda}\right)\right) \mid \lambda,\ \mu \in (0,1)\right\}.$$

In the following example, we will find the Nash equilibrium strategy for the proportional matrix game.

Example 2 Consider the fuzzy two-person zero-sum game $\tilde{G}$ with trapezoidal fuzzy payoff matrix $\tilde{A}$ given by

$$\tilde{A} = \begin{pmatrix} (50,60,10,20) & (80,96,16,32) \\ (100,120,20,40) & (20,24,4,8) \end{pmatrix}.$$

By Definition 3.4, we note that $\gamma_1 = 0.5$ and $\gamma_2 = 0.3$. Therefore, the fuzzy game $\tilde{A}$ is a proportional fuzzy matrix game. From Theorem 3.2, it is evident that Nash equilibrium strategy of game $\tilde{A}$ can be obtained by solving a bi-matrix game whose payoff matrices are

$$A = \begin{pmatrix} 50 & 80 \\ 100 & 20 \end{pmatrix}, \qquad B = \begin{pmatrix} 60 & 96 \\ 120 & 24 \end{pmatrix}.$$

Following the same procedure as in Example 1, the Nash equilibrium strategy is

$$(\bar{x}^T, \bar{y}^T) = ((\frac{8}{11}, \frac{3}{11}), (\frac{6}{11}, \frac{5}{11}))$$

and expected fuzzy value of the game

$$\tilde{v} = (\frac{8}{11}, \frac{3}{11})\tilde{A}(\frac{6}{11}, \frac{5}{11}) = (\frac{7700}{121}, \frac{9240}{121}, \frac{1540}{121}, \frac{3080}{121}).$$

The next example describes the finding the Nash equilibrium strategy of constant matrix game.

Example 3 Let us consider the following fuzzy two-person zero-sum game $\tilde{G}$ with trapezoidal fuzzy payoff matrix $\tilde{A}$:

$$\tilde{A} = \begin{pmatrix} (30, 40, 10, 20) & (100, 110, 10, 20) \\ (120, 130, 10, 20) & (70, 80, 10, 20) \end{pmatrix}.$$

From Definition 3.5, we note here that $h = 10$ and $k = 20$ for fuzzy matrix game $\tilde{G}$. Therefore, $\tilde{G}$ is a constant fuzzy matrix game whose unique Nash equilibrium strategy exists and is given by

$$(\bar{x}^T, \bar{y}^T) = ((5/12, 7/12), (1/4, 3/4)),$$

and expected fuzzy value of the game

$$\tilde{v} = (5/12, 7/12)\tilde{A}(1/4, 3/4) = (330/4, 370/4, 10, 20).$$

5. Conclusion

In this paper, we focus on two-person zero-sum games with fuzzy payoffs. Our study extend Maeda's [17] and Cunlin and Qiang's [20] fuzzy matrix game models with symmetric and asymmetric triangular fuzzy number. We generalize the existence conditions for all three types of Nash equilibrium strategies from symmetric and asymmetric fuzzy numbers to trapezoidal fuzzy numbers. We define two new kinds of fuzzy matrix games where finding the Nash equilibrium strategies of fuzzy matrix game is equivalent to finding the Nash equilibrium strategies of a crisp bi-matrix game. By constructing the parametric bi-matrix game from fuzzy matrix game, we present a method to find Pareto Nash equilibrium strategies. Some numerical illustrations are also given to find Pareto and weak Pareto Nash equilibrium strategies. One can easily verify that all the generalizations of Nash equilibrium strategies are natural extension of crisp matrix games, and our approach for representing payoffs by trapezoidal fuzzy numbers provides experts more flexibility in expressing their uncertainty and ambiguity in subjective judgments in more reliable way than Maeda's and Cunlin and Qiang's models and therefore our model is more general one than Meada's and Cunlin and Qiang's models. In further research it would be interesting to analyze the Nash equilibrium condition of two-person zero-sum game with different types of uncertain payoffs, such as generalized fuzzy numbers, intuitionistic fuzzy numbers, hesitant fuzzy sets, etc.

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