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Relative elimination of quantifiers for Henselian valued fields

Serban A. Basarab
Institute of Mathematics, Bdul Pacii. 220, 79622 Bucharest, Romania

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Abstract

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A general result on relative elimination of quantifiers for Henselian valued fields of characteristic zero is proved by algebraic and basic model-theoretic methods.

0. Introduction

In a well-known series of papers, Ax and Kochen [2–4] and Ershov [17–20] initiated a metamathematical approach to some basic problems in the theory of Henselian valued fields. These papers were followed by other works [1, 19, 21, 23, 26, 27], which continued the investigation of Henselian fields using methods of model theory, recursive function theory and nonstandard arithmetic. Further refinements of these results were obtained in [5–9, 15, 32, 35]. An account of model-theoretic and algorithmic results in the elementary theory of valued fields, in an approach that uses explicit, primitive recursive quantifier elimination procedures as a unifying principle, is given in [34].

For p-adic fields $\mathbb{Q}_p$ and the power series fields $F((t))$ over a decidable field F of characteristic zero, Ax and Kochen [4] proved by model-theoretic methods decidability and relative quantifier elmination, when a cross-section is included in the language of valued fields. A quantifier elimination with cross-section and a decision method for $\mathbb{Q}_p$ were also given by Cohen [14] using primitive recursive methods. Later Macintyre [23] showed that quantifier elimination for $\mathbb{Q}_p$ can be obtained without cross-section, when more natural root-predicates are included in the language. In a more algebraic approach, this result was extended by Prestel and Roquette [24] to p-adically closed fields. Extensions of Macintyre's result are also contained in [13, 15]. Some general results concerning the transfer of model completion for Henselian fields with finite absolute ramification index were obtained in [7,35]. Elementary invariants for Henselian valued fields of mixed characteristic and arbitrary ramification are investigated by van den Dries [31].

On the other hand, Cohen's ideas were strongly generalized in [32,34] in order to get primitive recursive relative quantifier elimination procedures for Henselian fields of characteristic zero subject to a condition of moderate generality on the value group.

In the last time the quantifier elimination problem for valued fields gained popularity thanks to its interest for computer scientists under the aspect of feasability as well as for deep applications in diophantine questions. Concerning the diophantine applications, let us mention here Cantor and Roquette's [11,28] and Rumely's work, on Hilbert's tenth problem for the ring of algebraic integers, Denef's paper [16] on the rationality of certain Poincaré series over $\mathbb{Q}_p$ and Weispfenning's result [33] on the primitive recursive decidability of the adele ring and idele group of an algebraic number field.

The present paper, which appeared as an INCREST-Bucharest preprint in February 1986, is devoted to the proof by algebraic and basic model-theoretic methods of a general result on relative quantifier elimination for Henselian fields of characteristic zero.

Given a valued field $\boldsymbol{K} = (K, v)$ let us denote by $O_{\boldsymbol{K}}$ the valuation ring, by $\bar{\boldsymbol{K}}$ the residue field and by vK the value group. Assume that the characteristic of $\boldsymbol{K}$ is zero and let p be the characteristic exponent of $\bar{\boldsymbol{K}}$. For $k \in \mathbb{N}$, let $\mathfrak{m}_{\boldsymbol{K},k}$ be the ideal $\{a \in O_{\boldsymbol{K}}: vu > kvp\}$ of $O_{\boldsymbol{K}}$. In particular, $\mathfrak{m}_{\boldsymbol{K},0} = \mathfrak{m}_{\boldsymbol{K}}$ is the maximal ideal of the valuation ring $O_{\boldsymbol{K}}$. Denote by $O_{\boldsymbol{K},k}$ the factor ring $O_{\boldsymbol{K}}/\mathfrak{m}_{\boldsymbol{K},k}$; $O_{\boldsymbol{K},k}$ is a local ring with maximal ideal $\mathfrak{m}_{\boldsymbol{K}}/\mathfrak{m}_{\boldsymbol{K},k}$. In particular, for $p = 1$, $O_{\boldsymbol{K},k} = \bar{\boldsymbol{K}}$ for all $k \in \mathbb{N}$.

On the other hand, consider the multiplicative groups $G_{\boldsymbol{K},k} = K^x/1 + \mathfrak{m}_{\boldsymbol{K},k}$ for $k \in N$. If $p = 1$, then $G_{\boldsymbol{K},k} = G_{\boldsymbol{K},0} = G_{\boldsymbol{K}} = K^x/1 + \mathfrak{m}_{\boldsymbol{K}}$ for all $k \in N$. Given $k \in \mathbb{N}$, the ring $O_{\boldsymbol{K},2k}$ and the group $G_{\boldsymbol{K},k}$ are related through a natural map θ_k defined on the subset

$$\{a \in O_{\boldsymbol{K},2k}: a \mid p^k\} = O_{\boldsymbol{K},2k} \backslash (\mathfrak{m}_{\boldsymbol{K},k}/\mathfrak{m}_{\boldsymbol{K},2k}) \quad \text{of } O_{\boldsymbol{K},2k}$$

with values in $G_{\boldsymbol{K},k}$: $\theta_k(a + \mathfrak{m}_{\boldsymbol{K},2k}) = a(1 + \mathfrak{m}_{\boldsymbol{K},k})$ for $a \in O_{\boldsymbol{K}}$ subject to $vu \leqslant kvp$.

For $k \in \mathbb{N}$, the valuation v induces a map $v_k: O_{\boldsymbol{K},2k} \,\dot{\cup}\, G_{\boldsymbol{K},k} \to vK \cup \{\infty\}$; the image $v_k(O_{\boldsymbol{K},2k}\backslash\{0\})$ is the convex subset $vK_{2k} = \{\alpha \in vK: 0 \leqslant \alpha \leqslant 2kvp\}$ of the ordered group vK and the restriction $v_k|_{G_{\boldsymbol{K},k}}: G_{\boldsymbol{K},k} \to vK$ is a group epimorphism. Among other properties, the map θ_k as defined above satisfies the following valuation-theoretic condition of compatibility: the diagram

$$\begin{tikzcd}
O_{\boldsymbol{K},2k}\backslash\{0\} \arrow[r, "v_k"] & vK_{2k} \hookrightarrow vK \\
O_{\boldsymbol{K},2k}\backslash(m_{\boldsymbol{K},k}/m_{\boldsymbol{K},2k}) \arrow[u, hook] \arrow[r, "\theta_k"] & G_{\boldsymbol{K},k} \arrow[u, "v_k"']
\end{tikzcd}$$

commutes.

For $k \in \mathbb{N}$, consider the system $\boldsymbol{K}_k = (O_{\boldsymbol{K},2k}, G_{\boldsymbol{K},k}, vK, \theta_k, v_k)$ with $O_{\boldsymbol{K},2k}, G_{\boldsymbol{K},k}$, vK, θ_k, v_k as above and call it the ***mixed k-structure*** assigned to the valued field

K. In particular, for $p = 1$, $\boldsymbol{K}_k = \boldsymbol{K}_0$ is the triple $(\bar{K}, G_{\boldsymbol{K}}, vK)$ together with the exact sequence

$$1 \to \bar{K}^{\times} \to G_{\boldsymbol{K}} \to vK \to 0.$$

The mixed k-structures introduced above play a key role in the model theory of Henselian fields of characteristic zero as shows the following elementary equivalence theorem.

Theorem A. ***Given the valued field extensions LIK and FIK, where K is of characteristic zero and L, F are Henselian, the necessary and sufficient condition for the valued fields L, F to be elementarily equivalent over K is that for all $k \in \mathbb{N}$, the corresponding mixed k-structures $\boldsymbol{L}_k$, $\boldsymbol{F}_k$ are elementarily equivalent over $\boldsymbol{K}_k$.***

Given a sentence φ in the language $\mathfrak{L}_k$[1] of mixed k-structures, $k \in \mathbb{N}$, one may assign effectively to φ a formula $\mathrm{tr}_k(\varphi)(z)$, with one variable z, in the language $\mathfrak{L}^1$ of valued fields in such a way that for every valued field $\boldsymbol{K}$ of characteristic zero and residue characteristic exponent p, $\boldsymbol{K}$ satisfies $\mathrm{tr}_k(\varphi)(p)$, written $\boldsymbol{K} \models \mathrm{tr}_k(\varphi)(p)$, iff $\boldsymbol{K}_k$ satisfies φ, written $\boldsymbol{K}_k \models \varphi$. The correspondence above $\varphi \mapsto \mathrm{tr}_k(\varphi)$ extends naturally to a translation map tr_k from the arbitrary formulas in $\mathfrak{L}_k$ to formulas in $\mathfrak{L}$.

As a consequence of Theorem A we get the second main result of the paper concerning the relative elimination of quantifiers for Henselian fields of characteristic zero.

Theorem B. ***Let p be either 1 or a prime number, and denote by $\mathfrak{T}_p$ the theory of Henselian valued fields of characteristic zero and residue characteristic exponent p. For every formula $\varphi(\boldsymbol{x}), \boldsymbol{x} = (x_1, \ldots, x_n)$ in $\mathfrak{L}$ there exist $k \in \mathbb{N}$, formulas $\psi_1(\boldsymbol{y}), \ldots, \psi_l(\boldsymbol{y})$ in $\mathfrak{L}_k$, $\boldsymbol{y} = (y_1, \ldots, y_m)$, quantifierless formulas A,(x), …, $\lambda_l(\boldsymbol{x})$ in $\mathfrak{L}$ and polynomials $f_i, g_i \in \mathbb{Z}[\boldsymbol{x}]$, $1 \leq i \leq m$, such that $\varphi(\boldsymbol{x})$ is equivalent in $\mathfrak{T}_p$ with the following formula:***

$$\bigvee_{j=1}^{l} [\lambda_j(\boldsymbol{x}) \wedge \mathrm{tr}_k(\psi_j)(f_i(\boldsymbol{x})g_i(\boldsymbol{x})^{-1}: 1 \leq i \leq m; p)].$$

One may derive Weispfenning's main theorem [34, Theorem 4.3] (with recursive instead of primitive-recursive) from Theorem B above; details will be contained in a forthcoming paper. In the last section of the present work we shall show only that the Prestel-Roquette theorem [24] on quantifier elimination for p-adically closed fields is a consequence of Theorem B.

1. The radical structure theorem

Consider a valued field $\boldsymbol{K} = (K, v)$ of characteristic zero and residue characteristic exponent p. We define the ***canonical decomposition*** of the valuation v as

[1] **The languages $\mathfrak{L}_k$ and $\mathfrak{L}$ are natural. For details see Sections 3, 4.**

follows. Denote by $A = \Delta_K$ the smallest convex subgroup of vK containing vp. $A = 0$ iff $p = 1$, i.e., $\bar{K}$ is of characteristic zero. Let $\dot{v}K$ be the factor group vK/Δ, and $\dot{v}: K^\times \to \dot{v}K : a \mapsto \dot{v}a$ be the group epimorphism induced by v $: K^\times \to vK$. Since A is convex in $vK, \dot{v}K$ inherits from vK the structure of a totally ordered group and hence the map $\dot{v}$ is a valuation of the field K, called the ***coarse valuation*** assigned to v. Denote by $\dot{K}$ the valued field $(K, \dot{v})$. The valuation ring $O_{\dot{K}}$ of $\dot{K}$ is characterized as the smallest overring of O_K in which p becomes a unit, i.e., $O_{\dot{K}}$ is the ring of fractions of O_K with respect to the multiplicatively closed set $\{p^k: k \in \mathbb{N}\}$. Note that $\dot{v}$ = v iff $p = 1$, and $\dot{v}$ is trivial iff $vK = A$.

Let $\mathfrak{m}_{\dot{K}}$ be the maximal ideal of $O_{\dot{K}}$; then $\mathfrak{m}_{\dot{K}} \subset \mathfrak{m}_K \subset O_{\dot{K}} \subset O_{\dot{K}}$. Denote by K° the residue field $O_{\dot{K}}/\mathfrak{m}_{\dot{K}}$ of the valued field $\dot{K}$. For $a \in O_{\dot{K}}$ let a° be its residue in K°. The field K°, called the *core field* of the valuation v of K, carries naturally a valuation whose valuation ring is the image $O_K/\mathfrak{m}_{\dot{K}}$ of O_K. Denote also by v this valuation and by K° the core ***valued field (K”, v).*** The value group vK° is identified with the convex subgroup A of vK and the residue field $\bar{K}^\circ$ is identified with the residue field $\bar{K}$ of K. Thus the core valued field K° is of characteristic zero and residue characteristic exponent p; $K^\circ = \bar{K}$ iff $p = 1$.

Consider a valued field extension $L = (L, v)$ of K. Then the coarse valuation of L is a prolongation of the coarse valuation of K; hence both may be denoted by the same symbol $\dot{v}$. The core valued field L° is an extension of the core valued field K°.

We say that the extension L/K is ***core-dense*** if for every $\alpha \in A = \Delta_K$ and for every $b \in L$ there exists $a \in K$ such that $v(b - a) > a$; in particular, $\Delta_L = A$,. This is equivalent with the fact that for every $k \in \mathbb{N}$, the ring embedding $O_{K,k} = 0_{K^\circ,k} \to O_{L^\circ,k} = O_{L,k}$ is an isomorphism. In other words, L/K is core-dense iff the core extension L°/K° is dense. If $p = 1$, then L/K is core-dense iff $\bar{L} = \bar{K}$.

A main ingredient in the proof of Theorem A is the following natural generalization of the Prestel-Roquette radical structure theorem (24, Theorem 3.81.

Proposition 1.1. ***Let K = (K, v) be a Henselian valued field of characteristic zero and L be an algebraic core-dense extension of K. Then L/K is generated by radicals, i.e., L = K(T) where $T = T_{L/K} = \{t \in L”: \bigvee_{n \geq 1} t^n \in K\}$ is the multiplicative group of radical elements of LIK. The radical value group vT equals the full value group vL of L and the valuation map v $: T \to vL$ induces a group isomorphism T/K” $\simeq vL/vK$. If L/K is a finite extension, then [L: K] = (T : K”).***

Proof. As K is Henselian and L/K is algebraic, L is Henselian too and vL/vK is a torsion group. First of all let us show that $L^\circ = K^\circ$ and $vL = vT$. Let $a \in O_{L^\circ}$. We have to show that $a \in O_{K^\circ}$. As L°/K° is algebraic and K° is of characteistic zero, there exists $f \in O_{K^\circ}[X]$ such that $f(a) = 0$ and $f'(a) \neq 0$. Let $\alpha \in vK^\circ = vL^\circ$ be such that $\alpha \geq 2vf'(a)$. Since L°/K° is dense there exists c $\in O_{K^\circ}$ such that $v(a - c) > a$. Thus $vf(c) > \alpha \geq 2vf'(a) = 2vf'(c)$ and hence by Newton’s lemma

[24, p. 20] there is one and only one $b \in O_{K^\circ}$ such that $f(b) = 0$ and $v(b - c) > vf'(c) = vf'(a)$. (Note that K° is Henselian since K is Henselian). As f(a) = 0 and $v(a-c) > \alpha \geq 2vf'(a) \geq vf'(a)$, we get a = $b \in O_{K^\circ}$, as contended. Now let a εL”. *We* have to show that vu = vt for some $t \in T$. As vL/vK is a torsion group, $a^n = bu$ with $b \in K$, $n \geq 1$, u $\in O_L^x$. As $L^\circ = K^\circ$, there exists $u' \in O_K^x$ such that $\dot{v}(u -$ u') > 0. Consider the polynomial f(X) = $X^n - uu'^{-1} \in O_{\dot{L}}[X]$. Since $\dot{v}f(1) > 0 = \dot{v}n = \dot{v}f'(1)$ and $\dot{L}$ is Henselian (as L is Henselian), there exists $t' \in L^x$ such that $f(t') = 0$, i.e., $t'^n = a^n(bu')^{-1}$. Let $t = at'^{-1}$; then $t^n = bu' \in K^x$, i.e., $t \in T$, and $vt = vu$, as contended.

Consider the intermediate field $L' = K(T)$ between K and L. We have to show that $L' = L$. The value group vL' of $L' = (L', v)$ contains $vT = vL$ and hence $vL' = vL$; in particular, $\dot{v}L' = \dot{v}L$. On the other hand, $L'^\circ = L^\circ$ since K° = *L”.* Thus the valued field extension $\dot{L}/\dot{L}'$ is immediate. As an algebraic extension of the Henselian valued field $\dot{K}, \dot{L}'$ is Henselian and hence algebraically complete being of residue characteristic zero [1, Proposition 15]. Since $\dot{L}/\dot{L}'$ is algebraic immediate, we conclude that $L' = L$.

As $vT = vL$, the valuation v induces a group epimorphism v $: T/K^x \to vL/vK$. We have to show that this is an isomorphism. Let $t \in T$ be such that $vt \in vK$. Assume that the order of t modulo K^x is n and $t^n = a \in K$. As $vt \in vK$ by assumption, $t = bu$ with $b \in K$, $u \in O_L^x$. Since $K^\circ = L^\circ$, there exists $u' \in O_K^x$ such that $\dot{v}(u -$ u') > 0; therefore $t = (bu')(uu'^{-1}) \in K^x(1 + \mathfrak{m}_{\dot{L}})$. To show that $t \in K$ *we* may replace t by any other element in its coset modulo K^x, *so we* may assume without loss of generality that $\dot{v}(1 - t) > 0$ and hence $\dot{v}(1 - t^n) = \dot{v}(1 -$ a) > 0. Consider the polynomial f(X) = $X^n -$ a $\in O_{\dot{K}}[X]$. As $\dot{K}$ is Henselian and $\dot{v}f(1) = \dot{v}(1 - a) > 0 = \dot{v}(n) = \dot{v}f'(1)$, there exists one and only one c $\in K$ such that f(c)_= $c^n -$ a = 0 and $\dot{v}(1 -$ c) > 0. Since this uniqueness statement holds not only in K, but also in the Henselian field L and since $f(t) = 0$, $\dot{v}(1 - t) > 0$, *we* conclude that $t = c \in K$.

We have shown that *T/K”* $\simeq vL/vK$. If one of these groups is finite, then the other is finite too and *(T : K”)* = $(vL : vK)$. On the other hand, if *[L : K]* is finite then $[L:K] = (\dot{v}L : \dot{v}K) = (vL : vK)$, since $\dot{K}$ is algebraically complete and *[L”: K”]* = 1. We conclude that *[L: K] = (T: K”).* □

The following explicit description of the field structure of L/K in terms of radicals is an immediate consequence of Proposition 1.1; see the proof of [24, Corollary 3.91.

Corollary 1.2. In ***the same situation us in Proposition*** 1.1, *assume* ***that*** L/K ***is finite. Then Ll K can be generated by finitely many radicals such that the product of their radical exponents equals the field degree:***

$$L = K(t_1, \ldots, t_r),$$
$$t_i^{n_i} = a_i \in K^x, \quad 1 \leq i \leq r,$$
$$[L:K] = n_1 n_2 \cdots n_r.$$

The substitution $X_i \mapsto t_i$, $1 \le i \le r$, extends to a K-isomorphism of the factor algebra $K[X_1, \ldots, X_r]/I$, where Z is the ideal generated by the polynomials $X_i^{n_i} - a_i$, $1 \le i \le r$, onto the field L.

2. An embedding theorem for Henselian fields

The key role played by the embedding theorems in the investigation of the model-theoretic properties of the Henselian valued fields is well known (see, for instance, [21],[24, Theorem 4.11, [7, Theorem 1.21, [9, Proposition 2.21). In this section we prove a general embedding theorem for Henselian valued fields of characteristic zero that will be the main tool for the proof of Theorem A. A basic ingredient in the proof of this embedding theorem is Proposition 1.1.

Given a valued field $\boldsymbol{K} = (K, v)$ of characteristic zero, let us denote by $\dot{\boldsymbol{K}}_\circ$ the mixed structure assigned to the coarse valued field $\dot{\boldsymbol{K}} = (K, \dot{v})$, namely the system (K", $G_{\dot{K}} = K^x/1 + \mathfrak{m}_{\dot{K}}, \dot{v}K)$ together with the exact sequence

$$1 \to K^{\circ x} \to G_{\dot{K}} \xrightarrow{\dot{v}} \dot{v}K \to 0.$$

In fact we shall consider the core field K° not only as an abstract field but also as a valued field $\boldsymbol{K}^\circ = (K", v)$ with the valuation naturally induced by the valuation v of K. Thus it seems natural to consider systems $\mathfrak{S} = (M, H, \dot{\Gamma}, i, \dot{\Pi})$ where $\boldsymbol{M} = (M, v)$ is a valued field of characteristic zero whose value group vM equals the smallest convex subgroup containing vp, p = the characteristic exponent of the residue field $\bar{M}$, H is a multiplicative Abelian group, $\dot{\Gamma}$ is an additive totally ordered group, $i: M^x \to H$ is a group monomorphism and $\dot{\Pi}: H \to \dot{\Gamma}$ is a group epimorphism such that the sequence

$$1 \to M^x \xrightarrow{i} H \xrightarrow{\Pi} \dot{\Gamma} \to 0$$

is exact. Call such a system a ***minced w-structure.*** Thus the system $\dot{\boldsymbol{K}}_\circ$ above is the mixed w-structure naturally associated to K. If the residue characteristic of $\boldsymbol{K}$ is zero, then $\dot{\boldsymbol{K}}_\circ = \boldsymbol{K}_\circ$ is the mixed O-structure

$$(\bar{K}, G_{\boldsymbol{K}} = K^x/1 + \mathfrak{m}_{\boldsymbol{K}}, vK; 1 \to \bar{K}^x \to G_{\boldsymbol{K}} \to vK \to 0)$$

assigned to K.

Given a mixed w-structure $\mathfrak{S} = (M, H, \dot{\Gamma}, i, \dot{\Pi})$, we obtain the canonic exact sequence

$$0 \to vM \cong M^x/O_M^x \to \Gamma \dashrightarrow \dot{\Gamma} \to 0,$$

where $\Gamma = H/i(O_M^x)$. Thus Γ inherits a natural structure of a totally ordered group with the order given by $\alpha \le \beta$ iff either $\dot{\alpha} < \dot{\beta}$ or $0 \le \beta - \alpha \in M$; with respect to this order, vM is identified with a convex subgroup of Γ and $\dot{\Gamma} \simeq \Gamma/vM$ as ordered group. Thus the mixed o-structure $\mathfrak{S}$ may be seen also as a system $\mathfrak{S}' =$

(M, H, Γ, i, Π) where M, H, i are as above, Γ is an additive totally ordered group and $\Pi: H \to \Gamma$ is a group epimorphism such that the kernel of $\Pi \circ i: M^x \to \Gamma$ is the group O_M^x of units of O_M and the value group $vM \simeq M^x/O_M^x$ is identified through $\Pi \circ i$ with a convex subgroup of Γ. In the present section we prefer to use mixed o-structures in the first acceptation reserving the second equivalent definition for the next sections of the work.

Given two mixed w-structures $\mathfrak{S}_i$, $i = 1, 2$, an embedding $\mu: \mathfrak{S}_1 \to \mathfrak{S}_2$ will be a system (μ', μ, μ'') where $\mu': M_1 \to M_2$ is a valued field embedding, $\mu: H_1 \to H_2$ is a group monomorphism and $\mu'': \dot{\Gamma}_1 \to \dot{\Gamma}_2$ is a monomorphism of ordered groups such that

$$\begin{array}{ccccc} M_1^X & \xrightarrow{i_1} & H_1 & \xrightarrow{\Pi_1} & \dot{\Gamma}_1 \\ {\scriptstyle \mu'}\downarrow & & {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu''} \\ M_2^x & \xrightarrow{i_2} & H_2 & \xrightarrow{\dot{\Pi}_2} & \dot{\Gamma}_2 \end{array}$$

is commutative.

With this preparation we are now able to state the general embedding theorem for Henselian valued fields of characteristic zero.

Theorem 2.1. *Let $K = (K, v)$ be a valued field of characteristic zero and $L = (L, v)$, $F = (F, v)$ be Henselian valued fields extending K. Assume that $\dot{F}$ is $|L|$-pseudocomplete, where $|L|$ denotes the cardinality of L. Given a &-embedding $\mu: \dot{L}_\circ \to \dot{F}_\circ$ of mixed w-structures, there exists a K-embedding $\eta: L \to F$ of valued fields inducing the given &-embedding μ; in other words, the canonic map* $\mathrm{Hom}_K(L, F) \to \mathrm{Hom}_{\dot{K}_\circ}(L_\circ, \dot{F}_\circ)$ *is onto.*

Proof. Consider the family A of pairs (L', η) where L' is an intermediate field between K and L, and $\eta: L' \to F$ is a K-embedding of valued fields inducing the restriction embedding $\mu|_{\dot{L}'_\circ}: \dot{L}'_\circ \to \dot{F}_\circ$ of mixed w-structures. A is nonempty since the pair $(K, K \hookrightarrow F)$ belongs to A. Consider the partial order on A:

$$(L', \eta') \leqslant (L'', \eta'') \quad \text{iff} \quad L' \subset L'' \text{ and } \eta' = \eta''|_{L'}.$$

As the nonempty partial ordered set $(A, \leqslant)$ is inductive, there exists by Zorn's lemma a maximal pair $(L', \eta) \in A$. We have to show that $L' = L$.

Without loss of generality we may asssume that $L' = K$, i.e., A is the singleton $\{(K, K \hookrightarrow F)\}$; *so we* have to show that $K = L$. We proceed step by step as follows. Using well-known facts concerning Henselian valued fields and Corollary 1.2, we show in the first three steps of the proof that the field extensions L/K and L°/K° are regular.

(1) First let us show that $\dot{K} = (K, \dot{v})$ is Henselian. Let $\dot{K}' = (K', \dot{v})$ be the Henselization of $\dot{K}$. As L is Henselian, $\dot{L} = (L, \dot{v})$ is Henselian too and hence $\dot{K}'$ is identified with a subextension of $\dot{L}/\dot{K}$. Let $K' = (K', v)$ be the field K' with the valuation induced by the valuation v of L. As the residue field K'° of $\dot{K}'$ equals

K°, the valuation of K' is exactly the valuation on the field K' induced by the valuation $\dot{v}$ of $\dot{K}'$ and the valuation v of $\dot{K}^\circ$. Since F is Henselian, $\dot{F}$ is Henselian too and hence we get a canonic k-embedding $\eta:\dot{K}'\rightarrow\dot{F}$. In fact η is a K-embedding of K' into F. It remains to check that $\dot{K}'_\circ=\dot{K}_\circ$ in order to conclude that (K’, η)$\in$ A and hence K’ = K by maximality of K. The equalities $\dot{K}'^\circ=\dot{K}^\circ$, $\dot{v}K'=\dot{v}K$ are trivial, so it remain to verify that $G_{\dot{K}'}=G_{\dot{K}}$. Let $x\in K'$“. A s $\dot{v}K'=\dot{v}K, x=ay$ with $a\in K^x, y\in O^x_{\dot{K}'}$. Since $K'^\circ=K^\circ$, there is $b\in O^x_{\dot{K}}$ such that $\dot{v}(y-b)>0$. It follows: $x=(ab)(yb^{-1})\in K^x(1+\mathfrak{m}_{\dot{K}'})$. Therefore the canonic morphism $G_{\dot{K}}\rightarrow G_{\dot{K}'}$ is an isomorphism as contended.

Now since $\dot{K}$ is Henselian of residue characteristic zero, we may assume by [1, Proposition 16] that we have the following commutative diagram of valued fields:

$$\begin{tikzcd}
 & & F \\
F^\circ \arrow[urr] & K^\circ \arrow[l] \arrow[r] \arrow[d] & K \arrow[u] \arrow[d] \\
 & L^\circ \arrow[ul, "\mu'"] \arrow[r] & L
\end{tikzcd}$$

Note also that K, L° and respectively K, F° are linearly disjoint over K”.

(2) Let us show that K° is algebraically closed in L”. Let $x\in L^\circ$ be algebraic over K° and K' be the field $K(x)$ with the valuation v induced from L. By linear disjointness we get $[K':K]=[K''(x):K'']$ and hence $\dot{v}K'=\dot{v}K$ and $K'''=K''(x)$. The correspondence $x\mapsto\mu'(x)$ defines a field K-embedding $\eta:K'\rightarrow F$. Moreover, η is a K-embedding of valued fields. Indeed, let w be the valuation of K' induced through η by the valuation of F. As the residue field of K' with respect to the coarse valuation $\dot{w}$ equals K'°, the valuations w and v of K' induce the same valuation on K''' (μ' is a K°-embedding) and $\dot{w}$ equals the valuation $\dot{v}$ of K', the unique valuation of K' extending the Henselian valuation $\dot{v}$ of K, it follows that w equals the valuation v of K’, as contended. Thus it remains to show that the canonic &-embedding $\hat{\eta}:\dot{K}'_\circ\rightarrow\dot{F}_\circ$ of mixed w-structures induced by η coincides with the restriction of μ to $\dot{K}'_\circ$. As $\hat{\eta}$ coincides with μ' on K'° by definition of η and $\dot{v}K'=\dot{v}K$, it remains to show that the canonic monomorphism $\hat{\eta}:G_{\dot{K}'}\rightarrow G_{\dot{F}}$ induced by η equals the restriction of $\mu:G_{\dot{L}}\rightarrow G_{\dot{F}}$ to $G_{\dot{K}'}$. Let us show that $G_{\dot{K}'}=G_{\dot{K}}(K'^\circ)^x$. Indeed, let $z\in K'^x$. As $\dot{v}K'=\dot{v}K, z=au$ with $a\in K^x, u\in O^x_{\dot{K}'}$ and hence $z(1+\mathfrak{m}_{\dot{K}'})=[a(1+\mathfrak{m}_{\dot{K}})][u(1+\mathfrak{m}_{\dot{K}})]$; therefore $z(1+\mathfrak{m}_{\dot{K}'})\in$ G&K’“)“, as contended. Now the statement above is immediate since μ and $\hat{\eta}$ coincide on $G_{\dot{K}}$ and $(K'^\circ)^x$ by definition of η. Consequently, $(K', \eta)\in A$ and hence $K'=K$, since A = $\{(K, K\hookrightarrow F)\}$ by assumption. As x is arbitrary, we conclude that K° is algebraically closed in the Henselian field L”. In particular, K° is Henselian.

(3) Moreover, we claim that K is algebraically closed in L. First let us observe that K is Henselian since $\dot{K}$ is Henselian by (1) and K° is Henselian by (2). Let $K'\subset L$ be a finite extension of K. Note that $K'''=K^\circ$ by (2). Here is the point

where we use the radical field structure of K'/K given by Corollary 1.2: $K' = K(t_1, \ldots, t_r) \simeq K[X_1, \ldots, K_r]/I$, where I is the ideal generated by r polynomials $c_i X_i^{n_i} - 1$, $1 \leqslant i \leqslant r$, with $c_i \in K^\times$ such that $c_i t_i^{n_i} = 1$, $1 \leqslant i \leqslant r$. Let $T = K^\times t_1^{\mathbb{Z}} t_2^{\mathbb{Z}} \cdots t_r^{\mathbb{Z}}$ be the multiplicative group of radicals of K'/K. Consider the chain of isomorphisms

$$T/K^\times \simeq vK'/vK \simeq \dot{v}K/\dot{v}K \simeq \prod_{1 \leqslant i \leqslant r} \mathbb{Z}/n_i\mathbb{Z}.$$

Let us choose $y_1, \ldots, y_r \in F$ such that $y_i(1 + \mathfrak{m}_{\dot F}) = \mu(t_i(1 + \mathfrak{m}_{\dot L}))$, $1 \leqslant i \leqslant r$. We get $c_i y_i^{n_i} \in 1 + \mathfrak{m}_{\dot F}$ since $c_i t^{n_i} = 1$, $1 \leqslant i \leqslant r$. Consider the polynomials f,(X) $= X^{n_i} - c_i v_i^{n_i} \in O_{\dot F}[X]$, $1 \leqslant i \leqslant r$. As $\dot{v} f_i(1) > 0$, $\dot{v} f_i'(1) = \dot{v}(n_i) = 0$, we get by the Hensel lemma applied to $\dot{F}$ some elements $x_1, \ldots, x_r \in F$ uniquely determined by the conditions $x_i^{n_i} = c_i y_i^{n_i}$, $\dot{v}(x_i - 1) > 0, 1 \leqslant i \leqslant r$. Let us put $z_i = y_i x_i^{-1}, 1 \leqslant i \leqslant r$. The substitution $t_i \mapsto z_i$, $1 \leqslant i \leqslant r$, defines a field K-embedding $\eta : K' \to F$. Moreover, $\eta : K' \to F$ is a valued field K-embedding since K is Henselian and K'/K is algebraic. Let us show that the $\dot{K}_\circ$-embedding $\hat{\eta} : \dot{K}'_\circ \to \dot{F}_\circ$ of mixed u-structures induced by η equals the restriction of $\mu : \dot{L}_\circ \to \dot{F}_\circ$ to $\dot{K}'_\circ$. First note that $K'^\circ = K^\circ$ and $\dot{v}K'/\dot{v}K$ is finite since the extension K'/K is finite by assumption. Consequently, the embedding of totally ordered groups $\dot{v}K' \to \dot{v}F$ induced by η equals the restriction of $\mu'' : \dot{v}L \to \dot{v}F$ to $\dot{v}K'$, so it remains to show that the group monomorphism $G_{\dot K'} \to G_{\dot F}$ induced by η equals the restriction of $\mu : G_{\dot L} \to G_{\dot F}$ to $G_{\dot K'}$. By construction of η it suffices to verify the equality $G_{\dot K'} = G_{\dot K} \cdot \hat{t}_1^{\mathbb{Z}} \hat{t}_2^{\mathbb{Z}} \ldots \hat{t}_r^{\mathbb{Z}}$, where $\hat{t}_i$ is the coset of t_i modulo $1 + \mathfrak{m}_{\dot K'}$, $1 \leqslant i \leqslant r$. In order to do this we have to show that the canonic group morphism $T \to G_{\dot K'}$ is onto. Let z be an element of $K'^\times$. As $T/K” \simeq \dot{v}K'/\dot{v}K$, $z = tu$ with $t \in T$, $u \in O_{\dot K'}^\times$. Since $K'^\circ = K^\circ$, $u = u'a$ with $u' \in O_{\dot K}^\times$, $a \in 1 + \mathfrak{m}_{\dot K'}$. Thus $z = (tu')a \in$ T(l $+ \mathfrak{m}_{\dot K'})$, as contended.

We have shown that $(K', \eta) \in A$ and hence $K' = K$ since $A = \{(K, K \hookrightarrow F)\}$. K' being arbitrary we conclude that K is algebraically closed in L. The goal of the next two steps of the proof is to show that the regular valued field extension $\dot{L}/\dot{K}$ is immediate.

(4) Let us show that $K^\circ = L$”. Assuming the contrary, let $x \in L^\circ \setminus K$”. As K° is algebraically closed in L° by (2), x is transcendental over K”. Let K' be the rational function field $K(x)$ with the valuation induced by the valuation v of L. According to [10, Chapter 6, §10, Proposition 2], the restriction of $\dot{v}$ is K' is the unique valuation of K' extending the valuation $\dot{v}$ of K subject to $\dot{v}x = 0$ and $x^\circ = x$ transcendental over K”. Note that $\dot{v}K' = \dot{v}K$ and K’” $= K$”(x); therefore the correspondence $x \mapsto \mu'(x)$ defines a valued field K-embedding $\eta : K' \to F$. It remains to verify that the $\dot{K}_\circ$-embedding $\hat{\eta} : \dot{K}'_\circ \to \dot{F}_\circ$ induced by η equals the restriction of μ to $\dot{K}'_\circ$ in order to get $(K', \eta) \in A$, contrary to the fact that $\Lambda = \{(K, K \hookrightarrow F)\}$. As $\hat{\eta}$ and μ coincide on K'° by definition of η, and $\dot{v}K' = \dot{v}K$, it suffices to observe that $G_{\dot K'} = G_{\dot K}(K'^\circ)^\times$. Indeed, any element $z \in K$’” *can be* written in the form $z = au$ with $a \in K^\times, u \in O_{\dot K'}^\times$, since $\dot{v}K' = \dot{v}K$.

(5) Let us show that $vL = vK$. As $K^\circ = L^\circ$ by (4), it suffices to verify that $\dot vL = \dot vK$. Assuming the contrary, let $x \in L \setminus K$ be such that $\dot vx \notin \dot vK$. As K is algebraically closed in L by (3) and $K^\circ = L^\circ$, the factor group $\dot vL/\dot vK$ is torsion free. Indeed, let v EL" be such that $n\dot vy = \dot va$ with $a \in K^x, n \geqslant 1$, i.e., $\dot v(y^n a^{-1}) = 0$. Since $K^\circ = L^\circ, y^n a^{-1} = bu$ with $b \in O_{\dot K}^x$, $u \in 1 + \mathfrak{m}_{\dot L}$. As $\dot L$ is Henselian of residue characteristic zero, $u = z^n$ for some z $\in O_{\dot L}^x$; therefore $(yz^{-1})^n = ab \in K$. Since K is algebraically closed in L, $yz^{-1} \in K$ and $\dot vy = \dot v(yz^{-1}) \in \dot vK$, as contended.

Let K' be the rational function field $K(x)$ with the valuation induced by the valuation v of L. Since $\dot vK'/\dot vK \neq 0$ is torsion free as a subgroup of $\dot vL/\dot vK$, it follows by [10, Chapter 6, §10, Proposition 1], that

$$\dot vK' = \dot vK \oplus \mathbb{Z}\dot vx \quad \text{and} \quad \dot v\Big(\sum_{i=1}^{l} a_i x^i\Big) = \min_i (\dot va_i + i\dot vx)$$

for arbitrary $a_i \in K$, $1 \leqslant i \leqslant 1$. Let us choose an element y $\in F^x$ whose coset modulo $1 + \mathfrak{m}_{\dot F}$ is $\mu(x(1 + \mathfrak{m}_{\dot L}))$; in particular, $\dot vy = \mu''(\dot vx)$ is of infinite order modulo $\dot vK$ and hence y is transcendental over K. Thus the substitution $x \mapsto y$ defines a field K-embedding $\eta : K' \to F$. We claim that $\mu : \dot K' \to \dot F$ is a valuation field k-embedding such that the induced $\dot vK$-embedding $\dot vK' \to \dot vF$ equals restriction of μ'' to $\dot vK'$. Indeed, let f(x) = $\sum a_i x^i \in K$'". Then $\dot vf(x) = \min_i(\dot va_i + i\dot vx)$. On the other hand, [10, Chapter 6, §10, Proposition 1] may also be applied to $K(y)$ instead of $K' = K(x)$; we get $\dot vf(y) = \min_i(\dot va_i + i\dot vy)$, i.e., $\dot v(\eta(f(x)) = \mu''(\dot vf(x))$, as contended. Moreover, as $K^\circ = K'^\circ = L^\circ, \eta$ is in fact a K-embedding of the valued field K' into F. Thus it remains to show that the $\dot K_\circ$-embedding $\hat\eta : \dot K'_\circ \to \dot F_\circ$ of mixed w-structures induced by η equals the restriction of μ to $\dot K'_\circ$ in order to obtain $(K', \eta) \in A$ contrary to the assumption that A = $\{(K, K \hookrightarrow F)\}$. Since $K'^\circ = K^\circ$ and $\dot vK' = \dot vK \oplus \mathbb{Z}\dot vx$, we obtain $G_{\dot K'} = G'_{\dot K} \cdot \hat x^{\mathbb{Z}}$, where $\hat x$ is the coset of x modulo $1 + \mathfrak{m}_{\dot K'}$; therefore $\hat\eta$ and μ coincide on $G_{\dot K'}$ by definition of η. Note that till now we did not use the assumption that $\dot F$ is $|L|$-pseudocomplete.

(6) Finally we are ready to prove that $K = L$. Assuming the contrary, let $x \in L \setminus K$ and $K' = K(x)$. The element x is transcendental over K by (3), $K^\circ = K'^\circ$ by (4), $vK = vK'$ by (5); consequently, we get also $G_{\dot K} = G_{\dot K'}$ and hence $\dot K_\circ = \dot K'_\circ$. Thus it remains to show that there exists a K-embedding $\eta : K' \to F$ in order to get $(K', \eta) \in A$ contrary to the assumption that A = $\{(K, K \hookrightarrow F)\}$. As $K^\circ = K'^\circ$, it suffices to show that there exists a k-embedding $\eta : \dot K' \to \dot F$.

As the proper valued field extension $\dot K'/\dot K$ is immediate, there exists a pseudoconvergent sequence a = $(a_\xi)_{\xi<\lambda}$ in $\dot K$, A $\leqslant |K|$, without pseudolimits in K such that x is a pseudolimit of a. Moreover, since $\dot K$ is algebraically complete (as a Henselian field of residue characteristic zero), it follows by usual arguments [25] that the sequence a is of transcendental type, i.e., f(a) = $(f(a_\xi))_{\xi<\lambda} \not\to 0$ for all $f \in K[X] \setminus \{0\}$.

Here is the point where we use the assumption that $\dot F$ is $|L|$-pseudocomplete; therefore a has a pseudolimit y in F. As a is transcendental, its pseudolimit y is transcendental over K; thus the substitution $x \mapsto y$ defines a field K-embedding $\eta : K' \to F$. We claim that η is a g-embedding of the valued field $\dot{K}'$ into $\dot{F}$. Indeed, let $f \in [X] \setminus \{0\}$. As a is transcendental, $f(a) \not\to 0$ and hence there is $\tau <$ A such that $\dot{v}f(x) = \dot{v}f(a_\xi) = \dot{v}f(y)$ for all $\tau \leqslant \xi < \lambda$. □

The next embedding criterion for Henselian fields of characteristic zero is an immediate consequence of Theorem 2.1.

Corollary 2.2. ***Let K = (K, v) be a valued field of characteristic zero and L = (L, v), F = (F, v) be Henselian valued fields extending K. Suppose that $\dot{F}$ is $|L|$ -pseudocomplete. The necessary and sufficient condition for L to be K-isomorphically embeddable into F is that the mixed o-structure $\dot{L}_\circ$ is $\dot{K}_\circ$-embeddable in $\dot{F}_\circ$.***

3. Mixed structures

We have introduced in the previous section the so-called mixed o-structures and we have seen the key role played by these ones in embedding problems for Henselian fields of characteristic zero. The mixed o-structure $\dot{K}_\circ$ assigned to a valued field $K = (K, v)$ of characteristic zero is constructed with the help of the coarse valuation $\dot{v}$ induced by v and hence it is not an elementary object assigned to K, except the special case when the residue characteristic of K is zero. As the model-theoretic investigation of Henselian fields requires more elementary invariants, it seem natural to approximate the global object $\dot{K}_\circ$ above by a family of suitable objects that are definable in elementary terms. Posible candidates for these elementary objects are the mixed k-structures $K_k, k \in \mathbb{N}$, defined in the Introduction.

In abstract terms, a ***mixed k-structure,*** $k \in \mathbb{N}$, is a system $\mathfrak{A} = (A, H, \Gamma, \theta, v)$, where A is a commutative ring with 1, H is a multiplicative Abelian group, Γ is an additive totally ordered group, θ is a partial map from A into H and $v : (A \setminus \{0\}) \cup H \to \Gamma$ is a map subject to the next conditions:

(1.1) For a, $b \in A \setminus \{0\}$, a divides b, (written $a \mid b$) iff $va \leqslant vb$;

(1.2) For a, $b \in A \setminus \{0\}$ such that $ab \neq 0, v(ab) = vu + vb$; in particular, vu $\geqslant 0$ for all a $\in A \setminus \{0\}$ and vu = 0 for all units $u \in A''$ by (1.1); set by convention $v0 = \infty$ with the usual rules for the symbol ∞;

(1.3) For a, $b \in A$, $v(a + b) \geqslant \min(va, vb)$; thus A is a local ring with maximal ideal $\mathfrak{m}_A = \{a \in A\colon vu > 0\}$; let p be the characteristic exponent of the residue field $A = A/m,$;

(1.4) The image $vA := v(A \setminus \{0\})$ equals the convex subset $\Gamma_{2k} = \{\alpha \in \Gamma\colon 0 \leqslant \alpha \leqslant vp^{2k}\}$ of Γ; consequently, $A \setminus \{0\} = \{a \in A\colon a \mid p^{2k}\}$, the characteristic exponent of A is p^{2k+1} and $A = \bar{A}$ iff $p = 1$;

(2) The restriction $v|_H : H \to \Gamma$ is a group epimorphism;

(3.1) The domain of the partial map θ is the complement in A of the ideal $\mathfrak{m}_{A,k} := \{a \in A: va > vp^k\}$;

(3.2) Diagram

$$\begin{tikzcd} A\setminus\{0\} \arrow[r, "v|_A"] & \Gamma_{2k} \arrow[r, hook] & \Gamma \\ A\setminus m_{A,k} \arrow[u, hook] \arrow[rr, "\theta"] & & H \arrow[u, "v|_H"'] \end{tikzcd}$$

commutes;

(3.3) For a, $b \in A\setminus\mathfrak{m}_{A,k}$ such that $ab \notin \mathfrak{m}_{A,k}, \theta(ab) = \theta(a)\theta(b)$; in particular, $\theta|_A x: A" \to H$ is a group morphism;

(3.4) $\mathrm{Ker}(v|_H) \subset \theta(A^x)$; in fact we have equality by (3.2);

(3.5) For a, $b \in A\setminus\mathfrak{m}_{A,k}$, $\theta(a) = \theta(b)$ iff $v(a-b) > vp^k + vb;$ thus $A^x/1 + \mathfrak{m}_{A,k} \simeq \mathrm{Ker}(v|_H)$ by (3.4).

Note also that the image of the map θ equals $\{h \in H: 0 \leqslant vh \leqslant up"\}$. For suppose a $\bullet$ A\rnA,k, i.e., $va \leqslant vp^k$; *we* get $v\theta(a) = va \leqslant vp^k$ by (3.2). Conversely, let $h \in H$ be such that $0 \leqslant vh \leqslant vp^k$. By (1.4) there is a $\in A$ such that $vu = vh$. As $a \notin \mathfrak{m}_{A,k}$, we get $v\theta(a) = vu = vh$ by (3.2). Thus $h\theta(a)^{-1} \in \mathrm{Ker}(v|_H)$ and hence $h\theta(a)^1 = 8(b)$ for some $b \in A^x$ by (3.4). It follows: $h = \theta(a)\theta(b) = \theta(ab)$ by (3.3).

Given $k, k' \in \mathbb{N}, k \geqslant k'$, an **Abelian** totally ordered group Γ, a mixed k-structure $\mathfrak{A} = (A, H, \Gamma, \theta, v)$ and a mixed k'-structure $\mathfrak{A}' = (A', H', I', \theta', v')$ such that A and A' have the same residue characteristic exponent, say p, a ***restriction map*** Π: $\mathfrak{A} \to \mathfrak{A}'$ is a map $\Pi: A \cup H$-t $A' \cup H'$ subject to the next conditions:

(i) $\Pi|_A$ is a unitary ring morphism from A onto A' which induces an isomorphism $A/\mathfrak{m}_{A,2k} \simeq A'$; in particular, $\Pi|_A$ induces an isomorphism of residue fields $\bar{A} \simeq$ A';

(ii) if $a \in A\setminus\mathfrak{m}_{A,2k'}$ then vu $= v'\Pi(a)$; in particular, for $k' > 0$, *we* get $vp = v'p;$

(iii) $\Pi|_H$ is a group epimorphism from H onto H' such that $vh = v'\Pi(h)$ for all $h \in H$;

(iv) $\Pi(\theta(a)) = \theta'(\Pi(a))$ for all $a \in A \setminus\mathfrak{m}_{A,k'}$.

Note that the restriction map Π induces an isomorphism $H/\theta(1 + \mathfrak{m}_{A,k'}) \simeq H'$. For suppose a $\in 1 + \mathfrak{m}_{A,k'}$, i.e., $v(a-1) > vp^{k'}$. By (ii) we get $v'\Pi(a-1) > vp^{k'}$, i.e., $\Pi(a) \bullet 1 + \mathfrak{m}_{A',k'}$ and hence $\theta'(\Pi(a)) = 1$ since $A'^x/1 + \mathfrak{m}_{A',k'} \simeq \mathrm{Ker}(v'|_{H'})$ by (3.5). On the other hand, $\Pi(\theta(a)) = \theta'(\Pi(a)) = 1$ by (iv). Conversely, let $h \in H$ be such that $\Pi(h) = 1$. In particular, $vh = 0$ by (iii) and hence there is a $\in A^x$ such that $h = \theta(a)$ since $A^x/1 + \mathfrak{m}_{A,k} \simeq \mathrm{Ker}(v|_H)$. By (iv), we obtain $\theta'(\Pi(a)) = \Pi(\theta(a)) = II(h) = 1$; therefore $\Pi(a) \in 1 + \mathfrak{m}_{A',k'}$. According to (ii) $a \in 1 + \mathfrak{m}_{A,k'}$ follows, as contended.

By a ***projective system of mixed structures we*** mean a system $\mathfrak{A} = (\mathfrak{A}_k; \Pi_{k,l})_{k \in \mathbb{N}, k \leqslant l}$, where $\mathfrak{A}_k = (A_k, H_k, \Gamma_k, \theta_k, v_k)$ is a mixed k-structure with

$\Gamma_k = \Gamma$ for all k $\in \mathbb{N}$, and $\Pi_{k,l}: \mathfrak{A}_l \to \mathfrak{A}_k$ for $k \leq l$ is a restriction map as defined above such that $\Pi_{k,l} = \Pi_{k,s} \circ \Pi_{s,l}$ for $k \leq s \leq l$ and $\Pi_{k,k} = 1_{\mathfrak{A}_k}$.

To a mixed w-structure $\mathfrak{S} = (M, H, \Gamma, \theta, v)$ one assigns naturally a projective system of mixed structures $\mathfrak{P}(\mathfrak{S}) = (\mathfrak{S}_k; \Pi_{k,l})_{k\in\mathbb{N},k\leq l}$, where $\mathfrak{S}_k = (O_{M,2k}, H_k, \Gamma, \theta_k, v_k)$ with $O_{M,2k} = O_M/\mathfrak{m}_{M,2k}$, H_k = H/8(1 + $\mathfrak{m}_{M,k}$) and $\theta_k : O_M \setminus (\mathfrak{m}_{M,k}/\mathfrak{m}_{M,2k}) \to H_k$, $v_k : O_{M,2k} \dot\cup H_k \to \Gamma \cup \{\infty\}$, induced respectively by $\theta : M^{\times} \hookrightarrow H$, $v : H \to \Gamma$, and $\Pi_{k,l}: \mathfrak{S}_l \to \mathfrak{S}_k$ for $k \leq l$ are the canonic restriction maps. Note that $\mathfrak{P}(\mathfrak{S})$ is identified with $\mathfrak{S}$ if the residue characteristic of M is zero.

As we are interested in organizing the projective systems of mixed structures into a category, we have to define morphisms between such objects.

First we define the embeddings of mixed k-structures for given $k \in N$. Let $\mathfrak{A} = (A, H, \Gamma, \theta, v)$ and $\mathfrak{A}' = (A', H', \Gamma', \theta', v')$ be mixed k-structures. An embedding $\varphi: \mathfrak{A} \to \mathfrak{A}'$ is a map $\varphi: A \dot\cup H \dot\cup \Gamma \to A' \dot\cup H' \dot\cup \Gamma'$ subject to the next conditions:

(i) $\varphi|_A$ is a ring embedding of A into A';
(ii) $\varphi|_H$ is a group embedding of H into H';
(iii) $\varphi|_\Gamma$ is an ordered group embedding of Γ into Γ';
(iv) $v'\varphi(a) = \varphi(va)$ for all a $\in A$;
(v) $v'\varphi(h) = \varphi(vh)$ for all $h \in H$;
(vi) $\theta'(\varphi(a)) = \varphi(\theta(a))$ for all $a \in A \setminus \mathfrak{m}_{A,k}$.

Now, given the projective systems of mixed structures $\mathfrak{A} = (\mathfrak{A}_k; \Pi_{k,l})_{k\in N,k\leq l}$ and $\mathfrak{A}' = (\mathfrak{A}'_k; \Pi'_{k,l})_{k\in N,k\leq l}$, an embedding of $\mathfrak{A}$ into $\mathfrak{A}'$ is a family $\varphi_k: \mathfrak{A}_k \to \mathfrak{A}'_k$ of embeddings of mixed k-structures, $k \in \mathbb{N}$, subject to the compatibility condition $\varphi_k \Pi_{k,l} = \Pi'_{k,l}\varphi_l$ for $k \leq l$.

If $\mathfrak{A}, \mathfrak{B}$ are respectively mixed k-structures, o-structures, projective systems of mixed structures, denote by Hom($\mathfrak{A}$, '23) the set of embeddings of $\mathfrak{A}$ into $\mathfrak{B}$.

Call a mixed o-structure $\mathfrak{S} = (M, H, \Gamma, \theta, v)$ ***complete*** if M is Cauchy complete, i.e., $O_M \simeq \varprojlim O_{M,k}$, and H is Cauchy complete with respect to the topology given by the system of open neighbourhoods of 1,

$$\{\theta(1 + \mathfrak{m}_{M,k})\}_{k\in\mathbb{N}}, \quad \text{i.e., } H \simeq \varprojlim H/\theta(1 + \mathfrak{m}_{M,k}).$$

The next lemmas are immediate.

Lemma 3.1. ***Given a mixed w-structure*** $\mathfrak{S}$, ***there exists a complete mixed o-structure*** $\hat{\mathfrak{S}}$ ***and an embedding*** $i : \mathfrak{S} \to \hat{\mathfrak{S}}$ ***such that for every embedding*** $\varphi: \mathfrak{S} \to \mathfrak{S}'$ ***with*** $\mathfrak{S}'$ ***complete, there is a unique embedding*** $\psi: \hat{\mathfrak{S}} \to \mathfrak{S}'$ ***subject to*** $\varphi = \psi i$.

Call the mixed o-structure $\hat{\mathfrak{S}}$ above, the ***completion*** of $\mathfrak{S}$; $\hat{\mathfrak{S}}$ is unique up to an isomorphism over $\mathfrak{S}$.

Lemma 3.2. ***Let*** $\mathfrak{S}$ ***and*** $\mathfrak{S}'$ ***be mixed w-structures and assume*** $\mathfrak{S}'$ ***is complete. The cunonic map*** $\mathrm{Hom}(\mathfrak{S}, \mathfrak{S}') \to \mathrm{Hom}(\mathfrak{P}(\mathfrak{S}), \mathfrak{P}(\mathfrak{S}'))$ ***is bijective.***

Given $k \in \mathbb{N}$, the category of mixed k-structures, with embeddings as morphisms, can be seen as the category of models of a theory T_k in a suitable first-order language $\mathfrak{L}_k$. We may consider the mixed k-structures as one-sorted structures, as well as (finitely) many-sorted structures with sorts for the ring A, the group H and the ordered group Γ; we choose in the following the many-sorted approach. The languages $\mathfrak{L}_k$, $k \in \mathbb{N}$, are related each to other thanks to the next translation procedure.

Lemma 3.3. ***Let $k, l \in \mathbb{N}$ be such that $k \leqslant l$. Given an &-formula $\psi(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, one may assign effectively to it an $\mathfrak{L}_l$-formula $\mathrm{tr}_{k,l}(\psi)$ (x; y), where y is a new variable of sort A, in such a way that for every restriction map Π defined on a mixed l-structure $\mathfrak{A}$ of residue characteristic exponent p with values into a mixed k-structure $\mathfrak{A}'$ and for every $\boldsymbol{a} = (a_1, \ldots, a,)$ in $\mathfrak{A}$ of suitable sorts,***

$$\mathfrak{A} \models \psi(\Pi(\boldsymbol{a})) \quad \textit{iff} \quad \mathfrak{A}' \models \mathrm{tr}_{k,l}(\psi)(\boldsymbol{a}; p).$$

Proof. We define the translation map $\mathrm{tr}_{k,l} : \mathfrak{L}_k \to \mathfrak{L}_l$ by induction on the complexity of the formula ψ.

(1) Assuming ψ atomic, we distinguish the next cases:

(i) $\psi := (f(\boldsymbol{x}) = 0)$, where $f \in \mathbb{Z}[\boldsymbol{x}]$, $\boldsymbol{x} = (x_1, \ldots, x_n)$ variables of sort A; let us put $\mathrm{tr}_{k,l}(\psi) := vf(\boldsymbol{x}) > vy^{2k}$;

(ii) $\psi := (t = 1)$, where t is a term of sort H, i.e., t is a word $w_1^{n_1} \cdots w_m^{n_m}$ with $n_1, \ldots, n_m \in \mathbb{Z}$ and w_i has either the form $\theta(f(x))$ with $f \in \mathbb{Z}[\boldsymbol{x}]$, $\boldsymbol{x} = (x_1, \ldots, x_n)$ variables of sort A, or it is a variable of sort H; define $\mathrm{tr}_{k,l}(\psi) := [(\exists z)(v(z - 1) > vy^k) \wedge (\theta(z) = t)]$, where z is a new variable of sort A;

(iii) $\psi := (\xi = 0)$, where ξ is a term of sort Γ, i.e., $\xi = \sum_{i=1}^m n_i \alpha_i$ with $n_1, \ldots, n_m \in \mathbb{Z}$ and α_i has either the form $vf(\boldsymbol{x})$ with $f \in \mathbb{Z}[\boldsymbol{x}]$, $\boldsymbol{x} = (x_1, \ldots, x_n)$ variables of sort A, or the form vt, where t is a term of sort H or is a variable of sort Γ; let us put $\mathrm{tr}_{k,l}(\psi) := (\xi = 0)$;

(iv) $\psi := (\xi > 0)$, where ξ is a term of sort Γ; set $\mathrm{tr}_{k,l}(\psi) := (\xi > 0)$.

(2) $\mathrm{tr}_{k,l}(\psi_1 \wedge \psi_2) := \mathrm{tr}_{k,l}(\psi_1) \wedge \mathrm{tr}_{k,l}(\psi_2)$;

(3) $\mathrm{tr}_{k,l}(\neg\psi) := \neg\mathrm{tr}_{k,l}(\psi)$;

(4) $\mathrm{tr}_{k,l}((\exists z)\, \psi(z)) := (\exists z)\, \mathrm{tr}_{k,l}(\psi(z))$. ❑

Similarly we may interpret the category of projective systems of mixed structures, with embeddings as morphisms, as the category of models of a theory T_ω in a suitable first-order w-sorted language $\mathfrak{L}_\omega$ with sorts for the rings A_k, $k \in \mathbb{N}$, the groups H_k, $k \in N$, and the ordered group Γ. The next lemma is an immediate consequence of Lemma 3.3.

Lemma 3.4. ***Let $\psi(x_1, \ldots, x_n)$ be an &-formula. For $1 \leqslant i \leqslant n$, let***

$$m_i = \begin{cases} k, & \text{if } x_i \text{ is of sort } A_k \text{ or } H_k, \\ 0, & \text{if } x_i \text{ is of sort } \Gamma. \end{cases}$$

One may assign effectively to ψ a natural number $m \geqslant \max_i(m_i)$ and an $\mathfrak{L}_m$-formula $\varphi(y_1, \ldots, y_n; z)$ in such a way that for every projective system of mixed structures $\mathfrak{A} = (\mathfrak{A}_k; \Pi_{k,l})_{k\in\mathbb{N},k\leqslant l}$ of residue characteristic exponent p and for arbitrary $a_1, \ldots, a_n$ in $\mathfrak{A}_m$ of suitable sorts,

$$\mathfrak{A} \models \psi(\Pi_{m_1,m}(a_1), \ldots, \Pi_{m_n,m}(a_n)) \quad \textit{iff} \quad \mathfrak{A}_m \models \varphi(a_1, \ldots, a_n; p).$$

Given an uncountable cardinal κ, a projective system of mixed structures $\mathfrak{A} = (\mathfrak{A}_k; \Pi_{k,1})_{k\in\mathbb{N},k\leqslant 1}$ is *κ-saturated* if for every set Φ of $\mathfrak{L}_\omega$-formulas with parameters from $\mathfrak{A}$ (i.e., involving constants for elements $\mathfrak{A}$) the following holds: if Φ has cardinality less than κ and every finite set of formulas from Φ is realized in $\mathfrak{A}$, then all formulas from Φ are simultaneously realized in $\mathfrak{A}$.

Lemma *3.5.* ***Let $\mathfrak{S} = (M, H, \Gamma, \theta, v) \hookrightarrow \mathfrak{S}^{(i)} = (M^{(i)}, H^{(i)}, \Gamma^{(i)}, \theta^{(i)}, v^{(i)})$, $i = 1, 2$, be embeddings of mixed q-structures. Suppose that $\mathfrak{S}^{(2)}$ is complete and $\mathfrak{P}(\mathfrak{S}^{(2)})$ is K-saturated for some cardinal $\kappa > |\mathfrak{S}^{(1)}| = |H^{(1)}|$. The next statements are equivalent:***

(i) $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{S}^{(1)}, \mathfrak{S}^{(2)})$ ***is nonempty;***
(ii) $\mathrm{Hom}_{\mathfrak{S}_k}(\mathfrak{S}_k^{(1)}, \mathfrak{S}_k^{(2)})$ ***is nonempty for all $k \in \mathbb{N}$.***

Proof. The implication (i)$\to$(ii) is obvious.

(ii)$\to$(i) As $\mathfrak{S}^{(2)}$ is complete, it suffices to show that $\mathrm{Hom}_{\mathfrak{P}(\mathfrak{S})}(\mathfrak{P}(\mathfrak{S}^{(1)}), \mathfrak{P}(\mathfrak{S}^{(2)}))$ is nonempty, according to Lemma 3.2. Let $X = \{x_a: a \in \mathfrak{P}(\mathfrak{S}^{(1)}) \setminus \mathfrak{P}(\mathfrak{S})\}$ be a set of $\mathfrak{L}_\omega$-variables such that x_a has the same sort as a, and consider the set Φ of all basic $\mathfrak{L}_\omega$-formulas (atomic and negated atomic formulas) $\varphi(x_{a_1}, \ldots, x_{a_n})$ with parameters in $\mathfrak{P}(\mathfrak{S})$ subject to $\mathfrak{P}(\mathfrak{S}^{(1)}) \models \varphi(a_1, \ldots, a_n)$. Obviously, $\mathrm{Hom}_{\mathfrak{P}(\mathfrak{S})}(\mathfrak{P}(\mathfrak{S}^{(1)}), \mathfrak{P}(\mathfrak{S}^{(2)}))$ is nonempty iff Φ is realized in $\mathfrak{P}(\mathfrak{S}^{(2)})$. By the K-saturation property of $\mathfrak{P}(\mathfrak{S}^{(2)})$ it suffices to show that each finite subset of Φ is realized on $\mathfrak{P}(\mathfrak{S}^{(2)})$. Let $\varphi_i(x_{a_1}, \ldots, x_{a_n})$, $1 \leqslant i \leqslant m$, be formulas from Φ. According to Lemma 3.3, we may assume without loss of generality that there is $k \in \mathbb{N}$ such that all parameters occurring in φ_i, $1 \leqslant i \leqslant m$, denote constants from $\mathfrak{S}_k$; we may also assume that $a_1, \ldots, a_n$ belong to $\mathfrak{S}_k^{(1)}$. Since $\mathrm{Hom}_{\mathfrak{S}_k}(\mathfrak{S}_k^{(1)}, \mathfrak{S}_k^{(2)})$ is nonempty by assumption, we conclude that the formula $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_m$ is realized on $\mathfrak{S}_k^{(2)}$ and hence it is also realized on $\mathfrak{P}(\mathfrak{S}^{(2)})$. $\square$

4. Proof of the main results

Denote by $\mathfrak{L}$ the first-order language of valued fields, whose vocabulary contains, besides the logical symbols, constants 0, 1, function symbols for the field operations $(+, \cdot, -, ^{-1})$ (with convention $0^{-1} = 0$) and one unary predicate $\mathfrak{O}$ which in a valued field is interpreted as the valuation ring. The axioms of valued fields, as well as the axioms of Henselian valued fields can be formulated in this language.

The language $\mathfrak{L}$ and the languages $\mathfrak{L}_k$ of mixed k-structures, $k \in \mathbb{N}$, introduced in Section 3, are related via the following translation procedure.

Given a valued field $K = (K, v)$ of characteristic zero and a natural number k, let us consider the canonic maps $\Pi_{A,k}: O_K \to O_{K,k} = O_K/\mathfrak{m}_{K,k}$, $\Pi_{H,k}: K^x \to K^x/1 + \mathfrak{m}_{K,k}$.

Lemma 4.1. ***Let** $k \in \mathbb{N}$ **and** $\psi(x_1, \ldots, x_n; y_1, \ldots, y_m; \xi_1, \ldots, \xi_l)$ **be an** $\mathfrak{L}_k$**-formula, where the** x_i**'s are variables of sort A, the** y_i**'s are variables of sort H and the** ξ_i**'s are variables of sort** Γ**. One assigns effectively to** ψ **an** $\mathfrak{L}$**-formula** $\mathrm{tr}_k(\psi)$ $(z_1, z_2, \ldots, z_{n+m+1+1})$ **in such a way that for every valued field K of characteristic zero and residue characteristic exponent p, and for arbitrary** $a_i \in O_K$, $1 \leq i \leq n$, $a_i \in K^x$, $n+1 \leq i \leq n+m+l$,*

$$K_k \vDash \psi(\Pi_{A,2k}(a_i): 1 \leq i \leq n;\ \Pi_{H,k}(a_i): n+1 \leq i \leq n+m;$$
$$va_i: n+m+1 \leq i \leq n+m+l)$$

iff $K \vDash \mathrm{tr}_k(\psi)(a_1, \ldots, a_{n+m+l}, p)$.

The proof is immediate.

The next embedding theorem for Henselian fields of characteristic zero is a consequence of Corollary 2.2 and Lemma 3.5.

Theorem 4.2. ***Let K = (K, v) be a valued field of characteristic zero and L, F be Henselian valued fields extending K. Suppose that the valued field F is K-saturated for some cardinal** $\kappa > |L|$**. The necessary and sufficient condition for the valued field L to be K-embeddable into F is that the mixed k-structure** L_k **is** K_k**-embeddable into** F_k **for all** $k \in N$**.***

Proof. An implication is trivial. Conversely, assume that $\mathrm{Hom}_{K_k}(L_k, F_k)$ is nonempty for all $k \in \mathbb{N}$. By the K-saturation property of F, it follows that the mixed o-structure $F_\circ$ is complete, the projective system $\mathfrak{P}(\dot{F}_\circ)$ is K-saturated and the valued field $\dot{F}$ is $|L|$-pseudocomplete. According to Lemma 3.5, $\mathrm{Hom}_{\mathring{\dot{K}}}(\dot{L}_\circ, \dot{F}_\circ)$ is nonempty and hence L is K-embeddable into F by Corollary 2.2. □

Proof of Theorem A. An implication is obvious. Conversely, let us assume that the mixed k-structures L_k, F_k are elementarily equivalent over K_k (written $L_k \equiv_{K_k} F_k$) for all $k \in \mathbb{N}$. We may assume without loss of generality that F is K-saturated for some cardinal $\kappa > |L|$. For otherwise we may consider a K-saturated elementary extension F' of F according to [30, Theorem 16.41. In order to get $L \equiv_K F$ it suffices to show that there exists a K-embedding $\varphi: L \to F$ such that the induced K_k-embedding of mixed k-structures $\varphi_k: L_k \to F_k$ is an elementary one for all $k \in \mathbb{N}$. For in this case we may construct by iteration an infinite commutative diagram

$$\begin{tikzcd}
 & F^{(0)} = F \arrow[r] \arrow[dr, "\psi^{(1)}"] & F^{(1)} \arrow[r] \arrow[dr, "\psi^{(2)}"] & F^{(2)} \arrow[r] & \cdots \\
K \arrow[r] \arrow[ur] & L^{(0)} = L \arrow[r] \arrow[u, "\varphi^{(0)} = \varphi"'] & L^{(1)} \arrow[r] \arrow[u, "\varphi^{(1)}"'] & L^{(2)} \arrow[r] \arrow[u, "\varphi^{(2)}"'] & \cdots
\end{tikzcd}$$

where $\boldsymbol{L}^{(i+1)}/\boldsymbol{L}^{(i)}$, $\boldsymbol{F}^{(i+1)}/\boldsymbol{F}^{(i)}$ are elementary extensions, $\boldsymbol{L}^{(i+1)}$ is $|F_i|^+$-saturated, $\boldsymbol{F}^{(i)}$ is $|L_i|^+$-saturated and $\varphi_k^{(i)}: \boldsymbol{L}_k^{(i)} \to \boldsymbol{F}_k^{(i)}$, $\psi_k^{(i)}: \boldsymbol{F}_k^{(i)} \to \boldsymbol{L}_k^{(i+1)}$ are elementary embeddings for all $k \in N$. Let $\boldsymbol{F}^{(\omega)} = \varinjlim \boldsymbol{F}^{(i)} = \varinjlim \boldsymbol{L}^{(i)}$. By the Tarski-Vaught theorem [30, Theorem 10.11, $\boldsymbol{F}^{(\omega)}/\boldsymbol{L}$ and $\boldsymbol{F}^{(\omega)}/\boldsymbol{F}$ are elementary extensions and hence $\boldsymbol{L} \equiv_{\boldsymbol{K}} \boldsymbol{F}$.

Thus it remains to show that there exists a K-embedding $\varphi : \boldsymbol{L} \to \boldsymbol{F}$ with the required properties. As $\boldsymbol{L}_k \equiv_{\boldsymbol{K}_k} \boldsymbol{F}_k$ is K-saturated, $\kappa > |L| \geq |\boldsymbol{L}_k|$, there exist elementary $\boldsymbol{K}_k$-embeddings $\boldsymbol{L}_k \to \boldsymbol{F}_k$ for all $k \in \mathbb{N}$; therefore, by a slight adjustment of Lemma 3.5, we obtain a $\dot{\boldsymbol{K}}_\circ$-embedding $\varphi_\omega : \dot{\boldsymbol{L}}_\circ \to \dot{\boldsymbol{F}}_\circ$ of mixed ω-structures inducing elementary $\boldsymbol{K}_k$-embeddings $\varphi_k : \boldsymbol{L}_k \to \boldsymbol{F}_k$ for all $k \in N$. It remains to extend φ_ω to a K-embedding $\varphi : \boldsymbol{L} \to \boldsymbol{F}$ by Theorem 2.1 □

Proof of Theorem B. Let p be either 1 or a fixed prime number. Denote by $\mathfrak{T}_p$ the Z-theory of Henselian valued fields of characteristic zero and residue characteristic exponent p. Let $\varphi(\boldsymbol{x}), \boldsymbol{x} = (x_1, \ldots, x_n)$, be an L-formula. Denote by $\boldsymbol{W}$ the set of sentences in the language $\mathfrak{L}$ augmented with the constants $\boldsymbol{c} = (c_1, \ldots, c_n)$, consisting of:

(i) the axioms of $\mathfrak{T}_p$;

(ii) the sentence $\varphi(\mathbf{c})$;

(iii) the sentences of the form

$$\neg[\lambda(\boldsymbol{c}) \wedge \mathrm{tr}_k(\psi)(f_i(\boldsymbol{c})g_i(\boldsymbol{c})^{-1}: 1 \leq i \leq m; p)],$$

where $\lambda(\boldsymbol{x})$ is a quantifierless C-formula, k is a natural number, $\psi(\boldsymbol{y})$ with Y = $(y_1, \ldots, y_m)$ is an X$-formula and $f_i, g_i \in \mathbb{Z}[\boldsymbol{x}]$, $1 \leq i \leq m$, such that

$$\mathfrak{T}_p \vdash [\lambda(\boldsymbol{x}) \text{ A } \mathrm{tr}_k(\psi)(f_i(\boldsymbol{x})g_i(\boldsymbol{x})^{-1}: 1 \leq i \leq m; p)] \to \varphi(\boldsymbol{x}).$$

We claim that $\boldsymbol{W}$ is inconsistent. Assuming the contrary, let us choose, by Gödel's completeness theorem [30, Theorem 7.11, a model *(L; c)* of $\boldsymbol{W}$. In particular, $\boldsymbol{L}$ = *(L,* $v)$ is a Henselian valued field of characteristic zero and residue characteristic exponent p, and c = $(c_1, \ldots, c_n) \in L$”. Let $\boldsymbol{K} = (\mathbb{Q}(c), v)$ be the smallest valued subfield of $\boldsymbol{L}$ containing the c_i's. Denote by D(K) the diagram of $\boldsymbol{K}$, i.e., the set of basic sentences (atomic and negated atomic) in the language of $\boldsymbol{K}$ that are true on $\boldsymbol{K}$, and by $D(\boldsymbol{L}/\boldsymbol{K})$ the set of all sentences having the form $\mathrm{tr}_k(\psi)(\boldsymbol{a}; p)$, where $k \in \mathbb{N}, \psi(\boldsymbol{y})$ with y = $(y_1, \ldots, y_m)$ is an &-formula and $\boldsymbol{a} = (a_1, \ldots, a_m) \in K^m$ such that $\Pi_k(\boldsymbol{a})$ makes sense (i.e., $a_i \in O_K$ if y_i is a variable of sort A, $a_i \in K^\times$ if y_i is a variable of sort H or Γ; then $\Pi_k(a_i)$ is respectively $\Pi_{A,2k}(a_i)$, $\Pi_{H,k}(a_i)$, $va_i)$ and $\boldsymbol{L}_k \models (\Pi_k(\boldsymbol{a}))$. According to Theorem A, the theory

$S = \mathfrak{T}_p \cup D(K) \cup D(L/K)$ is complete; therefore $S \vdash \varphi(c)$, since $L \models S$ and $L \models \varphi(c)$. Consequently, there are finitely many sentences $\lambda_1, \ldots, \lambda_s \in D(K)$, $\eta_1, \ldots, \eta_t \in D(L/K)$ such that

$$\mathfrak{T}_p \vdash \bigwedge_i \lambda_i \wedge \bigwedge_j \eta_j \rightarrow \varphi(c).$$

Let $\lambda(x)$ be a quantifier less &formula such that $\lambda(c) \leftrightarrow \bigwedge \lambda_i$. On the other hand, by Lemma 3.3, there exist $k \in \mathbb{N}$, an &-formula $\psi(y)$, $y = (y_1, \ldots, y_m)$, and some polynomials $f_i, g_i \in \mathbb{Z}[x]$ such that

$$\mathrm{tr}_k(\psi)(f_i(c)g_i(c)^{-1}\colon 1 \leq i \leq m; p) \leftrightarrow \bigwedge \eta_j.$$

Thus

$$\mathfrak{T}_p \vdash [\lambda(x) \wedge \mathrm{tr}_k(\psi)(f_i(x)g_i(x)^{-1}\colon 1 \leq i \leq m; p)] \rightarrow \varphi(x)$$

since c does not ocur in $\mathfrak{T}_p$; therefore

$$L \models \neg[\lambda(c) \wedge \mathrm{tr}_k(\psi)(f_i(c)g_i(c)^{-1}\colon 1 \leq i \leq m; p)]$$

since (L, c) is a model of W, a contradiction.

We conclude that W is inconsistent and hence there exist quantifierless L-formulas $\lambda_1(x), \ldots, \lambda_l(x)$, natural numbers $k_1, \ldots, k_l$, &,-formulas $\psi_j(y)$, $1 \leq j \leq l$, $y = (y_1, \ldots, y_m)$, and polynomials $f_i, g_i \in \mathbb{Z}[x]$, $1 \leq i \leq m$, such that

$$\mathfrak{T}_p \vdash \varphi(x) \leftrightarrow \bigvee_{1 \leq j \leq l} [\lambda_j(x) \wedge \mathrm{tr}_{k_j}(\psi_j)(f_i(x)g_i(x)^{-1}\colon 1 \leq i \leq m;\ p)].$$

Using again Lemma 3.3, we may assume that $k_1 = k_2 = \cdots = k_l := k$. □

5. Application to p-adically closed fields

In the last section of the work we shall show that Macintyre's theorem [23] on quantifier elimination for p-adic fields and its generalization to p-adically closed fields [24, Theorem 5.6], can be obtained as consequences of the general Theorem B.

Let us fix in the following a prime number p. A valued field $K = (K, v)$ is called ***p-valued*** if K is of characteristic zero, the residue field $\bar{K}$ is of characteristic p and the $\mathbb{F}_p$-space O_K/pO_K is finite. Call the dimension of the space O_K/pO_K the ***p-rank*** of the p-valued field K.

The above condition implies that the residue field $\bar{K}$ is finite, say $\bar{K} \simeq \mathbb{F}_q$ with $q = p^f$, $f = [\bar{K} : \mathbb{F}_p]$ and the absolute ramification index e of K, i.e., the number of positive elements in vK which are $\leq vp$, is finite. The p-rank d of the p-valued field K satisfies the relation $d = ef$.

A p-valued field K of p-rank d is called ***p-adically closed*** if K does not admit any proper p-valued algebraic extension of the same p-rank. It turns out by [24, Theorem 3.11, that a p-valued field K is p-adically closed iff K is Henselian and its value group vK is a Z-group, i.e., the coarse value group $\dot{v}K$ is divisible.

Given p and $d \geqslant 1$, let us consider the augmentation $\mathfrak{L}^{(d)}$ of the language $\mathfrak{L}$ of valued fields with $d-1$ new constants $u_2, \ldots, u_d$. The class of p-valued fields of p-rank d is axiomatizable by universal axioms in the language $\mathfrak{L}^{(d)}$ in such a way that for any p-valued field K of p-rank d the constants $u_1 = 1, u_2, \ldots, u_d$ denote an $\mathbb{F}_p$-basis of O_K/pO_K. With respect to the modified language $\mathfrak{L}^{(d)}$, every substructure of a p-valued field of p-rank d is again a p-valued field of p-rank d. Note also, by [24, Theorem 3.11, that the class of p-adically closed fields of p-rank d is axiomatizable in $\mathfrak{L}$ and hence in $\mathfrak{L}^{(d)}$.

Now extend again the language $\mathfrak{L}^{(d)}$ by unary predicates P_n, $n \in \mathbb{N}$, and add to the $\mathfrak{L}^{(d)}$-axioms for the theory $T_{p,d}$ of p-adically closed fields of p-rank d the new defining axioms

$$P_n(x) \leftrightarrow (\exists y)\, xy^n = 1\,.$$

The basic result of Prestel and Roquette [24, Theorem 5.61, including Macintyre's result [23] as a special case, can be stated as follows.

Theorem 5.1. In *the language $\mathfrak{L}^{(d)}$ extended by unary predicates $P_n, n \in \mathbb{N}$, the theory $T_{p,d}$ together with the defining axioms above admits elimination of quantifiers.*

The main goal of this section is to show that Theorem 5.1 is a consequence of Theorem B. First of all we state an equivalent version of Theorem 5.1.

Extend the language $\mathfrak{L}^{(d)}$ by unary predicates $W_{n,k}$, $n, k \in \mathbb{N}$, and add to the $\mathfrak{L}^{(d)}$-theory $T_{p,d}$ of p-adically closed fields of p-rank d the defining axioms

$$W_{n,k}(x) \leftrightarrow (\exists y)(1 - xy^n)^{-1}p^k \notin 0.$$

We show that Theorem 5.1 is equivalent with the following statement.

Theorem 5.1.a. In *the language $\mathfrak{L}^{(d)}$ extended by unary predicates $W_{n,k}$, $n, k \in \mathbb{N}$, the theory $T_{p,d}$ together with the defining axioms above admits elimination of quantifiers.*

Proof of the equivalence (5.1) $\leftrightarrow$ (5.1.a). (5.1)$\rightarrow$ (5.1.a): Given $n \in \mathbb{N}$, let $l \in \mathbb{N}$ be such that $l \geqslant 2v_p(n)$, where $v_p(n)$ is the p-adic value of n, i.e., $n = p^{v_p(n)}m$ with $(m, p) = 1$. By Newton's lemma, $P_n(x) \leftrightarrow W_{n,l}(x)$ on any Henselian valued field of characteristic zero and residue characteristic p; in particular we obtain $T_{p,d} \vdash P_n(x) \leftrightarrow W_{n,l}(x)$.

(5.1.a)$\rightarrow$ (5.1): Given $n, k \in \mathbb{N}$, let $l \in \mathbb{N}$ be such that $l > \max(k, 2v_p(n))$. It suffices to show that

$$T_{p,d} \vdash W_{n,k}(x) \leftrightarrow \bigvee_{(\alpha_1,\ldots,\alpha_d)\in S} \left[P_n\Big(\sum_{i=1}^{d} \alpha_i u_i x\Big) \wedge v\Big(1 - \sum_{i=1}^{d} \alpha_i u_i\Big) > kvp \right],$$

where $S=\{(\alpha_1,\ldots,\alpha_d)\in\mathbb{Z}^d\setminus(0,\ldots,0): 0\leqslant\alpha_i\leqslant p^l-1, 1\leqslant i\leqslant d\}$. Let $\boldsymbol{K}=(K,v)$ be a p-adically closed field of p-rank d and $u_1=1, u_2,\ldots,u_d$ be the given constants in $O_{\boldsymbol{K}}$ defining a basis of $O_{\boldsymbol{K}}/pO_{\boldsymbol{K}}$. Clearly, the u_i's define also a $\mathbb{Z}/p^l\mathbb{Z}$-basis of $O_{\boldsymbol{K}}/p^lO_{\boldsymbol{K}}$. Given a $\in K$, let us asume $\boldsymbol{K}\models W_{,,}(a)$, i.e., $v(1-ab")>kvp$ for some $b\in K$. As $ab"\in O_K^x$, there exists $(\alpha_1,\ldots,\alpha_d)\in S$ such that

$$v\left(1-\sum_{i=1}^{d}\alpha_iu_iab^n\right)\geqslant lvp.$$

As $l>2v_p(n)$, it follows by Newton's lemma that

$$\left(\sum_{i=1}^{d}\alpha_iu_ia\right)(bc)^n=1 \quad \text{for fome } c\in K,$$

i.e., $K\models P_n(\sum_{i=1}^d\alpha_iu_ia)$. On the other hand,

$$v\left(1-\sum_{i=l}^{d}\alpha_iu_i\right)=v\left[\left(1-\sum_{i=1}^{d}\alpha_iu_iab^n\right)-\left(\sum_{i=1}^{d}\alpha_iu_i\right)(1-ab^n)\right]>kvp$$

since $v(\sum_{i=1}^d\alpha_iu_i)=0$ and $l>k$.

Conversely, let us assume that there exists $(a,, \ldots,\alpha_d)\in S$ such that

$$v\left(1-\sum_{i=l}^{d}\alpha_iu_i\right)>kvp \quad\text{and}\quad \sum_{i=1}^{d}\alpha_iu_iab^n=1 \text{ for some } b\in K.$$

Then $ab"\in O_K^x$ and

$$v(1-ab^n)=v\left(ab^n\left(\sum_{i=1}^{d}\alpha_iu_i-1\right)\right)>kvp,$$

i.e., $K\models W_{n,k}(a)$. □

As the residue rings $O_{\boldsymbol{K},k}=O_{\boldsymbol{K}}/\mathfrak{m}_{\boldsymbol{K},k}$, $k\in\mathbb{N}$, are finite with $|O_{\boldsymbol{K},k}|\leqslant p^{(k+1)d}$, any formula about such rings is equivalent to a quantifierless formula involving the constants $u_1=1, u_2,\ldots,u_d$. On the other hand, it follows by induction that on the mixed k-structures $\boldsymbol{K}_k$, an arbitrary formula is equivalent with a Boolean combination of formulas of the ring language, involving only variables of $O_{\boldsymbol{K},2k}$, and formulas of the system $G_{\boldsymbol{K},k}\xrightarrow{v_k}vK$ involving variables of this system and some of the finitely many constants $\theta_k(a)$ for $a\in O_{\boldsymbol{K},2k}\setminus(\mathfrak{m}_{\boldsymbol{K},k}/\mathfrak{m}_{\boldsymbol{K},2k})$ and $v_k(a)$ for $a\in O_{\boldsymbol{K},2k}\setminus\{0\}$.

Consequently, Theorem 5.1.a is an immediate consequence of Theorem B and of the elimination of quantifiers for the structures defined as follows.

Given a finite Abelian group A, let us consider the systems (H, Γ, v, t) where $(H,\cdot,1)$ is an Abelian group, $(\Gamma,+,\leqslant,0)$ is a totally ordered Abelian group, $v: H\to\Gamma$ is a group epimorphism and t is an element of H such that Ker $v\simeq A$ and vt is the smallest positive element of Γ. The class of these systems can be axiomatized by universal axioms in a first-order language $\mathfrak{L}_A$ whose vocabulary contains symbols for group operations on H and Γ, a predicate for the order on Γ, a function symbol for the map v and constants for the elements of A, the neutral element 0 of Γ and the distinguished element t of H.

Denote by $\mathfrak{T}_A$ and $\mathfrak{L}_A$-**theory** whose models are the systems (H, Γ, v, t) above satisfying the supplementary condition that Γ is a Z-group, i.e., $\Gamma/\mathbb{Z}vt$ is divisible.

Now extend the language $\mathfrak{L}_A$ by unary predicates $R_n, n \in \mathbb{N}$, and add to the axioms for the theory $\mathfrak{T}_A$ the defining axioms

$$R,(x) \leftrightarrow (\exists y \in H)\ x = y".$$

Proposition 5.2. In *the language $\mathfrak{L}_A$ extended with unary predicates R_n,* n $\in \mathbb{N}$, *the theory $\mathfrak{T}_A$ together with the defining axioms above admits elimination of quantifiers.*

Proof. Let $H' = (H', \Gamma', v, t)$, $H'' = (H'', \Gamma'', v, t)$ be models of $\mathfrak{T}_A$ and $H = (H, \Gamma, v, t)$ be a common $\mathfrak{L}_A$-**substructure** of H', H'' such that $H'^n \cap H = H''^n$ ıI H for all n $\in \mathbb{N}$. According to [30, Theorem 13.21 we have to show that $H' \equiv_H H''$.

Denote by $T(H'/H) = \{x \in H': \bigvee_{n\geqslant 1} x^n \in H\}$ the group of the torsion elements of H' over H. As an $\mathfrak{L}_A$-**structure**, $T(H'/H)$ is a model of $\mathbb{T}_A$ since $H'/T(H'/H)$ is torsion free and $\Gamma'/n\Gamma' \simeq \mathbb{Z}/n\mathbb{Z}$ for all n $\in \mathbb{N}$. First of all we show that $T(H'/H) \simeq_H T(H''/H)$. It suffices to show that $T(H'/H)$ is H-embeddable into H”. For suppose that $\varphi: T(H'/H) \to H''$ is an H-embedding, and let y $\in$ $T(H''/H)$; *we* have to show that y $\in \mathrm{Im}(\varphi)$. Let $a \in H$, $n \geqslant 1$ be such that $y^n = a$. As $H'^n \cap H = H''^n \cap H$, there is $x \in T(H'/H)$ such that $x^n = a$. It follows that $(\varphi(x)y^{-1})^n = 1$ and **hence** $\varphi(x)y^{-1} \in A$ = Ker v. As $A \subset H$, *we* get y $\in \mathrm{Im}(\varphi)$, as contended.

Thus it remains to show that $T(H'/H)$ is H-embeddable into H”. Since the equation $x^n = a$ with $a \in H$ has finitely many solutions in H' (assume $x_0 \in H'$ is a solution of the equation above; then any other solution has the form $x_0 u$ with u $\in A$), it suffices, by a standard compactness argument (the projective limit of a directed projective system of **nonempty** finite sets is nonempty), to show that any intermediate group G between H and $T(H'/H)$ which is finitely generated over H can be embedded over H (as an $\mathfrak{L}_A$-**structure**) into H”. Let G be such a group. As G/H is finite, let us consider a direct decomposition into finite cyclic groups:

$$G/H = C_1 \times C_2 \times \cdots \times C_l.$$

Let n_i be the order of C_i; then $n_1 \cdots n_l = n := (G:H)$. Let $x_i \in G$ be such **that** x_i generates C_i modulo H; then x_i is of order n_i modulo H and hence $x_i^{n_i} = a_i \in H$, $1 \leqslant i \leqslant 1$. By assumption there exist $y_i \in H''$ such that $y_i^{n_i} = a_i$ for $1 \leqslant i \leqslant 1$. The substitution $x_i \mapsto y_i, 1 \leqslant i \leqslant l$, defines a group morphism over H, $\Phi: G \to H''$. **First** note that φ is **injective**. Indeed, let $z \in$ G be such that $\varphi(z) = 1$. In particular, $\varphi(z^n) = 1$ and hence $z^n = \varphi(z^n) = 1$ since $z^n \in H$; it follows $z \in A \subset H$ and z = $\varphi(z) = 1$, as contended. Next let us show that φ is an $\mathfrak{L}_A$-**embedding**, i.e.,

$v\varphi(z) \geq 0$ for all $z \in G$ subject to $vz \geq 0$. Consider such an element z; as $z^n = b \in H$ and $vz \geq 0$, we get $vb \geq 0$, $\varphi(z)^n = b$ and hence $v\varphi(z) \geq 0$.

Thus we have shown that $T(H'/H)$ and $T(H''/H)$ are H-isomorphically as $\mathfrak{L}_A$-structures. Identifying $T(H'/H) = T(H''/H)$, it suffices to show that H' and H'' are elementarily equivalent over $T(H'/H)$ (rather than over H). Hence after replacing H by $T(H'/H)$ we may assume that H is a model of $\mathfrak{T}_A$. The required fact $H' \equiv_H H''$ is an immediate consequence of the following model completeness result.

Proposition 5.3. *The $\mathfrak{L}_A$-theory $\mathfrak{T}_A$ is model complete,* i.e., *if $H \hookrightarrow G$ is an $\mathfrak{L}_A$-embedding of models of $\mathfrak{T}_A$, then G is an elementary extension of H.*

Proof. By Robinson's test [12, Proposition 3.171 it suffices to show for each such embedding $H \hookrightarrow G$ that any primitive existential sentence with parameters from H which holds in G also holds in H.

First let us show that each primitive existential sentence with parameters from H has the form

$$(\exists x)\, \varphi(x), \quad \text{where } x = (x_1, \ldots, x_n)$$

$$\text{and } \varphi(x) : \Big(\bigwedge_{1 \leq i \leq l} W_i(x) = a_i \Big) \wedge \Big(\bigwedge_{l+1 \leq i \leq m} vW_i(x) \geq va_i \Big)$$

$$\text{with } W_i(x) = \prod_{1 \leq i \leq n} x_j^{s_{ij}},\ s_{ij} \in \mathbb{Z},\ a_i \in H.$$

It suffices to observe that

$$\mathfrak{T}_A \vdash x \neq y \leftrightarrow (vx \neq vy) \vee \Big(\bigvee_{1 \notin u \in A} y = xu \Big) \leftrightarrow (v(xy^{-1}) \geq vt)$$

$$\vee\ (v(yx^{-1}) \geq vt) \vee \Big(\bigvee_{1 \neq u \in A} yx^{-1} = u \Big)$$

and

$$\mathfrak{T}_A \vdash vx < 0 \leftrightarrow vx^{-1} \geq vt.$$

So let $\varphi(x)$ be as above and assume $G \models (\exists x)\, \varphi(x)$; let $b = (b_1, \ldots, b_n) \in G^n$ be such that $G \models \varphi(b)$. Consider the subgroup $G' = H \cdot b_1^{\mathbb{Z}} \cdot b_2^{\mathbb{Z}} \cdots b_n^{\mathbb{Z}}$ of G. As G/H is torsion free, the finitely generated subgroup G'/H is free by [22, Chapter 1, §10, Theorem 7]; in particular, G' has a direct decomposition $G' \simeq H \oplus G'/H$. Let $c = (c_1, \ldots, c_k) \in G'^k$ be such that $G' \simeq H \oplus \bigoplus_{j=1}^k c_j^{\mathbb{Z}}$. Thus each b_i admits a unique representation

$$b_i = a_i \prod_{j=1}^{k} c_j^{\alpha_{ij}}, \quad 1 \leq i \leq n,$$

with $a_i \in H, \alpha_{ij} \in \mathbb{Z}$.

Putting $x_i = a_i \prod_{j=1}^k y_j^{\alpha_{ij}}$, $1 \leq i \leq n$, the equalities which occur in $\varphi(x)$ become identities trivially satisfied thanks to the above direct decomposition of G'. The inequalities which occur in $\varphi(\boldsymbol{x})$ become inequalities in the new variables $y_1, \ldots, y_k$ of the form

$$\sum_{j=1}^{k} \beta_{ij} v y_j \geqslant v a_i, \quad l+1 \leqslant i \leqslant m, \tag{$*$}$$

with $\beta_{ij} \in \mathbb{Z}, a_i \in \boldsymbol{H}$. The system of inequalities $(*)$ admits the solution c ∈ G'". By model completeness of the theory of Z-groups [28], we obtain $\gamma_1, \ldots, \gamma_k \in vH$ such that $\sum_{j=1}^{k} \beta_{ij}\gamma_j \geqslant va_i, l+1 \leqslant i \leqslant m$. Let $d_1, \ldots, d_k \in \boldsymbol{H}$ be such that $vd_j = \gamma_j$ for $1 \leqslant j \leqslant k$, and $e_i = a_i \prod_{j=1}^{k} d_j^{\alpha_{ij}}$ for $1 \leqslant i \leqslant n$. Then $\boldsymbol{H} \models \varphi(e_1, \ldots, e_n)$ and hence $H \models (3x)\ \varphi(\boldsymbol{x})$, as contended. □

References

[1] J. Ax, A metamathematical approach to some problems in number theory, AMS Proc. Symp. Pure Math., 20 (1971) 161-190.

[2] J. Ax and S. Kochen, Diophantine problems over local fields I, Amer. J. Math. 87 (1965) 605–630.

[3] J. Ax and S. Kochen, Diophantine problems over local fields II, Amer. J. Math. 87 (1965) 631-648.

[4] J. Ax and S. Kochen, Diophantine problems over local fields III, Decidable fields, Ann. Math. 83 (1966) 437-456.

[5] S. Basarab, Some metamathematical aspects of the theory of Henselian fields, Studii Cercetari Matematice 25 (1973) 1449-1459 (in Romanian).

[6] S. Basarab, Some model theory for Henselian valued fields, J. Algebra 55 (1978) 191-212.

[7] S. Basarab, A model-theoretic transfer theorem for Henselian valued fields, J. Reine Angew. Math. 311/312 (1979) l-30.

[8] S. Basarab, Model-theoretic methods in the theory of Henselian valued fields I, Studii Cercetari Matematice 31 (1979) 3-39 (in Romanian).

[9] S. Basarab, Model-theoretic methods in the theory of Henselian valued fields II, Studii Cercetari Matematice 31 (1979) 617-656 (in Romanian).

[10] N. Bourbaki, Commutative Algebra (Mir, Moskow, 1971, in Russian).

[11] D. Cantor and P. Roquette, On diophantine equations over the ring of all algebraic integers, J. Number Theory 18 (1984) l-26.

[12] C.C. Chang and H.J. Keisler, Mode1 Theory, Stud. Logic Found. Math. 73 (North-Holland, Amsterdam, 1973).

[13] G. Cherlin and M.A. Dickmann, Real closed rings II. Model theory, Ann. Pure Appl. Logic 25 (3) (1983) 213-231.

[14] P. Cohen, Decision procedures for real and p-adic fields, Comm. Pure Appl. Math. 22 (1969) 131-151.

[15] F. Delon, Quelques proprietés des corps values en théorie des modèles, These, Paris (1981).

[16] J. Denef, The rationality of the Poincaré series associated to the p-adic points on a variety, Invent. Math. 77 (1984) l-23.

[17] Ju. Ershov, On the elementary theory of maximal valued fields, Algebra i Logika Sem. 4 (1965) 31-70 (in Russian).

[18] Ju. Ershov, On the elementary theory of maximal valued fields II, Algebra i Logika Sem. 5 (1966) 5-40 (in Russian).

[19] Ju. Ershov, On the elementary theory of maximal valued fields III, Algebra i Logika Sem. 6 (1967) 31-38 (in Russian).

[20] Ju. Ershov, On rational points over Henselian fields, Algebra i Logika Sem. 6 (1967) 39-49.

[21] S. Kochen, The model theory of local fields, in: Lecture Notes in Math. 499 (Springer, Berlin, 1975) 384-425.
[22] S. Lang, Algebra (Addison-Wesley, Reading, MA, 1965).
[23] A. Macintyre, On definable subsets of p-adic fields, J. Symbolic Logic 41 (1976) 605-611.
[24] A. Prestel and P. Roquette, Formally p-adic Fields, Lecture Notes in Math. 1050 (Springer, Berlin, 1984).
[25] P. Ribenboim, Théorie des Valuations (Les Presses de l'Université de Montreal, 2nd ed., 1968).
[26] A. Robinson, Problems and methods in model theory, Lecture Notes, Varrena, CIME (1968) 183-266.
[27] A. Robinson, Elementary embeddings of fields of power series, J. Number Theory 2 (1970) 237-247.
[28] A. Robinson and E. Zakon, Elementary properties of ordered Abelian groups, Trans. Amer. Math. Soc. 96 (1960) 222-236.
[29] P. Roquette, Some tendencies in contemporary algebra, to appear.
[30] G. Sacks, Saturated Model Theory (Addison-Wesley, Reading, MA, 1972).
[31] L. van den Dries, Elementary invariants for Henselian valuation rings of mixed characteristic, Manuscript notes.
[32] V. Weispfenning, On the elementary theory of Hensel fields, Ann. Math. Logic 10 (1) (1976) 59-93.
[33] V. Weispfenning, Model theory of lattice products, Habilitationsschrift, Heidelberg (1978).
[34] V. Weispfenning, Quantifier elimination and decision procedures for valued fields, in: Logic Colloquium, Aachen 1983, Lecture Notes in Math., to appear.
[35] M. Ziegler, Die elementare Theorie der Henselschen Körper, Dissertation, Köln (1972).