ELSEVIER
Discrete Applied Mathematics 79 (1997) 127–135
DISCRETE APPLIED MATHEMATICS

Bypaths in tournaments

Yubao Guo*, Lutz Volkmann

Lehrstuhl II für Mathematik, RWTH Aachen, Templergraben 55, 52056 Aachen, Germany

Received 30 August 1995; received in revised form 18 September 1996; accepted 5 March 1997

Abstract

If every arc of a 3-connected tournament T is contained in a cycle of length 3, then every arc of T has a bypath of length k for each $k \geqslant 3$, unless T is isomorphic to two tournaments, each of which has exactly 8 vertices. This extends the corresponding result for regular tournaments, due to Alspach, Reid and Roselle (1974).

1. Introduction

A digraph D on $n \geqslant 3$ vertices is said to be *arc-k-cyclic* ($3 \leqslant k \leqslant n$) if every arc of D is contained in a cycle of length k. An arc of D is *pancyclic* if it is contained in a cycle of length ℓ for each $\ell \in \{3, 4, \dots, n\}$. A digraph D is *arc-pancyclic* if every arc of D is pancyclic.

A *bypath* of an arc xy is a path from x to y. A digraph D is said to be *arc-k-anticyclic* if every arc of D has a bypath of length $k-1$.

If a digraph D is arc-pancyclic and arc-k-anticyclic for each $k \geqslant 3$, then we say that D is *completely strong path-connected.*

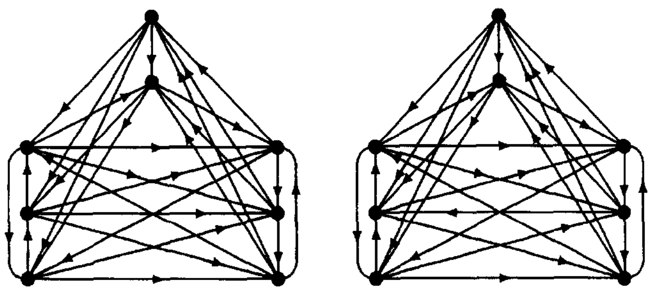
In 1967, Alspach [1] proved that every regular tournament is arc-pancyclic. Since then, many mathematicians have studied the path-connectivity in tournaments. Most of them used the following two different types of conditions.

The first one is in terms of degree (see [2, 4, 5, 10]). For example, Alspach, Reid and Roselle [2] proved that every arc of a regular tournament with $n \geqslant 7$ has bypaths of all lengths $k \geqslant 3$.

The second one is the arc-3-cyclicity condition (see [6–9]). For example, it has been proved in [6] that except for T_8^1, T_8^2 (see Fig. 1 below) and a certain family of 2-connected counterexamples, every arc-3-cyclic tournament is arc-pancyclic. In 1982, Zhang [8] presented that a tournament T is completely strong path-connected if and only if T is arc-3-cyclic, arc-3-anticyclic and T is 2-connected.

* Corresponding author. E-mail: guo@mathc.rwth-aachen.de.

0166-218X/97/$17.00
PII S0166-218X(97)00038-3



T_8^1 T_8^2

Fig. 1.

In this paper we shall prove that every arc of an arc-3-cyclic and 3-connected tournament T has bypaths of all lengths $k \geqslant 3$, unless T is isomorphic to T_8^1 or to T_8^2 (see Fig. 1).

It is easy to see from our result that the arc-3-anticyclicity condition for an arc-3-cyclic tournament to be completely strong path-connected is of consequence only for those tournaments that are exactly 2-connected (see Corollary 3.5 below). Furthermore, since every regular tournament on at least 7 vertices is arc-3-cyclic and 3-connected (by Lemma 2.2 below), our result extends that one of [2].

2. Terminology and preliminaries

Let D be a digraph on n vertices. We denote by $V(D)$ and $E(D)$ the vertex set and the arc set of D, respectively. A subdigraph induced by a subset $A \subseteq V(D)$ is denoted by $D[A]$. If xy is an arc of D, then we say that x *dominates* y and we sometimes use the notation $x \rightarrow y$ to denote this arc. More generally, if A and B are two disjoint subdigraphs of D such that every vertex of A dominates every vertex of B, then we say that A *dominates* B, denoted by $A \rightarrow B$.

The *outset* of a vertex $x \in V(D)$ is the set $N^+(x) = \{y \mid xy \in E(D)\}$. Similarly, $N^-(x) = \{y \mid yx \in E(D)\}$ is the *inset* of x. More generally, for a subdigraph A of D, we define its *outset* by $N^+(A) = \bigcup_{x \in V(A)} N^+(x) - A$ and its *inset* by $N^-(A) = \bigcup_{x \in V(A)} N^-(x) - A$.

The numbers $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ are called *outdegree* and *indegree* of x, respectively. If every vertex of a digraph D has outdegree r and indegree r, then we say that D is *r-regular*. The *irregularity* of a digraph D is the number $i(D) = \max\{|d^+(x) - d^-(x)| \mid x \in V(D)\}$.

Paths and cycles in a digraph are always directed. A cycle with the vertices $x_1, x_2, \ldots, x_k$ and the arcs $x_1x_2, x_2x_3, \ldots, x_kx_1$ is called a *k-cycle* and it is denoted by $x_1x_2 \ldots x_kx_1$. A path from x to y is called an *(x,y)-path*.

A *strong component* H of D is a maximal subdigraph such that for any two vertices $x, y \in V(H)$, the subdigraph H contains a path from x to y and a path from y to x. The digraph D is *strong* or *strongly connected*, if it has only one strong component, and D is k-connected if for any set A of at most $k-1$ vertices, the subdigraph $D - A$ is strong.

If we replace every arc xy of D by yx, then we call the resulting digraph (denoted by D^{-1}) the *converse digraph* of D.

A digraph is *acyclic* if it contains no cycle. A subdigraph of D is called a *spanning subdigraph* if it contains all vertices of D.

The next result plays an important role in our proofs.

Theorem 2.1 (Thomassen [5]). *Let T be a tournament and x, y distinct vertices of T. Then T has a Hamiltonian path from x to y if and only if T has a spanning acyclic subdigraph D such that for each vertex z of T, D contains an (x,z)-path and a (z,y)-path.*

For the proof of Corollary 3.4 we need the following Lemma, which is a part of Lemma 4.1 in [5].

Lemma 2.2. *Let T be a tournament on n vertices and $\mathrm{i}(T) \leqslant k$. Then T is $\lceil (n-2k)/3 \rceil$-connected.*

3. Main results

Before we present our main theorem, we prove two lemmas.

Lemma 3.1. *Let T be an arc-3-cyclic tournament on $n \geqslant 4$ vertices. If T contains a path $P = a_1 a_2 \ldots a_k$ with $3 \leqslant k \leqslant n-1$, but there is no path from a_1 to a_k of length k, then for every vertex $v \notin V(P)$, there exists an integer $\mu(v)$ with $1 \leqslant \mu(v) < k$ such that $N^+(v) \cap V(P) = \{a_1, a_2, \ldots, a_{\mu(v)}\}$.*

Proof. Let $H = V(T - V(P))$. First we show that $N^+(v) \cap V(P) \neq \emptyset$ and $N^-(v) \cap V(P) \neq \emptyset$ for each $v \in H$. Suppose that there is a vertex $v \in H$ with $N^+(v) \cap V(P) = \emptyset$. This means that $P \to v$. Since $a_k v$ is in a 3-cycle, there is a vertex w with $v \to w \to a_k$. Clearly, $w \in H$. Now we see that $a_1 a_2 \ldots a_{k-2} v w a_k$ is of length k, a contradiction. Therefore, every vertex $v \in H$ has at least one out-neighbour in P. Similarly, it can be shown that $N^-(v) \cap V(P) \neq \emptyset$.

Let $\mu(v) = \max\{i \mid v \to a_i\}$. From $N^-(v) \cap V(P) \neq \emptyset$ and the assumption that T contains no path from a_1 to a_k of length k, it is easy to check that

$$1 \leqslant \mu(v) < k \quad \text{and} \quad N^+(v) \cap V(P) = \{a_1, a_2, \ldots, a_{\mu(v)}\}.$$

This completes the proof of the lemma. □

Lemma 3.2. *Let T be an arc-3-cyclic and 3-connected tournament on n vertices. Suppose that T contains a path $P=a_1a_2\ldots a_k$ with $4 \leqslant k \leqslant n-1$ and $a_1 \to a_k$, but T contains no (a_1,a_k)-path of length k, then the following statements are valid:*
(a) *$k \geqslant 6$ and there exists an integer μ with $3 \leqslant \mu \leqslant k-3$ such that*

$$B \to (T - V(P)) \to A,$$

where $A=\{a_1,a_2,\ldots,a_\mu\}$ and $B=\{a_{\mu+1},a_{\mu+2},\ldots,a_k\}$.
(b) *$N^+(a_\mu)\cap B=\{a_{\mu+1}\}$ or $N^-(a_{\mu+1})\cap A=\{a_\mu\}$.*
(c) *If $N^+(a_\mu)\cap B=\{a_{\mu+1}\}$, then $T[B]$ contains a unique $(a_{\mu+1},a_k)$-path. If $N^-(a_{\mu+1})\cap A=\{a_\mu\}$, then $T[A]$ contains a unique (a_1,a_μ)-path.*

Proof. Let $H=V(T-V(P))$. According to Lemma 3.1, for every vertex v of H, there is an integer $\mu(v)$ with $1 \leqslant \mu(v)<k$ such that

$$N^+(v)\cap V(P)=\{a_1,a_2,\ldots,a_{\mu(v)}\}.$$

Suppose $k=4$. Because $a_4 \to H \to a_1$ and T is 3-connected, $T[V(P)]$ is a transitive tournament. Since the arc a_2a_3 is in a 3-cycle, there is a vertex $w\in H$ such that $a_3 \to w \to a_2$. Thus, $a_1a_3wa_2a_4$ is a path of length 4, a contradiction to the assumption that a_1a_k has no bypath of length k. So we have $k \geqslant 5$.

Let $H_i=\{v \mid v\in H,\ |N^+(v)\cap V(P)|=i\}$ and

$$\alpha=\min\{i \mid H_i\neq\emptyset\} \quad\text{and}\quad \beta=\max\{i \mid H_i\neq\emptyset\}.$$

From the assumption that T is 3-connected, we see that $k-\alpha,\ \beta \geqslant 3$.

In the following, we shall show that if $\alpha<\beta$, then there is always an (a_1,a_k)-path of length k.

Suppose that $\alpha=1$. Since T is 3-connected, there is an arc uv from H_1 to H_γ for some $\gamma \geqslant 2$. Obviously, we may assume $N^+(H_1)\cap H_i=\emptyset$ for all $i \geqslant 4$. It follows that $2 \leqslant \gamma \leqslant 3$. Since a_4u is in a 3-cycle, we obtain $a_1 \to a_4$.

At first we consider the case when $\beta \geqslant 4$. From the assumption above, we have $H_\beta \to H_1$. Because every arc from H_β to a_2 is in a 3-cycle, it is clear that $a_2 \to a_t$ for some $t>\beta$. Now we see that $a_1a_4\ldots a_{t-1}uva_2a_t\ldots a_k$ is a path of length k.

Thus we consider the case when $\beta\leqslant 3$. In fact, $\beta=3$ holds. Similar to the discussion above, we may assume that $a_t \to a_2$ for all $t \geqslant 5$. Since T is 3-connected, the set $\{a_5,\ldots,a_k\}$ has at least three in-neighbours. It follows that $a_3 \to a_\ell$ for some $\ell \geqslant 5$. If there is an arc wz from H_i to H_3 with $i \leqslant 2$, then $a_1a_4\ldots a_{\ell-1}wza_3a_\ell\ldots a_k$ is a path of length k. So we assume that $N^-(H_3)\cap H_i=\emptyset$ for $i=1,2$. But now, any arc from H_3 to H_1 is not in a 3-cycle, a contradiction.

Suppose now that $\alpha\geqslant 2$. Moreover, we may assume that $\beta\leqslant k-2$ (otherwise, one can consider the tournament T^{-1}). Since a_1 has at least three out-neighbours and $H \to a_1$, we have $a_1 \to a_p$ for some p with $3 \leqslant p \leqslant k-1$. Let $u\in H_\alpha$ and $v\in H_\beta$. Assume that $3 \leqslant p \leqslant \beta$. Since va_{p-1} is in a 3-cycle, there is a vertex z with $a_{p-1} \to z \to v$. If $z\in V(P)$, then $z=a_i$ for some $i>\beta$ and $a_1a_p\ldots a_{i-1}ua_2\ldots a_{p-1}a_i\ldots a_k$ is a path of length k. If $z \in H$, then we conclude from $\alpha \geqslant 2$ that $p \geqslant 4$. Since the arc va_{p-2} is in a 3-cycle, there is a vertex z' with $a_{p-2} \to z' \to v$. In the case $z' \in H$,

$$a_1 a_2 \dots a_{p-2} z' v a_p a_{p+1} \dots a_{k-1} a_k$$

is a desired path. In the other case, it is obvious that $z' = a_j$ for some $j > \beta$. Now we see that $a_1 a_p \dots a_{j-1} z v a_2 \dots a_{p-2} a_j \dots a_k$ is a path of length k. Hence, we may assume that $\beta < p < k$. By considering the converse digraph T^{-1}, it can be assumed that $a_q \to a_k$ for some q with $2 \leqslant q \leqslant \alpha$. Obviously, $p - q \geqslant 2$ holds.

If $p - q = 2$, then $a_1 a_p \dots a_{k-1} u v a_2 \dots a_q a_k$ or $a_1 a_p \dots a_{k-1} v u a_2 \dots a_q a_k$ is a path of length k. So we assume that $p - q \geqslant 3$. In the case $a_{q+1} \to u$,

$$a_1 a_p \dots a_{k-1} v a_{q+1} \dots a_{p-2} u a_2 \dots a_q a_k$$

is of length k. In the other case $u \to a_{q+1}$, the path $a_1 a_p \dots a_{k-1} v a_{q+2} \dots a_{p-1} u a_2 \dots a_q a_k$ is of length k.

From the proof above and the assumption that T contains no (a_1, a_k)-path of length k, we conclude that $\alpha = \beta$. This means that all vertices of H have the same out-neighbours and the same in-neighbours in P. Let

$$\mu = \max\{i \mid a_i \in N^+(H)\},$$

$$A = \{a_1, a_2, \dots, a_\mu\} \quad \text{and} \quad B = \{a_{\mu+1}, a_{\mu+2}, \dots, a_k\}.$$

Then $N^-(H) \cap V(P) = B \to H \to A = N^+(H) \cap V(P)$. By the assumption that T is 3-connected, it follows that $3 \leqslant \mu \leqslant k - 3$. This proves (a).

To prove (b), let z be a vertex of H. Suppose that $a_i \to a_{\mu+1}$ and $a_\mu \to a_j$ for some i, j satisfying $1 \leqslant i < \mu$ and $\mu + 1 < j \leqslant k$, respectively. Then

$$a_1 \dots a_i a_{\mu+1} \dots a_{j-1} z a_{i+1} \dots a_\mu a_j \dots a_k$$

is of length k, a contradiction. Hence, $N^+(a_\mu) \cap B = \{a_{\mu+1}\}$ or $N^-(a_{\mu+1}) \cap A = \{a_\mu\}$.

Now we prove (c). We consider the case $N^+(a_\mu) \cap B = \{a_{\mu+1}\}$. Suppose that $T[B]$ contains an arc $a_i a_j$ with $j \geqslant i + 2$. Since the arc $a_{i+1} z$ with $z \in H$ is in a 3-cycle, there is a vertex a_ℓ with $z \to a_\ell \to a_{i+1}$. From the assumption $N^+(a_\mu) \cap B = \{a_{\mu+1}\}$, we see that $\ell < \mu$. But now, $a_1 \dots a_\ell a_{i+1} \dots a_{j-1} z a_{\ell+1} \dots a_i a_j \dots a_k$ is a path of length k, a contradiction. Therefore, $T[B]$ contains the unique $(a_{\mu+1}, a_k)$-path $a_{\mu+1} a_{\mu+2} \dots a_k$. Similarly, it can be shown that if $N^-(a_{\mu+1}) \cap A = \{a_\mu\}$, then $T[A]$ is a tournament containing a unique (a_1, a_μ)-path. □

Theorem 3.3. *Let T be an arc-3-cyclic and 3-connected tournament on n vertices. Then every arc of T has bypaths of all lengths $m \geqslant 3$, unless T is isomorphic to T_8^1 or to T_8^2 (see Fig. 1).*

Proof. Let x, y be two distinct vertices of T with $x \to y$. First we shall show that xy has a bypath of length 3. If $|N^+(x) \cap N^-(y)| \geqslant 2$, then T contains an (x, y)-path of length 3. So we assume that $|N^+(x) \cap N^-(y)| \leqslant 1$. But then the set $Q = N^+(x) \cap N^+(y)$

is nonempty since T is 3-connected. Let u be a vertex of Q. Because yu is in a 3-cycle, there is a vertex v with $u \to v \to y$. Now we see that $xuvy$ is a desired path.

Suppose that T contains (x, y)-paths of all lengths from 3 to $k-1$, but T contains no (x, y)-path of length k with $4 \leqslant k \leqslant n-1$. Let $P = a_1 a_2 \ldots a_k$ be an (x, y)-path of length $k-1$ and $H = V(T - V(P))$. According to Lemma 3.2 (a), $k \geqslant 6$ and there is an integer μ with $3 \leqslant \mu \leqslant k-3$ such that $\{a_{\mu+1}, a_{\mu+2}, \ldots, a_k\} \to H \to \{a_1, a_2, \ldots, a_\mu\}$. We denote $A = \{a_1, a_2, \ldots, a_\mu\}$ and $B = \{a_{\mu+1}, a_{\mu+2} \ldots, a_k\}$.

From Lemma 3.2 (b), $N^+(a_\mu) \cap B = \{a_{\mu+1}\}$ or $N^-(a_{\mu+1}) \cap A = \{a_\mu\}$ holds. At first we investigate the case

$$N^+(a_\mu) \cap B = \{a_{\mu+1}\}. \tag{1}$$

Since T is 3-connected, the set $A - \{a_\mu\}$ contains at least two vertices which belong to $N^-(B - \{a_{\mu+1}\})$. Let

$$p = \max\{i \mid a_i \in N^-(B - a_{\mu+1}) \cap A\} \quad \text{and} \quad q = \min\{j \mid \mu + 2 \leqslant j \leqslant k,\ a_p \to a_j\}.$$

Then, we have $1 < p < \mu$. Let $F = T[\{a_{p+1}, \ldots, a_\mu\}]$. It is a simple matter to verify that the strong components of F can be ordered in a unique way $F_1, \ldots, F_\alpha$ ($\alpha \geqslant 1$) such that $F_i \to F_j$ for $j > i$ if $\alpha \geqslant 2$.

In the following proof, we always note that F_i consists of a single vertex or it has a Hamiltonian cycle (see [3]).

From the definition of the integer p and the fact that every arc from H to F is contained in a 3-cycle, we deduce that

$$(B - \{a_{\mu+1}\}) \to F \to a_{\mu+1}. \tag{2}$$

In addition, we conclude from the assumption that xy has no bypath of length k that

$$F_1 \to a_i \quad \text{for all } i < p. \tag{3}$$

Since x has at least three out-neighbours, there is an integer m with $3 \leqslant m \leqslant k-1$ such that $x \to a_m$. On the other hand, we deduce from Lemma 3.2 (c) that $a_\ell \to y$ for some ℓ satisfying $2 \leqslant \ell \leqslant p$. Let z be a vertex of H. We consider the following two cases.

Case 1: $\alpha \geqslant 2$. Since every arc from F_1 to F_i ($i \geqslant 2$) is in a 3-cycle, it is easy to check from (2) and (3) that $a_p \to F_1 \to F_i \to a_p$.

Assume $\alpha \geqslant 3$. Since every arc from F_2 to F_3 is in a 3-cycle, there is a vertex $a_i \in N^-(F_2)$ with $1 \leqslant i < p$. Thus, the two paths, obtained in order of

$$a_1, \ldots, a_i, F_2, a_{\mu+1}, \ldots, a_{q-1}, z, a_{i+1}, \ldots, a_p, a_q, \ldots, a_k$$

and in order of $z, F_1, F_3, \ldots, F_\alpha, a_p$, respectively, satisfy the conditions of Theorem 2.1, and hence T contains an (x, y)-path of length k, a contradiction.

Assume now that $\alpha = 2$. If there is a vertex a_i with $1 \leqslant i < p$ such that $a_i \to a_{\mu+1}$, then the two paths, obtained in order of

$$a_1, \ldots, a_i, a_{\mu+1}, \ldots, a_{q-1}, z, a_{i+1}, \ldots, a_p, a_q, \ldots, a_k$$

and in order of z, F_1, F_2, a_p, respectively, satisfy the condition of Thorem 2.1, and hence T contains an (x, y)-path of length k, a contradiction. Therefore, $a_{\mu+1} \to a_i$ for all $i < p$. In particular, $a_{\mu+1} \to x$.

Suppose that $\mu+2 \leqslant m \leqslant k-1$. If $\ell = p$, then the path obtained in order of $x, a_m, \dots, a_{k-1}, F_1, F_2, a_{\mu+1}, \dots, a_{m-1}, z, a_2, \dots, a_p, y$ yields a contradiction. If $\ell < p$, then the two paths, obtained in order of

$$x, a_m, \dots, a_{k-1}, F_2, a_{\mu+1}, \dots, a_{m-1}, z, a_2, \dots, a_\ell, y$$

and in order of $z, a_{\ell+1}, \dots, a_p, F_1, a_\ell$, respectively, satisfy the condition of Theorem 2.1, and hence T contains an (x, y)-path of length k, a contradiction. Therefore, x has no out-neighbour in $B - \{y\}$.

Suppose now that $3 \leqslant m \leqslant p$. Since the arc za_{m-1} is in a 3-cycle and $a_{\mu+1} \to a_i$ for all $i < p$, $a_{m-1} \to a_t$ for some $t \geqslant \mu+2$. But now, the (x, y)-path obtained in order of $x, a_m, \dots, a_p, F_1, F_2, a_{\mu+1}, \dots, a_{t-1}, z, a_2, \dots, a_{m-1}, a_t, \dots, a_k$ yields a contradiction.

From (3), it remains to consider the case $a_m \in V(F_2)$. If $|V(F_2)| \geqslant 3$, then the two paths, obtained in order of $a_1, a_m, a_{\mu+1}, \dots, a_{q-1}, z, a_2, \dots, a_p, a_q, \dots, a_k$ and in order of $z, F_1, F_2 - a_m, a_p$, respectively, satisfy the condition of Theorem 2.1, and hence T contains an (x, y)-path of length k, a contradiction. So, we assume that F_2 consists of the unique vertex a_m. Since a_m has at least three out-neighbours, $a_m \to a_i$ for some i with $2 \leqslant i \leqslant p-1$. This implies that $p \geqslant 3$. Thus, a path obtained in order of $a_1, a_m, a_{\mu+1}, \dots, a_{q-1}, z, F_1, a_2, \dots, a_p, a_q, \dots, a_k$ is an (x, y)-path of length k, a contradiction.

Case 2: $\alpha = 1$. Suppose that $3 \leqslant m \leqslant p$. Since za_{m-1} is in a 3-cycle, $a_{m-1} \to a_t$ for some $t \geqslant \mu+1$.

If $t \geqslant \mu+2$, then $a_1 a_m a_{m+1} \dots a_{t-1} z a_2 \dots a_{m-1} a_t \dots a_k$ is an (x, y)-path of length k, a contradiction. Hence, we have $N^+(a_{m-1}) \cap B = \{a_{\mu+1}\}$.

If $m < p$, then we see from (3) that the path

$$a_1 \dots a_{m-1} a_{\mu+1} \dots a_{q-1} z a_{p+1} \dots a_\mu a_m \dots a_p a_q \dots a_k$$

is of length k, a contradiction. Therefore, $m = p$. In fact, we have proved that $a_{\mu+1} \to a_i$ for all $i \leqslant p-2$ if $p \geqslant 3$.

Assume $p \geqslant 4$. Since the arc za_{p-2} is in a 3-cycle, then $a_{p-2} \to a_j$ with $j \geqslant \mu+2$. By (3) and the conclusions above, $a_1 a_p a_{p+1} \dots a_\mu a_{p-1} a_{\mu+1} \dots a_{j-1} z a_2 \dots a_{p-2} a_j \dots a_k$ is a path of length k in T, a contradiction.

Assume now that $p = 3$. According to (3) and $N^+(a_{m-1}) \cap B = \{a_{\mu+1}\}$, the vertex a_2 has exactly two out-neighbours a_3 and $a_{\mu+1}$. This contradicts the assumption that T is 3-connected.

Suppose now that $m = \mu+1$. It is a simple matter to verify that $p = 2$ and $a_2 \to F$. But now, $N^+(F) = \{a_1, a_{\mu+1}\}$, a contradiction.

Suppose finally that $m \geqslant \mu+2$ and $a_i \to x$ for all $3 \leqslant i \leqslant \mu+1$. If $a_p \to y$, then we see from (2) that the path $xa_m \dots a_{k-1} a_{p+1} a_{p+2} \dots a_{m-1} z a_2 \dots a_p y$ yields a contradiction. So, we may assume that $y \to a_p$. It follows that $2 \leqslant \ell < p$ with $a_\ell \to y$ and $q < k$.

If $k \geqslant \mu+4$, then we see from Lemma 3.2 (c) and (2) that

$$a_1 a_m \ldots a_{k-1} a_{\mu+1} \ldots a_{m-1} z a_{\ell+1} a_{\ell+2} \ldots a_\mu a_2 \ldots a_\ell y$$

is a path of length k in T, a contradiction.

Assume thus that $k=\mu+3$. It is not difficult to check that $a_{\ell+1} \to a_{k-1}$ and $q=k-1$. If there is a vertex a_i with $i<p-1$ such that $a_i \to a_{\mu+1}$, then

$$a_1 \ldots a_i a_{\mu+1} z a_{p+1} \ldots a_\mu a_{i+1} a_{i+2} \ldots a_p a_{k-1} a_k$$

is of length k, a contradiction. Hence, we have $a_{\mu+1} \to a_i$ for all $i \leqslant p-2$.

If $\ell \leqslant p-2$ or $\ell \geqslant 3$, then the path

$$a_1 a_{k-1} z a_{\ell+1} a_{\ell+2} \ldots a_{\mu+1} a_2 \ldots a_\ell y$$

is of length k, a contradiction. Therefore, $p=3$ and $\ell=2$. In the case $|V(F)| \geqslant 3$,

$$a_1 a_{k-1} a_5 a_6 \ldots a_{\mu+1} z a_3 a_4 a_2 a_k$$

yields a contradiction. In the other case $|V(F)|=1$ we have $k=7$. If $D[H]$ contains an arc zz', then $a_1 a_6 a_4 a_5 z z' a_2 a_7$ is a path of length 7 in T, again a contradiction. So, T has exactly 8 vertices. It is easy to see that $a_2 \to a_5$. If $a_3 \to a_5$ and $a_2 \to a_6$, then T is isomorphic to T_8^1. If $a_3 \to a_5$ and $a_6 \to a_2$, then T is isomorphic to T_8^2. We note that $a_1 a_6 z a_3 a_4 a_2 a_7$ is also a bypath of the arc xy. If $a_5 \to a_3$ and $a_6 \to a_2$, then T is isomorphic to T_8^1. If $a_5 \to a_3$ and $a_2 \to a_6$, then T is isomorphic to T_8^2.

If $N^-(a_{\mu+1}) \cap A = \{a_\mu\}$, then we consider the converse digraph D^{-1} of D. Since the converse digraph of T_8^i is isomorphic to T_8^{3-i} for $i=1,2$, the proof of the theorem is complete. □

Corollary 3.4 (Alspach, Reid and Roselle [2]). *Every arc of a regular tournament T on at least 7 vertices has bypaths of all lengths $m \geqslant 3$.*

Proof. It is easy to see that T is arc-3-cyclic. By Lemma 2.2, T is 3-connected. So the corollary follows from Theorem 3.3. □

Combining Theorem 3.3 and the main result of [6], which is mentioned in the first section of this paper, we obtain the following statement.

Corollary 3.5. *An arc-3-cyclic and 3-connected tournament T is arc-pancyclic and arc-k-anticyclic for each $k \geqslant 4$, unless T is isomorphic to T_8^1 or to T_8^2.*

References

[1] B. Alspach, Cycles of each length in regular tournaments, Canad. Math. Bull. 10 (1967) 283–285.

[2] B. Alspach, K.B. Reid, D.P. Roselle, Bypasses in asymmetric digraphs, J. Combin. Theory Ser. B 17 (1974) 11–18.

[3] P. Camion, Chemins et circuits hamiltoniens des graphs complets, C. Roy. Acad. Sci. Paris 249 (1959) 2151–2152.
[4] O.S. Jakobsen, Cycles and paths in tournaments, Thesis, University of Aarhus, Denmark, 1972.
[5] C. Thomassen, Hamiltonian-connected tournaments, J. Combin. Theory Ser. B 28 (1980) 142–163.
[6] F. Tian, Z.S. Wu, C.Q. Zhang, Cycles of each length in tournaments, J. Combin. Theory Ser. B 33 (1982) 245–255.
[7] Z.S. Wu, K.M. Zhang, Y. Zou, A necessary and sufficient condition for arc-pancyclicity of tournaments, Sci. Sinica Ser. A 25 (1982) 249–254.
[8] K.M. Zhang, Completely strong path-connected tournaments, J. Combin. Theory Ser. B 33 (1982) 166–177.
[9] K.M. Zhang, Z.S. Wu, A necessary and sufficient condition for completely strong path-connected tournaments, Chinese Ann. Math. Ser. A4 (3) (1983) 385–392.
[10] Y. Zhu, F. Tian, On the strong path connectivity of tournaments, Sci. Sinica (special issue) (II) (1979) 18–28.