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An Expansion Formula for the Askey–Wilson Function

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The Askey–Wilson function transform is a q-analogue of the Jacobi function transform with kernel given by an explicit non-polynomial eigenfunction of the Askey–Wilson second order q-difference operator. The kernel is called the

of an Askey–Wilson polynomial multiplied by an analogue of the Gaussian is computed explicitly. As a special case of these formulas a q-analogue (in one variable) of the Macdonald–Mehta integral is obtained, for which also two alternative, direct proofs are presented.

1. INTRODUCTION

The Macdonald polynomials and their orthogonality relations have an harmonic analytic interpretation on quantum compact Riemannian symmetric spaces; see, e.g., Noumi [16]. In particular, the spherical Fourier transform on the quantum $SU(2)$ group can be identified with the polynomial Askey–Wilson transform, which is the transform naturally associated to the orthogonality relations of the Askey–Wilson polynomials (see Koornwinder [13]).

In the non-compact set-up, only harmonic analysis on the quantum $SU(1, 1)$ group has been well understood by now; see, e.g., [10, 11]. This has led to the study of an explicit generalized Fourier transform in [11, 12], called the Askey–Wilson function transform. The kernel of this transform is called the Askey–Wilson function. It is a non-polynomial eigenfunction of the Askey–Wilson second-order q-difference operator, given explicitly as a very-well-poised ${}_8\phi_7$ series.

On the other hand, Cherednik [5] discussed several types of difference Fourier transforms, which are naturally related to the spectral theory of

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Macdonald polynomials. Cherednik [5] showed that a particular theta-function plays a role in the theory of the difference transforms which is similar to the role of the Gaussian in the theory of Hankel transforms, see also [7]. This led to explicit formulas for the image under the difference Fourier transforms of a Macdonald polynomial multiplied by (the inverse of) the analogue of the Gaussian. Furthermore, in the difference set-up Cherednik [5] defined certain "non-polynomial" spherical functions as explicit series expansions in terms of "polynomial" spherical functions (= Macdonald polynomials), which seems to be purely a quantum phenomenon.

The purpose of the present paper is to incorporate the above mentioned ideas and constructions of Cherednik into the theory of the polynomial Askey–Wilson transform, and into the theory of the Askey–Wilson function transform. We first give an explicit expansion formula for the Askey–Wilson function in terms of Askey–Wilson polynomials. This expansion formula provides an explicit link between Cherednik's construction [5] of non-polynomial eigenfunctions of q-difference operators with the constructions of Suslov [19], Ismail and Rahman [9] using the theory of basic hypergeometric series. We introduce the proper analogue of the Gaussian for the Askey–Wilson theory, and we compute the image under the polynomial Askey–Wilson transform of an Askey–Wilson polynomial multiplied by the inverse of the Gaussian. In the special case of continuous q-ultraspherical polynomials, these formulas were derived by Cherednik in [6]. Furthermore, we compute the image under the Askey–Wilson function transform of an Askey–Wilson polynomial multiplied by the Gaussian. A special case leads to the evaluation of a q-analogue (in one variable) of the Macdonald–Mehta integral (cf. Macdonald [14]).

The techniques employed in this paper are entirely based on basic hypergeometric series manipulations in the spirit of Gasper and Rahman's book [8]. The two main ingredients are the orthogonality relations for the Askey–Wilson polynomials (see [2]), and the inversion formula for the Askey–Wilson function transform (see [12]).

A generalization of Cherednik's affine Hecke algebra approach [5] to the Askey–Wilson level leads to independent proofs of the Plancherel and inversion formula for the Askey–Wilson function transform, and to independent proofs of most of the formulas presented in this paper. In fact, the affine Hecke algebra approach reduces the problem to the explicit evaluation of the q-analogue of the (one variable) Macdonald–Mehta integral. I therefore have added two alternative proofs of the evaluation of the (one variable) q-Macdonald–Mehta integral in this paper, which do not make use of the properties of the Askey–Wilson function transform. I will discuss the affine Hecke algebra approach in a future paper.

The plan of the paper is as follows. In Section 2 we recall the basic properties of the Askey–Wilson polynomials. In Section 3 we give the definition of the Askey–Wilson function. The expansion formula for the Askey–Wilson function in terms of Askey–Wilson polynomials is formulated in Section 4. In Section 4 we also introduce the analogue of the Gaussian, and we explicitly compute the image under the polynomial Askey–Wilson transform of an Askey–Wilson polynomial multiplied by the inverse of the Gaussian. In Section 5 the Askey–Wilson function transform and its basic properties are recalled, and the image under the Askey–Wilson function transform of an Askey–Wilson polynomial multiplied by the Gaussian is computed explicitly. We also show how this leads to the evaluation of a q-analogue (in one variable) of the Macdonald–Mehta integral. In Section 6 some density results are discussed, which are relevant for the L^2-theory of the Askey–Wilson function transform. This leads to explicit parameter restraints for which the formulas derived in Section 5 completely determine the Askey–Wilson function transform. Appendix A contains a proof of (a reformulation of) the expansion formula for the Askey–Wilson function. Appendix B contains two direct proofs for the evaluation of the q-analogue of the (one variable) Macdonald–Mehta integral.

Notations and Conventions. Throughout the paper we fix $0<q<1$. The notation $\mathbb{C}^\times$ and $\mathbb{R}^\times$ is used for $\mathbb{C}\setminus\{0\}$ and $\mathbb{R}\setminus\{0\}$, respectively. The nonnegative integers $\{0, 1, 2, ...\}$ are denoted by $\mathbb{Z}_+$. The book [8] of Gasper and Rahman is used as main reference for notations and results concerning basic hypergeometric series. For $k\in\mathbb{Z}\cup\{\infty\}$ we write $(x_1, ..., x_r; q)_k = \prod_{i=1}^r (x_i; q)_k$ with $(x; q)_\infty = \prod_{i=0}^\infty (1-xq^i)$ for $k=\infty$ and $(x; q)_k = (x; q)_\infty / (xq^k; q)_\infty$ for $k\in\mathbb{Z}$. Similarly, we write $\theta(a_1, ..., a_r) = \prod_{i=1}^r \theta(a_i)$ with $\theta(a) = (a, q/a; q)_\infty$ for (products) of renormalized Jacobi theta functions. The series expansion

$$
{}_r\phi_s\left(\begin{matrix} a_1, a_2, ..., a_r \\ b_1, b_2, ..., b_s \end{matrix}; q, z\right) = \sum_{k=0}^\infty \frac{(a_1, a_2, ..., a_r; q)_k}{(q, b_1, ..., b_s; q)_k} \left[(-1)^k q^{\frac{1}{2}k(k-1)}\right]^{1+s-r} z^k
$$

defines the ${}_r\phi_s$ basic hypergeometric series. The very-well-poised ${}_8\phi_7$ basic hypergeometric series is defined by

$$
\begin{aligned}
&{}_8W_7(a; b, c, d, e, f; q, z) \\
&\quad = \sum_{k=0}^\infty \frac{1-aq^{2k}}{1-a} \frac{(a, b, c, d, e, f; q)_k\, z^k}{(q, qa/b, qa/c, qa/d, qa/e, qa/f; q)_k}.
\end{aligned}
$$

The bilateral basic hypergeometric series ${}_r\psi_s$ is defined by

$$
{}_r\psi_s\left(\begin{matrix} a_1, a_2, ..., a_r \\ b_1, b_2, ..., b_s \end{matrix}; q, z\right) = \sum_{n\in\mathbb{Z}} \frac{(a_1, a_2, ..., a_r; q)_n}{(b_1, b_2, ..., b_s; q)_n} \left[(-1)^n q^{\frac{1}{2}n(n-1)}\right]^{s-r} z^n.
$$

We use the branch of the square root $\sqrt{\cdot}$ which is positive on $\mathbb{R}_{>0}$, with branch cut along the half-line $(-\infty, 0)$ of the complex plane.

2. THE ASKEY–WILSON POLYNOMIALS

In order to fix notations, we recall the basic properties of the Askey–Wilson polynomials in this section.

The Askey–Wilson polynomials depend, besides q, on four parameters a, b, c, d. To simplify notations it is convenient to use the short-hand notation

$$\alpha = (a, b, c, d)$$

for the four-tuple of parameters a, b, c, d, which we assume throughout this section to be generically complex and subject to the condition $\mathrm{Re}(a) > 0$. We define dual parameters

$$\alpha_\sigma = (a_\sigma, b_\sigma, c_\sigma, d_\sigma) \tag{2.1}$$

by

$$a_\sigma = \sqrt{q^{-1}abcd}\,, \qquad b_\sigma = ab/a_\sigma, \qquad c_\sigma = ac/a_\sigma, \qquad d_\sigma = ad/a_\sigma.$$

This notation turns out to be quite useful later on when we have to compose involutions on parameter sets. Since dual parameters play an important role throughout the paper, it is convenient to have a second, less cumbersome notation at hand. This second notation is

$$\alpha_\sigma = \tilde{\alpha}, \qquad (a_\sigma, b_\sigma, c_\sigma, d_\sigma) = (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}),$$

in accordance with [12]. The map $\alpha \mapsto \tilde{\alpha}$ defines an involution on the four tuple of parameters α. Here the condition $\mathrm{Re}(a) > 0$ is needed in view of the chosen branch for $\sqrt{\cdot}$, see the conventions at the end of the introduction. Observe in particular that $\mathrm{Re}(\tilde{a}) = \mathrm{Re}(\sqrt{q^{-1}abcd}) > 0$ for generic parameters α in view of the chosen branch for $\sqrt{\cdot}$.

Remark 2.1. Throughout the paper we formulate the results under the assumption $\mathrm{Re}(a) > 0$ in order to be able to use the duality involution σ without worrying about the chosen branch of the square-root. In most formulas the condition $\mathrm{Re}(a) > 0$ can be easily removed by analytic continuation.

We define a discrete subset $\mathcal{S}=\mathcal{S}(\alpha;q)\subset\mathbb{C}^\times$ by

$$\mathcal{S}=\{s_m \mid m\in\mathbb{Z}_+\},\qquad s_m=s_m(\alpha;q)=\tilde{a}q^m.$$

The Askey–Wilson polynomials $E_s(x)=E_s(x;\alpha;q)\quad(s\in\mathcal{S})$ are defined by the series expansion

$$\begin{aligned}E_{s_n}(x)&={}_4\phi_3\left(\begin{matrix}\tilde{a}\,s_n,\ \tilde{a}/s_n,\ ax,\ a/x\\ ab,\ ac,\ ad\end{matrix};q,q\right)\\ &={}_4\phi_3\left(\begin{matrix}q^{-n},\ q^{n-1}abcd,\ ax,\ a/x\\ ab,\ ac,\ ad\end{matrix};q,q\right)\end{aligned}\tag{2.2}$$

for $n\in\mathbb{Z}_+$; see [2]. For fixed $s\in\mathcal{S}$, the Askey–Wilson polynomial $E_s(x)$ is an eigenfunction of the Askey–Wilson second-order q-difference operator $L=L(\alpha;q)$,

$$\begin{aligned}(Lp)(x)&=C(x)(p(qx)-p(x))+C(x^{-1})(p(q^{-1}x)-p(x)),\\ C(x)&=\frac{(1-ax)(1-bx)(1-cx)(1-dx)}{(1-x^2)(1-qx^2)},\end{aligned}\tag{2.3}$$

with eigenvalue $\mu(s)$, where

$$\mu(\gamma)=-1-\tilde{a}^2+\tilde{a}(\gamma+\gamma^{-1}).$$

Furthermore, the Askey–Wilson polynomial $E_s(x)$ has the duality property

$$E_s(v)=\tilde{E}_v(s),\qquad s\in\mathcal{S},\qquad v\in\tilde{\mathcal{S}},\tag{2.4}$$

where $\tilde{\mathcal{S}}=\mathcal{S}(\tilde{\alpha};q)$ and $\tilde{E}_v(\cdot)=E_v(\cdot\,;\tilde{\alpha};q)$ for $v\in\tilde{\mathcal{S}}$, since $\tilde{a}\tilde{b}=ab$, $\tilde{a}\tilde{c}=ac$ and $\tilde{a}\tilde{d}=ad$. Let $\mathcal{T}=\mathcal{T}_{\alpha,q}$ be a closed, counterclockwise oriented contour in the complex plane, for which the sequences $eq^{\mathbb{Z}_+}$ (respectively $e^{-1}q^{-\mathbb{Z}_+}$) are in the interior (respectively exterior) of $\mathcal{T}$ for all $e=a,b,c,d$. In case $|a|,|b|,|c|,|d|<1$, one can for instance take for $\mathcal{T}$ the unit circle $\mathbb{T}$ in the complex plane.

We call a function $f\colon\mathbb{C}^\times\to\mathbb{C}$ *inversion-invariant* if $f(x^{-1})=f(x)$ for all $x\in\mathbb{C}^\times$. For "sufficiently nice" inversion-invariant functions f we define the *polynomial Askey–Wilson transform* $(\mathbb{F}f)(s)=(\mathbb{F}(\alpha,q)f)(s)$ of f at $s\in\mathcal{S}$ by

$$(\mathbb{F}f)(s)=\frac{1}{4\pi iN}\int_{\mathcal{T}}f(x)\,E_s(x)\,\Delta(x)\,\frac{dx}{x},$$

where $\Delta(x)=\Delta(x;\alpha;q)$ is the weight function

$$\Delta(x)=\frac{(x^2,1/x^2;q)_\infty}{(ax,a/x,bx,b/x,cx,c/x,dx,d/x;q)_\infty},$$

and the constant $N=N(\alpha;q)$ is the Askey–Wilson integral

$$N=\frac{1}{4\pi i}\int_{\mathscr{T}}\Delta(x)\,\frac{dx}{x} = \frac{(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}; \tag{2.5}$$

see [2]. Conversely, for "sufficiently nice" functions $g\colon \mathscr{S}\to\mathbb{C}$ we define the transform $(\mathbb{I}g)(x)=(\mathbb{I}(\alpha,q)\,g)(x)$ $(x\in\mathbb{C}^\times)$ by

$$(\mathbb{I}g)(x)=\sum_{s\in\mathscr{S}} g(s)\,E_s(x)\,\tilde{h}(s),$$

with the weight $\tilde{h}(s)=h(s;\tilde{\alpha};q)$ for $s\in\mathscr{S}$ given by

$$\tilde{h}(s_m)=\frac{\operatorname*{Res}_{x=s_m}\left(\frac{\tilde{\Delta}(x)}{x}\right)}{\operatorname*{Res}_{x=s_0}\left(\frac{\tilde{\Delta}(x)}{x}\right)} = \frac{(1-q^{2m-1}abcd)(q^{-1}abcd,ab,ac,ad;q)_m}{(1-q^{-1}abcd)(q,bc,bd,cd;q)_m}\,a^{-2m} \tag{2.6}$$

for $m\in\mathbb{Z}_+$; cf. [17]. Here $\tilde{\Delta}(x)=\Delta(x;\tilde{\alpha};q)$ is the weight function $\Delta(x)$ with respect to dual parameters. In this paper, we consider the transform $\mathbb{F}$ respectively $\mathbb{I}$ with respect to two classes of functions f respectively g. We first consider the function space $\mathscr{A}=\mathbb{C}[x+x^{-1}]$ consisting of inversion-invariant Laurent polynomials in the variable x for $\mathbb{F}$. Observe that the Askey–Wilson polynomials $\{E_s\mid s\in\mathscr{S}\}$ form a linear basis of $\mathscr{A}$. The corresponding function space $\mathscr{F}_0(\mathscr{S})$ for $\mathbb{I}$ consists of functions $g\colon\mathscr{S}\to\mathbb{C}$ with finite support. The set of delta-functions $\{\delta_s\mid s\in\mathscr{S}\}$, with $\delta_s(v)=\delta_{s,v}$ for $s,v\in\mathscr{S}$, forms a linear basis of $\mathscr{F}_0(\mathscr{S})$. The orthogonality relations

$$\frac{1}{4\pi i N}\int_{\mathscr{T}} E_s(x)\,E_v(x)\,\Delta(x)\,\frac{dx}{x}=\delta_{s,v}\,\frac{1}{\tilde{h}(s)},\qquad s,v\in\mathscr{S}, \tag{2.7}$$

for the Askey–Wilson polynomials (see [2, Theorem 2.3]), imply that $\mathbb{F}(E_s)=\tilde{h}(s)^{-1}\,\delta_s$ for $s\in\mathscr{S}$. On the other hand, $\mathbb{I}(\delta_s)=\tilde{h}(s)\,E_s$ by the definition of $\mathbb{I}$. This immediately leads to the following theorem.

THEOREM 2.2. *$\mathbb{F}$ defines a linear bijection $\mathbb{F}: \mathscr{A} \to \mathscr{F}_0(\mathscr{S})$. Its inverse is given by $\mathbb{I}: \mathscr{F}_0(\mathscr{S}) \to \mathscr{A}$.*

3. THE ASKEY–WILSON FUNCTION

In this section we recall the definition of the Askey–Wilson function (see, e.g., [9, 12, 19, 20, 21]), and give some of its basic properties. The Askey–Wilson function is a non-polynomial eigenfunction of the Askey–Wilson second order q-difference operator, given explicitly as a very-well-poised ${}_8\phi_7$ series. Recall that an explicit basis of eigenfunctions for the Askey–Wilson second-order q-difference operator in terms of very-well-poised ${}_8\phi_7$ series is known; see Ismail and Rahman [9] (compare also with Suslov [19]). See Ruijsenaars [18] for the case $|q|=1$, which requires a completely different approach.

We assume that the parameters $\alpha=(a,b,c,d)$ are generically complex, and subject to the condition $\mathrm{Re}(a)>0$. The Askey–Wilson function $\phi_\gamma(x)=\phi_\gamma(x;\alpha;q)$ is defined by

$$\phi_\gamma(x)=\frac{(qax\gamma/\tilde{d},\, qa\gamma/\tilde{d}x,\, qabc/d;\, q)_\infty}{(\tilde{a}\tilde{b}\tilde{c}\gamma,\, q\gamma/\tilde{d},\, qx/d,\, q/dx,\, bc,\, qb/d,\, qc/d;\, q)_\infty} \times {}_8W_7(\tilde{a}\tilde{b}\tilde{c}\gamma/q;\, ax,\, a/x,\, \tilde{a}\gamma,\, \tilde{b}\gamma,\, \tilde{c}\gamma\,;\, q,\, q/\tilde{d}\gamma), \qquad |q/\tilde{d}\gamma|<1. \tag{3.1}$$

Note that $\phi_\gamma(x)$ is normalized differently compared with [12]. It is known that

$$(L\phi_\gamma)(x)=\mu(\gamma)\,\phi_\gamma(x),$$

where L is the Askey–Wilson second-order q-difference operator (2.3), see e.g. [9, 12, 20]. In view of Bailey's formula [8, (2.10.10)] we can write

$$\begin{aligned}\phi_\gamma(x)=&\frac{(qabc/d;\, q)_\infty}{(bc,\, qa/d,\, qb/d,\, qc/d,\, q/ad;\, q)_\infty}\, {}_4\phi_3\left(\begin{matrix} ax,\, a/x,\, \tilde{a}\gamma,\, \tilde{a}/\gamma \\ ab,\, ac,\, ad\end{matrix};\, q,\, q\right)\\ &+\frac{(ax,\, a/x,\, \tilde{a}\gamma,\, \tilde{a}/\gamma,\, qabc/d;\, q)_\infty}{(qx/d,\, q/dx,\, q\gamma/\tilde{d},\, q/\tilde{d}\gamma,\, ab,\, ac,\, bc,\, qa/d,\, ad/q;\, q)_\infty}\\ &\times {}_4\phi_3\left(\begin{matrix} qx/d,\, q/dx,\, q\gamma/\tilde{d},\, q/\tilde{d}\gamma \\ qb/d,\, qc/d,\, q^2/ad\end{matrix};\, q,\, q\right).\end{aligned} \tag{3.2}$$

In particular, $\phi_\gamma(x)$ extends to a meromorphic function in $(\gamma, x) \in \mathbb{C}^\times \times \mathbb{C}^\times$ and is inversion-invariant in both x and γ. The possible poles of $\phi_\gamma(x)$ are simple and are located at $x^{\pm 1} = q^{1+k}/d$ $(k \in \mathbb{Z}_+)$ and $\gamma^{\pm 1} = q^{1+k}/\tilde{d}$ $(k \in \mathbb{Z}_+)$. It follows from (3.2) that

$$\phi_s(x) = \frac{(qabc/d; q)_\infty}{(bc, q/ad, qa/d, qb/d, qc/d; q)_\infty} E_s(x), \qquad s \in \mathcal{S}, \tag{3.3}$$

and that

$$\phi_\gamma(x) = \tilde{\phi}_x(\gamma), \tag{3.4}$$

where $\tilde{\phi}_x(\gamma) = \phi_x(\gamma; \tilde{\alpha}; q)$ is the Askey–Wilson function with respect to dual parameters. Formula (3.3) implies that the Askey–Wilson function is a meromorphic continuation of the Askey–Wilson polynomial in its degree. We will refer to formula (3.3) as the *polynomial reduction* of the Askey–Wilson function. Formula (3.4) implies that the geometric parameter x and the spectral parameter γ of the Askey–Wilson function are interchangeable in a suitable sense. We will refer to formula (3.4) as the *duality* of the Askey–Wilson function. It extends the duality (2.4) of the Askey–Wilson polynomial.

4. THE EXPANSION FORMULA

We assume in this section that the parameters $\alpha = (a, b, c, d)$ are generically complex and subject to the condition $\mathrm{Re}(a) > 0$. In order to formulate the expansion formula for the Askey–Wilson function in terms of Askey–Wilson polynomials, it is important to keep track of two involutions on the four tuples $\alpha = (a, b, c, d)$. First we have the concept of dual parameters, which we have already used extensively in the previous sections. It is now more convenient to write the dual parameter with sub-index σ, so

$$\alpha_\sigma = (a_\sigma, b_\sigma, c_\sigma, d_\sigma) = (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = \tilde{\alpha},$$

with α_σ defined by (2.1). Second, we define τ by

$$\alpha_\tau = (a_\tau, b_\tau, c_\tau, d_\tau) = (a, b, c, q/d). \tag{4.1}$$

We admit compositions of σ and τ, for instance we write

$$(\alpha_\sigma)_\tau = \alpha_{\sigma\tau} = (a_{\sigma\tau}, b_{\sigma\tau}, c_{\sigma\tau}, d_{\sigma\tau})$$

for first applying σ to α, and then applying τ to α_σ, i.e.,

$$\alpha_{\sigma\tau} = (\tilde{a}, \tilde{b}, \tilde{c}, q/\tilde{d}).$$

Observe that

$$\alpha_{\sigma\tau\sigma} = \alpha_{\tau\sigma\tau}, \qquad \alpha_{\tau\tau} = \alpha. \tag{4.2}$$

Furthermore, we have seen in Section 2 that $\alpha_{\sigma\sigma} = \alpha$ since $\mathrm{Re}(a) > 0$.

Finally we use the convention that if $H = H(\alpha)$ is an object depending on α, then e.g. $H^{\sigma\tau}$, or $H_{\sigma\tau}$, denotes the same object in which the four tuple α is replaced by $\alpha_{\sigma\tau}$. We sometimes write $\tilde{H} = H^\sigma$ to simplify notations.

We define the *Gaussian* $G(x) = G(x; \alpha; q)$ by

$$G(x) = (dx, d/x; q)_\infty^{-1}. \tag{4.3}$$

The terminology stems from Cherednik's work [5, 6] on Gaussians associated with Macdonald polynomials.

The *analytic part* $\phi_\gamma^{an}(x) = \phi_\gamma^{an}(x; \alpha; q)$ of the Askey–Wilson function $\phi_\gamma(x) = \phi_\gamma(x; \alpha; q)$ is defined by

$$\begin{aligned}\phi_\gamma^{an}(x) &= G^\tau(x)^{-1}\, G^{\sigma\tau}(\gamma)^{-1}\, \phi_\gamma(x)\\ &= (qx/d, q/dx, q\gamma/\tilde{d}, q/\tilde{d}\gamma; q)_\infty\, \phi_\gamma(x).\end{aligned} \tag{4.4}$$

The properties of $\phi_\gamma(x)$ as described in Section 3 imply that $\phi_\gamma^{an}(x)$ is analytic in $(\gamma, x) \in \mathbb{C}^\times \times \mathbb{C}^\times$.

Finally we observe that $\mathcal{S}^\tau = \mathcal{S}^{\sigma\tau}$, since

$$s_m^\tau = s_m^{\sigma\tau} = q^m \sqrt{abc/d}\,, \qquad m \in \mathbb{Z}_+. \tag{4.5}$$

We can now formulate the following key proposition.

PROPOSITION 4.1. *For $s \in \mathcal{S}^\tau = \mathcal{S}^{\sigma\tau}$ and $\gamma \in \mathbb{C}^\times$,*

$$(\mathbb{F}^\tau(\phi_\gamma^{an}))(s) = \frac{G^{\tau\sigma\tau}(s)}{G^{\tau\sigma\tau}(s_0^\tau)}\, E_s^{\sigma\tau}(\gamma). \tag{4.6}$$

The proof of the proposition, which is based on direct calculations using the theory of basic hypergeometric series, is given in Appendix A. Proposition 4.1 leads to the following expansion formula for the analytic part $\phi_\gamma^{an}(x)$ of the Askey–Wilson function.

Theorem 4.2 (The Expansion Formula).

$$
\begin{aligned}
\phi_\gamma^{an}(x) &= G^{\tau\sigma\tau}(s_0^\tau)^{-1} \sum_{s \in \mathcal{S}^\tau} h^{\tau\sigma}(s)\, G^{\tau\sigma\tau}(s)\, E_s^{\sigma\tau}(\gamma)\, E_s^\tau(x) \\
&= \sum_{m=0}^{\infty} {}_4\phi_3\left(\begin{matrix} q^{-m}, q^m abc/d, ax, a/x \\ ab, ac, qa/d \end{matrix} ; q, q \right) \\
&\quad \times {}_4\phi_3\left(\begin{matrix} q^{-m}, q^m abc/d, \tilde{a}\gamma, \tilde{a}/\gamma \\ ab, ac, bc \end{matrix} ; q, q \right) \\
&\quad \times \frac{(1-abcq^{2m}/d)(abc/d, ab, ac; q)_m}{(1-abc/d)(q, qb/d, qc/d; q)_m} \left(\frac{-1}{ad} \right)^m q^{m(m+1)/2}
\end{aligned}
\tag{4.7}
$$

for all $(\gamma, x) \in \mathbb{C}^\times \times \mathbb{C}^\times$.

Proof. First observe that the terms $h^{\tau\sigma}(s)\, E_s^{\sigma\tau}(\gamma)\, E_s^\tau(x)$ for $s \in \mathcal{S}^\tau$ in the expansion sum (4.7) are well defined in view of (4.5). Observe furthermore that the second expansion sum in (4.7) converges absolutely and uniformly for (γ, x) in compacta of $\mathbb{C}^\times \times \mathbb{C}^\times$ due to the Gaussian type factor $q^{m(m+1)/2}$ (use, e.g., [8, (7.5.13)] to control the convergence of the ${}_4\phi_3$'s in the expansion sum). In particular, the second expansion sum in (4.7) is analytic in $(\gamma, x) \in \mathbb{C}^\times \times \mathbb{C}^\times$. We can now verify the second identity in (4.7) termwise using the explicit expression for the Askey–Wilson polynomial as a balanced ${}_4\phi_3$ series (see (2.2)), and using the identities

$$
\frac{G^{\tau\sigma\tau}(s_m^\tau)}{G^{\tau\sigma\tau}(s_0^\tau)} = \frac{(bc; q)_m}{(qa/d; q)_m} \left(\frac{-a}{d} \right)^m q^{m(m+1)/2}
$$

and

$$
h^{\tau\sigma}(s_m^\tau) = \frac{(1-abcq^{2m}/d)(abc/d, ab, ac, qa/d; q)_m}{(1-abc/d)(q, bc, qb/d, qc/d; q)_m} a^{-2m}
$$

for $m \in \mathbb{Z}_+$. So it remains to prove the first identity in (4.7). Denote $\psi_\gamma(x)$ for the right hand side of (4.7), which we consider for arbitrary, fixed $\gamma \in \mathbb{C}^\times$ as an analytic, inversion-invariant function in $x \in \mathbb{C}^\times$. Recall that the defining expansion sum for $\psi_\gamma(x)$ converges absolutely and uniformly for x in compacta of $\mathbb{C}^\times$. In particular, when applying the polynomial Askey–Wilson transform $\mathbb{F}^\tau$ to ψ_γ, it is allowed to interchange summation and integration. Combined with the orthogonality relations (2.7) for the Askey–Wilson polynomials, we obtain for $s \in \mathscr{S}^{\tau}$,

$$
\begin{aligned}
(\mathbb{F}^{\tau}(\psi_{\gamma}))(s) &= G^{\tau\sigma\tau}(s_0^{\tau})^{-1} \sum_{v \in \mathscr{S}^{\tau}} h^{\tau\sigma}(v)\, G^{\tau\sigma\tau}(v)\, E_v^{\sigma\tau}(\gamma)(\mathbb{F}^{\tau}(E_v^{\tau}))(s) \\
&= \frac{G^{\tau\sigma\tau}(s)}{G^{\tau\sigma\tau}(s_0^{\tau})}\, E_s^{\sigma\tau}(\gamma) \\
&= (\mathbb{F}^{\tau}(\phi_{\gamma}^{an}))(s),
\end{aligned}
$$

where the last equality follows from Proposition 4.1. Since any analytic, inversion-invariant function $f\colon \mathbb{C}^{\times} \to \mathbb{C}$ is uniquely determined by its image under the polynomial Askey–Wilson transform $\mathbb{F}^{\tau}$, we conclude that $\psi_{\gamma}(x) = \phi_{\gamma}^{an}(x)$ for all $x \in \mathbb{C}^{\times}$, as desired. ∎

Remark 4.3. The right hand side of the expansion formula (4.7) resembles the (non-symmetric) Poisson-kernel for Askey–Wilson polynomials, see [1]. The essential difference is the occurrence of the Gaussian $G^{\tau\sigma\tau}(s)$ in the expansion sum (4.7), which is not present in the Poisson-kernels. It is also due to this extra factor that the expansion sum (4.7) has better convergence properties.

We explore now the implications of formula (4.6) for the polynomial Askey–Wilson transform $\mathbb{F}$ and its inverse $\mathbb{I}$. Let $\mathscr{F}(\mathscr{S})$ be the space of functions $g\colon \mathscr{S} \to \mathbb{C}$.

PROPOSITION 4.4. *For $s \in \mathscr{S}^{\tau} = \mathscr{S}^{\sigma\tau}$, we have*

$$
\mathbb{F}(E_s^{\tau} G^{-1}) = \frac{(bc, bc, d/a, ad, q/ad, bd, cd; q)_{\infty}}{(abcd; q)_{\infty}}\, G^{\tau\sigma\tau}(s)\, E_s^{\sigma\tau} G^{\sigma\tau} \tag{4.8}
$$

as functions in $\mathscr{F}(\mathscr{S})$.

Proof. Let $s \in \mathscr{S}^{\tau}$ and $v \in \mathscr{S}$. It follows from the polynomial reduction (3.3) of the Askey–Wilson function, (4.4) and (4.6), that

$$
\begin{aligned}
(\mathbb{F}(E_s^{\tau} G^{-1}))(v) &= \frac{(bc, d/a, ad, bd, cd; q)_{\infty}}{(abcd; q)_{\infty}}\, (\mathbb{F}(\phi_s^{\tau} G^{-1}))(v) \\
&= \frac{(bc, d/a, ad, bd, cd; q)_{\infty}}{(abcd; q)_{\infty}}\, G^{\tau\sigma\tau}(s)(\mathbb{F}(\phi_s^{an,\tau}))(v) \\
&= \frac{(bc, d/a, ad, bd, cd; q)_{\infty}}{(abcd; q)_{\infty}}\, \frac{G^{\tau\sigma\tau}(s)\, G^{\sigma\tau}(v)}{G^{\sigma\tau}(s_0)}\, E_v^{\tau\sigma\tau}(s).
\end{aligned}
$$

Now $\mathscr{S} = \mathscr{S}^{\sigma\tau\sigma}$ and $\mathscr{S}^{\tau} = \mathscr{S}^{\sigma\tau} = \mathscr{S}^{\sigma\tau\sigma\sigma}$, so we conclude from (4.2) and from the duality (2.4) of the Askey–Wilson polynomials that

$$E_v^{\tau\sigma\tau}(s) = E_v^{\sigma\tau\sigma}(s) = E_s^{\sigma\tau}(v). \tag{4.9}$$

Furthermore, $G^{\sigma\tau}(s_0) = (bc, q/ad; q)_\infty^{-1}$, hence

$$(\mathbb{F}(E_s^{\tau} G^{-1}))(v) = \frac{(bc, bc, d/a, ad, q/ad, bd, cd; q)_\infty}{(abcd; q)_\infty} G^{\tau\sigma\tau}(s)\, E_s^{\sigma\tau}(v)\, G^{\sigma\tau}(v),$$

as desired. ∎

Remark 4.5. Observe that

$$G(x)^{-1} \Delta(x) = \frac{(x^2, 1/x^2; q)_\infty}{(ax, a/x, bx, b/x, cx, c/x; q)_\infty}$$

is the orthogonality weight function for the continuous dual q-Hahn polynomials. Formula (4.8) thus computes the constant term of the product of two Askey–Wilson polynomials (with parameters $(a, b, c, q/d)$ and (a, b, c, d) respectively) with respect to the continuous dual q-Hahn orthogonality measure. This observation leads to the following alternative way to prove Proposition 4.4. First use the explicit expansion of the Askey–Wilson polynomial as linear combination of continuous dual q-Hahn polynomials (see [8, (7.6.8) and (7.6.9)], and be aware of the fact that a factor $(q; q)_n$ is missing in the numerator of [8, (7.6.9)]), and substitute these for the two Askey–Wilson polynomials E_s^{τ} $(s \in \mathscr{S}^{\tau})$ and E_v $(v \in \mathscr{S})$ in $(\mathbb{F}(E_s^{\tau} G^{-1}))(v)$. Using the orthogonality relations for the continuous dual q-Hahn polynomials we arrive at a single sum, which is easily seen to give the same result as (4.8).

Cherednik's formulas [6, (1.15)] involving continuous q-ultraspherical polynomials can now be generalized to the level of Askey–Wilson polynomials as follows.

THEOREM 4.6. *The polynomial Askey–Wilson transform* $\mathbb{F} = \mathbb{F}(\alpha; q)$ *defines a linear bijection*

$$\mathbb{F}\colon \mathscr{A} G^{-1} \to (\mathscr{A} G^{\sigma\tau})|_{\mathscr{S}} \subset \mathscr{F}(\mathscr{S}),$$

with inverse

$$\mathbb{I}\colon (\mathscr{A} G^{\sigma\tau})|_{\mathscr{S}} \to \mathscr{A} G^{-1}.$$

Explicitly, we have

$$
\begin{aligned}
\mathbb{F}(E_s^{\tau} G^{-1}) &= \frac{(bc, bc, d/a, ad, q/ad, bd, cd; q)_\infty}{(abcd; q)_\infty} G^{\tau\sigma\tau}(s)\, E_s^{\sigma\tau} G^{\sigma\tau},\\
\mathbb{I}(E_s^{\sigma\tau} G^{\sigma\tau}) &= \frac{(abcd; q)_\infty}{(bc, bc, d/a, ad, q/ad, bd, cd; q)_\infty} G^{\tau\sigma\tau}(s)^{-1}\, E_s^{\tau} G^{-1},
\end{aligned}
\tag{4.10}
$$

for $s \in \mathcal{S}^{\tau} = \mathcal{S}^{\sigma\tau}$.

Proof. In view of the previous proposition, it suffices to prove the explicit formula for $\mathbb{I}(E_s^{\sigma\tau} G^{\sigma\tau})$.

Let $s \in \mathcal{S}^{\tau} = \mathcal{S}^{\sigma\tau}$, then $\mathbb{I}(E_s^{\sigma\tau} G^{\sigma\tau})(x)$ is given by a series expansion in Askey–Wilson polynomials $E_v(x)$ $(v \in \mathcal{S})$ which converges absolutely and uniformly on compacta of $x \in \mathbb{C}^{\times}$, compare with the proof of Theorem 4.2. In particular, $\mathbb{I}(E_s^{\sigma\tau} G^{\sigma\tau})$ is an inversion-invariant, analytic function. Furthermore, when applying $\mathbb{F}$ to $\mathbb{I}(E_s^{\sigma\tau} G^{\sigma\tau})$ we may interchange summation and integration. The orthogonality relations (2.7) for the Askey–Wilson polynomials then show that

$$\mathbb{F}(\mathbb{I}(E_s^{\sigma\tau} G^{\sigma\tau})) = E_s^{\sigma\tau} G^{\sigma\tau}.$$

On the other hand, (4.8) shows that $E_s^{\sigma\tau} G^{\sigma\tau} \in \mathcal{F}(\mathcal{S})$ is the image under $\mathbb{F}$ of the analytic, inversion-invariant function

$$\frac{(abcd; q)_\infty}{(bc, bc, d/a, ad, q/ad, bd, cd; q)_\infty} G^{\tau\sigma\tau}(s)^{-1}\, E_s^{\tau} G^{-1}.$$

Since any inversion-invariant, analytic function $f\colon \mathbb{C}^{\times} \to \mathbb{C}$ is uniquely determined by its image under the polynomial Askey–Wilson transform $\mathbb{F}$, we conclude that

$$\mathbb{I}(E_s^{\sigma\tau} G^{\sigma\tau}) = \frac{(abcd; q)_\infty}{(bc, bc, d/a, ad, q/ad, bd, cd; q)_\infty} G^{\tau\sigma\tau}(s)^{-1}\, E_s^{\tau} G^{-1},$$

as desired. ∎

Remark 4.7. The explicit formula (4.10) for $\mathbb{I}(E_s^{\sigma\tau} G^{\sigma\tau})$ $(s \in \mathcal{S}^{\sigma\tau})$ can also be derived from the polynomial reduction (3.3) for the Askey–Wilson function and from the expansion formula (4.7), since

$$
\begin{aligned}
E_s^{\tau} G^{-1} &= \frac{(bc, d/a, ad, bd, cd; q)_\infty}{(abcd; q)_\infty} G^{-1} \phi_s^{\tau}\\
&= \frac{(bc, d/a, ad, bd, cd; q)_\infty}{(abcd; q)_\infty} G^{\tau\sigma\tau}(s)\, \phi_s^{an,\tau}
\end{aligned}
$$

$$= \frac{(bc, d/a, ad, bd, cd; q)_\infty}{(abcd; q)_\infty \, G^{\sigma\tau}(s_0)} G^{\tau\sigma\tau}(s) \sum_{v \in \mathscr{S}} h^\sigma(v) \, G^{\sigma\tau}(v) \, E_v^{\tau\sigma\tau}(s) \, E_v$$

$$= \frac{(bc, bc, d/a, ad, q/ad, bd, cd; q)_\infty}{(abcd; q)_\infty} G^{\tau\sigma\tau}(s) \sum_{v \in \mathscr{S}} h^\sigma(v) \, G^{\sigma\tau}(v) \, E_s^{\sigma\tau}(v) \, E_v$$

$$= \frac{(bc, bc, d/a, ad, q/ad, bd, cd; q)_\infty}{(abcd; q)_\infty} G^{\tau\sigma\tau}(s) \, \mathbb{I}(E_s^{\sigma\tau} G^{\sigma\tau}),$$

where the fourth equality follows from (4.9).

Remark 4.8. The special case $s = s_0^\tau$ in the formula (4.10) for $\mathbb{I}(E_s^{\sigma\tau} G^{\sigma\tau})$ gives

$$(dx, d/x; q)_\infty = \frac{(bc, ad, q/ad, bd, cd; q)_\infty}{(abcd; q)_\infty} (\mathbb{I}(G^{\sigma\tau}))(x)$$

$$= \frac{(ad, bd, cd; q)_\infty}{(abcd; q)_\infty}$$

$$\times \sum_{m=0}^{\infty} \frac{(1 - q^{2m-1} abcd)(q^{-1} abcd, ab, ac; q)_m}{(1 - q^{-1} abcd)(q, bd, cd; q)_m} \left(\frac{-d}{q^{\frac{1}{2}} a} \right)^m q^{\frac{m^2}{2}} E_{s_m}(x),$$

which may be viewed as a generalization of Jacobi's triple product identity [8, (1.6.1)] to the level of the Askey–Wilson polynomials. Specializing $x = s_0^\sigma = a$ in this formula leads to a limiting case of Rogers' [8, (2.7.1)] summation formula of a very-well-poised ${}_6\phi_5$ series,

$${}_5\phi_5 \left(\begin{matrix} \tilde{a}^2, q\tilde{a}, -q\tilde{a}, ab, ac \\ \tilde{a}, -\tilde{a}, bd, cd, 0 \end{matrix} ; q, d/a \right) = \frac{(abcd, d/a; q)_\infty}{(bd, cd; q)_\infty}.$$

The special case $(\mathbb{F}(G^{-1}))(\tilde{a})$ of (4.10) is the evaluation of the Askey–Wilson integral with one of the four parameters equal to zero; cf. Remark 4.5. In Cherednik's terminology [5], both $(\mathbb{F}(G^{-1}))(\tilde{a})$ and $(\mathbb{I}(G^{\sigma\tau}))(a)$ are (polynomial) q-analogues of the (one variable) Macdonald–Mehta integral.

5. THE ASKEY–WILSON FUNCTION TRANSFORM

In [12], Koelink and the author defined and studied a generalized Fourier transform called the Askey–Wilson function transform, whose kernel is given by the Askey–Wilson function. In this section we show that Proposition 4.1, together with the inversion formula [12, Theorem 1] for the Askey–Wilson function transform, leads to an explicit expression for the image under the Askey–Wilson function transform of an Askey–Wilson polynomial multiplied by a Gaussian. These explicit formulas lead to a non-polynomial analogue of Theorem 4.6.

We start by recalling the definition and the main properties of the Askey–Wilson function transform, see [12] for more details. We use slightly different conventions and normalizations compared with [12].

We fix a five tuple

$$\beta=(\alpha, t)=(a, b, c, d, t)\in\mathbb{R}^5$$

satisfying the conditions

$$\begin{aligned} t<0, &\qquad 0<b, c\leqslant a<d/q,\\ bd, cd\geqslant q, &\qquad ab, ac<1. \end{aligned}\tag{5.1}$$

The parameters b, c, d then automatically satisfy $b, c<1$ and $d>q$. The dual parameter $t_\sigma=\tilde{t}$ is defined by

$$t_\sigma=\tilde{t}=\frac{1}{adt}.\tag{5.2}$$

Note here the slightly different convention compared with [12, (4.4)]. The dual parameters $\beta_\sigma=\tilde{\beta}=(\tilde{\alpha}, \tilde{t})$ satisfy the same conditions (5.1) as $\beta=(\alpha, t)$, see [12, Lemma 1]. In fact, $\beta\mapsto\beta_\sigma$ defines an involution on the five tuples $\beta=(\alpha, t)$ satisfying (5.1). For an object $H=H(\beta)$ depending on β, we write H^σ, or $\tilde{H}$, for $H(\beta_\sigma)$.

A new weight function $W(x)=W(x;\beta;q)$ is defined by

$$W(x)=\Delta(x)\,\Theta(x),$$

where $\Delta(x)=\Delta(x;\alpha;q)$ is the weight function for the Askey–Wilson polynomials and $\Theta(x)=\Theta(x;\beta;q)$ is the quasi-constant

$$\Theta(x)=\frac{\theta(dx, d/x)}{\theta(dtx, dt/x)}.\tag{5.3}$$

For generic parameters β such that the weight function W has simple poles, we define a measure $m=m(\cdot\,;\beta;q)$ by

$$\begin{aligned}\int f(x)\,dm(x)=&\frac{1}{4\pi i}\int_{x\in\mathbb{T}} f(x)\,W(x)\,\frac{dx}{x}\\ &+\frac{1}{2}\sum_{x\in D}\left(f(x)+f(x^{-1})\right)\operatorname*{Res}_{y=x}\left(\frac{W(y)}{y}\right),\end{aligned}\tag{5.4}$$

where $D=D_+\cup D_-$ is the infinite, discrete set given by

$$\begin{aligned} D_+&=\{aq^k\mid k\in\mathbb{Z}_+ : aq^k>1\},\\ D_-&=\{dtq^k\mid k\in\mathbb{Z} : dtq^k<-1\}.\end{aligned}\tag{5.5}$$

We can extend the definition of the measure m to a positive measure for all parameters β satisfying (5.1), using the fact that the discrete weights $m(\{x\})=m(\{x^{-1}\})$ for $x\in D$ depend continuously on β; see [12, (5.7) and (5.8)].

Let $L^2_+(m)$ be the Hilbert space of L^2-functions f with respect to the measure m satisfying $f(x)=f(x^{-1})$ m-a.e. The Askey–Wilson function transform $\mathcal{J}=\mathcal{J}(\beta;q)$ is now defined by

$$(\mathcal{J}f)(\gamma)=\frac{1}{K}\int f(x)\,\phi_\gamma(x)\,dm(x)$$

for compactly supported functions $f\in L^2_+(m)$, with $K=K(\beta;q)$ the positive constant

$$\begin{aligned} K&=\frac{N^\tau}{\sqrt{\theta(adt,bdt,cdt,qt)}}\\ &=\frac{(qabc/d;q)_\infty}{(q,ab,ac,bc,qa/d,qb/d,qc/d;q)_\infty}\frac{1}{\sqrt{\theta(adt,bdt,cdt,qt)}}, \end{aligned} \tag{5.6}$$

where N is the Askey–Wilson integral, see (2.5).

We can now restate [12, Theorem 1] as follows.

Theorem 5.1. *The Askey–Wilson function transform $\mathcal{J}$ uniquely extends by continuity to an isometric isomorphism*

$$\mathcal{J}\colon L^2_+(m)\to L^2_+(m^\sigma).$$

The inverse of $\mathcal{J}$ is given by $\mathcal{J}^\sigma\colon L^2_+(m^\sigma)\to L^2_+(m)$.

Combined with (4.6), we obtain the following main result of this section.

Theorem 5.2. *Suppose that the parameters $\beta=(\alpha,t)$ satisfy the conditions* (5.1). *Let $s\in\mathcal{S}^\tau=\mathcal{S}^{\sigma\tau}$, then*

$$\begin{aligned} (\mathcal{J}(E^\tau_sG^\tau))(\gamma)&=\frac{1}{\sqrt{\theta(adt,bdt,cdt,qt)}}\frac{G^{\tau\sigma}(s^\tau_0)}{G^{\tau\sigma}(s)}E^{\sigma\tau}_s(\gamma)\,G^\sigma(\gamma)^{-1}\,\Theta^\sigma(\gamma)^{-1},\\ (\mathcal{J}(E^\tau_sG^{-1}\Theta^{-1}))(\gamma)&=\sqrt{\theta(adt,bdt,cdt,qt)}\frac{G^{\sigma\tau\sigma}(s)}{G^{\sigma\tau\sigma}(s^\tau_0)}E^{\sigma\tau}_s(\gamma)\,G^{\sigma\tau}(\gamma), \end{aligned} \tag{5.7}$$

as identities in $L^2_+(m^\sigma)$.

Proof. Observe that the factor $\theta(adt, bdt, cdt, qt)$ appearing in the formulas is invariant under the parameter involution σ. In view of Theorem 5.1 it thus suffices to prove the explicit evaluation formula for $\mathscr{J}(E_s^\tau G^{-1}\Theta^{-1})$ with $s \in \mathscr{S}^\tau$. We show that it is in fact a reformulation of (4.6).

Observe that $E_s^\tau G^{-1}\Theta^{-1} \in L_+^2(m)$, since it is a compactly supported function (use here that Θ^{-1} vanishes on the discrete mass points D_- of the measure m). For the moment we assume that the parameters $\beta = (\alpha, t)$ satisfy the conditions (5.1) and that they are generic. Recall that the conditions (5.1) on the parameters β imply that $0 < b, c < 1$ and $d > q$. By Cauchy's Theorem we thus conclude that

$$\begin{aligned}\frac{1}{4\pi i}\int_{\mathscr{T}^\tau} f(x)\,\Delta^\tau(x)\,\frac{dx}{x} &= \frac{1}{4\pi i}\int_{\mathbb{T}} f(x)\,\Delta^\tau(x)\,\frac{dx}{x}\\ &\quad+\frac{1}{2}\sum_{x\in D_+}(f(x)+f(x^{-1}))\,\operatorname*{Res}_{y=x}\left(\frac{\Delta^\tau(y)}{y}\right)\end{aligned}$$

for analytic, inversion-invariant functions $f\colon \mathbb{C}^\times \to \mathbb{C}$, where D_+ is given by (5.5). Since Θ^{-1} vanishes on the discrete mass points D_- of the measure m, and since

$$\begin{aligned}\phi_\gamma(x)\,\Theta(x)^{-1}\,G(x)^{-1}\,W(x) &= \phi_\gamma(x)\,G^\tau(x)^{-1}\,\Delta^\tau(x)\\ &= G^{\sigma\tau}(\gamma)\,\phi_\gamma^{an}(x)\,\Delta^\tau(x),\end{aligned}$$

we obtain for $s \in \mathscr{S}^\tau$,

$$\begin{aligned}(\mathscr{J}(E_s^\tau G^{-1}\Theta^{-1}))(\gamma) &= \frac{1}{K}\left\{\frac{1}{4\pi i}\int_{\mathscr{T}^\tau} E_s^\tau(x)\,\phi_\gamma^{an}(x)\,\Delta^\tau(x)\,\frac{dx}{x}\right\} G^{\sigma\tau}(\gamma)\\ &= \frac{N^\tau}{K}\,(\mathbb{F}^\tau(\phi_\gamma^{an}))(s)\,G^{\sigma\tau}(\gamma)\\ &= \frac{G^{\tau\sigma\tau}(s)\,N^\tau}{G^{\tau\sigma\tau}(s_0^\tau)\,K}\,E_s^{\sigma\tau}(\gamma)\,G^{\sigma\tau}(\gamma),\end{aligned}$$

where the last equality is due to (4.6). The desired identity now follows from (4.2) in view of the explicit expression (5.6) of the constant K. The generic conditions on the parameters β can be removed by continuity. ∎

Consider the sub-spaces $V_{cl} = V_{cl}(\beta; q)$ and $V_{str} = V_{str}(\beta; q)$ of $L_+^2(m)$ defined by

$$V_{cl} = \mathscr{A}G^{-1}\Theta^{-1}, \qquad V_{str} = \mathscr{A}G^\tau,$$

cf. (the proof of) Theorem 5.2. The subscripts *cl* and *str* stand for "classical" and "strange," respectively. This terminology is justified by the harmonic analytic interpretation of the Askey–Wilson function transform, see [11]. Indeed, when we regard m as the Plancherel measure for the quantum $SU(1,1)$ group, then the functions $f \in V_{cl}$ are supported on the "classical part" of the measure m, which is the part of the measure representing the contributions of the principal unitary series representations and of the positive discrete series representations to the Plancherel measure. This translates to the property that all functions $f \in V_{cl}$ vanish on the discrete mass points D_- of the measure m. The functions V_{str} on the other hand also have support on the "strange part" D_- of the measure m, which is the part of the measure representing the contributions of the strange series representations to the Plancherel measure.

Obviously $V_{cl} \cap V_{str} = \{0\}$ since any $f \in V_{cl}$ is zero on D_-. Note that $V_{cl} \oplus V_{str} \subset L^2_+(m)$ is *not* an orthogonal direct sum decomposition, since for all $s, u \in \mathscr{S}^\tau$,

$$\int \frac{E^\tau_u(x)}{G(x)\,\Theta(x)}\, E^\tau_s(x)\, G^\tau(x)\, dm(x) = \frac{N^\tau}{h^{\tau\sigma}(s)}\, \delta_{s,u}$$

by the orthogonality relations (2.7) for the Askey–Wilson polynomials; cf. the proof of Theorem 5.2. In the next section it is shown that $V_{cl} \oplus V_{str}$ is a dense sub-space of $L^2_+(m)$ if we impose an extra condition on the allowed parameter values for β (see Proposition 6.4). For these parameters β, the explicit formulas (5.7) thus completely determine the Askey–Wilson function transform. In particular, V_{str} then completely takes care of the "strange part" D_- of the measure m.

The formulas (5.7) immediately lead to the following result.

Corollary 5.3. *The restriction of the Askey–Wilson function transform* $\mathscr{J}$ *to the sub-space* $V_{cl} \subset L^2_+(m)$ *defines a linear bijection* $\mathscr{J}|_{V_{cl}}: V_{cl} \to V^\sigma_{str}$. *The inverse of* $\mathscr{J}|_{V_{cl}}$ *is given by* $\mathscr{J}^\sigma|_{V^\sigma_{str}}$.

The transform $\mathscr{J}|_{V_{cl}}$ is closely related to the polynomial Askey–Wilson transform $\mathbb{F}^\tau$ acting upon the sub-space $\mathscr{A}G^{\tau^{-1}}$. In fact, for generic β satisfying the conditions (5.1), we have

$$(\mathscr{J}(E^\tau_s G^{-1}\Theta^{-1}))(v) = \frac{(qabc/d;q)_\infty\, N^\tau}{(bc, q/ad, qa/d, qb/d, qc/d;q)_\infty\, K}\,(\mathbb{F}^\tau(E_v\, G^{\tau^{-1}}))(s) \tag{5.8}$$

for all $v \in \mathscr{S}$ and $s \in \mathscr{S}^\tau$ by the polynomial reduction (3.3) of the Askey–Wilson function, compare with the proof of Theorem 5.2. Observe in particular that the explicit formulas (4.10) for the polynomial Askey–Wilson transform $\mathbb{F}^{\tau}$ acting on $\mathscr{A}G^{\tau^{-1}}$ are direct consequence of (5.8) and the explicit formulas (5.7) for $\mathscr{J}|_{V_{cl}}$.

On the other hand, by the strong convergence of the Gaussian $G_{\tau}(x)$ as $|x|$ tends to zero it is possible to rewrite $\mathscr{J}|_{V_{str}}$ as a completely discrete transform by shrinking the radius of the integration circle $\mathbb{T}$ in m to zero while picking up residues.

These remarks show that the Askey–Wilson function transform $\mathscr{J}$ contains a continuous, polynomial ("compact") type transform and a discrete, non-polynomial ("non-compact") type transform in a natural way, which are essentially each-others inverses. In the opposite direction, one may view the Askey–Wilson function transform $\mathscr{J}$ as a *self-dual* transform obtained by glueing a continuous, compact type transform and a discrete, non-compact type transform together.

Remark 5.4. Cherednik's affine Hecke algebra approach [5] extended to the present Askey–Wilson set-up shows that there is a natural flexibility in the choice of the measure m. More precisely, it turns out that for several different choices of measure m, the associated Fourier transform $\mathscr{J}_m$ admits explicit formulas which have a similar structure as the formulas (5.7) for the Askey–Wilson function transform. This provides another explanation for the similarities between Theorem 4.6 and Theorem 5.2. The "proper choice" of measure m (and hence of Fourier transform $\mathscr{J}_m$) thus depends on the applications which one has in mind. For harmonic analysis on the quantum $SU(2)$ and quantum $SU(1,1)$ group, the "proper choice" is the polynomial Askey–Wilson transform $\mathbb{F}$ and the Askey–Wilson function transform $\mathscr{J}$, respectively (see [11, 13], respectively).

We have used the inversion formula for the Askey–Wilson function transform (see Theorem 5.1) to explicitly compute the Askey–Wilson function transform $\mathscr{J}$ acting upon functions $f \in V_{str}$, see Theorem 5.2. On the other hand, the formulas (5.7) can be used to reprove the inversion formula and Plancherel formula for the Askey–Wilson function transform for those parameter values β such that $V_{cl} \oplus V_{str} \subset L^2_+(m)$ is dense (see Proposition 6.7 for the related density result). It would therefore be of interest to have an alternative proof of the formulas (5.7) without referring to the main results of [12]. This is also of interest for the study of multivariable generalizations of the present results.

An alternative proof for the key formulas (5.7) is indeed possible using Cherednik's theory [5, 6] of affine Hecke algebras, together with some elementary elliptic function theory. We do not go into details here, but only remark that these techniques reduce the explicit computation of $\mathscr{J}$ on V_{str} to the evaluation of $(\mathscr{J}(G^{\tau}))(\tilde{a})$, which is a q-analogue of the (one variable) Macdonald–Mehta integral; cf. Macdonald [14]. The integral $(\mathscr{J}(G^{\tau}))(\tilde{a})$

can also be viewed as a "non-polynomial" analogue of the (one variable) q-Macdonald–Mehta integrals as discussed in Remark 4.8. The evaluation of $(\mathcal{J}(G^\tau))(\tilde{a})$ is equivalent to the following explicit identity.

THEOREM 5.5 (One variable q-Macdonald–Mehta integral). *For generic parameters $a, b, c, u \in \mathbb{C}^\times$ with $q < |u| < 1$ and $|a|, |b|, |c| < 1$, we have*

$$\begin{aligned}\frac{1}{4\pi i}\int_{\mathbb{T}} &\frac{(x^2, 1/x^2; q)_\infty}{(ax, a/x, bx, b/x, cx, c/x; q)_\infty\, \theta(ux, u/x)}\frac{dx}{x}\\ &+\sum_{k=1}^{\infty}\operatorname*{Res}_{x=uq^{-k}}\left(\frac{(x^2, 1/x^2; q)_\infty}{(ax, a/x, bx, b/x, cx, c/x; q)_\infty\, \theta(ux, u/x)\, x}\right)\\ =&\frac{1}{(q, ab, ac, bc; q)_\infty}\frac{\theta(abcu)}{\theta(au, bu, cu)}. \end{aligned}\tag{5.9}$$

Proof. For generic parameters β satisfying (5.1) and satisfying $|a|, |b|, |c| < 1$ and $q < |dt| < 1$, we compute $(\mathcal{J}(G^\tau))(\tilde{a})$ in two different ways. The first way is by substituting the definition of the Askey–Wilson function transform $\mathcal{J}$ and using the polynomial reduction (3.3) for the Askey–Wilson function. This gives the left hand side of (5.9) with $u = dt$, multiplied by the constant

$$\frac{(q, ab, ac; q)_\infty}{(q/ad; q)_\infty}\sqrt{\theta(adt, bdt, cdt, qt)}\,.$$

The second way to compute $(\mathcal{J}(G^\tau))(\tilde{a})$ is by using (5.7) (take $s = s_0^\tau$ and $\gamma = \tilde{a}$ in the first formula of (5.7)). This gives the explicit infinite product evaluation

$$(\mathcal{J}(G^\tau))(\tilde{a}) = \frac{1}{\sqrt{\theta(adt, bdt, cdt, qt)}}\frac{\theta(qt, abcdt)}{(bc, q/ad; q)_\infty}.$$

Combining both expressions for $(\mathcal{J}(G^\tau))(\tilde{a})$ gives (5.9) with $u = dt$. The conditions on the parameters can be removed by analytic continuation. ∎

Due to its independent interest, two alternative, direct proofs of (5.9) are given in Appendix B.

6. DENSITY RESULTS

We next address the question whether $V_{cl} \oplus V_{str}$ is a dense sub-space of $L^2_+(m)$, i.e., whether the explicit formulas (5.7) completely determine the Askey–Wilson function transform. Surprisingly, the solution to this problem can be derived from density results related to the polynomial

Askey–Wilson transform $\mathbb{F}$. We therefore first discuss the L^2-theory of $\mathbb{F}$ and of its inverse $\mathbb{I}$.

For our purposes it suffices to restrict attention to parameters $\alpha = (a, b, c, d)$ satisfying the conditions

(i) $a, b, c, d \in \mathbb{R}$,

(ii) $d < 0 < a$ and $abcd > 0$,

(iii) $ab, ac, ad, bc, bd, cd < 1$.

Under these conditions, at most two of the four parameters have moduli $\geqslant 1$. If two parameters have moduli $\geqslant 1$, then they have opposite sign. For future reference, we write V_{pol} for the set of four-tuples $\alpha = (a, b, c, d)$ satisfying the conditions (i), (ii), and (iii).

For generic $\alpha \in V_{pol}$ and for sufficiently regular, inversion-invariant functions f (e.g. for $f \in \mathscr{A}$ or $f \in \mathscr{A}G^{-1}$), we can rewrite the Fourier transform $(\mathbb{F}f)(s)$ $(s \in \mathscr{S})$ by Cauchy's Theorem as

$$(\mathbb{F}f)(s) = \int f(x)\, E_s(x)\, dv(x), \tag{6.1}$$

with $v = v(\cdot\,; \alpha; q)$ the positive measure

$$\int f(x)\, dv(x) = \frac{1}{4\pi i N} \int_{\mathbb{T}} f(x)\, \Delta(x) \frac{dx}{x} + \frac{1}{2N} \sum_{x \in F} (f(x) + f(x^{-1}))\, \operatorname*{Res}_{y = x} \left(\frac{\Delta(y)}{y} \right), \tag{6.2}$$

with $F = F(\alpha; q)$ the finite, discrete set

$$F = \{ eq^k \mid e \in \{a, b, c, d\}, k \in \mathbb{Z}_+ \text{ such that } |eq^k| > 1 \}$$

and with N the Askey–Wilson integral (2.5). By continuity in the parameters α, both (6.1) and (6.2) may be extended to all parameters $\alpha \in V_{pol}$. We use the notation $L^2_+(v)$ for the L^2-functions $f\colon \mathbb{C}^\times \to \mathbb{C}$ with respect to the measure v satisfying $f(x) = f(x^{-1})$ v-a.e. Sometimes it is convenient to think of the measure v as a positive measure $\hat{v}$ on the real line supported on

$$[-2, 2] \cup \hat{F}, \qquad \hat{F} = \{ x + x^{-1} \mid x \in F \},$$

by the change of variable $y = x + x^{-1}$. The measure $\hat{v}$ is then given by

$$\int f(y)\, d\hat{v}(y) = \frac{1}{4\pi N} \int_{-2}^{2} f(y) \frac{\hat{\Delta}(y)}{(1 - y^2/4)^{\frac{1}{2}}}\, dy + \sum_{y \in \hat{F}} f(y)\, \hat{v}(\{y\}),$$

with the weight function $\hat{\Delta}$ satisfying $\hat{\Delta}(x+x^{-1})=\Delta(x)$ and with the discrete weights

$$\hat{v}(\{x+x^{-1}\}):=v(\{x\})+v(\{x^{-1}\})=2v(\{x\}), \qquad x\in F.$$

The Hilbert space $L^2_+(v)$ can then be identified with the Hilbert space $L^2(\hat{v})$ of L^2-functions with respect to the measure $\hat{v}$ (cf. [2]). Observe that the space $\mathbb{C}[x]$ of polynomials with complex coefficients is dense in $L^2(\hat{v})$ since $\hat{v}$ is compactly supported. Equivalently, $\mathscr{A}\subset L^2_+(v)$ is a dense sub-space.

For $\alpha\in V_{pol}$ we define the discrete measure $\mu=\mu(\cdot\,;\alpha;q)$ supported on $\mathscr{S}$ by

$$\int g(x)\,d\mu(x)=\sum_{s\in\mathscr{S}} g(s)\,\tilde{h}(s).$$

Observe that μ is a positive measure since the inverse quadratic norm $\tilde{h}(s)$ (see (2.6)) of the Askey–Wilson polynomial E_s is strictly positive for $\alpha\in V_{pol}$. Let $L^2(\mu)$ be the corresponding L^2-space. By continuity, Theorem 2.2 implies the following result.

COROLLARY 6.1. *Let $\alpha\in V_{pol}$. The polynomial Askey–Wilson transform $\mathbb{F}$ and its inverse $\mathbb{I}$ extend by continuity to isometric isomorphisms*

$$\mathbb{F}\colon L^2_+(v)\to L^2(\mu),$$
$$\mathbb{I}\colon L^2(\mu)\to L^2_+(v).$$

Furthermore, $\mathbb{I}: L^2(\mu)\to L^2_+(v)$ is the inverse of $\mathbb{F}: L^2_+(v)\to L^2(\mu)$.

Combined with Theorem 4.6 we obtain the following lemma.

LEMMA 6.2. *Let $\alpha\in V_{pol}$. The orthocomplement of $\mathscr{A}G^{-1}$ in $L^2_+(v)$ (respectively of $(\mathscr{A}G^{\sigma\tau})|_{\mathscr{S}}$ in $L^2(\mu)$) is finite dimensional. In both cases, the dimension of the orthocomplement is*

$$\#\{n\in\mathbb{Z}_+\mid |dq^n|>1\}.$$

In particular, $\mathscr{A}G^{-1}\subset L^2_+(v)$ (respectively $(\mathscr{A}G^{\sigma\tau})|_{\mathscr{S}}\subset L^2(\mu)$) is dense if and only if $|d|\leqslant 1$.

Proof. Let $\alpha=(a,b,c,d)\in V_{pol}$. Let $\hat{G}$ be the meromorphic function satisfying $\hat{G}(x+x^{-1})=G(x)$. Let $\hat{F}_0$ be the intersection of the polar divisor of $\hat{G}$ with the set $\hat{F}$ of discrete mass points of $\hat{v}$. It is easy to verify that

$$\hat{F}_0=\{dq^n+d^{-1}q^{-n}\mid n\in\mathbb{Z}_+ : |dq^n|>1\}.$$

The closure of $\mathbb{C}[x]\,\hat{G}^{-1}$ in $L^2(\hat{\nu})$ is exactly the sub-space of functions $f \in L^2(\hat{\nu})$ which vanish on $\hat{F}_0$, since $\hat{\nu}$ is compactly supported. Hence the orthocomplement of $\mathbb{C}[x]\,G^{-1}$ in $L^2(\hat{\nu})$ is a $\#\hat{F}_0$-dimensional sub-space of $L^2(\hat{\nu})$. Equivalently, the orthocomplement of $\mathscr{A}G^{-1}$ in $L^2_+(\nu)$ is a $\#\hat{F}_0$-dimensional sub-space of $L^2_+(\nu)$.

The identities in Theorem 4.6, which were proven for generic parameters α, are valid for all $\alpha \in V_{pol}$ since they are regular at $\alpha \in V_{pol}$. By Theorem 4.6 and Corollary 6.1 the previous results on $\mathscr{A}G^{-1} \subset L^2_+(\nu)$ thus imply that the orthocomplement of $(\mathscr{A}G^{\sigma\tau})|_{\mathscr{S}}$ in $L^2(\mu)$ is a $\#\hat{F}_0$-dimensional sub-space of $L^2(\mu)$. ∎

Remark 6.3. Suppose that $\alpha \in V_{pol}$ with $|d| > 1$. It follows from Corollary 6.1 and from the proof of Lemma 6.2 that the functions $f_n \in L^2(\mu)$ $(n \in \mathbb{Z}_+ \colon |dq^n| > 1)$ defined by

$$f_n(s) = E_s(dq^n) = {}_4\phi_3\left(\begin{matrix} \tilde{a}s,\, \tilde{a}/s,\, adq^n,\, q^{-n}a/d \\ ab,\, ac,\, ad \end{matrix};q,q\right), \qquad s \in \mathscr{S}$$

form an orthogonal basis for the orthocomplement of $(\mathscr{A}G^{\sigma\tau})|_{\mathscr{S}}$ in $L^2(\mu)$. The quadratic norm of f_n $(n \in \mathbb{Z}_+ : |dq^n| > 1)$ in $L^2(\mu)$ is given by

$$\int |f_n(s)|^2\, d\mu(s) = \frac{N}{\operatorname*{Res}_{y=dq^n}\left(\frac{\Delta(y)}{y}\right)}$$

for the generic parameters $\alpha \in V_{pol}$ such that the pole of $\Delta(y)$ at $y = dq^n$ is simple.

COROLLARY 6.4. *Let $u, v \in \mathbb{R}^\times$ with $|u| \leqslant 1$ and $v > 0$. Let $\rho = \rho_{u,v}$ be the positive discrete measure given by*

$$\int f(x)\, d\rho(x) = \sum_{k=0}^{\infty} f(uq^k + u^{-1}q^{-k})\, v^k q^{k(k-1)} \tag{6.3}$$

and let $L^2(\rho)$ be the associated L^2-space. Then $\mathbb{C}[x]$ is dense in $L^2(\rho_{u,v})$ if and only if $v \leqslant 1$.

Proof. Let $\alpha = (a, b, c, d) \in V_{pol}$. Observe that the condition $\alpha \in V_{pol}$ implies $\tilde{a} < q^{-\frac{1}{2}}$, so $s_n + s_n^{-1} = s_m + s_m^{-1}$ for $m, n \in \mathbb{Z}_+$ iff $n = m$, where (recall) $s_n = \tilde{a}q^n$. Hence we can define a positive, discrete measure $\hat{\rho}$ supported on $s_n + s_n^{-1}$ $(n \in \mathbb{Z}_+)$, with weights

$$\hat{\rho}(\{s_n + s_n^{-1}\}) = |G^{\sigma\tau}(s_n)|^2\, \mu(\{s_n\}), \qquad n \in \mathbb{Z}_+.$$

The corresponding L^2-space $L^2(\hat{\rho})$ is isomorphic to $L^2(\mu)$ via the surjective isometric isomorphism $T: L^2(\mu) \to L^2(\hat{\rho})$ defined by

$$(Tf)(s_n + s_n^{-1}) = G^{\sigma\tau}(s_n)^{-1} f(s_n), \qquad n \in \mathbb{Z}_+.$$

Furthermore, the image of $(\mathscr{A}G^{\sigma\tau})|_{\mathscr{S}}$ under T is exactly the space $\mathbb{C}[x] \subset L^2(\hat{\rho})$ of polynomials with complex coefficients. It thus follows from the previous lemma that $\mathbb{C}[x]$ is dense in $L^2(\hat{\rho})$ iff $|d| \leqslant 1$.

A direct computation using (2.6) shows that

$$\hat{\rho}(\{s_n + s_n^{-1}\}) = \gamma(s_n + s_n^{-1})\, \rho_{\tilde{a}, d^2}(\{s_n + s_n^{-1}\}), \qquad n \in \mathbb{Z}_+$$

with $\rho_{u, v}$ the positive, discrete measure defined by (6.3) and with

$$\gamma: \operatorname{supp}(\hat{\rho}) = \operatorname{supp}(\rho_{\tilde{a}, d^2}) \to \mathbb{R}_{>0}$$

a strictly positive, bounded function with bounded inverse. We conclude that $\mathbb{C}[x]$ is dense in $L^2(\rho_{\tilde{a}, d^2})$ iff $|d| \leqslant 1$.

Choose now $v > 0$ and $0 < u \leqslant 1$ arbitrarily. Then there exist parameters $\alpha = (a, b, c, d) \in V_{pol}$ such that

$$d^2 = v, \qquad \tilde{a} = u.$$

This proves the corollary in case $0 < u \leqslant 1$. The corollary for $-1 \leqslant u < 0$ follows from the result for $0 < u \leqslant 1$, using the surjective isometric isomorphism $S: L^2(\rho_{u, v}) \to L^2(\rho_{-u, v})$ defined by $(Sf)(x) = f(-x)$. ∎

Remark 6.5. Borichev and Sodin [4, Theorem A] formulated criteria for the density of the space of polynomials $\mathbb{C}[x]$ in a L^p-space ($p \geqslant 1$) when the associated measure is supported on the zero set of a Hamburger class function B; see [4] for more details. The measure $\rho_{u, v}$ is of this particular form, with the associated Hamburger class function B given by

$$B(z) = \prod_{\lambda \in \operatorname{supp}(\rho_{u, v})} \left(1 - \frac{z}{\lambda}\right).$$

It is a nice exercise to re-prove Corollary 6.4 using these general criteria of Borichev and Sodin.

Remark 6.6. By a result of Riesz (see, e.g., [3, Lemma A]), it is easy to verify that the measure $\rho_{u, v}$ for $u, v \in \mathbb{R}^\times$ with $|u| \leqslant 1$ and $v > 0$ corresponds to a determinate moment problem if and only if $0 < v \leqslant q^2$. In particular, $\rho_{u, v}$ is a N-extremal measure (or, in the terminology of [4], a canonical measure) if and only if $q^2 < v \leqslant 1$. If $\rho_{u, v}$ is determinate (i.e. if $v \leqslant q^2$), then $\rho_{u, v}$ has a finite index of determinacy, which can be computed explicitly (see Berg and Duran [3] for a detailed study of measures with finite index of determinacy). On the other hand, if $\rho_{u,v}$ is indeterminate and the polynomials are not dense in $L^2(\rho_{u,v})$ (i.e. if $v>1$), then the dimension of the orthocomplement of $\mathbb{C}[x]$ in $L^2(\rho_{u,v})$ equals $\#\{k\in\mathbb{Z}_+ \mid |vq^{2k}|>1\}$ by, e.g., [4, Proposition A1.4]. This observation nicely relates to Lemma 6.2.

We now use Corollary 6.4 to derive the following density result for the linear sub-space $V_{cl}\oplus V_{str}\subset L^2_+(m)$.

PROPOSITION 6.7. *Let the parameters* $\beta=(\alpha,t)$ *satisfy the conditions* (5.1). *Let* $k\in\mathbb{Z}$ *be the unique integer such that* $1<|dtq^k|\leqslant q^{-1}$.
The sub-space

$$V_{cl}\oplus V_{str}\subset L^2_+(m)$$

is dense if and only if $|\tilde{a}tq^k|\geqslant 1$. *Furthermore, the (non-empty) set of parameters* β *satisfying* (5.1) *and satisfying the condition* $|\tilde{a}tq^k|\geqslant 1$ *is invariant under the duality involution* σ.

Proof. We first prove the last part of the proposition. Suppose that the parameters β satisfy (5.1) and $|\tilde{a}tq^k|\geqslant 1$, where $k\in\mathbb{Z}$ is the unique integer such that $1<|dtq^k|\leqslant q^{-1}$. The dual parameters $\tilde{\beta}=\beta_\sigma$ then satisfy the conditions (5.1) in view of [12, Lemma 1]. It remains to verify the inequality $|a\tilde{t}q^r|\geqslant 1$, where $r\in\mathbb{Z}$ is the unique integer such that $1<|\tilde{d}\tilde{t}q^r|\leqslant q^{-1}$. By the definition of dual parameters the condition $1<|\tilde{d}\tilde{t}q^r|\leqslant q^{-1}$ is equivalent to the condition $q\leqslant|\tilde{a}tq^{-r}|<1$. In particular, $-r>k$. But then

$$|a\tilde{t}q^r|\geqslant q^{-1}\,|a\tilde{t}q^{-k}|=q^{-1}\,|d^{-1}t^{-1}q^{-k}|\geqslant 1,$$

which is the desired inequality.

We now focus on the first part of the statement. We fix parameters β satisfying the conditions (5.1). Via the change of variable $y=x+x^{-1}$ we can rewrite the measure m as a positive measure $\hat{m}$ on $\mathbb{R}$ supported on $\mathcal{D}_{cl}\cup\mathcal{D}_{str}$, where

$$\mathcal{D}_{cl}=[-2,2]\cup\{aq^n+a^{-1}q^{-n}\mid n\in\mathbb{Z}_+ : aq^n>1\},$$
$$\mathcal{D}_{str}=\{uq^n+u^{-1}q^{-n}\mid n\in\mathbb{Z}_+\},$$

with $u=d^{-1}t^{-1}q^{-k}$. Here $k\in\mathbb{Z}$ is the unique integer such that $1<|dtq^k|\leqslant q^{-1}$. Under the change of variable $y=x+x^{-1}$, the Hilbert space $L^2_+(m)$ is isomorphic to the Hilbert space $L^2(\hat{m})$ of L^2-functions with respect to the measure $\hat{m}$ (compare with the identification of $L^2_+(v)$ and $L^2(\hat{v})$ as discussed at the beginning of this section).

Consider $\hat{V}_{cl}=\mathbb{C}[x]\,\varphi$ and $\hat{V}_{str}=\mathbb{C}[x]\,\hat{G}^\tau$ as linear sub-spaces of $L^2(\hat{m})$, where φ and $\hat{G}^\tau$ are the meromorphic functions satisfying

$$\varphi(x+x^{-1})=\Theta(x)^{-1}\,G(x)^{-1},\qquad \hat{G}^\tau(x+x^{-1})=G^\tau(x).$$

Then $V_{cl}\oplus V_{str}\subset L^2_+(m)$ is dense iff $\hat{V}_{cl}\oplus\hat{V}_{str}\subset L^2(\hat{m})$ is dense.

Let $\hat{m}_{cl} = \hat{m}|_{\mathscr{D}_{cl}}$ (respectively $\hat{m}_{str} = \hat{m}|_{\mathscr{D}_{str}}$) be the restriction of the measure $\hat{m}$ to $\mathscr{D}_{cl}$ (respectively $\mathscr{D}_{str}$), and denote $L^2(\hat{m}_{cl})$ (respectively $L^2(\hat{m}_{str})$) for the associated L^2-space. We define surjective, continuous linear mappings

$$\pi_{cl} : L^2(\hat{m}) \to L^2(\hat{m}_{cl}), \qquad \pi_{str} : L^2(\hat{m}) \to L^2(\hat{m}_{str})$$

by $\pi_{cl}(f) = f|_{\mathscr{D}_{cl}}$ and $\pi_{str}(f) = f|_{\mathscr{D}_{str}}$.

Observe that $\pi_{cl}(\varphi)$ is non-zero $\hat{m}_{cl}$-a.e. due to the conditions (5.1) on the parameters β. Since the measure $\hat{m}_{cl}$ is compactly supported, we conclude that the sub-space $\pi_{cl}(\hat{V}_{cl}) \subset L^2(\hat{m}_{cl})$ is dense, compare with the proof of Lemma 6.2.

Let $H_{cl} \subset L^2(\hat{m})$ be the closed sub-space of functions $f \in L^2(\hat{m})$ with support contained in $\mathscr{D}_{cl}$. Then $\hat{V}_{cl} \subset H_{cl}$ since φ vanishes on $\mathscr{D}_{str}$, and $\pi_{cl}|_{H_{cl}} : H_{cl} \to L^2(\hat{m}_{cl})$ is a surjective isometric isomorphism. It follows that $\hat{V}_{cl} \subset H_{cl}$ is dense.

Since $\hat{V}_{cl} \subset H_{cl}$ is dense we have that $\hat{V}_{cl} \oplus \hat{V}_{str} \subset L^2(\hat{m})$ is dense iff $\pi_{str}(\hat{V}_{str}) \subset L^2(\hat{m}_{str})$ is dense. It thus suffices to prove that $\pi_{str}(\hat{V}_{str}) \subset L^2(\hat{m}_{str})$ is dense iff the parameters β satisfy the extra condition $|\tilde{a}tq^k| \geqslant 1$. Observe first that $-1 < u = d^{-1}t^{-1}q^{-k} < 0$ by the conditions (5.1) on the parameters β and by the definition of the integer k. Furthermore, for any discrete mass point $y_n = uq^n + u^{-1}q^{-n} \in \mathscr{D}_{str}$ $(n \in \mathbb{Z}_+)$, we have

$$|\hat{G}^\tau(y_n)|^2 \, \hat{m}_{str}(\{y_n\}) = \gamma(y_n) \, \rho_{u,v}(\{y_n\}), \qquad v := \tilde{a}^{-2}t^{-2}q^{-2k},$$

with $\rho = \rho_{u,v}$ the measure (6.3) and with $\gamma \colon \mathscr{D}_{str} \to \mathbb{R}_{>0}$ a bounded function with bounded inverse (we have used here the explicit expression for the weights $\hat{m}_{str}(\{y_n\})$, see [12, (5.8)]). It follows that $\pi_{str}(\hat{V}_{str}) \subset L^2(\hat{m}_{str})$ is dense iff $\mathbb{C}[x] \subset L^2(\rho_{u,v})$ is dense, cf. the proof of Corollary 6.4. The desired density result is now a direct consequence of Corollary 6.4. ∎

7. APPENDIX A

Proof of Proposition 4.1

In this appendix we give a proof of the formulas (4.6), which can be rewritten as

$$(\mathbb{F}^\tau(\phi_\gamma^{an}))(s_m^\tau) = \frac{(bc;q)_m}{(qa/d;q)_m} \left(\frac{-qa}{d}\right)^m q^{m(m-1)/2} \times {}_4\phi_3\left(\begin{matrix} q^{-m}, q^m abc/d, \tilde{a}\gamma, \tilde{a}/\gamma \\ ab, ac, bc \end{matrix}; q, q\right) \tag{7.1}$$

for $m \in \mathbb{Z}_+$.

Throughout the proof of (7.1) we fix $m \in \mathbb{Z}_+$, and we set $s = s_m^\tau$. We substitute the expression (3.2) for the Askey–Wilson function $\phi_\gamma(x)$ in the integral

$$(\mathbb{F}^\tau(\phi_\gamma^{an}))(s) = \frac{1}{4\pi i N^\tau} \int_{\mathcal{T}^\tau} \phi_\gamma^{an}(x)\, E_s^\tau(x)\, \Delta^\tau(x) \frac{dx}{x}$$
$$= \frac{1}{4\pi i N^\tau G^{\sigma\tau}(\gamma)} \int_{\mathcal{T}^\tau} \phi_\gamma(x)\, E_s^\tau(x) \frac{\Delta^\tau(x)}{G^\tau(x)} \frac{dx}{x}$$

and we use that

$$E_s^\tau(x) = \frac{(bc, qb/d; q)_m}{(ac, qa/d; q)_m} \left(\frac{a}{b}\right)^m {}_4\phi_3\left(\begin{matrix} q^{-m}, q^m abc/d, bx, b/x \\ ab, bc, qb/d \end{matrix}; q, q\right)$$

by Sear's transformation formula [8, (2.10.4)] with a, b, c, d, e and f in [8, (2.10.4)] taken to be $q^m abc/d$, ax, a/x, ab, ac and qa/d, respectively. We arrive at

$$(\mathbb{F}^\tau(\phi_\gamma^{an}))(s) = \frac{(bc, qb/d; q)_m}{(ac, qa/d; q)_m} \left(\frac{a}{b}\right)^m \{I_1(\gamma) + I_2(\gamma)\}, \tag{7.2}$$

with $I_1(\gamma)$ given by

$$I_1(\gamma) = \frac{(qabc/d, q\gamma/\tilde{d}, q/\tilde{d}\gamma; q)_\infty}{(bc, qa/d, qb/d, qc/d, q/ad; q)_\infty}$$
$$\times \sum_{n=0}^{m} \sum_{k=0}^{\infty} \frac{(\tilde{a}\gamma, \tilde{a}/\gamma; q)_k\, (q^{-m}, q^m abc/d; q)_n}{(q, ab, ac, ad; q)_k\, (q, ab, bc, qb/d; q)_n} q^{k+n}$$
$$\times \frac{1}{4\pi i N^\tau} \int_{\mathcal{T}^\tau} \frac{(x^2, 1/x^2; q)_\infty}{(q^k ax, q^k a/x, q^n bx, q^n b/x, cx, c/x; q)_\infty} \frac{dx}{x},$$

and with $I_2(\gamma)$ given by

$$I_2(\gamma) = \frac{(qabc/d, \tilde{a}\gamma, \tilde{a}/\gamma; q)_\infty}{(ab, ac, bc, qa/d, ad/q; q)_\infty}$$
$$\times \sum_{n=0}^{m} \sum_{k=0}^{\infty} \frac{(q\gamma/\tilde{d}, q/\tilde{d}\gamma; q)_k\, (q^{-m}, q^m abc/d; q)_n}{(q, qb/d, qc/d, q^2/ad; q)_k\, (q, ab, bc, qb/d; q)_n} q^{k+n}$$
$$\times \frac{1}{4\pi i N^\tau} \int_{\mathcal{T}^\tau} \frac{(x^2, 1/x^2; q)_\infty}{(q^n bx, q^n b/x, cx, c/x, q^{k+1}x/d, q^{k+1}/dx; q)_\infty} \frac{dx}{x}.$$

Now the integrals in the expressions for $I_1(\gamma)$ and $I_2(\gamma)$ can be evaluated as special case of the evaluation of the Askey–Wilson integral; see (2.5). The resulting sum over k in both $I_1(\gamma)$ and $I_2(\gamma)$ can then be rewritten as a non-terminating ${}_3\phi_2$. This leads to the identity

$$I_1(\gamma)+I_2(\gamma)=\sum_{n=0}^{m}\frac{(q^{-m},q^m abc/d;q)_n}{(q;q)_n\,(bc;q)_\infty}q^n \times\left\{\frac{(q\gamma/\tilde{d},q/\tilde{d}\gamma;q)_\infty}{(q/ad;q)_\infty\,(qb/d;q)_n}\,{}_3\phi_2\left(\begin{matrix}\tilde{a}\gamma,\tilde{a}/\gamma,q^n ab\\ ab,ad\end{matrix};q,q\right)+\frac{(\tilde{a}\gamma,\tilde{a}/\gamma;q)_\infty}{(ad/q;q)_\infty\,(ab;q)_n}\,{}_3\phi_2\left(\begin{matrix}q\gamma/\tilde{d},q/\tilde{d}\gamma,q^{n+1}b/d\\ q^2/ad,qb/d\end{matrix};q,q\right)\right\}.$$

Applying the three-term transformation formula [8, (3.3.1)] for ${}_3\phi_2$'s with parameters a, b, c, d, and e in [8, (3.3.1)] specialized to q^{-n}, $\tilde{a}\gamma$, $\tilde{a}/\gamma$, ab and $bc(=q\tilde{a}/\tilde{d})$ respectively, shows that

$$I_1(\gamma)+I_2(\gamma)=\sum_{n=0}^{m}\frac{(q^{-m},q^m abc/d;q)_n}{(q,qb/d;q)_n}q^n\,{}_3\phi_2\left(\begin{matrix}q^{-n},\tilde{a}\gamma,\tilde{a}/\gamma\\ ab,bc\end{matrix};q,q^{n+1}b/d\right). \tag{7.3}$$

Formula (7.1) now immediately follows from (7.2), (7.3), and the following lemma.

LEMMA 7.1. *The following identity is valid:*

$$\sum_{n=0}^{m}\frac{(q^{-m},q^m abc/d;q)_n}{(q,qb/d;q)_n}q^n\,{}_3\phi_2\left(\begin{matrix}q^{-n},\tilde{a}\gamma,\tilde{a}/\gamma\\ ab,bc\end{matrix};q,q^{n+1}b/d\right)=\left(\frac{-qb}{d}\right)^m q^{m(m-1)/2}\frac{(ac;q)_m}{(qb/d;q)_m}\,{}_4\phi_3\left(\begin{matrix}q^{-m},q^m abc/d,\tilde{a}\gamma,\tilde{a}/\gamma\\ ab,ac,bc\end{matrix};q,q\right).$$

Proof. Denote the left hand side of the desired identity by $f_m(\gamma)$. It is clear that $f_m(\gamma)$ is a polynomial of degree m in $\gamma+\gamma^{-1}$. In the expansion

$$f_m(\gamma)=\sum_{k=0}^{m}\alpha_k(\tilde{a}\gamma,\tilde{a}/\gamma;q)_k,$$

the coefficients α_k are explicitly given by

$$\alpha_k=\sum_{n=k}^{m}\frac{(q^{-m},q^m abc/d;q)_n\,(q^{-n};q)_k}{(q,qb/d;q)_n\,(q,ab,bc;q)_k}q^n\left(\frac{q^{n+1}b}{d}\right)^k.$$

Changing the summation variable to $r=n-k$ and simplifying the sum yields

$$\alpha_k=\frac{(q^{-m},q^m abc/d;q)_k}{(q,q,ab,bc,qb/d;q)_k}\left(\frac{q^{k+2}b}{d}\right)^k \times\sum_{r=0}^{m-k}\frac{(q^{k-m},q^{m+k}abc/d;q)_r}{(q^{k+1},q^{k+1}b/d;q)_r}(q^{-r-k};q)_k\,q^{(k+1)r}.$$

Now applying [8, (1.2.37)] to the q-shifted factorial $(q^{-r-k};q)_k$ appearing in the right hand side of the last formula for α_k, leads to

$$\alpha_k=\frac{(q^{-m},q^m abc/d;q)_k}{(q,ab,bc,qb/d;q)_k}\,q^{k(k-1)/2}\left(\frac{-q^2b}{d}\right)^k {}_2\phi_1\left(\begin{matrix}q^{k-m},q^{m+k}abc/d\\ q^{k+1}b/d\end{matrix};q,q\right).$$

The terminating ${}_2\phi_1$ is summable by the q-Vandermonde formula [8, (1.5.3)]. Simplification of the resulting expression then shows that

$$\alpha_k=\left(\frac{-qb}{d}\right)^m q^{m(m-1)/2}\frac{(ac;q)_m}{(qb/d;q)_m}\frac{(q^{-m},q^m abc/d;q)_k}{(q,ab,ac,bc;q)_k}\,q^k,$$

as desired. ▮

8. APPENDIX B

Evaluations of the One Variable q-Macdonald–Mehta Integral

In this appendix we give two alternative proofs of the q-analogue of the (one variable) Macdonald–Mehta integral, see Theorem 5.5. The first proof is based on Nassrallah and Rahman's integral representation [15] of the very-well-poised ${}_8\phi_7$ series. The second proof uses the fact that the q-analogue of the Macdonald–Mehta integral in one variable can be rewritten in a completely discrete form using Cauchy's Theorem. The evaluation then follows from limit cases of the summation formulas of the very-well-poised ${}_6\phi_5$ series and of the very-well-poised ${}_6\psi_6$ series, together with some elementary elliptic function theory (the second proof is in the spirit of Askey and Wilson's original proof [2] of the evaluation of the Askey–Wilson integral).

8.1. *First Direct Proof of Theorem* 5.5. We fix generic parameters $a,b,c,u\in\mathbb{C}^\times$ satisfying $|abcu|>q$ and $|a|,|b|,|c|,|u|<1$. We write

$$L_1=\frac{1}{4\pi i}\int_{\mathbb{T}}\frac{(x^2,1/x^2;q)_\infty}{(ax,a/x,bx,b/x,cx,c/x;q)_\infty\,\theta(ux,u/x)}\frac{dx}{x}$$

and

$$L_2 = \sum_{k=1}^{\infty} \operatorname*{Res}_{x=uq^{-k}} \left(\frac{(x^2, 1/x^2; q)_\infty}{(ax, a/x, bx, b/x, cx, c/x; q)_\infty \, \theta(ux, u/x)\, x} \right),$$

respectively. The aim is to evaluate $L_1 + L_2$. For L_1, we use the integral representation [8, (6.3.8)] of the very-well-poised ${}_8\phi_7$ series due to Nassrallah and Rahman [15], with parameters a, b, c, d, f, and g in [8, (6.3.8)] specialized to $a, b, c, u, q/u$ and 0, respectively. The ${}_8W_7$ then reduces to a ${}_3\phi_2$, and we obtain

$$L_1 = \frac{(abcu, qabc/u; q)_\infty}{(q, ab, ac, bc, au, qa/u, bu, qb/u, cu, qc/u; q)_\infty} \times {}_3\phi_2\left(\begin{matrix} ab, ac, bc \\ abcu, qabc/u \end{matrix} ; q, q \right). \tag{8.1}$$

On the other hand, a straightforward residue computation shows that

$$L_2 = -\frac{(u^2/q^2; q)_\infty}{(q, q, qa/u, qb/u, qc/u, au/q, bu/q, cu/q, u^2/q; q)_\infty} \times {}_7\phi_7\left(\begin{matrix} q^2/u^2, q^2/u, -q^2/u, q, qa/u, qb/u, qc/u \\ q/u, -q/u, q^2/u^2, q^2/au, q^2/bu, q^2/cu, 0 \end{matrix} ; q, \frac{q^2}{abcu} \right).$$

Applying [8, (3.8.9)] with the parameters a, c, d, e, and f in [8, (3.8.9)] replaced by q^2/u^2, q, qa/u, qb/u, and qc/u, respectively, we arrive at

$$L_2 = -\frac{(q/bc; q)_\infty \, \theta(u^2/q^2)}{(q, q, qa/u, qb/u, qc/u, au/q, u^2/q; q)_\infty \, \theta(bu/q, cu/q)} \times {}_3\phi_2\left(\begin{matrix} q/au, qb/u, qc/u \\ q^2/u^2, q^2/au \end{matrix} ; q, \frac{q}{bc} \right).$$

Now applying the transformation formula [8, (3.2.7)] for ${}_3\phi_2$'s with the parameters a, b, c, d, and e in [8, (3.2.7)] specialized to q/au, qb/u, qc/u, q^2/u^2, and q^2/au, respectively, we obtain

$$L_2 = -\frac{(q^2/abcu; q)_\infty \, \theta(u^2/q^2)}{(q, qa/u, qb/u, qc/u, u^2/q; q)_\infty \, \theta(au/q, bu/q, cu/q)} \times {}_3\phi_2\left(\begin{matrix} q/au, q/bu, q/cu \\ q^2/u^2, q^2/abcu \end{matrix} ; q, q \right). \tag{8.2}$$

Now combine (8.1) and (8.2), and simplify the terms using

$$\theta(qx^{-1}) = \theta(x), \qquad \theta(qx) = (-x)^{-1}\,\theta(x). \tag{8.3}$$

Then we obtain

$$\begin{aligned} L_1 + L_2 &= \frac{1}{(q, qa/u, qb/u, qc/u, au, bu, cu; q)_\infty} \\ &\quad \times \left\{ \frac{(abcu, qabc/u; q)_\infty}{(ab, ac, bc; q)_\infty}\, {}_3\phi_2\left(\begin{matrix} ab, ac, bc \\ abcu, qabc/u \end{matrix}; q, q \right) \right. \\ &\quad \left. - \frac{q}{abcu} \frac{(q^2/abcu, q^2/u^2; q)_\infty}{(q/au, q/bu, q/cu; q)_\infty}\, {}_3\phi_2\left(\begin{matrix} q/au, q/bu, q/cu \\ q^2/u^2, q^2/abcu \end{matrix}; q, q \right) \right\}. \end{aligned}$$

By the three term transformation formula [8, (3.3.1)] for the ${}_3\phi_2$ basic hypergeometric series with the parameters a, b, c, d, and e in [8, (3.3.1)] specialized to qc/u, ac, bc, $qabc/u$, and qc/u, respectively, we obtain

$$L_1 + L_2 = \frac{(qabc/u; q)_\infty\, \theta(abcu)}{(q, ab, ac, bc, cu, qa/u, qb/u; q)_\infty\, \theta(au, bu)}\, {}_2\phi_1\left(\begin{matrix} ac, bc \\ qabc/u \end{matrix}; q, \frac{q}{cu} \right).$$

Application of the q-Gauss sum [8, (1.5.1)] yields

$$L_1 + L_2 = \frac{1}{(q, ab, ac, bc; q)_\infty} \frac{\theta(abcu)}{\theta(au, bu, cu)}.$$

The evaluation of the one variable q-Macdonald–Mehta integral (5.9) follows from this last formula by analytic continuation.

8.2. *Second Direct Proof of Theorem* 5.5. For generic values of the parameters $a, b, c, u \in \mathbb{C}^\times$ satisfying $|a|, |b|, |c| < 1$ and $q < |u| < 1$, we can rewrite the left hand side of (5.9) as

$$\begin{aligned} &\frac{1}{2} \sum_{\substack{e \in \{a, b, c\} \\ k \in \mathbb{Z}_+}} \operatorname*{Res}_{x = eq^k} \left(\frac{(x^2, 1/x^2; q)_\infty}{(ax, a/x, bx, b/x, cx, c/x; q)_\infty\, \theta(ux, u/x)\, x} \right) \\ &\quad + \frac{1}{2} \sum_{k \in \mathbb{Z}} \operatorname*{Res}_{x = uq^k} \left(\frac{(x^2, 1/x^2; q)_\infty}{(ax, a/x, bx, b/x, cx, c/x; q)_\infty\, \theta(ux, u/x)\, x} \right) \end{aligned} \tag{8.4}$$

by shrinking the radius of the integration circle to zero while picking up residues, cf. [8, Sect. 4.10]. The first three sums over $k \in \mathbb{Z}_+$ with fixed $e \in \{a, b, c\}$ in (8.4) can be evaluated by the limit case $d \to \infty$ in Rogers' summation formula [8, (2.7.1)] of a very-well-poised ${}_6\phi_5$ series. For instance, the case $e=a$ yields

$$\sum_{k\in\mathbb{Z}_+}\operatorname*{Res}_{x=aq^k}\left(\frac{(x^2,1/x^2;q)_\infty}{(ax,a/x,bx,b/x,cx,c/x;q)_\infty\,\theta(ux,u/x)\,x}\right)$$
$$=\frac{(1/a^2;q)_\infty}{(q,ab,b/a,ac,c/a;q)_\infty\,\theta(ua,u/a)}\,{}_5\phi_5\left(\begin{matrix}a^2,qa,-qa,ab,ac\\a,-a,qa/b,qa/c,0\end{matrix};q,\frac{q}{bc}\right)$$
$$=\frac{(q/bc;q)_\infty}{(q,ab,ac;q)_\infty}\frac{\theta(1/a^2)}{\theta(b/a,c/a,au,u/a)}.$$

The sums for $e=b,c$ can be obtained by interchanging the role of a and e in the above formula. The fourth sum in (8.4) (over $k\in\mathbb{Z}$) can be evaluated by the limit case $e\to\infty$ in Bailey's summation formula [8, (5.3.1)] of a very-well-poised ${}_6\psi_6$ series, yielding

$$\sum_{k\in\mathbb{Z}}\operatorname*{Res}_{x=uq^k}\left(\frac{(x^2,1/x^2;q)_\infty}{(ax,a/x,bx,b/x,cx,c/x;q)_\infty\,\theta(ux,u/x)\,x}\right)$$
$$=\frac{(1-1/u^2)}{(q,q,au,bu,cu,a/u,b/u,c/u;q)_\infty}$$
$$\times\,{}_5\psi_6\left(\begin{matrix}q/u,-q/u,a/u,b/u,c/u\\1/u,-1/u,q/au,q/bu,q/cu,0\end{matrix};q,\frac{q}{abcu}\right)$$
$$=\frac{(q/ab,q/ac,q/bc;q)_\infty}{(q;q)_\infty}\frac{\theta(1/u^2)}{\theta(au,bu,cu,a/u,b/u,c/u)}.$$

Set $\chi=\log(u)$. Then it remains to evaluate

$$f(\chi)=\frac{(q/bc;q)_\infty}{(q,ab,ac;q)_\infty}\frac{\theta(1/a^2)}{\theta(b/a,c/a,ae^\chi,e^\chi/a)}$$
$$+\frac{(q/ac;q)_\infty}{(q,ab,bc;q)_\infty}\frac{\theta(1/b^2)}{\theta(a/b,c/b,be^\chi,e^\chi/b)}$$
$$+\frac{(q/ab;q)_\infty}{(q,bc,ac;q)_\infty}\frac{\theta(1/c^2)}{\theta(b/c,a/c,ce^\chi,e^\chi/c)}$$
$$+\frac{(q/ab,q/ac,q/bc;q)_\infty}{(q;q)_\infty}\frac{\theta(e^{-2\chi})}{\theta(ae^\chi,be^\chi,ce^\chi,ae^{-\chi},be^{-\chi},ce^{-\chi})},$$

which we consider as a meromorphic function in $\chi\in\mathbb{C}$, with fixed (generic) parameters a,b,c. We consider first the meromorphic function

$$g(\chi)=\theta(ae^\chi,be^\chi,ce^\chi)\,f(\chi).$$

Observe that the possible poles of g are located at $\log(e)+\Lambda$ for $e=a, b, c$, where $\Lambda\subset\mathbb{C}$ is the lattice

$$\Lambda=\mathbb{Z}\log(q)+\mathbb{Z}\,2\pi i.$$

Furthermore, the poles are at most simple for generic parameter values. We show now that g is in fact analytic. Observe that g is $2\pi i$-periodic. Furthermore, by (8.3),

$$g(\chi+\log(q))=\left(\frac{-1}{abce^{\chi}}\right)g(\chi),$$

i.e., g is quasi-periodic with quasi-period $\log(q)$. So g is quasi-periodic with respect to the period lattice Λ. In view of the symmetry of g in the parameters a, b and c, we conclude that g is analytic if the residue of $g(\chi)$ at $\alpha:=\log(a)$ is zero. This follows from the observation that

$$\begin{aligned}\lim_{\chi\to\alpha}(1-e^{\chi-\alpha})\,g(\chi)&=\frac{(q/bc;q)_\infty}{(q,q,q,ab,ac;q)_\infty}\frac{\theta(1/a^2,ab,ac)}{\theta(b/a,c/a)}\\&\quad-\frac{(q/ab,q/ac,q/bc;q)_\infty}{(q,q,q;q)_\infty}\frac{\theta(1/a^2)}{\theta(b/a,c/a)}\\&=0.\end{aligned}$$

We conclude that the function

$$\begin{aligned}h(\chi)&=\frac{g(\chi)}{\theta(abce^{\chi})}=\frac{\theta(ae^{\chi},be^{\chi},ce^{\chi})}{\theta(abce^{\chi})}f(\chi)\\&=\frac{(q/bc;q)_\infty}{(q,ab,ac;q)_\infty}\frac{\theta(1/a^2,be^{\chi},ce^{\chi})}{\theta(b/a,c/a,e^{\chi}/a,abce^{\chi})}\\&\quad+\frac{(q/ac;q)_\infty}{(q,ab,bc;q)_\infty}\frac{\theta(1/b^2,ae^{\chi},ce^{\chi})}{\theta(a/b,c/b,e^{\chi}/b,abce^{\chi})}\\&\quad+\frac{(q/ab;q)_\infty}{(q,bc,ac;q)_\infty}\frac{\theta(1/c^2,ae^{\chi},be^{\chi})}{\theta(b/c,a/c,e^{\chi}/c,abce^{\chi})}\\&\quad+\frac{(q/ab,q/ac,q/bc;q)_\infty}{(q;q)_\infty}\frac{\theta(e^{-2\chi})}{\theta(abce^{\chi},ae^{-\chi},be^{-\chi},ce^{-\chi})}\end{aligned}$$

defines an elliptic function with respect to the period lattice Λ, with at most one pole in a fundamental domain of $\mathbb{C}/\Lambda$. Thus h is constant, so in particular,

$$f(\chi)=\frac{\theta(abce^{\chi})}{\theta(ae^{\chi},be^{\chi},ce^{\chi})}h(-\alpha),\qquad\alpha=\log(a).$$

By the explicit expression for h we have

$$h(-\alpha) = \frac{2}{(q, ab, ac, bc; q)_\infty},$$

hence the left hand side of (5.9) is equal to

$$\frac{1}{2} f(\log(u)) = \frac{1}{(q, ab, ac, bc; q)_\infty} \frac{\theta(abcu)}{\theta(au, bu, cu)},$$

which completes the proof of Theorem 5.5.

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