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Extending lazy 2-cocycles on Hopf algebras and lifting projective representations afforded by them

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Abstract

We study some problems related to lazy 2-cocycles, such as: extension of (lazy) 2-cocycles to a Drinfeld double and to a Radford biproduct, Yetter–Drinfeld data obtained from lazy 2-cocycles, lifting of projective representations afforded by lazy 2-cocycles.

Keywords: Hopf algebra; Lazy cocycles; Drinfeld double; Radford biproduct; Projective representations

Introduction

A left 2-cocycle $\sigma : H \otimes H \to k$ on a Hopf algebra H is called *lazy* if it satisfies the condition

$$\sigma\left(h_1, h_1'\right)h_2h_2' = \sigma\left(h_2, h_2'\right)h_1h_1', \quad \forall h, h' \in H.$$

This kind of cocycles were used in [7] as a tool to compare the Brauer groups of Sweedler's Hopf algebra with respect to the different quasitriangular structures. See also [9] and [10] for an

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application of this technique to other sort of Hopf algebras. Lazy cocycles and lazy cohomology were also used in [20] to give a generalized version of Kac's exact sequence. A general theory of lazy cocycles and lazy cohomology started to be developed recently in [2]. The remarkable fact is that the set $Z_L^2(H)$ of normalized and convolution invertible lazy 2-cocycles on H form a group (this was noted in [11]), and that one can also define lazy 2-coboundaries $B_L^2(H)$ and the second lazy cohomology group $H_L^2(H) = Z_L^2(H)/B_L^2(H)$, generalizing the second Sweedler cohomology group of a cocommutative Hopf algebra (note that for cocommutative Hopf algebras *any* 2-cocycle is lazy). The group $H_L^2(H)$ can be regarded as a subgroup of $Bigal(H)$, the group of Bigalois objects of H, and the examples in [2] show that it is much easier to compute $H_L^2(H)$ than $Bigal(H)$.

In general, the results in [2] suggest that, for an arbitrary Hopf algebra, lazy cocycles are much closer to the cocommutative case than general left cocycles. Hence, a sort of general principle is suggested: results that hold for an *arbitrary* 2-cocycle on a *cocommutative* Hopf algebra are likely to hold also for a *lazy* 2-cocycle on an *arbitrary* Hopf algebra. A good example of this principle is the extension of Schur–Yamazaki formula in [2] that allows to describe the second lazy cohomology group of a tensor product of Hopf algebras. Throughout this paper we will verify this principle several times.

This paper is a contribution to the study of lazy cocycles and lazy cohomology, in three different directions: the problem of extending (lazy) 2-cocycles to a Drinfeld double and to a Radford biproduct; Yetter–Drinfeld data obtained from lazy 2-cocycles; lifting of projective representations afforded by lazy 2-cocycles. As we will see below, each of these directions has its own (natural) motivations and possible applications.

We describe now in some detail the contents of the paper. After presenting in Section 1 some preliminaries, in Section 2 we provide some new properties of lazy 2-cocycles that are needed in the next sections, but which could also be of independent interest. Among these properties is the following formula:

$$\sigma\big(h_1, S(h_2)\big) = \sigma\big(S(h_1), h_2\big), \quad \forall h \in H,$$

for a lazy 2-cocycle σ on a Hopf algebra H; this formula is important and well known for group algebras, and we show that in general it is false if σ is not lazy.

In Section 3 we prove that any lazy 2-cocycle σ on a finite dimensional Hopf algebra H can be extended to a lazy 2-cocycle $\bar{\sigma}$ on the Drinfeld double $D(H)$ (this property can be obtained also from results in [2], where moreover a complete description of $H_L^2(D(H))$ is given). We point out that this extension is canonical in a certain sense (expressed in terms of the so-called *diagonal crossed product*, a construction introduced in [14]; actually, the relation between lazy 2-cocycles and the diagonal crossed product was our starting point for this article). Section 4 is devoted to the extension of cocycles on a Radford biproduct. We consider a Radford biproduct $B \times H$, with H a Hopf algebra and B a Hopf algebra in the Yetter–Drinfeld category ${}_H^H\mathcal{YD}$. Cocycles and the second lazy cohomology group $H_L^2(B)$ may be defined in the category ${}_H^H\mathcal{YD}$. We prove that, if σ is a left 2-cocycle on B in ${}_H^H\mathcal{YD}$, it can be extended canonically to a left 2-cocycle $\bar{\sigma}$ on $B \times H$, σ lazy in ${}_H^H\mathcal{YD}$ implies $\bar{\sigma}$ lazy and the map $\sigma \mapsto \bar{\sigma}$ induces a group morphism $H_L^2(B) \to H_L^2(B \times H)$.

In Section 5 we study Yetter–Drinfeld data obtained from lazy 2-cocycles. Namely, if $\sigma : H \otimes H \to k$ is a normalized and convolution invertible lazy 2-cocycle, we have the H-bicomodule algebra $H(\sigma) = {}_\sigma H = H_\sigma$, hence the Yetter–Drinfeld category ${}_{H(\sigma)}\mathcal{YD}(H)^H$. We prove that, if M is a finite dimensional object in this category, then $End(M)$ and $End(M)^{op}$ are algebras in ${}_H\mathcal{YD}^H$. More can be said if H is finite dimensional (for this we use again the diagonal crossed product and results from Section 3).

In Section 6 we prove that any Hopf algebra H admits a central extension B with the property that any projective representation of H afforded by a *lazy* 2-cocycle can be lifted to an ordinary representation of B. The case when H is cocommutative was done by I. Boca (generalizing in turn the classical case of groups, due to Schur); our proof follows closely the one of Boca. This section could be regarded as a good illustration of the general principle we mentioned before (replacement of cocommutativity by laziness).

Finally, in Section 7 we verify once more this principle, by generalizing a certain result from [5] concerning $H(\sigma)$-central elements; our proof uses several results obtained in this paper, and it is much easier than the one in [5] (which moreover works only in the cocommutative case).

1. Preliminaries

In this section we recall some definitions and results and we fix some notation that will be used throughout the paper. For unexplained terminology we refer to [15,16,18,23].

We will work over a ground field k. All algebras, linear spaces, etc., will be over k; unadorned $\otimes$ means $\bigotimes_k$. For a Hopf algebra H with comultiplication Δ we use the version of Sweedler's sigma notation: $\Delta(h) = h_1 \otimes h_2$. Unless otherwise stated, H will denote a Hopf algebra with bijective antipode S. For a linear map $\sigma : H \otimes H \to k$ we will use either the notation $\sigma(h, h')$ or $\sigma(h \otimes h')$.

A linear map $\sigma : H \otimes H \to k$ is called a *left 2-cocycle* if it satisfies the condition

$$\sigma(a_1, b_1)\sigma(a_2b_2, c) = \sigma(b_1, c_1)\sigma(a, b_2c_2),$$

for all $a, b, c \in H$, and it is called a *right 2-cocycle* if it satisfies the condition

$$\sigma(a_1b_1, c)\sigma(a_2, b_2) = \sigma(a, b_1c_1)\sigma(b_2, c_2).$$

Given a linear map $\sigma : H \otimes H \to k$, define a product $\cdot_\sigma$ on H by

$$h \cdot_\sigma h' = \sigma\big(h_1, h'_1\big)h_2h'_2, \quad \forall h, h' \in H.$$

Then $\cdot_\sigma$ is associative if and only if σ is a left 2-cocycle. If we define $\cdot_\sigma$ by

$$h \cdot_\sigma h' = h_1h'_1\sigma\big(h_2, h'_2\big), \quad \forall h, h' \in H,$$

then $\cdot_\sigma$ is associative if and only if σ is a right 2-cocycle. In any of the two cases, σ is normalized (i.e. $\sigma(1, h) = \sigma(h, 1) = \varepsilon(h)$ for all $h \in H$) if and only if 1_H is the unit for $\cdot_\sigma$. If σ is a normalized left (respectively right) 2-cocycle, we denote the algebra $(H, \cdot_\sigma)$ by ${}_\sigma H$ (respectively H_σ). It is well known that ${}_\sigma H$ (respectively H_σ) is a right (respectively left) H-comodule algebra via the comultiplication Δ of H. If $\sigma : H \otimes H \to k$ is normalized and convolution invertible, then σ is a left 2-cocycle if and only if σ^{-1} is a right 2-cocycle.

If $\gamma : H \to k$ is linear, normalized (i.e. $\gamma(1) = 1$) and convolution invertible, define $D^1(\gamma) : H \otimes H \to k$ by

$$D^1(\gamma)(h,h') = \gamma(h_1)\gamma\big(h'_1\big)\gamma^{-1}\big(h_2h'_2\big), \quad \forall h,h' \in H.$$

Then $D^1(\gamma)$ is a normalized and convolution invertible left 2-cocycle. If $\sigma, \sigma' : H \otimes H \to k$ are normalized and convolution invertible left 2-cocycles, they are called *cohomologous* if there exists $\gamma : H \to k$ normalized and convolution invertible such that

$$\sigma'(h,h') = \gamma(h_1)\gamma\big(h'_1\big)\sigma\big(h_2,h'_2\big)\gamma^{-1}\big(h_3h'_3\big), \quad \forall h,h' \in H.$$

We recall now from [2] some facts about lazy cocycles and lazy cohomology. The set $Reg^1(H)$ (respectively $Reg^2(H)$) consisting of normalized and convolution invertible linear maps $\gamma : H \to k$ (respectively $\sigma : H \otimes H \to k$), is a group under the convolution product. An element $\gamma \in Reg^1(H)$ is called *lazy* if

$$\gamma(h_1)h_2 = h_1\gamma(h_2), \quad \forall h \in H.$$

The set of lazy elements of $Reg^1(H)$, denoted by $Reg^1_L(H)$, is a central subgroup of $Reg^1(H)$. An element $\sigma \in Reg^2(H)$ is called *lazy* if

$$\sigma\big(h_1,h'_1\big)h_2h'_2 = h_1h'_1\sigma\big(h_2,h'_2\big), \quad \forall h,h' \in H.$$

The set of lazy elements of $Reg^2(H)$, denoted by $Reg^2_L(H)$, is a subgroup of $Reg^2(H)$. We denote by $Z^2(H)$ the set of left 2-cocycles on H and by $Z^2_L(H)$ the set $Z^2(H) \cap Reg^2_L(H)$ of normalized and convolution invertible lazy 2-cocycles. If $\sigma \in Z^2_L(H)$, then the algebras ${}_\sigma H$ and H_σ coincide and will be denoted by $H(\sigma)$; moreover, $H(\sigma)$ is an H-bicomodule algebra via Δ.

It is well known that in general the set $Z^2(H)$ of left 2-cocycles is not closed under convolution. One of the main features of lazy 2-cocycles is that the set $Z^2_L(H)$ is closed under convolution, and that the convolution inverse of an element $\sigma \in Z^2_L(H)$ is again a lazy 2-cocycle, so $Z^2_L(H)$ is a group under convolution. In particular, a lazy 2-cocycle is also a right 2-cocycle. Consider now the map $D^1 : Reg^1(H) \to Reg^2(H)$, $D^1(\gamma)(h,h') = \gamma(h_1)\gamma(h'_1)\gamma^{-1}(h_2h'_2)$, for all $h,h' \in H$. Then, by [2], the map D^1 induces a group morphism $Reg^1_L(H) \to Z^2_L(H)$, whose image is contained in the center of $Z^2_L(H)$; denote by $B^2_L(H)$ this central subgroup $D^1(Reg^1_L(H))$ of $Z^2_L(H)$ (its elements are called *lazy 2-coboundaries*). Finally, define the *second lazy cohomology group* $H^2_L(H) = Z^2_L(H)/B^2_L(H)$ (most likely nonabelian in general). Lazy 2-cocycles belonging to the same class in $H^2_L(H)$ (we call them *lazy cohomologous*) are in particular cohomologous in the sense recalled before.

2. Some properties of lazy 2-cocycles

The aim of this section is to give some general properties of lazy 2-cocycles needed in the next sections although they could also be of independent interest.

Let $\sigma : H \otimes H \to k$ be a normalized and convolution invertible left 2-cocycle. It is well known (see [18], [8]) that the following formulae hold:

$$\sigma\big(h_1,S(h_2)\big)\sigma^{-1}\big(S(h_3),h_4\big) = \varepsilon(h), \tag{2.1}$$

$$\sigma\big(S^{-1}(h_2),h_1\big)\sigma^{-1}\big(h_4,S^{-1}(h_3)\big) = \varepsilon(h), \tag{2.2}$$

for all $h \in H$, but in general we see no reason to have formulae of the type

$$\sigma\big(h_1, S(h_2)\big) = \sigma\big(S(h_1), h_2\big), \tag{2.3}$$

$$\sigma\big(S^{-1}(h_2), h_1\big) = \sigma\big(h_2, S^{-1}(h_1)\big), \tag{2.4}$$

even if for group algebras these formulae are true and well known. We have searched through the literature to find an explicit counterexample, but we could not find any, so we are going to provide here one. The Hopf algebra will be the Taft Hopf algebra H_9 of dimension 9.

Recall that $H_9 = k\langle X, Y \mid X^3 = 1,\ Y^3 = 0,\ YX = qXY\rangle$, where q is a primitive 3rd root of unity, $\Delta(X) = X \otimes X$, $\Delta(Y) = 1 \otimes Y + Y \otimes X$, $S(X) = X^2$, $S(Y) = -q^2X^2Y$. The cleft extensions for any H_{n^2} have been classified in [12,17]; we use here the form in [12]. We will construct a certain H_9-cleft datum over k (in the terminology of [12]). Namely, in the notation of [12, Theorem 3.5], we choose $F = id_k$, $D = 0$ and $\alpha, \beta, \gamma \in k$ with $\alpha, \gamma \neq 0$. Then, also in the notation of [12], one computes easily that:

$$\begin{aligned}
&\gamma_2 = (1+q)\gamma,\\
&\gamma_3 = \big(1+q+q^2\big)\gamma = 0,\\
&\theta_2 = (1+q)\gamma^2,\\
&\theta_3 = D(\theta_2) + \theta_2\gamma_3 = 0,\\
&D_3 = 0.
\end{aligned}$$

Using these formulae, one can see that the conditions (1)–(9) in [12, Theorem 3.5], are satisfied, so indeed $(id, 0, \alpha, \beta, \gamma)$ is an H_9-cleft datum. The table for the left 2-cocycle corresponding to any H_9-cleft datum is given in [12, Example 3.6]. For our datum, we get from the table:

$$\sigma(Y, XY) = 0, \qquad \sigma\big(Y^2, X\big) = \theta_2\alpha = \gamma^2(1+q)\alpha,$$
$$\sigma\big(X^2Y, XY\big) = 0, \qquad \sigma\big(XY^2, X^2\big) = FD(\gamma_2\alpha) = 0.$$

From the equalities

$$\Delta\big(Y^2\big) = 1 \otimes Y^2 + (1+q)Y \otimes XY + Y^2 \otimes X^2,$$
$$S\big(Y^2\big) = XY^2, \qquad S(XY) = -qXY, \qquad S\big(X^2\big) = X,$$

we compute $\sigma(h_1, S(h_2))$ and $\sigma(S(h_1), h_2)$ for the element $h := Y^2$ and we obtain:

$$\sigma\big(h_1, S(h_2)\big) = (1+q)\sigma(Y, -qXY) + \sigma\big(Y^2, X\big) = \gamma^2(1+q)\alpha \neq 0,$$
$$\sigma\big(S(h_1), h_2\big) = (1+q)\sigma\big(-q^2X^2Y, XY\big) + \sigma\big(XY^2, X^2\big) = 0,$$

so the two terms cannot be equal.

However, we have the following very useful result.

Lemma 2.1. *If σ is lazy, then formulae* (2.3) *and* (2.4) *hold.*

Proof. Since σ is lazy, the left cocycle condition can be written as

$$\sigma(a_1, b_1)\sigma(a_2b_2, c) = \sigma(a, b_1c_1)\sigma(b_2, c_2).$$

By taking $a = h_1$, $b = S(h_2)$, $c = h_3$ in this formula, we obtain (2.3). Since σ is lazy, it is also a right 2-cocycle, and the right cocycle condition can be written, using the laziness of σ, as

$$\sigma(a_1b_1, c)\sigma(a_2, b_2) = \sigma(b_1, c_1)\sigma(a, b_2c_2).$$

By taking in this formula $a = h_3$, $b = S^{-1}(h_2)$, $c = h_1$, we obtain (2.4). □

There exist certain relations (suggested by the referee) between (2.1) and (2.2) or between (2.3) and (2.4). Namely, for a linear map $\sigma : H \otimes H \to k$, define ${}^t\sigma : H \otimes H \to k$ by ${}^t\sigma(h, h') = \sigma(h', h)$. It is easy to see that if σ is a left 2-cocycle for H then ${}^t\sigma$ is a left 2-cocycle for the opposite Hopf algebra H^{op} and if σ is lazy then ${}^t\sigma$ is also lazy. Then one can see that (2.2) is obtained from (2.1) applied to ${}^t\sigma$ and (2.4) is obtained from (2.3) applied to ${}^t\sigma$.

We give now some more useful formulae.

Lemma 2.2. *Let $\sigma : H \otimes H \to k$ be a normalized and convolution invertible lazy 2-cocycle. Then we have*:

$$\sigma^{-1}\big(h_3, S^{-1}(h_2)\big)h_4S^{-1}(h_1) = \sigma^{-1}\big(h_2, S^{-1}(h_1)\big)1, \tag{2.5}$$

$$\sigma^{-1}\big(S^{-1}(h_3), h_2\big)S^{-1}(h_4)h_1 = \sigma^{-1}\big(S^{-1}(h_2), h_1\big)1, \tag{2.6}$$

$$\sigma^{-1}\big(S(h_2), h_3\big)S(h_1)h_4 = \sigma^{-1}\big(S(h_1), h_2\big)1, \tag{2.7}$$

$$\sigma^{-1}\big(S(h_2), h_3\big)S(h_1) = \sigma^{-1}\big(S(h_1), h_2\big)S(h_3), \tag{2.8}$$

$$\sigma^{-1}\big(h_2, S(h_3)\big)h_1S(h_4) = \sigma^{-1}\big(h_1, S(h_2)\big)1, \tag{2.9}$$

$$\sigma^{-1}\big(S(h_2), h_3\big)h_4S^{-1}(h_1) = \sigma^{-1}\big(S(h_1), h_2\big)1. \tag{2.10}$$

Proof. For (2.5), apply the lazy condition for σ^{-1} to the elements h_2 and $S^{-1}(h_1)$; for (2.6), apply the lazy condition for σ^{-1} to the elements $S^{-1}(h_2)$ and h_1; for (2.7), apply the lazy condition for σ^{-1} to the elements $S(h_1)$ and h_2; (2.8) is obtained from (2.7) by making convolution to the right with S; for (2.9), apply the lazy condition for σ^{-1} to the elements h_1 and $S(h_2)$; finally, (2.10) is obtained from (2.9) by using (2.3) and then applying S^{-1}. □

Let $\sigma : H \otimes H \to k$ be a normalized and convolution invertible left 2-cocycle. Let us recall from [2] that the linear map $\phi_\sigma : {}_\sigma H \to H_{\sigma^{-1}}$ defined by

$$\phi_\sigma(h) = \sigma\big(h_1, S(h_2)\big)S(h_3) \tag{2.11}$$

is an algebra antimorphism, and moreover it satisfies, for all $h \in H$:

$$\phi_\sigma(h_1) \cdot_{\sigma^{-1}} h_2 = \varepsilon(h)1 = h_1 \cdot_{\sigma^{-1}} \phi_\sigma(h_2).$$

Also, let us recall from [8] the maps $S_1, S_2 : H \to H$ given for all $h \in H$ by

$$S_1(h)=\sigma^{-1}\big(S(h_2),h_3\big)S(h_1), \tag{2.12}$$

$$S_2(h)=\sigma^{-1}\big(h_3,S^{-1}(h_2)\big)S^{-1}(h_1). \tag{2.13}$$

From (2.3) and (2.8) it follows immediately that:

Proposition 2.3. *If σ is lazy, then $S_1=\phi_{\sigma^{-1}}$.*

There exists also a relation between S_2 and ϕ_σ, which holds in general.

Proposition 2.4. *If σ is a normalized and convolution invertible left 2-cocycle on H, then S_2 is the composition inverse of ϕ_σ. In particular, it follows that ϕ_σ is bijective.*

Proof. That $S_2\circ\phi_\sigma=id$ and $\phi_\sigma\circ S_2=id$ reduce respectively to formulae (2.1) and (2.2). □

If σ is lazy, since $S_1=\phi_{\sigma^{-1}}$ and S_2 is the composition inverse of ϕ_σ, from the properties of ϕ_σ we obtain:

Proposition 2.5. *If σ is lazy, then $S_1,S_2:H(\sigma^{-1})\to H(\sigma)$ are algebra antiisomorphisms, and we have, for all $h\in H$:*

$$S_1(h_1)\cdot_\sigma h_2=\varepsilon(h)1=h_1\cdot_\sigma S_1(h_2), \tag{2.14}$$

$$S_2(h_2)\cdot_\sigma h_1=\varepsilon(h)1=h_2\cdot_\sigma S_2(h_1). \tag{2.15}$$

Let us note that (2.14) and (2.15) appear also in [8], in a slightly different form, and they actually hold for any left 2-cocycle, not necessarily lazy.

Proposition 2.6. *Let σ be a normalized and convolution invertible left 2-cocycle on H. Then we have, for all $h\in H$:*

$$\Delta\big(S_1(h)\big)=S_1(h_2)\otimes S(h_1), \tag{2.16}$$

$$\Delta\big(S_2(h)\big)=S_2(h_2)\otimes S^{-1}(h_1). \tag{2.17}$$

Proof. An easy computation. □

3. Extending lazy 2-cocycles to a Drinfeld double

Throughout this section, H will be a finite dimensional Hopf algebra and we will denote the Drinfeld double of H by $D(H)$. A complete description of $H^2_L(D(H))$ in terms of $H^2_L(H)$ and $H^2_L(H^*)$ was given in [2]. In particular, it follows from [2] that if σ is a normalized and convolution invertible lazy 2-cocycle on H, then it can be extended to a normalized and convolution invertible lazy 2-cocycle $\bar\sigma$ on $D(H)$. In this section we provide an alternative approach to the problem of extending a lazy 2-cocycle from H to $D(H)$, based on the so-called diagonal crossed product construction. The results in this section will be also used in Section 5.

Recall that the Drinfeld double of H is a quasitriangular Hopf algebra realized on the k-linear space $H^* \otimes H$; its coalgebra structure is $H^{*cop} \otimes H$ and the algebra structure is given by

$$(p \otimes h)(q \otimes l) = p\big(h_1 \rightharpoonup q \leftharpoonup S^{-1}(h_3)\big) \otimes h_2 l,$$

for all $p, q \in H^*$ and $h, l \in H$, where $\rightharpoonup$ and $\leftharpoonup$ are the left and right regular actions of H on H^* given by $(h \rightharpoonup p)(l) = p(lh)$ and $(p \leftharpoonup h)(l) = p(hl)$ for all $h, l \in H$ and $p \in H^*$. Let now A be an H-bicomodule algebra, with comodule structures $A \to A \otimes H$, $a \mapsto a_{\langle 0\rangle} \otimes a_{\langle 1\rangle}$ and $A \to H \otimes A$, $a \mapsto a_{[-1]} \otimes a_{[0]}$, and denote, for $a \in A$,

$$a_{\{-1\}} \otimes a_{\{0\}} \otimes a_{\{1\}} = a_{\langle 0\rangle_{[-1]}} \otimes a_{\langle 0\rangle_{[0]}} \otimes a_{\langle 1\rangle} = a_{[-1]} \otimes a_{[0]_{\langle 0\rangle}} \otimes a_{[0]_{\langle 1\rangle}},$$

as an element in $H \otimes A \otimes H$. Recall from [14] that the (left) *diagonal crossed product* $H^* \bowtie A$ is equal to $H^* \otimes A$ as a k-space, but with multiplication given by

$$(p \bowtie a)(q \bowtie b) = p\big(a_{\{-1\}} \rightharpoonup q \leftharpoonup S^{-1}(a_{\{1\}})\big) \bowtie a_{\{0\}} b,$$

for all $a, b \in A$ and $p, q \in H^*$, and with unit $\varepsilon_H \bowtie 1_A$. The space $H^* \bowtie A$ becomes a $D(H)$-bicomodule algebra, with structures

$$\begin{aligned} H^* \bowtie A &\to (H^* \bowtie A) \otimes D(H), & p \bowtie a &\mapsto (p_2 \bowtie a_{\langle 0\rangle}) \otimes (p_1 \otimes a_{\langle 1\rangle}),\\ H^* \bowtie A &\to D(H) \otimes (H^* \bowtie A), & p \bowtie a &\mapsto (p_2 \otimes a_{[-1]}) \otimes (p_1 \bowtie a_{[0]}), \end{aligned}$$

for all $p \in H^*$, $a \in A$. If $A = H$ then $H^* \bowtie A$ is just $D(H)$, with bicomodule algebra structure over itself given by its comultiplication. It is well known (see [13]) that the Drinfeld double can be expressed as a twisting of $H^{*cop} \otimes H$. Similarly, using the framework and notation of [21], one can prove that

$$H^* \bowtie A = H^{*cop} \mathop{\#}_{\tau}^{\tau} A,$$

where $\tau : H \otimes H^{*cop} \to k$ is the skew-pairing given by $\tau(h, p) = p(h)$.

Let $\sigma : H \otimes H \to k$ be a normalized and invertible lazy 2-cocyle. Either as a consequence of the proof in [2], or by direct means, one can see that the extended cocycle $\bar{\sigma} : D(H) \otimes D(H) \to k$ and its convolution inverse are given by the formulae

$$\bar{\sigma}(p \otimes h, q \otimes l) = p(1) q\big(S^{-1}(h_3) h_1\big) \sigma(h_2, l), \tag{3.1}$$

$$\bar{\sigma}^{-1}(p \otimes h, q \otimes l) = p(1) q\big(S^{-1}(h_3) h_1\big) \sigma^{-1}(h_2, l), \tag{3.2}$$

for all $p, q \in H^*$ and $h, l \in H$.

In view of the above description of the diagonal crossed product as a twisting and of the nature of the proof for the description of $H^2_L(D(H))$ in [2], it is likely that the following result can be proved using the approach in [2]. But we prefer to give a direct proof, because this is how we discovered it (actually, how we got the formula (3.1) for $\bar{\sigma}$).

Proposition 3.1. *Let $\sigma : H \otimes H \to k$ be a normalized and convolution invertible lazy 2-cocycle. Consider the H-bicomodule algebra $H(\sigma)$. Then $H^* \bowtie H(\sigma) = D(H)(\bar{\sigma})$, as $D(H)$-bicomodule algebras. Moreover, $\bar{\sigma}$ is unique with this property.*

Proof. We compute the multiplications in the two algebras and show that they coincide.

$$\begin{aligned}
(p \bowtie h)(q \bowtie l) &= p\big(h_1 \rightharpoonup q \leftharpoonup S^{-1}(h_3)\big) \bowtie h_2 \cdot_\sigma l \\
&= \sigma(h_2, l_1) p\big(h_1 \rightharpoonup q \leftharpoonup S^{-1}(h_4)\big) \bowtie h_3 l_2 \\
&= p\big(h_4 S^{-1}(h_3) h_1 \rightharpoonup q \leftharpoonup S^{-1}(h_6)\big) \sigma(h_2, l_1) \otimes h_5 l_2 \\
&= p\big(h_4 \rightharpoonup q_1 \leftharpoonup S^{-1}(h_6)\big) q_2\big(S^{-1}(h_3) h_1\big) \sigma(h_2, l_1) \otimes h_5 l_2 \\
&= p_2(1) q_2\big(S^{-1}(h_{(1,3)}) h_{(1,1)}\big) \sigma(h_{(1,2)}, l_1) \\
&\quad \times p_1\big(h_{(2,1)} \rightharpoonup q_1 \leftharpoonup S^{-1}(h_{(2,3)})\big) \otimes h_{(2,2)} l_2 \\
&= \bar{\sigma}(p_2 \otimes h_1, q_2 \otimes l_1)(p_1 \otimes h_2)(q_1 \otimes l_2) \\
&= (p \otimes h) \cdot_{\bar{\sigma}} (q \otimes l).
\end{aligned}$$

Clearly $H^* \bowtie H(\sigma)$ and $D(H)(\bar{\sigma})$ have the same $D(H)$-bicomodule structure. For the uniqueness of $\bar{\sigma}$, we write down the fact that the multiplications in $H^* \bowtie H(\sigma)$ and $D(H)(\bar{\sigma})$ coincide, then we evaluate this equality on $1 \otimes \varepsilon$ and we obtain that $\bar{\sigma}$ has to be given by (3.1). □

It was proved in [2] that $H_L^2(H)$ can be embedded as a subgroup in $Bigal(H)$, the group of Bigalois objects of H introduced in [22,25].

Proposition 3.2. *The map $A \mapsto H^* \bowtie A$ gives an embedding of $Bigal(H)$ into $Bigal(D(H))$, whose restriction to $H_L^2(H)$ is the embedding of $H_L^2(H)$ into $H_L^2(D(H))$ from* [2].

Proof. The fact that the map $A \mapsto H^* \bowtie A$ gives the desired embedding between Bigalois groups is contained, even if not explicitly stated, in Schauenburg's paper [21], and the compatibility between the two embeddings, at the levels of Bigalois groups and lazy cohomologies, follows from the compatibility between the proof in [21] and the one in [2]. □

The antipode of $D(H)$ is given by the formula

$$S_{D(H)}(p \otimes h) = \big(\varepsilon \otimes S(h)\big)\big(S^{*-1}(p) \otimes 1\big),$$

for all $h \in H$, $p \in H^*$. Denote $S_{D(H)}$ by $\overline{S}$. One can easily check that its inverse is given by

$$S_{D(H)}^{-1}(p \otimes h) = \big(\varepsilon \otimes S^{-1}(h)\big)\big(S^*(p) \otimes 1\big).$$

Let now $\sigma : H \otimes H \to k$ be a normalized and convolution invertible lazy 2-cocycle, and $\bar{\sigma}$ its extension to $D(H)$, given by the formula (3.1). Denote by $S_1, S_2 : H \to H$ the maps given by the formulae (2.12), (2.13), and by $\overline{S}_1, \overline{S}_2 : D(H) \to D(H)$ the analogous maps for $D(H)$ corresponding to $\bar{\sigma}$, that is:

$$\overline{S}_1(p\otimes h)=\bar{\sigma}^{-1}\big(\overline{S}\big((p\otimes h)_2\big),(p\otimes h)_3\big)\overline{S}\big((p\otimes h)_1\big),\tag{3.3}$$

$$\overline{S}_2(p\otimes h)=\bar{\sigma}^{-1}\big((p\otimes h)_3,\overline{S}^{-1}\big((p\otimes h)_2\big)\big)\overline{S}^{-1}\big((p\otimes h)_1\big).\tag{3.4}$$

The following result will be needed in a subsequent section.

Proposition 3.3. *$\overline{S}_1$ and $\overline{S}_2$ can be computed as*:

$$\overline{S}_1(p\otimes h)=\big(\varepsilon\otimes S_1(h)\big)\big(S^{*-1}(p)\otimes 1\big),\tag{3.5}$$

$$\overline{S}_2(p\otimes h)=\big(\varepsilon\otimes S_2(h)\big)\big(S^{*}(p)\otimes 1\big),\tag{3.6}$$

for all $h\in H$, $p\in H^*$.

Proof. We give the proof for $\overline{S}_1$, the one for $\overline{S}_2$ is similar (but for $\overline{S}_2$ one has to use the formula (2.5)). We compute:

$$\begin{aligned}
\overline{S}_1(p\otimes h)&=\bar{\sigma}^{-1}\big(\overline{S}(p_2\otimes h_2),p_1\otimes h_3\big)\overline{S}(p_3\otimes h_1)\\
&=\bar{\sigma}^{-1}\big(\big(\varepsilon\otimes S(h_2)\big)\big(S^{*-1}(p_2)\otimes 1\big),p_1\otimes h_3\big)\overline{S}(p_3\otimes h_1)\\
&=\bar{\sigma}^{-1}\big(S(h_2)_1\rightharpoonup S^{*-1}(p_2)\leftharpoonup S^{-1}\big(S(h_2)_3\big)\otimes S(h_2)_2,p_1\otimes h_3\big)\overline{S}(p_3\otimes h_1)\\
&=\bar{\sigma}^{-1}\big(S(h_4)\rightharpoonup S^{*-1}(p_2)\leftharpoonup h_2\otimes S(h_3),p_1\otimes h_5\big)\overline{S}(p_3\otimes h_1)\\
&=\big(S(h_4)\rightharpoonup S^{*-1}(p_2)\leftharpoonup h_2\big)(1)p_1\big(S^{-1}\big(S(h_3)_3\big)S(h_3)_1\big)\\
&\quad\times\sigma^{-1}\big(S(h_3)_2,h_5\big)\overline{S}(p_3\otimes h_1)\\
&=S^{*-1}(p_2)\big(h_2S(h_6)\big)p_1\big(h_3S(h_5)\big)\sigma^{-1}\big(S(h_4),h_7\big)\overline{S}(p_3\otimes h_1)\\
&=p_1\big(h_3S(h_5)h_6S^{-1}(h_2)\big)\sigma^{-1}\big(S(h_4),h_7\big)\overline{S}(p_2\otimes h_1)\\
&=\sigma^{-1}\big(S(h_2),h_3\big)\big(\varepsilon\otimes S(h_1)\big)\big(S^{*-1}(p)\otimes 1\big)\\
&=\big(\varepsilon\otimes S_1(h)\big)\big(S^{*-1}(p)\otimes 1\big),
\end{aligned}$$

which was what we had to prove. □

Remark 3.4. Using either the formula for $\bar{\sigma}$ or the identification $H^*\bowtie H(\sigma)=D(H)(\bar{\sigma})$, one can easily check that

$$(\varepsilon\otimes h)\cdot_{\bar{\sigma}}(p\otimes 1)=(\varepsilon\otimes h)(p\otimes 1),\tag{3.7}$$

for all $h\in H$ and $p\in H^*$ (we will use this later).

4. Extending (lazy) 2-cocycles to a Radford biproduct

For a Hopf algebra H and a Hopf algebra B in the category of left Yetter–Drinfeld modules ${}_H^H\mathcal{YD}$ it is possible to construct the Radford biproduct Hopf algebra $B \times H$. A second lazy cohomology group $H_L^2(B)$ can be defined for B inside the category ${}_H^H\mathcal{YD}$. In this section we find out a relation between $H_L^2(B)$ and $H_L^2(B \times H)$.

We start by recalling from [19] the construction of a Radford biproduct. Let H be a bialgebra and B a vector space such that $(B, 1_B)$ is an algebra (with multiplication denoted by $b \otimes c \mapsto bc$ for all $b, c \in B$) and $(B, \Delta_B, \varepsilon_B)$ is a coalgebra. The pair (H, B) is called *admissible* if B is endowed with a left H-module structure (denoted by $h \otimes b \mapsto h \cdot b$) and with a left H-comodule structure (denoted by $b \mapsto b^{(-1)} \otimes b^{(0)} \in H \otimes B$) such that:

(1) B is a left H-module algebra;
(2) B is a left H-comodule algebra;
(3) B is a left H-comodule coalgebra, that is, for all $b \in B$:

$$b_1^{(-1)} b_2^{(-1)} \otimes b_1^{(0)} \otimes b_2^{(0)} = b^{(-1)} \otimes \left(b^{(0)}\right)_1 \otimes \left(b^{(0)}\right)_2, \tag{4.1}$$

$$b^{(-1)} \varepsilon_B\left(b^{(0)}\right) = \varepsilon_B(b) 1_H; \tag{4.2}$$

(4) B is a left H-module coalgebra, that is, for all $h \in H$ and $b \in B$:

$$\Delta_B(h \cdot b) = h_1 \cdot b_1 \otimes h_2 \cdot b_2, \tag{4.3}$$

$$\varepsilon_B(h \cdot b) = \varepsilon_H(h) \varepsilon_B(b); \tag{4.4}$$

(5) ε_B is an algebra map and $\Delta_B(1_B) = 1_B \otimes 1_B$;
(6) the following relations hold for all $h \in H$ and $b, c \in B$:

$$\Delta_B(bc) = b_1\left(b_2^{(-1)} \cdot c_1\right) \otimes b_2^{(0)} c_2, \tag{4.5}$$

$$(h_1 \cdot b)^{(-1)} h_2 \otimes (h_1 \cdot b)^{(0)} = h_1 b^{(-1)} \otimes h_2 \cdot b^{(0)}. \tag{4.6}$$

If (H, B) is an admissible pair, then we know from [19] that the smash product algebra structure and smash coproduct coalgebra structure on $B \otimes H$ afford $B \otimes H$ a bialgebra structure, denoted by $B \times H$ and called the *smash biproduct* or *Radford biproduct*. Its comultiplication is given by

$$\Delta(b \times h) = \left(b_1 \times b_2^{(-1)} h_1\right) \otimes \left(b_2^{(0)} \times h_2\right), \tag{4.7}$$

for all $b \in B$, $h \in H$, and its counit is $\varepsilon_B \otimes \varepsilon_H$. Let us record the following formula:

$$\Delta_B\left(b(h \cdot c)\right) = b_1\left(b_2^{(-1)} h_1 \cdot c_1\right) \otimes b_2^{(0)} (h_2 \cdot c_2), \tag{4.8}$$

for all $h \in H$ and $b, c \in B$, which follows immediately from (4.5) and (4.3). If H is a Hopf algebra with antipode S_H and (H, B) is an admissible pair such that there exists $S_B \in Hom(B, B)$ a convolution inverse for id_B, then $B \times H$ is a Hopf algebra with antipode

$$S(b \times h) = \big(1 \times S_H\big(b^{(-1)}h\big)\big)\big(S_B\big(b^{(0)}\big) \times 1\big), \tag{4.9}$$

for all $h \in H$, $b \in B$. In this case, we will say that (H, B) is a *Hopf admissible pair*. For a Hopf algebra H, it is well known (see for instance [18], [16]) that (H, B) being an admissible pair (respectively Hopf admissible pair) is equivalent to B being a bialgebra (respectively Hopf algebra) in the Yetter–Drinfeld category ${}_H^H\mathcal{YD}$.

Recall now from [24] the so-called *generalized smash product*. If H is a bialgebra, B a left H-module algebra (with action $h \otimes b \mapsto h \cdot b$) and A a left H-comodule algebra (with coaction $a \mapsto a_{(-1)} \otimes a_{(0)} \in H \otimes A$), then on $B \otimes A$ we have an associative algebra structure, denoted by $B \blacktriangleright\!\!< A$, with unit $1_B \blacktriangleright\!\!< 1_A$ and multiplication

$$(b \blacktriangleright\!\!< a)(b' \blacktriangleright\!\!< a') = b(a_{(-1)} \cdot b') \blacktriangleright\!\!< a_{(0)}a', \tag{4.10}$$

for all $b, b' \in B$ and $a, a' \in A$.

As we have seen before, the relation between a Drinfeld double and a diagonal crossed product is that the diagonal crossed product becomes a bicomodule algebra over the Drinfeld double. The next result shows that a similar relation exists between a Radford biproduct and a generalized smash product.

Proposition 4.1. *If (H, B) is an admissible pair and A is a left H-comodule algebra, then $B \blacktriangleright\!\!< A$ becomes a left $B \times H$-comodule algebra, with coaction*

$$\lambda : B \blacktriangleright\!\!< A \to (B \times H) \otimes (B \blacktriangleright\!\!< A), \qquad \lambda(b \blacktriangleright\!\!< a) = \big(b_1 \times b_2^{(-1)} a_{(-1)}\big) \otimes \big(b_2^{(0)} \blacktriangleright\!\!< a_{(0)}\big),$$

for all $b \in B$ and $a \in A$.

Proof. We prove first that $(B \blacktriangleright\!\!< A, \lambda)$ is a left $B \times H$-comodule (for this part we only need A to be a left H-comodule). We compute:

$$\begin{aligned}
&(id \otimes \lambda)\big(\lambda(b \blacktriangleright\!\!< a)\big) \\
&\quad = \big(b_1 \times b_2^{(-1)} a_{(-1)}\big) \otimes \big(\big(b_2^{(0)}\big)_1 \times \big(b_2^{(0)}\big)_2^{(-1)} a_{(0)_{(-1)}}\big) \otimes \big(\big(b_2^{(0)}\big)_2^{(0)} \blacktriangleright\!\!< a_{(0)_{(0)}}\big) \\
&\quad = \big(b_1 \times b_2^{(-1)} a_{(-1)_1}\big) \otimes \big(\big(b_2^{(0)}\big)_1 \times \big(b_2^{(0)}\big)_2^{(-1)} a_{(-1)_2}\big) \otimes \big(\big(b_2^{(0)}\big)_2^{(0)} \blacktriangleright\!\!< a_{(0)}\big) \\
&\quad = \big(b_1 \times b_2^{(-1)} b_3^{(-1)} a_{(-1)_1}\big) \otimes \big(b_2^{(0)} \times b_3^{(0)^{(-1)}} a_{(-1)_2}\big) \otimes \big(b_3^{(0)^{(0)}} \blacktriangleright\!\!< a_{(0)}\big) \quad \big(\text{by } (4.1)\big) \\
&\quad = \big(b_1 \times b_2^{(-1)} \big(b_3^{(-1)}\big)_1 a_{(-1)_1}\big) \otimes \big(b_2^{(0)} \times \big(b_3^{(-1)}\big)_2 a_{(-1)_2}\big) \otimes \big(b_3^{(0)} \blacktriangleright\!\!< a_{(0)}\big) \\
&\quad = \big(b_{(1,1)} \times b_{(1,2)}^{(-1)} \big(b_2^{(-1)}\big)_1 a_{(-1)_1}\big) \otimes \big(b_{(1,2)}^{(0)} \times \big(b_2^{(-1)}\big)_2 a_{(-1)_2}\big) \otimes \big(b_2^{(0)} \blacktriangleright\!\!< a_{(0)}\big) \\
&\quad = \Delta\big(b_1 \times b_2^{(-1)} a_{(-1)}\big) \otimes \big(b_2^{(0)} \blacktriangleright\!\!< a_{(0)}\big) \\
&\quad = (\Delta \otimes id)\big(\lambda(b \blacktriangleright\!\!< a)\big).
\end{aligned}$$

Then obviously we have that $(\varepsilon \otimes id)\lambda = id$, so $B \blacktriangleright\!\!< A$ is indeed a left $B \times H$-comodule. We proceed to show that λ is an algebra map. First, by (5), we have $\lambda(1 \blacktriangleright\!\!< 1) = (1 \times 1) \otimes (1 \blacktriangleright\!\!< 1)$. For $b, b' \in B$ and $a, a' \in A$ we have:

$$
\begin{aligned}
&\lambda\big((b \blacktriangleright\!< a)(b' \blacktriangleright\!< a')\big)\\
&\quad = \lambda\big(b(a_{(-1)} \cdot b') \blacktriangleright\!< a_{(0)}a'\big)\\
&\quad = \big(\big(b(a_{(-1)} \cdot b')\big)_1 \times \big(b(a_{(-1)} \cdot b')\big)_2^{(-1)}(a_{(0)}a')_{(-1)}\big) \otimes \big(\big(b(a_{(-1)} \cdot b')\big)_2^{(0)} \blacktriangleright\!< (a_{(0)}a')_{(0)}\big)\\
&\quad = \big(b_1\big(b_2^{(-1)}a_{(-1)_1} \cdot b'_1\big) \times b_2^{(0)^{(-1)}}(a_{(-1)_2} \cdot b'_2)^{(-1)}a_{(0)_{(-1)}}a'_{(-1)}\big)\\
&\qquad \otimes \big(b_2^{(0)^{(0)}}\big(a_{(-1)_2} \cdot b'_2\big)^{(0)} \blacktriangleright\!< a_{(0)_{(0)}}a'_{(0)}\big) \quad \big(\text{by (4.8)}\big)\\
&\quad = \big(b_1\big(b_2^{(-1)}a_{(-1)_1} \cdot b'_1\big) \times b_2^{(0)^{(-1)}}\big(a_{(-1)_2} \cdot b'_2\big)^{(-1)}a_{(-1)_3}a'_{(-1)}\big)\\
&\qquad \otimes \big(b_2^{(0)^{(0)}}\big(a_{(-1)_2} \cdot b'_2\big)^{(0)} \blacktriangleright\!< a_{(0)}a'_{(0)}\big)\\
&\quad = \big(b_1\big(b_2^{(-1)}a_{(-1)_1} \cdot b'_1\big) \times b_2^{(0)^{(-1)}}a_{(-1)_2}b_2'^{(-1)}a'_{(-1)}\big)\\
&\qquad \otimes \big(b_2^{(0)^{(0)}}\big(a_{(-1)_3} \cdot b_2'^{(0)}\big) \blacktriangleright\!< a_{(0)}a'_{(0)}\big) \quad \big(\text{by (4.6)}\big)\\
&\quad = \big(b_1\big(b_2^{(-1)}a_{(-1)_1} \cdot b'_1\big) \times b_2^{(0)^{(-1)}}a_{(-1)_2}b_2'^{(-1)}a'_{(-1)}\big) \otimes \big(b_2^{(0)^{(0)}}\big(a_{(0)_{(-1)}} \cdot b_2'^{(0)}\big) \blacktriangleright\!< a_{(0)_{(0)}}a'_{(0)}\big)\\
&\quad = \big(b_1\big(\big(b_2^{(-1)}\big)_1a_{(-1)_1} \cdot b'_1\big) \times \big(b_2^{(-1)}\big)_2a_{(-1)_2}b_2'^{(-1)}a'_{(-1)}\big)\\
&\qquad \otimes \big(b_2^{(0)}\big(a_{(0)_{(-1)}} \cdot b_2'^{(0)}\big) \blacktriangleright\!< a_{(0)_{(0)}}a'_{(0)}\big)\\
&\quad = \lambda(b \blacktriangleright\!< a)\lambda(b' \blacktriangleright\!< a'),
\end{aligned}
$$

and the proof is finished. □

Now, let (H, B) be an admissible pair and $\sigma : H \otimes H \to k$ a normalized and convolution invertible right 2-cocycle, so that we can consider H_σ, which is a left H-comodule algebra, and we can make $B \blacktriangleright\!< H_\sigma$, which, by the above proposition, becomes a left $B \times H$-comodule algebra.

Proposition 4.2. *With notation as above, the map $\tilde{\sigma} : (B \times H) \otimes (B \times H) \to k$ defined by*

$$
\tilde{\sigma}(b \times h, b' \times h') = \varepsilon_B(b)\varepsilon_B(b')\sigma(h, h'), \quad \forall b, b' \in B \text{ and } h, h' \in H,
$$

is a normalized and convolution invertible right 2-cocycle on $B \times H$, and we have $(B \times H)_{\tilde{\sigma}} = B \blacktriangleright\!< H_\sigma$ as left $B \times H$-comodule algebras. Moreover, $\tilde{\sigma}$ is unique with this property.

Proof. We have:

$$
\begin{aligned}
(b \blacktriangleright\!< h)(b' \blacktriangleright\!< h') &= b(h_1 \cdot b') \blacktriangleright\!< h_2 \cdot_\sigma h'\\
&= b(h_1 \cdot b') \blacktriangleright\!< h_2h'_1\sigma\big(h_3, h'_2\big)\\
&= (b \times h_1)\big(b' \times h'_1\big)\sigma\big(h_2, h'_2\big)\\
&= \big(b_1 \times b_2^{(-1)}h_1\big)\big(b'_1 \times b_2'^{(-1)}h'_1\big)\varepsilon_B\big(b_2^{(0)}\big)\varepsilon_B\big(b_2'^{(0)}\big)\sigma\big(h_2, h'_2\big) \quad \big(\text{by (4.2)}\big)\\
&= \big(b_1 \times b_2^{(-1)}h_1\big)\big(b'_1 \times b_2'^{(-1)}h'_1\big)\tilde{\sigma}\big(b_2^{(0)} \times h_2, b_2'^{(0)} \times h'_2\big).
\end{aligned}
$$

$$
\begin{aligned}
&= (b \times h)_1 (b' \times h')_1 \tilde{\sigma}\big((b \times h)_2, (b' \times h')_2\big) \\
&= (b \times h) \cdot_{\tilde{\sigma}} (b' \times h').
\end{aligned}
$$

So, the multiplication in $(B \times H)_{\tilde{\sigma}}$ coincides with the one in $B \blacktriangleright\!< H_\sigma$, which is associative, so $\tilde{\sigma}$ is automatically a right 2-cocycle, and we have $(B \times H)_{\tilde{\sigma}} = B \blacktriangleright\!< H_\sigma$ as algebras; it is obvious that they coincide also as left $B \times H$-comodules, and is easy to prove that $\tilde{\sigma}$ is normalized and convolution invertible. To prove the uniqueness of $\tilde{\sigma}$, write that the multiplications in $B \blacktriangleright\!< H_\sigma$ and $(B \times H)_{\tilde{\sigma}}$ coincide, apply $\varepsilon_B \otimes \varepsilon_H$ and get $\tilde{\sigma}(b \times h, b' \times h') = \varepsilon_B(b)\varepsilon_B(b')\sigma(h, h')$. □

The map $\pi : B \times H \to H$, $b \times h \mapsto \varepsilon(b)h$ is a Hopf algebra map. Observe that $\tilde{\sigma}$ is just the cocycle obtained by pulling back through the map π.

Remark 4.3. With notation as above, we have:

$$
\begin{aligned}
(b \times h)_1 (b' \times h')_1 \tilde{\sigma}\big((b \times h)_2, (b' \times h')_2\big) &= b(h_1 \cdot b') \times h_2 h'_1 \sigma\big(h_3, h'_2\big), \\
(b \times h)_2 (b' \times h')_2 \tilde{\sigma}\big((b \times h)_1, (b' \times h')_1\big) &= \sigma\big(b^{(-1)} h_1, b'^{(-1)} h'_1\big) b^{(0)} \big(h_2 \cdot b'^{(0)}\big) \times h_3 h'_2,
\end{aligned}
$$

for all $b, b' \in B$ and $h, h' \in H$. Assume that $\tilde{\sigma}$ is lazy; then, by taking $b = b' = 1$ above, we obtain that σ is lazy. Conversely, if σ is lazy, then $\tilde{\sigma}$ is lazy if and only if

$$
\sigma(h_2, h') b(h_1 \cdot b') = \sigma\big(b^{(-1)} h_1, b'^{(-1)} h'\big) b^{(0)} \big(h_2 \cdot b'^{(0)}\big),
$$

for all $b, b' \in B$ and $h, h' \in H$, from which follow some necessary conditions for the laziness of $\tilde{\sigma}$, such as

$$
\begin{aligned}
h \cdot b &= \sigma\big(h_1, b^{(-1)}\big)\big(h_2 \cdot b^{(0)}\big), \\
bb' &= \sigma\big(b^{(-1)}, b'^{(-1)}\big) b^{(0)} b'^{(0)},
\end{aligned}
$$

for all $b, b' \in B$ and $h \in H$, which have no reason to hold in general.

We study now the problem of extending (lazy) 2-cocycles from B to $B \times H$.

Let $\mathcal{C}$ be a braided monoidal category and B a Hopf algebra in $\mathcal{C}$. Then, just as if B would be a usual Hopf algebra, one can define 2-cocycles, crossed products, Galois extensions, etc., for B in $\mathcal{C}$, see for instance [26], [1]. Also, one can define lazy 2-cocycles, lazy 2-coboundaries and the second lazy cohomology group $H^2_L(B) = Z^2_L(B)/B^2_L(B)$. Here, we will only be interested in the case when $\mathcal{C} = {}^H_H\mathcal{YD}$, the category of left Yetter–Drinfeld modules over a Hopf algebra H, and B a Hopf algebra in ${}^H_H\mathcal{YD}$ (that is, (H, B) is a Hopf admissible pair, so $B \times H$ is a Hopf algebra). For this category, one can prove by hand all the properties of lazy 2-cocycles that allow to define $H^2_L(B)$ (the most difficult is to prove that the product of two lazy 2-cocycles is a left 2-cocycle—we will give an easy alternative proof of this fact at the end of the section).

If $M, N \in {}^H_H\mathcal{YD}$, then $M \otimes N \in {}^H_H\mathcal{YD}$ with module structure $h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n$ and comodule structure $m \otimes n \mapsto m_{\langle -1\rangle} n_{\langle -1\rangle} \otimes (m_{\langle 0\rangle} \otimes n_{\langle 0\rangle})$, where $m \mapsto m_{\langle -1\rangle} \otimes m_{\langle 0\rangle}$ and $n \mapsto n_{\langle -1\rangle} \otimes n_{\langle 0\rangle}$ are the comodule structures of M and N, and the braiding is given by

$$
c_{M,N} : M \otimes N \to N \otimes M, \qquad c_{M,N}(m \otimes n) = m_{\langle -1\rangle} \cdot n \otimes m_{\langle 0\rangle}. \tag{4.11}
$$

Hence, the coalgebra structure of $B \otimes B$ in ${}_H^H\mathcal{YD}$ is given by

$$\begin{aligned}\Delta_{B\otimes B}(b \otimes b') &= (id \otimes c_{B,B} \otimes id) \circ (\Delta_B \otimes \Delta_B)(b \otimes b') \\ &= \big(b_1 \otimes b_2^{(-1)} \cdot b'_1\big) \otimes \big(b_2^{(0)} \otimes b'_2\big).\end{aligned}$$

So, if $\sigma, \tau : B \otimes B \to k$ are morphisms in ${}_H^H\mathcal{YD}$, their convolution in ${}_H^H\mathcal{YD}$ is given by

$$(\sigma * \tau)(b \otimes b') = \sigma\big(b_1 \otimes b_2^{(-1)} \cdot b'_1\big)\tau\big(b_2^{(0)} \otimes b'_2\big). \tag{4.12}$$

Let $\sigma : B \otimes B \to k$ be a morphism in ${}_H^H\mathcal{YD}$, that is, it satisfies the conditions:

$$\sigma(h_1 \cdot b \otimes h_2 \cdot b') = \varepsilon(h)\sigma(b \otimes b'), \tag{4.13}$$

$$\sigma\big(b^{(0)} \otimes b'^{(0)}\big)b^{(-1)}b'^{(-1)} = \sigma(b \otimes b')1_H, \tag{4.14}$$

for all $h \in H$ and $b, b' \in B$. Then σ is a lazy element if it satisfies the categorical laziness condition:

$$\sigma\big(b_1 \otimes b_2^{(-1)} \cdot b'_1\big)b_2^{(0)}b'_2 = \sigma\big(b_2^{(0)} \otimes b'_2\big)b_1\big(b_2^{(-1)} \cdot b'_1\big), \tag{4.15}$$

for all $b, b' \in B$.

Let $\sigma : B \otimes B \to k$ be a normalized left 2-cocycle in ${}_H^H\mathcal{YD}$, that is σ is a normalized morphism in ${}_H^H\mathcal{YD}$ satisfying the categorical left 2-cocycle condition

$$\sigma\big(a_1 \otimes a_2^{(-1)} \cdot b_1\big)\sigma\big(a_2^{(0)}b_2 \otimes c\big) = \sigma\big(b_1 \otimes b_2^{(-1)} \cdot c_1\big)\sigma\big(a \otimes b_2^{(0)}c_2\big), \tag{4.16}$$

for all $a, b, c \in B$. Then we can consider the crossed product ${}_\sigma B = k \#_\sigma B$ as in [26], which is an algebra in ${}_H^H\mathcal{YD}$, and whose multiplication is:

$$b \cdot b' = \sigma\big(b_1 \otimes b_2^{(-1)} \cdot b'_1\big)b_2^{(0)}b'_2. \tag{4.17}$$

Since ${}_\sigma B$ is an algebra in ${}_H^H\mathcal{YD}$, it is in particular a left H-module algebra, so we can consider the smash product ${}_\sigma B \# H$.

Let now $\gamma : B \to k$ be a morphism in ${}_H^H\mathcal{YD}$, that is

$$\gamma(h \cdot b) = \varepsilon(h)\gamma(b), \tag{4.18}$$

$$\gamma\big(b^{(0)}\big)b^{(-1)} = \gamma(b)1_H, \tag{4.19}$$

for all $h \in H$ and $b \in B$. If γ is normalized and convolution invertible in ${}_H^H\mathcal{YD}$, with convolution inverse γ^{-1} in ${}_H^H\mathcal{YD}$, the analogue of the operator D^1 is given in ${}_H^H\mathcal{YD}$ by

$$\begin{aligned}D^1(\gamma)(b \otimes b') &= \gamma(b_1)\gamma\big(b_2^{(-1)} \cdot b'_1\big)\gamma^{-1}\big(b_2^{(0)}b'_2\big) \\ &= \gamma(b_1)\gamma\big(b'_1\big)\gamma^{-1}\big(b_2b'_2\big) \quad \big(\text{by } (4.18)\big)\end{aligned}$$

that is, D^1 is given by the same formula as for ordinary Hopf algebras. For a morphism $\gamma : B \to k$ in ${}_H^H\mathcal{YD}$, the laziness condition is identical to the usual one: $\gamma(b_1)b_2 = b_1\gamma(b_2)$ for all $b \in B$.

Theorem 4.4. *Let (H, B) be a Hopf admissible pair.*

(i) *For a normalized left 2-cocycle $\sigma : B \otimes B \to k$ in ${}_H^H\mathcal{YD}$ define $\bar{\sigma} : (B \times H) \otimes (B \times H) \to k$,*

$$\bar{\sigma}(b \times h, b' \times h') = \sigma(b \otimes h \cdot b')\varepsilon(h'). \tag{4.20}$$

Then $\bar{\sigma}$ is a normalized left 2-cocycle on $B \times H$ and we have ${}_\sigma B \# H = {}_{\bar{\sigma}}(B \times H)$ as algebras. Moreover, $\bar{\sigma}$ is unique with this property.

(ii) *If σ is convolution invertible in ${}_H^H\mathcal{YD}$, then $\bar{\sigma}$ is convolution invertible, with inverse*

$$\bar{\sigma}^{-1}(b \times h, b' \times h') = \sigma^{-1}(b \otimes h \cdot b')\varepsilon(h'), \tag{4.21}$$

where σ^{-1} is the convolution inverse of σ in ${}_H^H\mathcal{YD}$.

(iii) *If σ is lazy in ${}_H^H\mathcal{YD}$, then $\bar{\sigma}$ is lazy.*

(iv) *If $\sigma, \tau : B \otimes B \to k$ are lazy 2-cocycles in ${}_H^H\mathcal{YD}$, then $\overline{\sigma * \tau} = \bar{\sigma} * \bar{\tau}$, hence the map $\sigma \mapsto \bar{\sigma}$ is a group homomorphism from $Z_L^2(B)$ to $Z_L^2(B \times H)$.*

(v) *If $\gamma : B \to k$ is a normalized and convolution invertible morphism in ${}_H^H\mathcal{YD}$, define $\bar{\gamma} : B \times H \to k$ by*

$$\bar{\gamma}(b \times h) = \gamma(b)\varepsilon(h). \tag{4.22}$$

Then $\bar{\gamma}$ is normalized and convolution invertible and $\overline{D^1(\gamma)} = D^1(\bar{\gamma})$. If γ is lazy in ${}_H^H\mathcal{YD}$, then $\bar{\gamma}$ is also lazy.

(vi) *If σ is a lazy 2-coboundary for B in ${}_H^H\mathcal{YD}$, then $\bar{\sigma}$ is a lazy 2-coboundary for $B \times H$, so the group homomorphism $Z_L^2(B) \to Z_L^2(B \times H)$, $\sigma \mapsto \bar{\sigma}$, factorizes to a group homomorphism $H_L^2(B) \to H_L^2(B \times H)$.*

Proof. (i) It is easy to see that $\bar{\sigma}$ is normalized. We will prove that the multiplications in ${}_\sigma B \# H$ and ${}_{\bar{\sigma}}(B \times H)$ coincide, and from the associativity of ${}_\sigma B \# H$ will follow automatically that $\bar{\sigma}$ is a left 2-cocycle on $B \times H$. We compute:

$$\begin{aligned}
(b \# h)(b' \# h') &= b \cdot (h_1 \cdot b') \# h_2 h' \\
&= \sigma\big(b_1 \otimes b_2^{(-1)} \cdot (h_1 \cdot b')_1\big) b_2^{(0)} (h_1 \cdot b')_2 \# h_2 h' \\
&= \sigma\big(b_1 \otimes b_2^{(-1)} h_1 \cdot b'_1\big) b_2^{(0)} \big(h_2 \cdot b'_2\big) \# h_3 h' \quad \big(\text{by (4.3)}\big) \\
&= \sigma\big(b_1 \otimes b_2^{(-1)} h_1 \cdot b'_1\big) \big(b_2^{(0)} \times h_2\big) \big(b'_2 \times h'\big) \\
&= \sigma\big(b_1 \otimes b_2^{(-1)} h_1 \cdot b'_1\big) \varepsilon\big(b_2'^{(-1)}\big) \varepsilon\big(h'_1\big) \big(b_2^{(0)} \times h_2\big) \big(b_2'^{(0)} \times h'_2\big) \\
&= \bar{\sigma}\big(b_1 \times b_2^{(-1)} h_1, b'_1 \times b_2'^{(-1)} h'_1\big) \big(b_2^{(0)} \times h_2\big) \big(b_2'^{(0)} \times h'_2\big) \\
&= \bar{\sigma}\big((b \times h)_1, (b' \times h')_1\big) (b \times h)_2 (b' \times h')_2 \\
&= (b \times h) \cdot_{\bar{\sigma}} (b' \times h').
\end{aligned}$$

The uniqueness of $\bar{\sigma}$ follows easily by applying $\varepsilon_B \otimes \varepsilon_H$ to the multiplications in ${}_\sigma B \# H$ and ${}_{\bar{\sigma}}(B \times H)$.

(ii) Follows by a direct computation, using the formula (4.12) for the convolution in ${}_H^H\mathcal{YD}$.

(iii) We have already seen that

$$\bar\sigma\big((b\times h)_1,(b'\times h')_1\big)(b\times h)_2(b'\times h')_2=\sigma\big(b_1\otimes b_2^{(-1)}h_1\cdot b'_1\big)b_2^{(0)}\big(h_2\cdot b'_2\big)\times h_3h'.$$

Now we compute:

$$\begin{aligned}
&\bar\sigma\big((b\times h)_2,(b'\times h')_2\big)(b\times h)_1(b'\times h')_1\\
&\quad=\bar\sigma\big(b_2^{(0)}\times h_2,b_2'^{(0)}\times h'_2\big)\big(b_1\times b_2^{(-1)}h_1\big)\big(b'_1\times b_2'^{(-1)}h'_1\big)\\
&\quad=\sigma\big(b_2^{(0)}\otimes h_2\cdot b_2'^{(0)}\big)\big(b_1\times b_2^{(-1)}h_1\big)\big(b'_1\times b_2'^{(-1)}h'\big)\\
&\quad=\sigma\big(b_2^{(0)}\otimes h_3\cdot b_2'^{(0)}\big)b_1\big(\big(b_2^{(-1)}\big)_1h_1\cdot b'_1\big)\times\big(b_2^{(-1)}\big)_2h_2b_2'^{(-1)}h'\\
&\quad=\sigma\big(b_2^{(0)^{(0)}}\otimes h_3\cdot b_2'^{(0)}\big)b_1\big(b_2^{(-1)}h_1\cdot b'_1\big)\times b_2^{(0)^{(-1)}}h_2b_2'^{(-1)}h'\\
&\quad=\sigma\big(b_2^{(0)^{(0)}}\otimes\big(h_2\cdot b'_2\big)^{(0)}\big)b_1\big(b_2^{(-1)}h_1\cdot b'_1\big)\times b_2^{(0)^{(-1)}}\big(h_2\cdot b'_2\big)^{(-1)}h_3h'\quad\big(\text{by (4.6)}\big)\\
&\quad=\sigma\big(b_2^{(0)}\otimes h_2\cdot b'_2\big)b_1\big(b_2^{(-1)}h_1\cdot b'_1\big)\times h_3h'\quad\big(\text{by (4.14)}\big)\\
&\quad=\sigma\big(b_2^{(0)}\otimes(h_1\cdot b')_2\big)b_1\big(b_2^{(-1)}\cdot(h_1\cdot b')_1\big)\times h_2h'\quad\big(\text{by (4.3)}\big)\\
&\quad=\sigma\big(b_1\otimes b_2^{(-1)}\cdot(h_1\cdot b')_1\big)b_2^{(0)}(h_1\cdot b')_2\times h_2h'\quad\big(\text{by (4.15)}\big)\\
&\quad=\sigma\big(b_1\otimes b_2^{(-1)}h_1\cdot b'_1\big)b_2^{(0)}\big(h_2\cdot b'_2\big)\times h_3h'\quad\big(\text{by (4.3)}\big)
\end{aligned}$$

which proves that $\bar\sigma$ is indeed lazy.

(iv) Using the formula (4.12) for the convolution in ${}_H^H\mathcal{YD}$, we compute:

$$\begin{aligned}
\overline{(\sigma*\tau)}(b\times h,b'\times h')&=(\sigma*\tau)(b\otimes h\cdot b')\varepsilon(h')\\
&=\sigma\big(b_1\otimes b_2^{(-1)}\cdot(h\cdot b')_1\big)\tau\big(b_2^{(0)}\otimes(h\cdot b')_2\big)\varepsilon(h')\\
&=\sigma\big(b_1\otimes b_2^{(-1)}h_1\cdot b'_1\big)\tau\big(b_2^{(0)}\otimes h_2\cdot b'_2\big)\varepsilon(h')\quad\big(\text{by (4.3)}\big)\\
&=\sigma\big(b_1\otimes b_2^{(-1)}h_1\cdot b'_1\big)\varepsilon\big(b_2'^{(-1)}\big)\varepsilon\big(h'_1\big)\tau\big(b_2^{(0)}\otimes h_2\cdot b_2'^{(0)}\big)\varepsilon\big(h'_2\big)\\
&=\bar\sigma\big(b_1\times b_2^{(-1)}h_1,b'_1\times b_2'^{(-1)}h'_1\big)\bar\tau\big(b_2^{(0)}\times h_2,b_2'^{(0)}\times h'_2\big)\\
&=(\bar\sigma*\bar\tau)(b\times h,b'\times h').
\end{aligned}$$

(v) Obviously $\bar\gamma$ is normalized, and it is easy to see that its convolution inverse is given by $\bar\gamma^{-1}(b\times h)=\gamma^{-1}(b)\varepsilon(h)$, where γ^{-1} is the convolution inverse of γ in ${}_H^H\mathcal{YD}$. Now we compute:

$$\begin{aligned}
\overline{D^1(\gamma)}(b\times h,b'\times h')&=D^1(\gamma)(b\otimes h\cdot b')\varepsilon(h')\\
&=\gamma(b_1)\gamma\big((h\cdot b')_1\big)\gamma^{-1}\big(b_2(h\cdot b')_2\big)\varepsilon(h')\\
&=\gamma(b_1)\gamma\big(h_1\cdot b'_1\big)\gamma^{-1}\big(b_2\big(h_2\cdot b'_2\big)\big)\varepsilon(h')\quad\big(\text{by (4.3)}\big)\\
&=\gamma(b_1)\gamma\big(b'_1\big)\gamma^{-1}\big(b_2\big(h\cdot b'_2\big)\big)\varepsilon(h')\quad\big(\text{by (4.18)}\big)\\
&=\gamma(b_1)\gamma\big(b'_1\big)\bar\gamma^{-1}\big(b_2\big(h_1\cdot b'_2\big)\times h_2h'\big)
\end{aligned}$$

$$
\begin{aligned}
&= \gamma(b_1)\gamma(b'_1)\bar{\gamma}^{-1}\big((b_2 \times h)\big(b'_2 \times h'\big)\big) \\
&= \bar{\gamma}\big(b_1 \times b_2^{(-1)}h_1\big)\bar{\gamma}\big(b'_1 \times b_2'^{(-1)}h'_1\big)\bar{\gamma}^{-1}\big(\big(b_2^{(0)} \times h_2\big)\big(b_2'^{(0)} \times h'_2\big)\big) \\
&= D^1(\bar{\gamma})(b \times h, b' \times h').
\end{aligned}
$$

Hence we have indeed $\overline{D^1(\gamma)} = D^1(\bar{\gamma})$. Finally, if γ is lazy in ${}_H^H\mathcal{YD}$, then we have:

$$
\begin{aligned}
\bar{\gamma}\big((b \times h)_1\big)(b \times h)_2 &= \gamma(b_1)(b_2 \times h) \\
&= \gamma(b_2)(b_1 \times h) \\
&= \gamma\big(b_2^{(0)}\big)\big(b_1 \times b_2^{(-1)}h\big) \quad \big(\text{by } (4.19)\big) \\
&= \bar{\gamma}\big((b \times h)_2\big)(b \times h)_1,
\end{aligned}
$$

where the second equality holds because γ is lazy, so $\bar{\gamma}$ is indeed lazy.

(vi) Follows immediately from (v). □

Remark 4.5. Let $\sigma : B \otimes B \to k$ be a normalized morphism in ${}_H^H\mathcal{YD}$, and define $\bar{\sigma} : (B \times H) \otimes (B \times H) \to k$ by the formula (4.20). Then one can easily prove that, conversely, if $\bar{\sigma}$ is a left 2-cocycle on $B \times H$, then σ is a left 2-cocycle on B in ${}_H^H\mathcal{YD}$. Together with (iii) and (iv) of Theorem 4.4, this proves easily that, if σ and τ are lazy 2-cocycles on B in ${}_H^H\mathcal{YD}$, then $\sigma * \tau$ is a left 2-cocycle on B in ${}_H^H\mathcal{YD}$ (as we mentioned before, this is quite difficult to prove by hand).

Example 4.6. Let H_4 be Sweedler's Hopf algebra. As an algebra, $H_4 = k\langle G, X \mid G^2 = 1, X^2 = 0, GX = -XG\rangle$. The comultiplication is given by $\Delta(G) = G \otimes G$, $\Delta(X) = 1 \otimes X + X \otimes G$, and the antipode is $S(G) = G$ and $S(X) = GX$. This Hopf algebra is a Radford biproduct of the Hopf algebra $H = k\mathbb{Z}_2$ and the Hopf algebra $B = k\langle x \mid x^2 = 0\rangle$ in ${}_H^H\mathcal{YD}$. Let g be the generator of the cyclic group of order two $\mathbb{Z}_2$. Then B is a left H-module algebra with the action $g \cdot x = -x$ and a left H-comodule (co)algebra with the coaction $\rho(x) = g \otimes x$. The comultiplication and counit of B are given by $\Delta(x) = 1 \otimes x + x \otimes 1$ and $\varepsilon(x) = 0$. The Radford biproduct $B \times H$ is isomorphic to H_4 via $1 \times g \mapsto G$, $x \times g \mapsto X$.

The group of lazy cocycles of H_4 is isomorphic to k. Any lazy cocycle σ of H_4 is of the form

σ	1	G	X	GX
1	1	1	0	0
G	1	1	0	0
X	0	0	$\frac{t}{2}$	$-\frac{t}{2}$
GX	0	0	$\frac{t}{2}$	$-\frac{t}{2}$

for some $t \in k$, see [2, Example 2.1]. The group $B_L(H_4)$ is trivial, so $H_L^2(H_4) = Z_L^2(H_4) \cong k$. One may check that any cocycle θ in B is of the form $\theta(1,1) = 1, \theta(1,x) = \theta(x,1) = 0$ and $\theta(x,x) = s$ for some $s \in k$. Denote this cocycle by θ_s. It is not difficult to verify that the map $H_L^2(B) \to H_L^2(H_4)$, $\theta_{-s/2} \mapsto \overline{\theta_{-s/2}}$ is a group isomorphism. Indeed this isomorphism holds more generally for Taft's Hopf algebras H_{n^2} and for the Hopf algebras $E(n)$. It would be interesting to find some sufficient conditions in a Radford biproduct $B \times H$ for the map $H_L^2(B) \to H_L^2(B \times H)$ to be an isomorphism.

5. Yetter–Drinfeld data obtained from lazy 2-cocycles

Let A be an H-bicomodule algebra, with comodule structures $A \to A \otimes H$, $a \mapsto a_{\langle 0\rangle} \otimes a_{\langle 1\rangle}$ and $A \to H \otimes A$, $a \mapsto a_{[-1]} \otimes a_{[0]}$. We can consider the *Yetter–Drinfeld datum* (H, A, H) as in [5] (the second H is regarded as an H-bimodule coalgebra), and the Yetter–Drinfeld category ${}_A\mathcal{YD}(H)^H$, whose objects are k-modules M endowed with a left A-action (denoted by $a \otimes m \mapsto a \cdot m$) and a right H-coaction (denoted by $m \mapsto m_{(0)} \otimes m_{(1)}$) satisfying the compatibility condition

$$a_{\langle 0\rangle} \cdot m_{(0)} \otimes a_{\langle 1\rangle} m_{(1)} = (a_{[0]} \cdot m)_{(0)} \otimes (a_{[0]} \cdot m)_{(1)} a_{[-1]}, \tag{5.1}$$

for all $a \in A$ and $m \in M$.

Let now σ be a normalized and convolution invertible lazy 2-cocycle on H, and consider the H-bicomodule algebra $H(\sigma)$ and the associated category ${}_{H(\sigma)}\mathcal{YD}(H)^H$; for an object M of this category, the compatibility (5.1) becomes

$$h_1 \cdot m_{(0)} \otimes h_2 m_{(1)} = (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)} h_1, \tag{5.2}$$

for all $h \in H(\sigma)$ and $m \in M$, which is identical to the compatibility in the usual Yetter–Drinfeld category ${}_H\mathcal{YD}^H$. Just as for ${}_H\mathcal{YD}^H$, it is easy to see that (5.2) is equivalent to

$$(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_2 \cdot m_{(0)} \otimes h_3 m_{(1)} S^{-1}(h_1). \tag{5.3}$$

Our aim will be to prove that, if M is a finite dimensional object in ${}_{H(\sigma)}\mathcal{YD}(H)^H$, then $End(M)$ and $End(M)^{op}$ are algebras in ${}_H\mathcal{YD}^H$.

Lemma 5.1.

(i) *The map Δ, regarded as a map $\Delta : H \to H(\sigma) \otimes H(\sigma^{-1})$, is an algebra map. Consequently, if $M \in {}_{H(\sigma)}\mathcal{M}$ and $N \in {}_{H(\sigma^{-1})}\mathcal{M}$ then $M \otimes N \in {}_H\mathcal{M}$ with action $h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n$.*

(ii) *If moreover $M \in {}_{H(\sigma)}\mathcal{YD}(H)^H$ and $N \in {}_{H(\sigma^{-1})}\mathcal{YD}(H)^H$, then $M \otimes N \in {}_H\mathcal{YD}^H$, with comodule structure $m \otimes n \mapsto (m_{(0)} \otimes n_{(0)}) \otimes n_{(1)} m_{(1)}$.*

Proof. A straightforward computation; note that (i) appears also in [2]. □

Proposition 5.2. *Let σ be a normalized and convolution invertible lazy 2-cocycle on H. Let $M \in {}_{H(\sigma)}\mathcal{YD}(H)^H$ finite dimensional. Then:*

(i) *M^* becomes an object in ${}_{H(\sigma^{-1})}\mathcal{YD}(H)^H$, with the following structures (called "of type 1"):*

$$(h \cdot m^*)(m) = m^*\big(S_1(h) \cdot m\big), \tag{5.4}$$

$$m^*_{(0)}(m) m^*_{(1)} = m^*(m_{(0)}) S^{-1}(m_{(1)}), \tag{5.5}$$

for all $h \in H$, $m \in M$, $m^* \in M^*$, where $S_1 : H \to H$ is the map given by (2.12);

(ii) *M^* becomes an object in ${}_{H(\sigma^{-1})}\mathcal{YD}(H)^H$, with the following structures* (*called* "*of type* 2"):

$$(h \cdot m^*)(m) = m^*\big(S_2(h) \cdot m\big), \tag{5.6}$$

$$m^*_{(0)}(m)m^*_{(1)} = m^*(m_{(0)})S(m_{(1)}), \tag{5.7}$$

for all $h \in H$, $m \in M$, $m^* \in M^*$, where $S_2 : H \to H$ is the map given by (2.13).

If σ is trivial, i.e. $M \in {}_H\mathcal{YD}^H$, these are the usual left and right duals of M in ${}_H\mathcal{YD}^H$, see [6].

Proof. We prove only (i), while (ii) is similar and left to the reader (for (i) we will use (2.10), for (ii) one has to use (2.5)). First, it is known that M^* is a right H-comodule with structure (5.5), and it is a left $H(\sigma^{-1})$-module with structure (5.4) because $S_1 : H(\sigma^{-1}) \to H(\sigma)$ is an algebra antihomomorphism. Hence, we only have to prove the Yetter–Drinfeld compatibility condition (5.2) for M^*. We compute, for all $h \in H$, $m \in M$, $m^* \in M^*$:

$$\begin{aligned}
\big(h_1 \cdot m^*_{(0)}\big)(m)h_2m^*_{(1)} &= m^*_{(0)}\big(S_1(h_1)\cdot m\big)h_2m^*_{(1)} && \big(\text{by } (5.4)\big)\\
&= m^*\big(\big(S_1(h_1)\cdot m\big)_{(0)}\big)h_2S^{-1}\big(\big(S_1(h_1)\cdot m\big)_{(1)}\big) && \big(\text{by } (5.5)\big)\\
&= m^*\big(S_1(h_1)_2\cdot m_{(0)}\big)h_2S^{-1}\big(S_1(h_1)_3m_{(1)}S^{-1}\big(S_1(h_1)_1\big)\big) && \big(\text{by } (5.3)\big)\\
&= m^*\big(S(h_2)\cdot m_{(0)}\big)h_4S^{-1}\big(S(h_1)m_{(1)}S^{-1}\big(S_1(h_3)\big)\big) && \big(\text{by } (2.16)\big)\\
&= m^*\big(S(h_2)\cdot m_{(0)}\big)\sigma^{-1}\big(S(h_4),h_5\big)h_6S^{-1}(h_3)S^{-1}(m_{(1)})h_1 && \big(\text{by } (2.12)\big)\\
&= m^*\big(S(h_2)\cdot m_{(0)}\big)\sigma^{-1}\big(S(h_3),h_4\big)S^{-1}(m_{(1)})h_1 && \big(\text{by } (2.10)\big)\\
&= m^*\big(S_1(h_2)\cdot m_{(0)}\big)S^{-1}(m_{(1)})h_1 && \big(\text{by } (2.12)\big)\\
&= (h_2\cdot m^*)(m_{(0)})S^{-1}(m_{(1)})h_1 && \big(\text{by } (5.4)\big)\\
&= (h_2\cdot m^*)_{(0)}(m)(h_2\cdot m^*)_{(1)}h_1 && \big(\text{by } (5.5)\big)
\end{aligned}$$

so (5.2) holds. $\square$

Remark 5.3. It was proved in [2] that, if $M \in {}_{H(\sigma)}\mathcal{M}$, then $M^* \in {}_{H(\sigma^{-1})}\mathcal{M}$ with action given by $(h \cdot m^*)(m) = m^*(\phi_{\sigma^{-1}}(h) \cdot m)$. Since we have proved that $\phi_{\sigma^{-1}} = S_1$, it follows that this action coincides with (5.4). Also, it should be clear that under our hypothesis that H has bijective antipode, the monoidal $H^2_L(H)$-category constructed in [2] has not only left duality, but also right duality, the right dual of an object M having the H-action given by (5.6).

We can prove now the following result, generalizing the well-known fact (see [6, Proposition 4.1]) that if M is a finite dimensional Yetter–Drinfeld module then $End(M)$ and $End(M)^{op}$ are Yetter–Drinfeld module algebras.

Proposition 5.4. *Let σ be a normalized and convolution invertible lazy 2-cocycle on H, and $M \in {}_{H(\sigma)}\mathcal{YD}(H)^H$ finite dimensional. Then*:

(i) *$End(M)$ becomes an algebra in ${}_H\mathcal{YD}^H$, with H-structures*:

$$(h \cdot f)(m) = h_1 \cdot f\big(S_1(h_2) \cdot m\big), \tag{5.8}$$

$$f_{(0)}(m) \otimes f_{(1)} = f(m_{(0)})_{(0)} \otimes S^{-1}(m_{(1)}) f(m_{(0)})_{(1)}, \tag{5.9}$$

for all $h \in H$, $m \in M$ and $f \in End(M)$;

(ii) *$End(M)^{op}$ becomes an algebra in ${}_H\mathcal{YD}^H$, with H-structures*:

$$(h \cdot f)(m) = h_2 \cdot f\big(S_2(h_1) \cdot m\big), \tag{5.10}$$

$$f_{(0)}(m) \otimes f_{(1)} = f(m_{(0)})_{(0)} \otimes f(m_{(0)})_{(1)} S(m_{(1)}). \tag{5.11}$$

Proof. (i) Since $M \in {}_{H(\sigma)}\mathcal{YD}(H)^H$ and $M^* \in {}_{H(\sigma^{-1})}\mathcal{YD}(H)^H$ with structures of type 1, $M \otimes M^*$ becomes an object in ${}_H\mathcal{YD}^H$, and by transferring its structure to $End(M)$ via the canonical isomorphism we get exactly (5.8) and (5.9), so $End(M) \in {}_H\mathcal{YD}^H$. It is clear that $End(M)$ is a right H^{op}-comodule algebra (its comodule and algebra structures do not depend on σ), so we only have to prove that $End(M)$ is a left H-module algebra. For $h \in H$, $f, f' \in End(M)$ and $m \in M$, we have:

$$\begin{aligned}
\big((h_1 \cdot f)(h_2 \cdot f')\big)(m) &= (h_1 \cdot f)\big(h_2 \cdot f'\big(S_1(h_3) \cdot m\big)\big) \\
&= h_1 \cdot f\big(S_1(h_2) \cdot \big(h_3 \cdot f'\big(S_1(h_4) \cdot m\big)\big)\big) \\
&= h_1 \cdot f\big(\big(S_1(h_2) \cdot_\sigma h_3\big) \cdot f'\big(S_1(h_4) \cdot m\big)\big) \\
&= h_1 \cdot f\big(f'\big(S_1(h_2) \cdot m\big)\big) \quad \big(\text{by } (2.14)\big) \\
&= \big(h \cdot (ff')\big)(m).
\end{aligned}$$

The relation $h \cdot id_M = \varepsilon(h)\, id_M$ follows immediately from (2.14).

(ii) The H-structures (5.10) and (5.11) come from the ones of $M^* \otimes M$ via the identification $End(M) = M^* \otimes M$, where M^* is regarded now as an object in ${}_{H(\sigma^{-1})}\mathcal{YD}(H)^H$ with structures of type 2. One can prove that $End(M)^{op}$ is an algebra in ${}_H\mathcal{YD}^H$ by a computation similar to the one in (i), using this time the relation (2.15). □

Let σ be as above and $M \in {}_{H(\sigma)}\mathcal{M}$, not necessarily finite dimensional. Define two actions of H on $End(M)$ by the formulae (5.8) and (5.10). Then one can check by direct computations that these actions give left H-module structures on $End(M)$, and the computations in the proof of the previous proposition show that actually $End(M)$ is a left H-module algebra with (5.8) and $End(M)^{op}$ is a left H-module algebra with (5.10). In particular, take $M = H(\sigma)$ and denote $End(H(\sigma))$ by A^σ. Then we recover the result in [10] that A^σ is a left H-module algebra, with action $(h \cdot f)(h') = h_1 \cdot_\sigma f(S_1(h_2) \cdot_\sigma h')$, for all $h, h' \in H$ and $f \in A^\sigma$. We will see below that if H is moreover finite dimensional then A^σ becomes an algebra in ${}_H\mathcal{YD}^H$.

Assume now that H is finite dimensional and A is an H-bicomodule algebra with notation as before. Then, by results in [5] or [4], the category ${}_A\mathcal{YD}(H)^H$ is isomorphic to the category ${}_{H^* \bowtie A}\mathcal{M}$ of left modules over the diagonal crossed product algebra $H^* \bowtie A$. If $M \in {}_A\mathcal{YD}(H)^H$, then M becomes a left $H^* \bowtie A$-module with structure

$$(p \bowtie a) \cdot m = p\big((a \cdot m)_{(1)}\big)(a \cdot m)_{(0)},$$

for all $p \in H^*$, $a \in A$ and $m \in M$. By taking $A = H(\sigma)$, where σ is a normalized and convolution invertible lazy 2-cocycle on H, we obtain that if $M \in {}_{H(\sigma)}\mathcal{YD}(H)^H$ then $M \in {}_{H^* \bowtie H(\sigma)}\mathcal{M}$, with $(p \bowtie h) \cdot m = p((h \cdot m)_{(1)})(h \cdot m)_{(0)}$.

On the other hand, we have seen in Proposition 3.1 that we have $H^* \bowtie H(\sigma) = D(H)(\bar{\sigma})$ as $D(H)$-bicomodule algebras, where $\bar{\sigma}$ is the extension of σ to $D(H)$ given by the formula (3.1). Hence, we get that $M \in {}_{D(H)(\bar{\sigma})}\mathcal{M}$. By the previous discussion, we obtain that $End(M)$ and $End(M)^{op}$ are left $D(H)$-module algebras, with $D(H)$-actions given respectively by

$$\big((p \otimes h) \cdot f\big)(m) = (p \otimes h)_1 \cdot f\big(\overline{S}_1\big((p \otimes h)_2\big) \cdot m\big), \tag{5.12}$$

$$\big((p \otimes h) \cdot f\big)(m) = (p \otimes h)_2 \cdot f\big(\overline{S}_2\big((p \otimes h)_1\big) \cdot m\big), \tag{5.13}$$

for all $p \in H^*$, $h \in H$, $f \in End(M)$, $m \in M$, where $\overline{S}_1, \overline{S}_2 : D(H) \to D(H)$ are the maps given by the formulae (3.3), (3.4).

If M is moreover assumed to be finite dimensional, then by Proposition 5.4, $End(M)$ and $End(M)^{op}$ are algebras in ${}_H\mathcal{YD}^H$. Hence they become left $D(H)$-module algebras, with $D(H)$-actions on $End(M)$ and $End(M)^{op}$ given by

$$(p \otimes h) \cdot f = p\big((h \cdot f)_{(1)}\big)(h \cdot f)_{(0)}, \tag{5.14}$$

where $h \cdot f$ is the action (5.8), respectively (5.10). So in this case we have two $D(H)$-module algebra structures on $End(M)$ and two on $End(M)^{op}$.

Proposition 5.5. *The two $D(H)$-module algebra structures as above on $End(M)$ (respectively on $End(M)^{op}$) coincide, and are given respectively by*

$$\big((p \otimes h) \cdot f\big)(m) = p\big(S^{-1}(m_{(1)})h_3 f\big(S_1(h_4) \cdot m_{(0)}\big)_{(1)} S^{-1}(h_1)\big)h_2 \cdot f\big(S_1(h_4) \cdot m_{(0)}\big)_{(0)},$$

$$\big((p \otimes h) \cdot f\big)(m) = p\big(h_4 f\big(S_2(h_1) \cdot m_{(0)}\big)_{(1)} S^{-1}(h_2) S(m_{(1)})\big)h_3 \cdot f\big(S_2(h_1) \cdot m_{(0)}\big)_{(0)},$$

for all $p \in H^*$, $h \in H$, $f \in End(M)$ and $m \in M$.

Proof. We give the proof only for $End(M)$, the one for $End(M)^{op}$ is similar. We compute first the $D(H)$-module structure of $End(M)$ obtained using $\bar{\sigma}$. We have:

$$\begin{aligned}
&\big((p \otimes h) \cdot f\big)(m) \\
&\quad = (p_2 \otimes h_1) \cdot f\big(\overline{S}_1(p_1 \otimes h_2) \cdot m\big) \\
&\quad = (p_2 \otimes h_1) \cdot f\big(\big(\varepsilon \otimes S_1(h_2)\big)\big(S^{*-1}(p_1) \otimes 1\big) \cdot m\big) \quad \big(\text{by (3.5)}\big) \\
&\quad = (p_2 \otimes h_1) \cdot f\big(\big(\varepsilon \otimes S_1(h_2)\big) \cdot \big(\big(S^{*-1}(p_1) \otimes 1\big) \cdot m\big)\big) \quad \big(\text{by (3.7)}\big) \\
&\quad = (p_2 \otimes h_1) \cdot f\big(\big(\varepsilon \otimes S_1(h_2)\big) \cdot S^{*-1}(p_1)(m_{(1)})m_{(0)}\big) \\
&\quad = (p_2 \otimes h_1) \cdot f\big(p_1\big(S^{-1}(m_{(1)})\big)S_1(h_2) \cdot m_{(0)}\big) \\
&\quad = \big(p \leftharpoonup S^{-1}(m_{(1)}) \otimes h_1\big) \cdot f\big(S_1(h_2) \cdot m_{(0)}\big)
\end{aligned}$$

$$= p \leftharpoonup S^{-1}(m_{(1)})\big(\big(h_1 \cdot f\big(S_1(h_2) \cdot m_{(0)}\big)\big)_{(1)}\big)\big(h_1 \cdot f\big(S_1(h_2) \cdot m_{(0)}\big)\big)_{(0)}$$
$$= p\big(S^{-1}(m_{(1)})h_3 f\big(S_1(h_4) \cdot m_{(0)}\big)_{(1)} S^{-1}(h_1)\big)h_2 \cdot f\big(S_1(h_4) \cdot m_{(0)}\big)_{(0)} \quad \big(\text{by (5.3)}\big).$$

We compute now the $D(H)$-module structure of $End(M)$ coming from ${}_H\mathcal{YD}^H$. We have:

$$\begin{aligned}
\big((p \otimes h) \cdot f\big)(m) &= \big(p\big((h \cdot f)_{(1)}\big)(h \cdot f)_{(0)}\big)(m) \\
&= p\big((h \cdot f)_{(1)}\big)(h \cdot f)_{(0)}(m) \\
&= p\big(S^{-1}(m_{(1)})(h \cdot f)(m_{(0)})_{(1)}\big)(h \cdot f)(m_{(0)})_{(0)} \quad \big(\text{by (5.9)}\big) \\
&= p\big(S^{-1}(m_{(1)})\big(h_1 \cdot f\big(S_1(h_2) \cdot m_{(0)}\big)\big)_{(1)}\big)\big(h_1 \cdot f\big(S_1(h_2) \cdot m_{(0)}\big)\big)_{(0)} \\
&= p\big(S^{-1}(m_{(1)})h_3 f\big(S_1(h_4) \cdot m_{(0)}\big)_{(1)} S^{-1}(h_1)\big)h_2 \cdot f\big(S_1(h_4) \cdot m_{(0)}\big)_{(0)},
\end{aligned}$$

so the two structures coincide. □

Let H be of finite dimension and A an H-bicomodule algebra with notation as before. Then one can check, by direct computation, that $A \in {}_A\mathcal{YD}(H)^H$, where A is a left A-module by the left regular action $a \cdot b = ab$ for all $a, b \in A$, and A is a right H-comodule with coaction $A \to A \otimes H$, $a \mapsto a_{\{0\}} \otimes a_{\{1\}} S^{-1}(a_{\{-1\}})$ for all $a \in A$. Hence, if σ is a normalized and convolution invertible lazy 2-cocycle on H, by taking $A = H(\sigma)$ we obtain that $H(\sigma) \in {}_{H(\sigma)}\mathcal{YD}(H)^H$, with $H(\sigma)$-action $h \cdot l = h \cdot_\sigma l$ for all $h, l \in H$, and right H-comodule structure $H(\sigma) \to H(\sigma) \otimes H$, $h \mapsto h_2 \otimes h_3 S^{-1}(h_1)$ for all $h \in H(\sigma)$. By applying all the above to $H(\sigma) \in {}_{H(\sigma)}\mathcal{YD}(H)^H$, we obtain that $A^\sigma = End(H(\sigma))$ and $End(H(\sigma))^{op}$ are algebras in ${}_H\mathcal{YD}^H$.

Proposition 5.6. *Let σ be a normalized and convolution invertible lazy 2-cocycle on H and $M \in {}_{H(\sigma)}\mathcal{M}$. If σ is a lazy 2-coboundary, then the H-module algebra structure of $End(M)$ given by* (5.8) *is strongly inner* (*afforded by some algebra map $G : H \to End(M)$*). *If moreover H is finite dimensional and $M \in {}_{H(\sigma)}\mathcal{YD}(H)^H$, then the $D(H)$-module structure of $End(M)$ given by* (5.12) *is also strongly inner.*

Proof. Since σ is a lazy 2-coboundary, there exists $\gamma : H \to k$ lazy, normalized and convolution invertible such that $\sigma = D^1(\gamma)$. Then, by [2], the map $\varphi : H(\sigma) \to H$, $\varphi(h) = \gamma(h_1)h_2$, is an isomorphism of H-bicomodule algebras. Define $F : H(\sigma) \to End(M)$, $F(h)(m) = h \cdot m$, which is obviously an algebra map. Hence, the map $G : H \to End(M)$, $G = F \circ \varphi^{-1}$, is also an algebra map. Using the laziness of γ, we can express F as $F(h) = G(\varphi(h)) = \gamma(h_1)G(h_2) = \gamma(h_2)G(h_1)$. Using (2.14), it is easy to see that F is convolution invertible with inverse $F^{-1} = F \circ S_1$, so the action (5.8) is just the inner action afforded by F. Hence, we can write (5.8) as follows:

$$\begin{aligned}
h \cdot f &= F(h_1) \circ f \circ F^{-1}(h_2) \\
&= G(h_1)\gamma(h_2) \circ f \circ \gamma^{-1}(h_3)G^{-1}(h_4) \\
&= G(h_1) \circ f \circ G^{-1}(h_2),
\end{aligned}$$

thus (5.8) is strongly inner, afforded by G.

Assume now that H is finite dimensional and $M \in {}_{H(\sigma)}\mathcal{YD}(H)^H$. Then we know that M becomes a left $D(H)(\bar{\sigma})$-module, and, due to the embedding $H_L^2(H) \to H_L^2(D(H)), \sigma \mapsto \bar{\sigma}$, since σ is a lazy 2-coboundary for H then $\bar{\sigma}$ is a lazy 2-coboundary for $D(H)$ (namely, $\bar{\sigma} = D^1(\bar{\gamma})$, where $\bar{\gamma} : D(H) \to k$, $\bar{\gamma}(p \otimes h) = p(1)\gamma(h)$). Hence, we can repeat the above proof for $\bar{\sigma}$ instead of σ and $D(H)$ instead of H, and we obtain that the $D(H)$-module structure on $End(M)$ given by (5.12) is also strongly inner. $\square$

We can prove also a partial converse of this result. Recall from [2] the normal subgroups $CoInt(H)$ and $CoInn(H)$ of $Aut_{Hopf}(H)$. If $\gamma \in Reg^1(H)$, define $ad(\gamma) : H \to H$ by $ad(\gamma) = \gamma^{-1} * id_H * \gamma$; then $ad(\gamma) \in Aut_{Hopf}(H)$ if and only if $D^1(\gamma)$ is lazy. $CoInt(H)$ is defined as the set of Hopf algebra automorphisms of H of the type $ad(\gamma)$. It contains the subgroup

$$CoInn(H) = \bigl\{ f \in Aut_{Hopf}(H) \bigm| \exists \phi \in Alg(H,k) \text{ with } f = (\phi \circ S) * id_H * \phi \bigr\}.$$

Suppose that, for a given Hopf algebra H, we have $CoInt(H) = CoInn(H)$, and we have $\sigma \in Z_L^2(H)$ of the form $\sigma = D^1(\gamma)$, with $\gamma \in Reg^1(H)$. Then, by [2, Lemma 1.12], it follows that $\sigma \in B_L^2(H)$, that is there exists $\chi \in Reg_L^1(H)$ such that $\sigma = D^1(\chi)$.

Proposition 5.7. *Let σ be as above and $M \in {}_{H(\sigma)}\mathcal{M}$ finite dimensional. If the action* (5.8) *of H on $End(M)$ is strongly inner* (*afforded by some algebra map $G : H \to End(M)$*), *then there exists $\gamma : H \to k$ normalized and convolution invertible such that $\sigma = D^1(\gamma)$. If moreover we have $CoInn(H) = CoInt(H)$, then σ is a lazy* 2*-coboundary.*

Proof. Denote as before $F : H(\sigma) \to End(M)$, $F(h)(m) = h \cdot m$, which is an algebra map. We have, for all $h \in H$ and $f \in End(M)$:

$$h \cdot f = F(h_1) \circ f \circ F^{-1}(h_2) = G(h_1) \circ f \circ G^{-1}(h_2).$$

Hence, if we define $\gamma : H \to End(M)$ by $\gamma(h) = G^{-1}(h_1) \circ F(h_2)$, we obtain that $\gamma(h) \circ f = f \circ \gamma(h)$, and since $End(M)$ is a central algebra and this relation holds for all $f \in End(M)$, it follows that actually γ is a map from H to k. Obviously γ is normalized and convolution invertible, so we only have to prove that $\sigma = D^1(\gamma)$.

First note that, since G is an algebra map, we have $G^{-1} = G \circ S$, so G^{-1} is an antialgebra map. Also, since $F : H(\sigma) \to End(M)$ is an algebra map, we have $F(hl) = \sigma^{-1}(h_1, l_1)F(h_2)F(l_2)$ for all $h, l \in H$. Now we compute:

$$\begin{aligned} \gamma(hl) &= G^{-1}(h_1 l_1)F(h_2 l_2) \\ &= G^{-1}(h_1 l_1)\sigma^{-1}(h_2, l_2)F(h_3)F(l_3) \\ &= \sigma^{-1}(h_1, l_1)G^{-1}(h_2 l_2)F(h_3)F(l_3) \end{aligned}$$

because σ^{-1} is lazy. Hence, we have:

$$\begin{aligned} \sigma(h_1, l_1)\gamma(h_2 l_2) &= \sigma(h_1, l_1)\sigma^{-1}(h_2, l_2)G^{-1}(l_3)G^{-1}(h_3)F(h_4)F(l_4) \\ &= G^{-1}(l_1)G^{-1}(h_1)F(h_2)F(l_2) \\ &= G^{-1}(l_1)\gamma(h)F(l_2) \end{aligned}$$

$$= \gamma(h)G^{-1}(l_1)F(l_2)$$
$$= \gamma(h)\gamma(l),$$

so we obtain $\sigma(h,l) = \gamma(h_1)\gamma(l_1)\gamma^{-1}(h_2l_2) = D^1(\gamma)(h,l)$. In general, we do not know whether γ is lazy or whether there exists another $\chi : H \to k$ *lazy* such that $\sigma = D^1(\chi)$. However, if $CoInn(H) = CoInt(H)$, then by [2] such a χ exists, so σ is a lazy 2-coboundary in this case. □

6. Lifting projective representations afforded by lazy 2-cocycles

A theorem of Schur asserts that for any finite group G there exists a finite central extension C such that any projective representation of G can be lifted to an ordinary representation of C. This theorem has been generalized by Ioana Boca in [3], who proved that any *cocommutative* Hopf algebra H admits a (cocommutative) central extension B such that any projective representation of H can be lifted to an ordinary representation of B. The aim of this section is to further generalize her result, by proving that *any* Hopf algebra H admits a central extension B such that any projective representation of H *afforded by a lazy 2-cocycle* can be lifted to an ordinary representation of B. Our proof follows closely the one of Boca, so many details will be skipped. The proof will reveal again how important is the fact that lazy 2-cocycles form a group.

If H is a Hopf algebra and K is a Hopf subalgebra of H, then K^+ is defined by $K^+ = K \cap Ker(\varepsilon)$. If K is a central Hopf subalgebra of H, then $HK^+ = K^+H$ and HK^+ is a Hopf ideal of H, so $\overline{H} = H/HK^+$ is a Hopf algebra. A *central extension* of H is a Hopf algebra B together with a central Hopf subalgebra A such that the Hopf algebra quotient B/BA^+ is isomorphic to H (we denote by π the surjection $B \to H$ with kernel BA^+). Recall now from [3, Definition 2.2] the concept of a projective representation for a Hopf algebra H.

Definition 6.1. If V is a vector space, a linear map $T : H \to End(V)$ is called a projective representation of H if:

(i) T is convolution invertible;
(ii) $T(1) = id_V$;
(iii) $T(h)T(l) = \alpha(h_1,l_1)T(h_2l_2)$ for all $h,l \in H$, where $\alpha \in Hom(H \otimes H,k)$ is convolution invertible.

It was proved in [3] that if T is a projective representation, then α is a normalized (and convolution invertible) left 2-cocycle and is uniquely determined by T (it will be called the *cocycle of* T, or we say that T *is afforded by* α). Conversely, one can see that, if a map T as above satisfies (ii) and (iii), where α is a normalized and convolution invertible left 2-cocycle, then it also satisfies (i), its convolution inverse being $T^{-1}(h) = T(S_1(h))$, where S_1 is the map defined by (2.12). Hence, T is a projective representation if and only if V is a left ${}_\alpha H$-module. Recall now from [3, Definition 2.11] the concept of *lifting* of a projective representation.

Definition 6.2. If (B,A) is a central extension of a Hopf algebra H and $T : H \to End(V)$ is a projective representation of H, then we say that T can be lifted to B if there exists an ordinary representation (algebra map) $X : B \to End(V)$ and an element $\gamma \in Reg(B,k)$, with $\gamma(1) = 1$, such that $X = \gamma * (T \circ \pi)$. Such a representation X is called a lift of T.

Lemma 6.3. *Let H and A be Hopf algebras with A commutative. If $\sigma : H \otimes H \to A$ is a normalized and convolution invertible left 2-cocycle (with respect to the trivial action of H on A) which is moreover a coalgebra map and is lazy in the sense that*

$$\sigma(h_1, l_1) \otimes h_2 l_2 = \sigma(h_2, l_2) \otimes h_1 l_1$$

in $A \otimes H$, for all $h, l \in H$, then the crossed product $B = A \#_\sigma H$ is a Hopf algebra with:

(1) $(a \# h)(c \# l) = ac\sigma(h_1, l_1) \# h_2 l_2$, *for all $a, c \in A$ and $h, l \in H$;*
(2) $\Delta(a \# h) = (a_1 \# h_1) \otimes (a_2 \# h_2)$;
(3) $\varepsilon(a \# h) = \varepsilon(a)\varepsilon(h)$;
(4) $S(a \# h) = (\sigma^{-1}(S(h_2), h_3) \# S(h_1))(S(a) \# 1)$;
(5) *The map $\pi : B \to H$, $\pi(a \# h) = \varepsilon(a)h$ is a Hopf algebra epimorphism with kernel BA^+;*
(6) $A \simeq A \# 1$ *is a central Hopf subalgebra of B and*

$$A = B^{co(H)} := \{b \in B \mid (id \otimes \pi)\Delta(b) = b \otimes 1\}.$$

Proof. We only show how to replace the cocommutativity of H in [3, Lemma 2.1], by the laziness of σ, the rest of the proof is identical to the one in [3]. Namely, one can compute as in [3] that

$$\Delta\big((a \# h)(c \# l)\big) = \big(a_1 c_1 \sigma(h_1, l_1) \# h_3 l_3\big) \otimes \big(a_2 c_2 \sigma(h_2, l_2) \# h_4 l_4\big),$$

for all $a, c \in A$ and $h, l \in H$, using the fact that σ is a coalgebra map, and

$$\Delta(a \# h)\Delta(c \# l) = \big(a_1 c_1 \sigma(h_1, l_1) \# h_2 l_2\big) \otimes \big(a_2 c_2 \sigma(h_3, l_3) \# h_4 l_4\big),$$

and these are equal because, since σ is lazy, we have $\sigma(h_2, l_2) \otimes h_3 l_3 = \sigma(h_3, l_3) \otimes h_2 l_2$. □

Denote by G the group $Z_L^2(H)$ of normalized and convolution invertible lazy 2-cocycles on H. Denote by A the finite dual $(kG)^0$ of the group algebra kG, so A is a commutative Hopf algebra. We can generalize [3, Lemma 3.1] as follows.

Lemma 6.4. *Let H, G, A be as above. Define $\sigma : H \otimes H \to (kG)^*$ by $\sigma(h, l)(\alpha) = \alpha(h, l)$, for all $h, l \in H$ and $\alpha \in G$. Then $Im(\sigma) \subseteq A$ and the corestriction $\sigma : H \otimes H \to A$ is a coalgebra map and a normalized and convolution invertible lazy 2-cocycle.*

Proof. We only prove that σ is lazy, the rest of the proof is identical to the one in [3]. Namely, we have to prove that for all $h, l \in H$ we have the equality $\sigma(h_1, l_1) \otimes h_2 l_2 = \sigma(h_2, l_2) \otimes h_1 l_1$ in $(kG)^0 \otimes H$. This is equivalent to proving that $\sigma(h_1, l_1)(\alpha)h_2 l_2 = \sigma(h_2, l_2)(\alpha)h_1 l_1$ for all $\alpha \in G$, that is, $\alpha(h_1, l_1)h_2 l_2 = \alpha(h_2, l_2)h_1 l_1$ for all $\alpha \in G$, which is obviously true because G consists exactly of lazy cocycles. □

The following result generalizes [3, Proposition 2.9].

Lemma 6.5. *Let H be a Hopf algebra and $T : H \to End(V)$ a projective representation afforded by a lazy 2-cocycle α and let $u \in Reg(H, k)$ with $u(1) = 1$. If $W := u * T$, then W is a projective representation with cocycle $\delta(u) * \alpha$, where $\delta(u)(h, l) = u(h_1)u(l_1)u^{-1}(h_2 l_2)$ for all $h, l \in H$.*

Proof. Obviously $W(1) = id_V$; then one computes immediately that

$$\begin{aligned} W(h)W(l) &= u(h_1)u(l_1)T(h_2)T(l_2) \\ &= u(h_1)u(l_1)\alpha(h_2, l_2)T(h_3l_3) \\ &= u(h_1)u(l_1)u^{-1}(h_2l_2)u(h_3l_3)\alpha(h_4, l_4)T(h_5l_5) \\ &= u(h_1)u(l_1)u^{-1}(h_2l_2)\alpha(h_3, l_3)u(h_4l_4)T(h_5l_5) \\ &= \big(\delta(u) * \alpha\big)(h_1, l_1)W(h_2l_2), \end{aligned}$$

where in the fourth equality we used the fact that α is lazy. □

The next result generalizes [3, Proposition 2.12].

Proposition 6.6. *Let H, A, σ, B be as in Lemma* 6.3. *Then*:

(i) *If X is an ordinary representation of B such that $\lambda := X_{/A}$ is a scalar function, then, if we define $T(h) = X(1 \# h)$, T is a projective representation of H afforded by the lazy* 2*-cocycle $\lambda \circ \sigma$ and moreover X is a lift of T*;
(ii) *If (T, α) is a projective representation of H afforded by the lazy* 2*-cocycle α and X is a lift of T, then $\lambda := X_{/A}$ is a scalar function. Moreover, the lazy* 2*-cocycles α and $\lambda \circ \sigma$ are cohomologous* (*but not necessarily lazy cohomologous*);
(iii) *Let (T, α) be a projective representation of H afforded by the lazy* 2*-cocycle α. Then T can be lifted to B if and only if there exists an algebra map $\lambda : A \to k$ such that α is cohomologous to $\lambda \circ \sigma$ (via a map $u \in Reg(H, k)$ with $u(1) = 1$, but u not necessarily lazy).*

Proof. Follows closely the proof in [3]. The laziness of α is used through the fact that $\delta(u) * \alpha$ can be written as $(\delta(u) * \alpha)(h, l) = u(h_1)u(l_1)u^{-1}(h_2l_2)\alpha(h_3, l_3) = u(h_1)u(l_1)\alpha(h_2, l_2)u^{-1}(h_3l_3)$, and through the fact that one has to use the previous lemma, where α is supposed to be lazy. □

We can finally obtain the desired result, generalizing [3, Theorem 3.2].

Theorem 6.7. *Let H be a Hopf algebra. Then there exists a central extension B of H such that any projective representation of H afforded by a lazy* 2*-cocycle can be lifted to B.*

Proof. Take as above $G = Z_L^2(H)$, $A = (kG)^0$, $\sigma : H \otimes H \to (kG)^0$, $\sigma(h, l)(\alpha) = \alpha(h, l)$ for all $h, l \in H$ and $\alpha \in G$. By Lemma 6.4, the hypotheses of Lemma 6.3 are satisfied, so we can consider the Hopf algebra $B = A \#_\sigma H$, a central extension of H. We prove that any projective representation T of H afforded by a lazy 2-cocycle α can be lifted to B. By the previous proposition, it is enough to find an algebra map $\lambda : A \to k$ such that α is cohomologous to $\lambda \circ \sigma$. As in [3], define $\lambda : A \to k$ by $\lambda(F) = F(\alpha)$, for all $F \in A = (kG)^0$. Then we have

$$(\lambda \circ \sigma)(h, l) = \lambda\big(\sigma(h, l)\big) = \sigma(h, l)(\alpha) = \alpha(h, l),$$

hence $\alpha = \lambda \circ \sigma$. Then, we have, for $F, G \in A = (kG)^0$:

$$\lambda(FG) = (FG)(\alpha) = F(\alpha)G(\alpha) = \lambda(F)\lambda(G)$$

because α is grouplike in kG, and $\lambda(\varepsilon) = \varepsilon(\alpha) = 1$, hence λ is an algebra map. □

7. Lazy cocycles and separable Hopf algebras

Let A be an algebra; we recall that an element $e = e^1 \otimes e^2 \in A \otimes A$ is called A-central if $ae^1 \otimes e^2 = e^1 \otimes e^2 a$, for all $a \in A$, and A is separable if and only if there exists an A-central element $e^1 \otimes e^2$ such that $e^1 e^2 = 1$ (such an element is called a separability idempotent).

It was proved in [5, Proposition 147], that if H is a *cocommutative* Hopf algebra, $\sigma : H \otimes H \to k$ is a 2-cocycle and $t \in H$ is a right integral, then the element $R_\sigma = \sigma^{-1}(S(t_2) \otimes t_3)S(t_1) \otimes t_4 \in H_\sigma \otimes H_\sigma$ is H_σ-central (and H separable implies H_σ separable). We can generalize this result (by replacing cocommutativity by laziness) as follows:

Proposition 7.1. *Let $\sigma : H \otimes H \to k$ be a normalized and convolution invertible lazy* 2*-cocycle, and $S_1 : H \to H$ the map given by* (2.12)*. Then*:

- (i) *If $t \in H$ is a right integral, then the element $R(\sigma) = S_1(t_1) \otimes t_2 \in H(\sigma) \otimes H(\sigma)$ is $H(\sigma)$-central.*
- (ii) *If $t \in H$ is a left integral, then the element $R(\sigma) = t_1 \otimes S_1(t_2) \in H(\sigma) \otimes H(\sigma)$ is $H(\sigma)$-central.*
- (iii) *Consequently, if H is separable, then $H(\sigma)$ is also separable.*

Proof. We prove (i), while (ii) is similar and left to the reader. We denote as before by $\cdot_\sigma$ the multiplication of $H(\sigma)$ and by $\cdot_{\sigma^{-1}}$ the one of $H(\sigma^{-1})$. Since t is a right integral, we have $th = \varepsilon(h)t$ for all $h \in H$, hence we can write $h \otimes t = h_1 \otimes th_2$. By applying Δ on the second component and using Lemma 5.1(i), with σ and σ^{-1} interchanged, we obtain

$$h \otimes t_1 \otimes t_2 = h_1 \otimes t_1 \cdot_{\sigma^{-1}} h_2 \otimes t_2 \cdot_\sigma h_3.$$

By applying S_1 on the second component and using Proposition 2.5, we obtain

$$h \otimes S_1(t_1) \otimes t_2 = h_1 \otimes S_1(h_2) \cdot_\sigma S_1(t_1) \otimes t_2 \cdot_\sigma h_3.$$

By multiplying (in $H(\sigma)$) the first two components we obtain

$$h \cdot_\sigma S_1(t_1) \otimes t_2 = h_1 \cdot_\sigma S_1(h_2) \cdot_\sigma S_1(t_1) \otimes t_2 \cdot_\sigma h_3,$$

and using formula (2.14) this becomes

$$h \cdot_\sigma S_1(t_1) \otimes t_2 = S_1(t_1) \otimes t_2 \cdot_\sigma h,$$

that is $R(\sigma)$ is $H(\sigma)$-central.

(iii) If H is separable, there exists a right integral $t \in H$ with $\varepsilon(t) = 1$, and by (i) and using again (2.14) it follows that $R(\sigma)$ is a separability idempotent for $H(\sigma)$. □

It is instructive to compare this very easy and conceptual proof with the one in [5], which is more involved though it works only in the cocommutative case.

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