Journal of Algebra 360 (2012) 87–91

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Quasi-projective modules over prime hereditary noetherian V-rings are projective or injective

Hai Q. Dinh [a], Christopher J. Holston [b], Dinh V. Huynh [b,*]

[a] *Department of Mathematical Sciences, Kent State University, Warren, OH, USA*
[b] *Department of Mathematics, Ohio University, Athens, OH, USA*

ARTICLE INFO

Article history:
Received 30 March 2012
Available online 12 April 2012
Communicated by Louis Rowen

MSC:
16U10
16D40
16P40

Keywords:
Hereditary noetherian ring
R-projective module
PCI-domain
V-domain
SI-ring
Nonsingular module
Singular module

ABSTRACT

Let $\mathbb{Q}$ be the field of rational numbers. As a module over the ring $\mathbb{Z}$ of integers, $\mathbb{Q}$ is $\mathbb{Z}$-projective, but $\mathbb{Q}_{\mathbb{Z}}$ is not a projective module. Contrary to this situation, we show that over a prime right noetherian right hereditary right V-ring R, a right module P is projective if and only if P is R-projective. As a consequence of this we obtain the result stated in the title. Furthermore, we apply this to affirmatively answer a question that was left open in a recent work of Holston, López-Permouth and Orhan Ertag (2012) [9] by showing that over a right noetherian prime right SI-ring, quasi-projective right modules are projective or semisimple.

1. Introduction

We consider associative rings with identity. All modules are unitary modules. Let M, N be right R-modules. The module M is called N-projective if for each exact sequence

$$0 \to H \to N \xrightarrow{g} K \to 0$$

* Corresponding author.
E-mail addresses: hdinh@kent.edu (H.Q. Dinh), holston@ohio.edu (C.J. Holston), huynh@ohio.edu (D.V. Huynh).

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.04.002

in Mod-R, and any homomorphism $f : M \to K$ there is a homomorphism $f' : M \to N$ such that $gf' = f$.

A right R-module P is defined to be a projective module if P is N-projective for any $N \in$ Mod-R. A ring R is right (left) hereditary if every right (left) ideal of R is projective as a right (left) R-module. A right and left hereditary (noetherian) ring is simply called a hereditary (noetherian) ring.

For basic properties of (quasi-)projective modules as well as concepts of modules and rings not defined here we refer to [1,7,11,12,14].

Unlike the injectivity of modules, an R-projective module may not be projective, in general. As an example for that, we consider the ring $\mathbb{Q}$ of rational numbers as a module over the ring $\mathbb{Z}$ of integers (cf. [1, 10(1), p. 190]). Then, since every nonzero homomorphic image of $\mathbb{Q}_{\mathbb{Z}}$ is infinite (and injective), there is no nonzero homomorphism from $\mathbb{Q}_{\mathbb{Z}}$ to $\mathbb{Z}/A$ for any ideal $A \subset \mathbb{Z}$. Hence $\mathbb{Q}_{\mathbb{Z}}$ is $\mathbb{Z}$-projective. But, obviously, $\mathbb{Q}_{\mathbb{Z}}$ is not projective. Note that $\mathbb{Z}$ is a noetherian hereditary domain, it is even a commutative PID.

Motivated by this we ask a question:

For which noetherian hereditary domains D, D-projectivity implies projectivity?

In this note we answer this question affirmatively for prime right noetherian right hereditary right V-rings (Corollary 4). Using this we show that the class (iii) of [9, Theorem 3.11] is not empty. This is what the authors of [9] wanted to see.

Note that a ring R is called a right V-ring (after Villamyor) if every simple right R-module is injective. For basic properties of V-rings we refer to [13].

2. Results

A submodule E of a module M is called an essential submodule if for any nonzero submodule $A \subseteq M$, $E \cap A \neq 0$. A nonzero submodule $U \subseteq M$ is called uniform if every nonzero submodule of U is essential in U.

A right R-module N is called nonsingular if for any nonzero element $x \in N$ the annihilator $ann_R(x)$ of x in R is not an essential right ideal of R. A right R-module S is called a singular module if the annihilator in R of each nonzero element of S is an essential right ideal of R. Every R-module M has a maximal singular submodule $Z(M)$ which contains all singular submodules of M. This is a fully invariant submodule of M and it is called the singular submodule of M. Clearly, M is nonsingular if and only if $Z(M) = 0$. For a ring R, if $Z(R_R) = 0$ (resp., $Z({}_RR) = 0$) then R is called right (left) nonsingular. (See, e.g., [8, p. 5].) To indicate that M is a right (left) module over R we write M_R (resp., ${}_RM$).

Lemma 1. *Let R be a right nonsingular right V-ring. Any nonzero R-projective right R-module M is nonsingular.*

Proof. Assume on the contrary that M contains a nonzero singular submodule T. As R is a right V-ring, T contains a maximal submodule V for which we have $M/V = (T/V) \oplus (L/V)$ for some submodule L of M with $V \subset L$. On the other hand, there exists a maximal right ideal $B \subset R$ such that $R/B \cong T/V$ as right R-modules. This means there is a homomorphism $f : M \to R/B$ with $\mathrm{Ker}(f) = L$. By the definition of the R-projectivity, there exists a homomorphism $f' : M \to R_R$ such that $gf' = f$ where g is the canonical homomorphism $R \to R/B$. However, this is impossible, because as R is right nonsingular, the kernel of f' must contain the singular submodule T which implies $\mathrm{Ker}(gf') \neq \mathrm{Ker}(f)$. Thus M does not contain a nonzero singular submodule, proving that M is nonsingular. □

We would like to remark that Lemma 1 does not hold if the ring R is not a right V-ring. As an example for this, we again take the ring $\mathbb{Z}$. Let $p \in \mathbb{Z}$ be a prime number, then the p-Prüfer group $C(p^\infty)$ is $\mathbb{Z}$-projective (cf. [1, 10(1), p. 190]), but $C(p^\infty)$ is a singular $\mathbb{Z}$-module.

If a module M has finite uniform dimension, we denote its dimension by u-dim(M) and call M a finite dimensional module.

Lemma 2. *Let D be a right hereditary domain. Then*:

(a) *Every projective right D-module is isomorphic to a direct sum of right ideals of D.*
(b) *In addition, if D is right noetherian and right V, then every finite dimensional submodule of a D-projective right D-module P is projective.*

Proof. (a) holds by a well-known result of Kaplansky [10] (see also [12, 2.24]).

(b) Let D be a right noetherian right hereditary right V-domain, and P be a D-projective right D-module. By Lemma 1, P is nonsingular. Let $M \subseteq P$ be a submodule with u-dim$(M_R) = k$ where k is a positive integer. Then there are k cyclic uniform independent submodules $X_i \subseteq M$ such that $X_1 \oplus \cdots \oplus X_k$ is essential in M. Let M_i be a maximal submodule of X_i, then $(X_1/M_1) \oplus \cdots \oplus (X_k/M_k)$ is injective and so $P/(\bigoplus_{i=1}^k M_i) = (\bigoplus_{i=1}^k X_i/M_i) \oplus L/(\bigoplus_{i=1}^k M_i)$ for some submodule $L \subseteq P$ containing $\bigoplus_{i=1}^k M_i$. Hence there is an epimorphism $f : P \to \bigoplus_{i=1}^k (X_i/M_i)$ with $\operatorname{Ker}(f) = L$. On the other hand, as D is a right noetherian domain, $X_i \cong D_D$, and P is $(\bigoplus_{i=1}^k X_i)$-projective (see, e.g., [1, 16.12]), there is a homomorphism $f' : P \to \bigoplus_{i=1}^k X_i$ such that $gf' = f$ where g is the canonical homomorphism $\bigoplus_{i=1}^k X_i \to \bigoplus_{i=1}^k (X_i/M_i)$.

As $X_i \nsubseteq \operatorname{Ker}(f)$ we see that for each i, $X_i \nsubseteq \operatorname{Ker}(f')$. It follows that $f'(M) \neq 0$, and as a submodule of the projective module $\bigoplus_{i=1}^k X_i$, $f'(M)$ is projective. Setting $K = \operatorname{Ker}(f'|_M)$ we have

$$M \cong f'(M) \oplus K, \quad \text{and} \quad f'(M) \neq 0. \tag{1}$$

Now we prove the claim by induction on k. For $k = 1$, M is uniform, and hence $K = 0$ and so M is projective. Now assume that the claim is true for any submodule with uniform dimension $< k$. If in (1), $K \neq 0$, then both $f'(M)$ and K have uniform dimension $< k$, and hence they are projective. Therefore, by (1), M is projective. If $K = 0$, then $M \cong f'(M)$, a projective module. □

Now we are ready to prove the main result.

Theorem 3. *For a right module P over a right noetherian right hereditary right V-domain D, the following conditions are equivalent*:

(i) *P is D-projective.*
(ii) *P is projective.*

Proof. (ii) $\Rightarrow$ (i) is clear.

(i) $\Rightarrow$ (ii): We consider the case $P \neq 0$. Let $\wp$ be the set of all nonzero projective submodules of P. By Lemma 1, P is nonsingular, and hence $\wp$ is not empty. Moreover, $\wp$ is partially ordered with respect to the set inclusion ($\subseteq$). We aim to show that $(\wp, \subseteq)$ satisfies the hypothesis of Zorn's Lemma. For this purpose, let $\{P_\lambda\}_{\lambda \in \Lambda}$ be an ascending chain in $\wp$, and set $Q = \bigcup_{\lambda \in \Lambda} P_\lambda$. We need to show that $Q \in \wp$, i.e., that Q is projective.

There is an epimorphism $\varphi : D^{(A)} \to Q_D$ where $D^{(A)}$ is the direct sum of $|A|$ copies of D_D, and A is a suitable index set. Let $T_\lambda = \{x \in D^{(A)} \mid \varphi(x) \in P_\lambda\}$, i.e., T_λ is the largest submodule of $D^{(A)}$ with $\varphi(T_\lambda) = P_\lambda$. It is clear that $\operatorname{Ker}(\varphi) \subseteq T_\lambda$. Hence $T_\lambda = V_\lambda \oplus \operatorname{Ker}(\varphi)$ for some submodule $V_\lambda \subseteq T_\lambda$ such that $\varphi(V_\lambda) = P_\lambda$. Observe that for any $\lambda, \lambda' \in \Lambda$, $P_\lambda \subseteq P_{\lambda'}$ if and only if $V_\lambda \subseteq V_{\lambda'}$. Hence the system $\{V_\lambda\}_{\lambda \in \Lambda}$ forms an ascending chain in $D^{(A)}$. Set $V = \bigcup_{\lambda \in \Lambda} V_\lambda$. We see that V is a submodule of $D^{(A)}$. As D is right hereditary, V is a projective module. It is obvious that $\varphi(V) \subseteq Q$. Now let $y \in Q$, then there is a P_λ such that $y \in P_\lambda$. By the construction of T_λ, there is an $x \in V_\lambda$ such that $\varphi(x) = y$. Hence $\varphi(V) \supseteq Q$, and so $\varphi(V) = Q$. Let $0 \neq z \in V$. There is a V_λ such that $z \in V_\lambda$. As $\varphi|_{V_\lambda} : V_\lambda \to P_\lambda$ is an isomorphism, we have $\varphi(z) \neq 0$. Thus $\varphi|_V$ is an isomorphism, or equivalently, $V \cong Q$. This shows that $Q \in \wp$.

Now we can apply Zorn's Lemma for the partially ordered set $(\wp, \subseteq)$ to see that $\wp$ contains a maximal element P'. We aim to show that $P' = P$, proving the projectivity of P, i.e., (ii) holds. We assume on the contrary that $P' \neq P$.

By Lemma 2(a), we have

$$P' = \bigoplus_{\alpha \in \Omega} P_\alpha \tag{2}$$

where each P_α is isomorphic to a right ideal of D, in particular, each P_α is uniform and noetherian. Let $t \in P \setminus P'$. Since P_D is nonsingular, $tD \cong D$, hence tD is uniform and noetherian.

There is a finite subset $F_1 \subseteq \Omega$, such that $P' \cap tD \subseteq (\bigoplus_{\alpha \in F_1} P_\alpha)$. Set $W = \bigoplus_{\alpha \in \Omega \setminus F_1} P_\alpha$. Again since $W \cap [(\bigoplus_{\alpha \in F_1} P_\alpha) + tD]$ is noetherian, there is a finite subset $F_2 \subseteq \Omega \setminus F_1$ such that $W \cap [(\bigoplus_{\alpha \in F_1} P_\alpha) + tD] \subseteq \bigoplus_{\alpha \in F_2} P_\alpha$. Set $F = F_1 \cup F_2$, then $(\bigoplus_{\alpha \in \Omega \setminus F} P_\alpha) \cap [(\bigoplus_{\alpha \in F} P_\alpha) + tD] = 0$. Namely, let $a \in (\bigoplus_{\alpha \in \Omega \setminus F} P_\alpha) \cap [(\bigoplus_{\alpha \in F} P_\alpha) + tD]$, then $a = b + tc$ for some $b \in \bigoplus_{\alpha \in F} P_\alpha$, $c \in D$. It follows that $a - b = tc \in P' \cap tD \subseteq \bigoplus_{\alpha \in F_1} P_\alpha$. This implies that $a \in (\bigoplus_{\alpha \in \Omega \setminus F} P_\alpha) \cap (\bigoplus_{\alpha \in F} P_\alpha) = 0$. Thus

$$P' + tD = \left(\bigoplus_{\alpha \in \Omega \setminus F} P_\alpha \right) \oplus \left[\left(\bigoplus_{\alpha \in F} P_\alpha \right) + tD \right]. \tag{3}$$

Now, as $(\bigoplus_{\alpha \in F} P_\alpha) + tD$ is a finitely generated submodule of P, $(\bigoplus_{\alpha \in F} P_\alpha) + tD$ is noetherian, hence projective by Lemma 2(b). Thus by (3), $P' + tD$ is projective. This is a contradiction to the maximality of P' in $\wp$. Thus $P' = P$. □

Following Faith [6] a ring R is a right PCI-ring if every cyclic right R-module is either injective or isomorphic to R_R. By [6] and [5], a right PCI-ring is either semisimple artinian or a right noetherian right hereditary right V-domain. Hence Theorem 3 holds for right PCI-domains. An example of a PCI-domain which is not a division ring was given by Cozzens [4]. If D is a right PCI-domain, then D/A is injective and semisimple for any nonzero right ideal $A \subseteq D$. In particular, every singular right D-module is injective and semisimple.

Ken Goodearl [8] introduced and investigated a class of rings over which singular right modules are injective and called these rings, right SI-rings. From the general structure theorem of right SI-rings in [8, 3.11], if R is a prime right SI-ring with zero right socle (i.e., $\mathrm{Soc}(R_R) = 0$), then R is Morita equivalent with a right SI-domain D. On the other hand, it is known that a domain D is right SI if and only if D is a right PCI-domain.

More generally, let R be a prime right noetherian right hereditary right V-ring. By [7, 7.36A], R is a simple ring. Hence R is Morita equivalent to a right noetherian right hereditary right V-domain (see e.g., [7, 4.23B]). In the light of [1, 21.6] about the relationship between Morita equivalent rings and Theorem 3, we obtain the following result which includes prime right noetherian right SI-rings.[1]

Corollary 4. *Over a prime right noetherian right hereditary right V-ring R, a right R-module P is projective if and only if P is R-projective.*

For a ring R we denote by $\Im$ the class of all semisimple right R-modules. In [9], a right R-module M is called p-poor if whenever M is N-projective, for some $N \in \text{Mod-}R$, then $N \in \Im$. They studied the class $\Re$ of rings over which every right module is either projective or p-poor, and asked a question whether any right PCI-domain does belong to $\Re$. Theorem 3 provides a positive answer for this when we apply it for right PCI-domains. Moreover the following result particularly shows that the class (iii) of [9, Theorem 3.11] is indeed *not empty*. This is what the authors of [9] wanted to see.

1 For a prime right SI-ring R, if $\mathrm{Soc}(R_R) = 0$ then, by [8, 3.11], R is right noetherian, right hereditary and right V.

Proposition 5. *For a prime right SI-ring R with $\mathrm{Soc}(R_R) = 0$, the following conditions are equivalent*:

(a) *Every right R-module is projective or p-poor.*
(b) *Every quasi-projective right R-module is semisimple or projective.*

In particular, every quasi-projective right R-module is either projective or injective and semisimple.

Proof. Note that every singular right R-module is injective and semisimple (cf., [8, 3.1]). Because of [1, 21.6] and since R is Morita equivalent to a right PCI-domain D, it is enough to prove our statements for D.

(a) $\Rightarrow$ (b): Let P be a quasi-projective right D-module. Assume that P_D is not semisimple. Then $P = P_1 \oplus S$ where S is the singular submodule of P, and P_1 is nonsingular and nonzero. For any $0 \neq x \in P_1$, $xD \cong D_D$ since D is a right noetherian domain. As P is xD-projective, P is then D-projective. Thus P is projective by Theorem 3. This proves (b).

(b) $\Rightarrow$ (a): Let Q_D be a module which is not p-poor. This means, there is a cyclic right D-module X with the property that X_D is not semisimple, and Q is X-projective. Write $X = X_1 \oplus S$ where S is the singular submodule of X. Hence $X_1 \neq 0$ and it is cyclic and nonsingular. Therefore, $X_1 \cong D_D$. As Q is X_1-projective, Q must be D-projective. By Theorem 3, Q is projective, proving (a).

For the last statement, let P be a quasi-projective right D-module. If P is not semisimple, then P contains a copy of D, hence D-projective. Therefore P is projective by Theorem 3. It is moreover clear that every semisimple right D-module is injective. □

Let R be a prime noetherian hereditary right V-ring. Then for each essential right (left) ideal $A \subseteq R$, R/A is an artinian right (left) R-module by [3] and [15]. Hence, as R is right V, R is a right SI-ring. By [2, Corollary 6], R is left SI. Thus by Proposition 5, every quasi-projective right (left) R-module is either projective or injective and semisimple. This is the result stated in the title of this note.

All rings considered in this note are in fact simple rings. From this nature, we would like to ask the following

Question 6. *Let R be a right noetherian right hereditary simple ring, and P be an R-projective nonsingular right R-module. Does this necessarily follow that P_R is a projective module?*

In the light of the proof of Theorem 3, to have a positive answer for Question 6 we need to show that all finite dimensional submodules of P are projective. Hence, P is projective if and only if every finite dimensional submodule of P is projective.

References

[1] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules, second ed., Grad. Texts in Math., vol. 13, Springer-Verlag, New York, Heidelberg, Berlin, 1992.
[2] A.K. Boyle, K.R. Goodearl, Rings over which certain modules are injective, Pacific J. Math. 58 (1975) 43–53.
[3] A.W. Chatters, The restricted minimum condition in Noetherian hereditary rings, J. Lond. Math. Soc. 4 (1971) 83–87.
[4] J.H. Cozzens, Homological properties of the ring of differential polynomials, Bull. Amer. Math. Soc. 76 (1970) 75–79.
[5] R.F. Damiano, A right PCI-ring is right Noetherian, Proc. Amer. Math. Soc. 77 (1979) 11–14.
[6] C. Faith, When are proper cyclics injective?, Pacific J. Math. 45 (1973) 97–112.
[7] C. Faith, Algebra: Rings, Modules and Categories I, Grundlehren Math. Wiss., vol. 190, Springer-Verlag, Berlin, Heidelberg, New York, 1973.
[8] K. Goodearl, Singular torsion and the splitting properties, Mem. Amer. Math. Soc. 124 (1972).
[9] C.F. Holston, S.R. López-Permouth, N. Orhan Ertag, Rings whose modules have maximal or minimal projectivity domain, J. Pure Appl. Algebra 216 (2012) 673–678.
[10] I. Kaplansky, Projective modules, Ann. Math. 68 (1958) 372–377.
[11] F. Kasch, Moduln und Ringe, B.G. Teubner Stuttgart, 1977.
[12] T.Y. Lam, Lectures on Modules and Rings, Grad. Texts in Math., vol. 189, Springer-Verlag, New York, Heidelberg, Berlin, 1999.
[13] G.O. Michler, O.E. Villamayor, On rings whose simple modules are injective, J. Algebra 25 (1973) 185–201.
[14] L.H. Rowen, Ring Theory, vol. I, Academic Press, New York, 1988.
[15] D.B. Webber, Ideals and modules of simple Noetherian hereditary rings, J. Algebra 16 (1970) 239–242.