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Physics Letters B

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Vacuum quantum effect for curved boundaries in static Robertson–Walker space–time

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ARTICLE INFO

Article history:

Received 5 June 2009

Received in revised form 7 August 2009

Accepted 17 August 2009

Available online 19 August 2009

ABSTRACT

The energy–momentum tensor for a massless conformally coupled scalar field in the region between two curved boundaries in $k = -1$ static Robertson–Walker space–time is investigated. We assume that the scalar field satisfies the Dirichlet boundary condition on the boundaries. $k = -1$ Robertson–Walker space is conformally related to the Rindler space, as a result we can obtain vacuum expectation values of energy–momentum tensor for conformally invariant field in Robertson–Walker space from the corresponding Rindler counterpart by the conformal transformation.

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1. Introduction

The Casimir effect is regarded as one of the most striking manifestation of vacuum fluctuations in quantum field theory. The presence of reflecting boundaries alters the zero-point modes of a quantized field, and results in the shifts in the vacuum expectation values of quantities quadratic in the field, such as the energy density and stresses. In particular, vacuum forces arise acting on constraining boundaries. The particular features of these forces depend on the nature of the quantum field, the type of space–time manifold and its dimensionality, the boundary geometries and the specific boundary conditions imposed on the field. Since the original work by Casimir in 1948 [1] many theoretical and experimental works have been done on this problem (see, e.g., [2–5] and references therein).

The Casimir effect can be viewed as a polarization of vacuum by boundary conditions. Another type of vacuum polarization arises in the case of an external gravitational fields [6,7]. Casimir stress for parallel plates in the background of static domain wall in four and two dimensions is calculated in [8,9]. Spherical bubbles immersed in different de Sitter spaces in- and out-side are considered in [10,11].

It is well known that the vacuum state for an uniformly accelerated observer, the Fulling–Rindler vacuum [12–16], turns out to be inequivalent to that for an inertial observer, the familiar Minkowski vacuum. Quantum field theory in accelerated systems contains many of special features produced by a gravitational field

avoiding some of the difficulties entailed by renormalization in a curved space–time. In particular, near the canonical horizon in the gravitational field, a static space–time may be regarded as a Rindler-like space–time. Rindler space is conformally related to the de Sitter space and to the Robertson–Walker space with negative spatial curvature. As a result the expectation values of the energy–momentum tensor for a conformally invariant field and for corresponding conformally transformed boundaries on the de Sitter and Robertson–Walker backgrounds can be derived from the corresponding Rindler counterpart by the standard transformation [6]. Vacuum expectation values of the energy–momentum tensor for the conformally coupled Dirichlet and Neumann massless scalar and electromagnetic fields in four-dimensional Rindler space–time was considered by Candelas and Deutsch [18]. In Ref. [19] the vacuum expectation value of the surface energy–momentum tensor is evaluated for a massless scalar field obeying a Robin boundary condition on an infinite plane moving by uniform proper acceleration through Fulling–Rindler vacuum. By using the conformal relation between the Rindler and de Sitter space–times, in Ref. [20] the vacuum energy–momentum tensor for a scalar field is evaluated in de Sitter space–time in presence of a curved brane on which the field obeys the Robin boundary condition with coordinate dependent coefficients.

In this Letter the vacuum expectation value of the energy–momentum tensor is investigated for a massless conformally coupled scalar field obeying the Dirichlet boundary condition on two curved boundaries in $k = -1$ static Robertson–Walker backgrounds. Here we use the results of Ref. [16] to generate vacuum energy–momentum tensor for the $k = -1$ Robertson–Walker background which is conformally related to the Rindler space–time. Previously this method has been used in [21] to derive the vacuum

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stress on parallel plates for scalar field with Dirichlet boundary condition in de Sitter space–time. Also this method has been used in [22] for the investigation of the vacuum characteristics of the Casimir configuration on background of conformally flat brane-world geometries for massless scalar field with Robin boundary condition on plates.

2. Vacuum expectation values for the energy–momentum tensor

We will consider a conformally coupled massless scalar field $\varphi(x)$ satisfying the following equation

$$\left(\nabla_\mu \nabla^\mu + \frac{1}{6}R\right)\varphi(x) = 0, \quad (1)$$

on the background of a static $k = -1$ Robertson–Walkers space–time. In Eq. (1) ∇_μ is the operator of the covariant derivative, and R is the Ricci scalar for the background space–time. The corresponding line element is given by the formula

$$ds_{\text{RW}}^2 = dt^2 - \gamma dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

where $\gamma = 1/(1+r^2)$. This geometry is conformally related to the Rindler space–time [6].

The four-dimensional Rindler line element can be written as [6]

$$ds_{\text{Rin}}^2 = \xi^2 d\eta^2 - d\xi^2 - dy^2 - dz^2, \quad 0 < \xi < \infty. \quad (3)$$

Under the coordinate transformation

$$\begin{aligned} \xi &= \gamma^{1/2}(1 - r\gamma^{1/2} \cos\theta)^{-1}, \\ y &= r\gamma^{1/2} \sin\theta \cos\phi (1 - r\gamma^{1/2} \cos\theta)^{-1}, \\ \eta &= t, \\ z &= r\gamma^{1/2} \sin\theta \sin\phi (1 - r\gamma^{1/2} \cos\theta)^{-1}, \end{aligned} \quad (4)$$

line element (3) takes the form

$$ds_{\text{Rin}}^2 = \xi^2 [dt^2 - \gamma dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)], \quad (5)$$

which explicitly shows the conformal relation between the Rindler and $k = -1$ Robertson–Walker space–times. Let us denote the energy–momentum tensor in coordinates (η, ξ, y, z) as $T_{\mu\nu}$, and the same tensor in coordinates (t, r, θ, ϕ) as $T'_{\mu\nu}$. Our main interest in the present Letter is to investigate the vacuum expectation values (VEVs) of the energy–momentum tensor (EMT) for the field $\varphi(x)$ in the background described by (2) induced by two curved boundaries. We will consider the case of a scalar field satisfying Dirichlet boundary condition on the surface of the plates:

$$\varphi|_{\xi=\xi_1} = \varphi|_{\xi=\xi_2} = 0. \quad (6)$$

The presence of boundaries modifies the spectrum of the zero-point fluctuations compared to the case without boundaries. This results in the shift in the VEVs of the physical quantities, such as vacuum energy density and stresses. This is the well-known Casimir effect. It can be shown that for a conformally coupled scalar by using field equation (1) the expression for the energy–momentum tensor can be presented in the form

$$T_{\mu\nu} = \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{6} \left[\frac{g_{\mu\nu}}{2} \nabla_\rho \nabla^\rho + \nabla_\mu \nabla_\nu + R_{\mu\nu} \right] \varphi^2, \quad (7)$$

where $R_{\mu\nu}$ is the Ricci tensor. The quantization of a scalar field on background of metric (2) is standard. Let $\{\varphi_\alpha(x), \varphi_\alpha^*(x)\}$ be a complete set of orthonormalized positive and negative frequency solutions to the field equation (1), obeying boundary condition (6). By expanding the field operator over these eigenfunctions, using

the standard commutation rules and the definition of the vacuum state for the vacuum expectation values of the energy–momentum tensor one obtains

$$\langle 0|T_{\mu\nu}(x)|0\rangle = \sum_\alpha T_{\mu\nu}\{\varphi_\alpha, \varphi_\alpha^*\}, \quad (8)$$

where $|0\rangle$ is the amplitude for the corresponding vacuum state, and the bilinear form $T_{\mu\nu}\{\varphi, \psi\}$ on the right is determined by the classical energy–momentum tensor (7). Instead of evaluating Eq. (8) directly on background of the curved metric, the vacuum expectation values can be obtained from the corresponding Rindler space–time results for a scalar field $\bar{\varphi}$ by using the conformal properties of the problem under consideration. Under the conformal transformation $g_{\mu\nu} = \Omega^2 \bar{g}_{\mu\nu}$ the $\bar{\varphi}$ field will change by the rule

$$\varphi(x) = \Omega^{-1} \bar{\varphi}(x), \quad (9)$$

where for metric (2) the conformal factor is given by $\Omega = \xi^{-1}$. In this Letter as a Rindler counterpart we will take the vacuum energy–momentum tensor induced by a couple of infinite plates moving by uniform proper acceleration through the Fulling–Rindler vacuum. In the corresponding RW problem the geometry of boundaries follows from (4) and is given by the equations

$$\frac{\gamma^{1/2}}{1 - r\gamma^{1/2} \cos\theta} = \xi_i, \quad i = 1, 2, \quad (10)$$

where $\xi = \xi_i$ are the locations of the plates in the Rindler problem.

The Casimir effect for two parallel plates moving with uniform proper acceleration on background of the Rindler space–time is investigated in Ref. [16] for a scalar field with Dirichlet and Neumann boundary conditions and in [17] for Robin boundary conditions. The expectation values of the energy–momentum tensor for a scalar field φ_R in the Fulling–Rindler vacuum can be presented in the form of the sum

$$\langle 0_R|T_i^k|0_R\rangle = \langle \tilde{0}_R|T_i^k|\tilde{0}_R\rangle + \langle T_i^k \rangle^{(b)}, \quad (11)$$

where $|0_R\rangle$ and $|\tilde{0}_R\rangle$ are the amplitudes for the vacuum states in the Rindler space in the presence and of absence of the plates, respectively, $\langle T_i^k \rangle^{(b)}$ is the part of the vacuum energy–momentum tensor induced by the plates. In the case of a conformally coupled massless scalar field for the part without boundaries one has

$$\langle \tilde{0}_R|T_i^k|\tilde{0}_R\rangle = \frac{-2\delta_i^k}{(4\pi)^{3/2} \Gamma(3/2) \xi^4} \int_0^\infty \frac{\omega^3 g^{(i)}(\omega) d\omega}{e^{2\pi\omega} - 1}, \quad (12)$$

where the expressions for the functions $g^{(i)}(\omega)$ are presented in Ref. [23].

For a scalar field φ_R , satisfying the Dirichlet boundary condition, the boundary induced part in the region between the plates has the form [16]

$$\begin{aligned} \langle T_i^k \rangle^{(b)} &= \langle T_i^k \rangle^{(b)}(\xi_1, \xi) \\ &- \frac{1}{2\pi^2} \delta_i^k \int_0^\infty dk k^3 \int_0^\infty d\omega \frac{I_\omega(k\xi_1) F^{(i)}[D_\omega(k\xi, k\xi_2)]}{I_\omega(k\xi_2) D_\omega(k\xi_1, k\xi_2)}, \end{aligned} \quad (13)$$

where

$$D_\omega(k\xi, k\xi_2) = I_\omega(k\xi_2) K_\omega(k\xi) - K_\omega(k\xi_2) I_\omega(k\xi), \quad (14)$$

and $I_\omega(z)$, $K_\omega(z)$ are the modified Bessel functions. The first term on the right-hand side of (13),

$$\langle T_i^k \rangle^{(b)}(\xi_1, \xi) = \frac{1}{2\pi^2} \delta_i^k \int_0^\infty dk k^3 \int_0^\infty d\omega \frac{I_\omega(k\xi_1)}{K_\omega(k\xi_1)} F^{(i)}[K_\omega(k\xi)], \quad (15)$$

is the part induced in the region $\xi > \xi_1$ by the presence of a single plane boundary located at $\xi = \xi_1$. The functions $F^{(i)}[G(z)]$, $i = 0, 1, 2, 3$, in the expressions above are defined by the formulae

$$F^{(0)}[G(z)] = \frac{1}{6} \left| \frac{dG(z)}{dz} \right|^2 + \frac{1}{6z} \frac{d}{dz} |G(z)|^2 + \frac{1}{6} \left(1 - \frac{5\omega^2}{z^2} \right) |G(z)|^2, \quad (16)$$

$$F^{(1)}[G(z)] = \frac{-1}{2} \left| \frac{dG(z)}{dz} \right|^2 - \frac{1}{6z} \frac{d}{dz} |G(z)|^2 + \frac{1}{2} \left(1 + \frac{\omega^2}{z^2} \right) |G(z)|^2, \quad (17)$$

$$F^{(2,3)}[G(z)] = \frac{-1}{2} \left| \frac{dG(z)}{dz} \right|^2 + \frac{1}{6} \left| \frac{dG(z)}{dz} \right|^2 + \frac{1}{6} \left(1 + \frac{\omega^2}{z^2} \right) |G(z)|^2, \quad (18)$$

and the indices 0, 1 correspond to the coordinates τ, ξ , respectively. The expressions in (13), (15) are given in the coordinate system (η, ξ, y, z) . To find the vacuum expectation values generated by the boundaries in the $k = -1$ static RW space-time, first we will consider the Rindler space-time quantities in the coordinates t, r, θ, ϕ with line element (5). So, it is necessary to make the coordinate transformation to obtain $\langle 0_R | \hat{T}_i^k | 0_R \rangle$. After long calculations for the part induced by the boundaries we find the following expressions

$$\langle \hat{T}_t^t \rangle^{(b)} = \langle T_\eta^\eta \rangle^{(b)}, \quad (19)$$

$$\langle \hat{T}_\theta^\theta \rangle^{(b)} = \left\{ \frac{b^2}{\cos^2 \phi} + \frac{r^2 \gamma^2 \sin^2 \theta}{a^2} (1 - r\gamma^{1/2} \cos \theta)^{-4} \times [\gamma^{1/2} \sin \theta \cos \phi (1 - r\gamma^{1/2} \cos \theta)^{-1} + ra \sin \theta \cos \phi]^2 (\cot^2 \phi - \tan^2 \phi) \right\}^{-1} \times \left[(1 - \cot^2 \phi) \langle T_y^y \rangle^{(b)} + \left(\frac{\gamma^{1/2} \sin \theta \cos \phi}{1 - r\gamma^{1/2} \cos \theta} + ra \sin \theta \cos \phi \right) \times \frac{\cot^2 \phi - \tan^2 \phi}{a^2} \langle T_\xi^\xi \rangle^{(b)} \right], \quad (20)$$

$$\langle \hat{T}_r^r \rangle^{(b)} = \frac{1}{a^2} [\langle T_\xi^\xi \rangle^{(b)} - r^2 \gamma^2 (1 - r\gamma^{1/2} \cos \theta)^{-4} \sin^2 \theta \langle \hat{T}_\theta^\theta \rangle^{(b)}], \quad (21)$$

$$\langle \hat{T}_\phi^\phi \rangle^{(b)} = \frac{(1 - r\gamma^{1/2} \cos \theta)^{-2}}{r^2 \gamma \sin^2 \theta \cos^2 \phi} \left\{ \langle T_z^z \rangle^{(b)} - [\gamma^{1/2} \sin \theta \sin \phi (1 - r\gamma^{1/2} \cos \theta)^{-1} + ra \sin \theta \sin \phi]^2 \langle \hat{T}_r^r \rangle^{(b)} - [r\gamma^{1/2} \cos \theta \sin \phi (1 - r\gamma^{1/2} \cos \theta)^{-1} - r^2 \gamma \sin^2 \theta \sin \phi (1 - r\gamma^{1/2} \cos \theta)^{-2}]^2 \langle \hat{T}_\theta^\theta \rangle^{(b)} \right\}, \quad (22)$$

where

$$a = \gamma (1 - r\gamma^{1/2} \cos \theta)^{-1} \times [r\gamma^{1/2} + \cos \theta (1 - r^2 \gamma) (1 - r\gamma^{1/2} \cos \theta)^{-1}], \quad (23)$$

$$b = r\gamma^{1/2} \cos \phi (1 - r\gamma^{1/2} \cos \theta)^{-1} \times [\cos \theta - r\gamma^{1/2} (1 - r\gamma^{1/2} \cos \theta)^{-1}], \quad (24)$$

Therefore, we can write the RW vacuum expectation values in the form similar to (11):

$$\langle 0_{RW} | T_i^k | 0_{RW} \rangle = \langle \tilde{0}_{RW} | T_i^k | \tilde{0}_{RW} \rangle + \langle T_i^k \rangle_{RW}^{(b)}, \quad (25)$$

where $\langle \tilde{0}_{RW} | T_i^k | \tilde{0}_{RW} \rangle$ are the vacuum expectation values in $k = -1$ RW space without boundaries and the part $\langle T_i^k \rangle_{RW}^{(b)}$ is induced by the plates. Conformally transforming the Rindler results one finds

$$\langle \tilde{0}_{RW} | T_i^k | \tilde{0}_{RW} \rangle = \Omega^{-4} \langle \tilde{0}_R | T_i^k | \tilde{0}_R \rangle + \langle T_i^k \rangle^{(an)}, \quad (26)$$

$$\langle T_i^k \rangle_{RW}^{(b)} = \Omega^{-4} \langle \hat{T}_i^k \rangle^{(b)}. \quad (27)$$

The second term in Eq. (26) can be rewritten in the form

$$\langle T_i^k \rangle^{(an)} = -\frac{1}{2880} \left[\frac{1}{6} \tilde{H}_\nu^{(1)\mu} - \tilde{H}_\nu^{(3)\mu} \right]. \quad (28)$$

Fortunately, for the static RW space-time with $k = -1$ the boundary free part is zero [6]

$$\langle \tilde{0}_{RW} | T_i^k | \tilde{0}_{RW} \rangle = 0. \quad (29)$$

Therefore, the vacuum energy-momentum tensor can be evaluated exactly as:

$$\langle 0_{RW} | T_i^k | 0_{RW} \rangle = \Omega^{-4} \langle \hat{T}_i^k \rangle^{(b)}, \quad (30)$$

where non-zero components of $\langle \hat{T}_i^k \rangle^{(b)}$ are given by Eqs. (19)–(22).

3. Conclusion

In the present Letter we have investigated the Casimir effect for a conformally coupled massless scalar field between two curved boundaries, on background of the static $k = -1$ Robertson–Walker space-time. We have assumed that the scalar field satisfies Dirichlet boundary condition on the curved boundaries. The $k = -1$ Robertson–Walker space-time is conformally related to the Rindler space-time, then the vacuum expectation values of the energy-momentum tensor are derived from the corresponding Rindler space-time results by using the conformal properties of the problem. The vacuum expectation value of the energy-momentum tensor for the curved boundaries in Robertson–Walker space-time consists of two parts given in Eq. (25). The first one corresponds to the purely RW contribution when the boundary is absent. For the static RW space-time with $k = -1$ this boundary free part is zero [6]. The second part in the vacuum energy-momentum tensor is due to the imposition of boundary conditions on the fluctuating quantum field. The corresponding components are related to the vacuum energy-momentum tensor in the Rindler space-time by Eq. (30) and the Rindler tensor is given by Eqs. (19)–(22).

Acknowledgement

The author are indebted to the referee for his comments that improved the Letter drastically.

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