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A Posynomial Geometric Programming restricted to a System of Fuzzy Relation Equations

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Abstract

A posynomial geometric optimization problem subjected to a system of max-min fuzzy relational equations (FRE) constraints is considered. The complete solution set of FRE is characterized by unique maximal solution and finite number of minimal solutions. A two stage procedure has been suggested to compute the optimal solution for the problem. Firstly all the minimal solutions of fuzzy relation equations are determined. Then a domain specific evolutionary algorithm (EA) is designed to solve the optimization problems obtained after considering the individual sub-feasible region formed with the help of unique maximum solution and each of the minimal solutions separately as the feasible domain with same objective function. A single optimal solution for the problem is determined after solving these optimization problems. The whole procedure is illustrated with a numerical example.

Keywords: Nonlinear programming; Genetic algorithm ; Fuzzy relation equations.

1. Introduction

Fuzzy relations and their calculation aspects offer a mathematical tool to model various real life and hypothetical systems with the help of fuzzy implications and approximate reasoning. Fuzzy information in relational structures is processed in terms of fuzzy relational equations. The notion of fuzzy relation equations was first investigated by Sanchez [1] in 1976 and then extended by Pedrycz [2, 3] and many others. The structure of the complete solution set of sup-TM equations was first introduced by Sanchez [4]. Peeva [5] proposed a method to obtain all the minimal solutions of max-min fuzzy relation equations.

Fang and Li [6] first considered the linear optimization problem with max-min composition based fuzzy relation equations constraints. Further Loetamonphong and Fang [7] studied the same problem with

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max-product composition. For both the compositions, optimization problem was divided into two sub-problems based on the negative and non-negative coefficients in the objective function. The former sub-problem was solved by the maximum solution while the latter was converted into a 0-1 integer programming problem and then solved by branch and bound method.

The extension to nonlinear optimization problem with fuzzy relation equations as constraints was first proposed by Lu and Fang [8]. Li, Fang and Zhang [9] considered a problem of minimizing a nonlinear objective function with system of fuzzy relational equations with max-min composition and reduced it to a 0-1 mixed integer programming problem. Markovskii [10] gave the concept of covering problem for fuzzy relation equations with max-product composition. Thapar, Pandey and Gaur [11] studied a linear optimization model subject to max-Archimedean fuzzy relation equations. The concept of covering problem was applied to establish 0-1 integer programming problem equivalent to linear programming problem and a binary coded genetic algorithm was proposed to obtain the optimal solution.

Geometric programming (GP) is a class of nonlinear, nonconvex optimization problems with objective function and constraints in a special form. Zener, Duffin and Peterson [12-14] were first to propose the geometric programming theory in 1961.

Further considering the wide applicability of geometric programming Cao (2001) proposed fuzzy relational geometric programming. Yang and Cao [15-19] made a significant contribution to the area of fuzzy relational geometric programming using a variety of fuzzy operators. Wu [20] also studied a geometric objective function subject to max-min fuzzy relational equations as constraints and gave a reduction procedure for solving the problem. In the same area Zhou and Ahat [21] considered a geometric programming problem with a system of max-product fuzzy relational equations as constraints and gave an efficient procedure to find optimal solution.

This paper considers a geometric objective function subjected to a system of max-min fuzzy relation equations. Geometric programming is referred as a special form of nonlinear programming. Research in direction of nonlinear optimization has always been very slow. The current paper is inspired by [17] and suggests a two stage procedure to find the optimal solution of the considered geometric optimization problem.

We consider the following fuzzy relational geometric optimization problem

$$\begin{aligned} &\min\ f(x) \\ &\text{s.t.}\ \max_{j=1}^{n} \min(a_{ij}, x_j) = b_i \quad \forall i = 1,2,\dots,m \\ &0 \le x_j \le 1 \quad , j \in J \end{aligned} \tag{1}$$

where $A=[a_{ij}]$, $0 \le a_{ij} \le 1$, be a $m \times n$ dimensional fuzzy matrix and right hand matrix $b=(b_1,b_2,\dots,b_m)^T$, $0 \le b_i \le 1$ be a m dimensional vector and $f(x)$ is the geometric function of x defined as $f(x)=\sum_{k=1}^{K} f_k(x)=\sum_{k=1}^{K} c_k \prod_{j=1}^{n} x_j^{r_{jk}}$, where each $f_k(x)$ represents the k^{th} monomial in x and each coefficient $c_k > 0$ and $r^{jk} \in R, (0 < k \le K, 1 \le j \le n)$ is corresponding exponent of variable x_j in the k^{th} monomial and $x=[x_1,x_2,\dots,x_n]^T$ is the solution vector. Let $I=\{1,2,\dots,m\}$ and $J=\{1,2,\dots,n\}$ be the index sets. In this optimization problem the objective function is nonconvex by nature and the feasible domain is also nonconvex, so the optimization problem can be categorized as a nonconvex programming problem. This characteristic of problem offers difficulty in employing the traditional methods for nonlinear programming to be applied directly to solve this optimization problem.

The paper is organized in five sections. Section 1 offers the basic motivation and the literature behind the problem including the definition of the problem considered. In section 2 structure of solution set of the problem (1) is introduced then consistency conditions are explained. Section 3 discusses the procedure used to determine the solution set of fuzzy relational system considered as the feasible domain of the problem (1). Section 4 describes the design of the evolutionary procedure applied. Section 5 presents a numerical example for describing the overall procedure to solve the problem (1). A concluding remark is given at the end.

2. Characteristics of the feasible domain

We have considered the following fuzzy relation equation defined over the residuated lattice $([0,1],\vee,\wedge,t,\Theta_t 0,1)$:

$$A \circ x = b \tag{2}$$

where "$\circ$" denotes the max-min composition of A and x .Let $X(A,b)=\{x\in[0,1]^n \mid A\circ x=b\}$ be the solution set of fuzzy relation equations (2). For any $x^1, x^2 \in X$, we say $x^1 \le x^2$ if and only if $x_j^{\,1} \le x_j^{\,2}, \forall\, j\in J$. Hence the operator "$\le$" establishes a partial order on X then the system $(X(A,b),\le)$ becomes a lattice. Over a lattice concepts of maximum and minimal solutions can be discussed. $\hat{x}\in X(A,b)$ is called as the maximum solution, if $x\le\hat{x}, \forall\, x\in X(A,b)$. Similarly, $\breve{x}\in X(A,b)$ is a minimal solution, if $x\le\breve{x}$ for any $\breve{x}\in X(A,b)$ implies $x=\breve{x}, \forall\, x\in X(A,b)$. A solution $x^*\in X(A,b)$ is said to be optimal solution of (1) when $f(x^*)\le f(x\ )$ for all $x\in X(A,b)$. When solution set of (2) is non-empty, the system is said to be consistent and inconsistent otherwise. If the system is consistent then in general, it can be completely determined by unique maximum solution $\hat{x}$ and possibly finite number of minimal solutions [1-5].

Definition 1: Let t be a continuous t -norm then there exists a unique operation Θ_t associated with t called as the implication operator defined as:

$$a\,\Theta_t b = \text{Sup}\,\{x\in[0,1] \mid\ t(a,x)\le b\},\ \forall a,b\in[0,1]$$

The Θ_t operator satisfies the following two properties:

1. $t((a\Theta_t b),a)\le b$ 3(a)
2. $t(a,x)\le b$ iff $x\le(a\,\Theta_t b)$ 3(b)

If the system $A\circ x=b$ is consistent, the maximum solution can be determined explicitly using the implication operator Θ_t as follows:

$$\hat{x} = A\Theta_t b = \left[\mathop{\wedge}_{i=1}^{m}(a_{ij}\Theta_t b_i)\right]_{j\in J} \tag{4}$$

Here $a_{ij}\Theta_t b_i = \begin{cases} 1, & \text{if } a_{ij} \le b_i \\ b_i, & \text{otherwise} \end{cases}$

Furthermore when $X(A,b) \neq \phi$, the solution set $X(A,b)$ is given as:

$$X(A,b) = \bigcup_{\breve{x} \in \breve{X}(A,b)} \{x \in [0,1]^n \mid \breve{x} \le x \le \hat{x}\}$$

where $\breve{X}(A,b)$ is the set of all minimal solutions of (2).

Lemma 1: If in the i^{th} equation $a_{ij} < b_i, \forall\ j \in J,$ then the solution set $X(A,b) = \phi.$

Proof: If in the i^{th} equation $a_{ij} < b_i$ holds for all $j \in J,$ then for $x_j \neq a_{ij},$ $\min(a_{ij}, x_j) \le \min(a_{ij}, 1) = a_{ij} < b_i$ and for $x_j = a_{ij}, \min(a_{ij}, x_j) = \min(a_{ij}, a_{ij}) = a_{ij} < b_i.$ Thus for both cases, $\min(a_{ij}, x_j) < b_i, \forall\ j \in J.$ Hence, $\max_{j \in J}(\min(a_{ij}, x_j)) < b_i$ which implies that the i^{th} equation remains dissatisfied by any variable then the system has no solution i.e., $X(A,b) = \phi.$

Definition 2: A continuous t -norm is said to be an Archimedean t -norm if and only if $t(x,x) < x$ for all $0 < x < 1$ and non-Archimedean otherwise. The *minimum* t-norm is a continuous but not Archimedean t-norm.

Lemma 2: A vector $x \in X(A,b)$ is a solution of system (2) if and only if, for each $i \in I$, there exists an index $j_i \in J$ such that $\min(a_{ij}, x_{j_i}) = b_i$ and $\min(a_{ij}, x_j) \le b_i, i \in I, j \in J$.

Proof: For the solvability of the system $A \circ x = b$ in (2) $\min(a_{ij}, x_j) \le b_i$ for all $j \in J$ $, \forall i \in I,$ and by the non-interactive nature of max operator $\exists j_i \in J,$s.t. $\min(a_{ij}, x_j) = b_i$ for $\forall\, i \in I$.

A system of fuzzy relation equations defined in (2) is said to be homogeneous if $b = 0,$ and non-homogeneous otherwise. Homogeneous system has the trivial solution. As in a system $A \circ x = b$ if $b_i = 0,$ for some $i \in I$ then $\max_{j \in J} \min(a_{ij}, x_j) = 0.$ If $b_{i_\circ} = 0$ for some $i_\circ \in I$ such that $\exists a_{i_\circ j} > 0,$ for some $j \in J, i_\circ \in I$ then x_j has to be zero. Hence for some $b_i = 0$ if $a_{ij} > 0, j \in J$, we can simply set $x_j = 0$ .So the value of all the variables that appear in the i^{th} equation must be 0. Hence the system can be reduced and simplified to a new system after deleting all such equations from A and corresponding component from b . Any solution of the original system can be obtained by simply setting variables x_j to zero wherever $a_{i_\circ j} > 0, j \in J.$

3. Reduction procedure

Consider the single equation from the system (2) given as:

$$(a_{i1} \vee x_1) \vee (a_{i2} \vee x_2) \vee \ldots \vee (a_{in} \vee x_n) = b_i$$
$$0 \le x_j \le 1 \quad , j \in J$$

Furthermore, from the property 3(b) of implication operator it is clear that $\min(a_{ij}, x_j) \le b_i$ iff $x \le (a_{ij}\Theta_t b_i)$ .The unit component equation $\min(a_{ij}, x_j) = b_i$ in above equation has a solution iff $b_i \le a_{ij}$ .The solution set of $\min(a_{ij}, x_j) = b_i$ can be discussed in the following cases:

- Case I: If $a_{ij} > b_i$ then we have $x_j = b_i$ as the solution of the unit equation $\min(a_{ij}, x_j) = b_i$.
- Case II: If $a_{ij} = b_i$ then $[b_i, a_{ij}\Theta_t b_i]$ is the solution set of the unit equation $\min(a_{ij}, x_j) = b_i$.
- Case III: If $a_{ij} < b_i$ then we have $x_j = \phi$, i.e. the equation $\min(a_{ij}, x_j) = b_i$ has no solution in this case.

The above discussion shows that the equation $\min(a, x) = b$ has a solution iff $b \le a$ and then the solution set of $\min(a, x) = b$ is given by $[b \ , a\Theta_t b]$ .For more details see [5,22-23].

With the help of computed maximum solution $\hat{x}$ the characteristic matrix $\hat{P} = (\hat{p}_{ij})_{m\times n}$ of the system $A \circ x = b$ can be defined as:

$$\hat{p}_{ij} = \begin{cases} [b_i, \hat{x}_j], & \text{if } \min(a_{ij}, \hat{x}_j) = b_i \\ \phi, & \text{otherwise} \end{cases} \tag{5}$$

It is clear from the above argument on the unit equation $\min(a_{ij}, x_j) = b_i$ that b_i presents the lower bound for a variable x_j to satisfy the i^{th} equation. Each nonempty element $\hat{p}_{ij}$ of the characteristic matrix $\hat{P}$ gives the whole range of possible values for the variable x_j to satisfy the i^{th} equation. The system $A \circ x = b$ is consistent if and only if $\hat{P}$ has no row with all elements as empty elements i.e. if there does not exist some equation not satisfied by any variable. In general for a continuous non-Archimedean t-norm the non-empty elements in characteristic matrix $\hat{P}$ might not be singleton while in the case of an Archimedean norm these entries are always singleton and are equal to the corresponding component of maximal solution.

Now if $\exists b_i = 0, i \in I$ then without loss of generality corresponding rows can be removed from the matrix $\hat{P}$. After removing these rows from matrix $\hat{P}$ then the matrix is simplified to matrix P . In the simplified matrix P there might be some variables that satisfy only those equations for which $b_i = 0$ .After removing these rows from $\hat{P}$, the columns corresponding to those variables have only zeros in the characteristic matrix P . Such variables are called *pseudo-essential* [22].

Definition 3: A variable is said to be *multi-essential* if its corresponding column in the characteristic matrix P contains a non-singleton element. Different values of a multi-essential variable can satisfy different no. of equations. The multi-essential variable can also assume some value other than 0 and corresponding maximal component value in the minimal solutions [22].

Definition 4: Let $P = (p_{ij})_{m\times n}$ be the simplified characteristic matrix of system (2) then a row i_1 dominates a row i_2 if $p_{i_2 j} \ne \phi$ imply $p_{i_2 j} \subseteq p_{i_1 j}$ for all $j \in J$ [22].

A row of P is redundant if and only if it dominates some other row. Moreover if a row i_1 dominates a row i_2 we have $b_{i_1} \le b_{i_2}$ since b_i represents the lower bound for a variable to satisfy the i^{th} equation. For the sake of simplification redundant rows can be removed from the modified matrix P . At this stage some columns can be zeroed out. It is clear that the variables of such columns cannot have non-zero values in minimal solutions. Such variables are called *semi-essential* [10]. After removing the redundant rows and the columns corresponding to semi-essential variables from matrix P the matrix transforms to the matrix P' .It is noteworthy that if there exists a row in P' having the unique nonempty entry corresponding to a column j , then the variable corresponding to that column is called *super-essential* [22]. If x_j is super-essential, it can assume different values in different minimal solutions of system (2). In the system with non-Archimedean based composition the super-essential variable can also attain some value other than its corresponding maximal component value while in case of Archimedean norm based composition they coincide with the corresponding component value in the maximal solution.

Once the matrix P' is obtained, we adopt the algebraic method for finding all minimal solutions of (2) by considering the simplified characteristic matrix P' associated with the formal logical expression. We follow the notations for the resolution as used in [23]. Each row of matrix P' is associated with logical sum, $u_i = \bigvee_j \left\langle \dfrac{b_i}{x_j} \right\rangle$ (DNF) for all $j \in J'$ where $J' = \{ j \in J \mid p_{ij} \neq \phi \} \forall i \in I$. The whole matrix P' corresponds with the logical product $P' = \bigwedge_i u_i$ (CNF). The whole truth function obtained in (CNF) can be reduced to DNF using laws of conversion of fuzzy truth function in CNF to DNF as defined in section 3.1. The truth function for finding all the minimal solutions can be given as:

$$F_{P'} = \bigwedge_{i \in P'} \bigvee_{j \in J'} \left\langle \frac{b_i}{x_j} \right\rangle$$

3.1. Rules to perform conversions form fuzzy truth function in CNF to DNF

1. $\left\langle \dfrac{b_{i_1}}{x_{j_1}} \right\rangle \left\langle \dfrac{b_{i_2}}{x_{j_2}} \right\rangle = \begin{cases} \left\langle \dfrac{b_{i_1} \vee b_{i_2}}{x_{j_1}} \right\rangle & \text{if } j_1 = j_2 \\ \text{unchanged} & \text{otherwise} \end{cases}$

2. $\left\langle \dfrac{b_{i_1}}{x_{j_1}} \right\rangle \left\langle \dfrac{b_{i_2}}{x_{j_2}} \right\rangle = \left\langle \dfrac{b_{i_2}}{x_{j_2}} \right\rangle \left\langle \dfrac{b_{i_1}}{x_{j_1}} \right\rangle, \quad \text{(commutative)}$

3. $\left\langle \dfrac{b_{i_1}}{x_{j_1}} \right\rangle \left(\left\langle \dfrac{b_{i_2}}{x_{j_2}} \right\rangle \vee \left\langle \dfrac{b_{i_3}}{x_{j_3}} \right\rangle \right) = \left\langle \dfrac{b_{i_1}}{x_{j_1}} \right\rangle \left\langle \dfrac{b_{i_2}}{x_{j_2}} \right\rangle \vee \left\langle \dfrac{b_{i_1}}{x_{j_1}} \right\rangle \left\langle \dfrac{b_{i_3}}{x_{j_3}} \right\rangle, \quad j_1, j_2, j_3 \in J'$

4. $\left\langle \dfrac{b_{i_1}}{x_{j_1}} \right\rangle \ldots\ldots\ldots\ldots \left\langle \dfrac{b_{i_m}}{x_{j_h}} \right\rangle \vee \left\langle \dfrac{b_{h_1}}{x_{j_1}} \right\rangle \ldots\ldots\ldots\ldots \left\langle \dfrac{b_{h_m}}{x_{j_n}} \right\rangle = \begin{cases} \left\langle \dfrac{b_{i_1}}{x_{j_1}} \right\rangle \ldots\ldots\ldots\ldots \left\langle \dfrac{b_{i_m}}{x_{j_h}} \right\rangle & \text{if } b_{i_t} \le b_{h_t}, \quad t = 1, 2, \ldots, m \\ \text{unchanged} & \text{otherwise} \end{cases}$

The whole procedure for finding all the minimal solutions of system (2) can be summarized in the *Algorithm 1* given below:

Algorithm 1: Finding all the minimal solutions

Step 1: Get the matrices A, b .
Step 2: Find the maximal solution $\hat{x}$ by (4).
Step 3: Check consistency of the system .If $A \circ \hat{x} \neq b$, system is inconsistent, stop the procedure.
Step 4: Find the characteristic matrix $\hat{P}$ of A using (5) and then find simplified characteristic matrix P .
Step 5: Find the reduced matrix P' .
Step 6: Find all the minimal solutions of (2) by applying logical rules of inferences defined in section 3.1.

Once the solution set of system (2) is determined from the *Algorithm 1*, we construct as many optimization problems as many minimal solutions are obtained with same objective function considering each of the convex sub-feasible regions formed by one minimal solution and unique maximal solution as the feasible region as follows:

$$\begin{aligned} &\min\ f(x) \\ &s.t.\ \check{x}^r \le x \le \hat{x}, \qquad r \in \{1,2,\ldots,|\check{X}(A,b)|\} \end{aligned} \tag{6}$$

Then the optimal solution x^* of the original optimization problem in (1) is obtained with the help of optimal solutions of the above optimization problems.

4. Evolutionary machinery to solve optimization problems in (6)

Genetic Algorithms (GA) are stochastic search techniques based on the ideas of natural selection and genetic inheritance. The basic idea of genetic algorithm is to maintain a population of candidate solutions for the problem at hand and making it evolve by iteratively applying genetic operators to produce (hopefully) better and better approximations to its solution.

Genetic Algorithm (GA) start operating on a set of potential solutions known as *population*. The potential solutions are termed as the *chromosomes* representing the candidate solutions of the problem under consideration. After representing the solutions into the decision variable domain, the performance of individual members of population is assessed by evaluating a fitness function characterizing an individual's performance in the problem domain. At each generation, a new set of approximate solutions is created by the process of selecting individuals according to their level of fitness in the problem domain and breeding them together using operators *Recombination and Mutation*. This process leads to the evolution of populations of individuals that are better suited to their environment than the individuals that already exist.

4.1. Selection

The main purpose of selection is to maintain good copies of individuals for the next generation. We have used the *tournament selection* as the selection strategy. Tournament selection starts by selecting a set of individuals at random and tournaments are played among them and the best fit player wins and chosen for mating. Similar process is repeated until a population of desired strength is selected. The number of players in a set denotes the tournament size. Bigger tournament size enhances the selection pressure so small tournament size is always a good choice. Selected individuals undergo crossover and mutation and new individuals are obtained.

Definition 5: Given a connected set T and any two points $x_1, x_2 \in T$, $0 \le \alpha \le 1, \beta \ge 1$.

(i) A linear contraction of x_1 supervised by x_2 is given as:

$$x_1 = \alpha x_1 + (1-\alpha) x_2$$

(ii) A linear extraction of x_1 supervised by x_2 is defined as:

$$x_1 = \beta x_1 - (\beta - 1) x_2$$

where α, β are the step lengths for the linear contraction and extraction respectively and are generally kept small.

4.2. Crossover

Crossover is the main operator that is responsible for bringing diversity in the next generation. In crossover, two selected good and feasible (satisfying constraints) individuals mate together and two new solutions called *offsprings* are created which typically share many of the characteristics of their “parents”. New parents are selected for two new children, and the process continues until a new population of appropriate size is created. We have adopted a line combination crossover operator. It is designed in such a way that children individuals remain in the feasible domain. The algorithm for the crossover is stated as follows:

Algorithm 2: Crossover

Get the matrices A, b and find the maximum solution $\hat{x}$ by (4) and set parameters $0 \le \alpha \le 1$, $\beta \ge 1$, $0 \le \zeta \le 1, 0 \le \delta \le 1$.

Randomly select two individuals x_1, x_2 from the selected population.
For *i*=1, 2
Generate a random number $\varepsilon \in [0,1]$

If $(\varepsilon \ge \zeta)$
$x_i = \beta x_i - (\beta - 1)\hat{x}$
Else
$x_i = \alpha x_i + (1-\alpha)\hat{x}$
Generate a random number $\varepsilon_2 \in [0,1]$

If $(\varepsilon_2 \ge \delta)$

Go to evaluation procedure

Else

$x_1^{next} \leftarrow x_1$
$x_1 = \alpha x_1 + (1-\alpha)x_2$

If $A \circ x_1 = b$

Go to evaluation procedure

Else

$x_1 \leftarrow x_1^{next}$
$x_1 = \beta x_1 - (\beta - 1)x_2$

If $A \circ x_1 \neq b$

$x_1 \leftarrow x_1^{next}$

Go to evaluation procedure.

End

Here, α, β are the probabilities of linear contraction and extraction respectively and are generally kept small. For our problem we are taking $\alpha = 0.99, \beta = 1.0085,\ \zeta = 0.012, \delta = 0.99$.

4.3. Mutation

Mutation randomly perturbs a candidate solution with a hope to create a better solution exploring the search space of the problem domain. We adopt the following mutation procedure for solving our problem:

Algorithm 3: Mutation

1. Get the matrices A, b and find the maximum solution $\hat{x}$ by (4) and set the mutation probability $\theta \in [0,1]$.
2. Generate $r_j \in [0,1]$ for each bit of every individual in the crossed population.
3. For $\forall j \in J$ if $r_j \leq \theta$, randomly assign x_j a number from $[0, \hat{x}_j]$.
4. For the modified $x = (x_1, x_2, ..., x_n)$ check $A \circ x = b$.
5. If $A \circ x = b$ go to the evaluation procedure.

The overall procedure applied for solving the considered optimization problems in model (6) can be summarized in the following algorithm:

Algorithm 4: Genetic Algorithm procedure

Step 1: Define maximum number of iterations as gen_max and set gen =1.
Step 2: Randomly generate initial population of size say k within the specified bounds of the decision variables.
Step 3: Check feasibility.
Step 4: Found feasible individual(s)? If yes then go to Step 5 else go to Step 2.
Step 5: Select the best individuals using objective function.
Step 6: Generate offsprings for next generations by applying crossover using Algorithm 2 and mutation using Algorithm 3.

Step 7: Select feasible individuals among offsprings and set gen = gen + 1.
Step 8: Found better individuals than parents among offsprings?
If yes then discard the offsprings and parent generation remains same for the next generation also, now go to Step 10 else go to Step 9.
Step 9: Update the optimal solution and value of objective function and now this generation becomes the parent generation.
Step 10: Is gen equal to gen_max? If yes, then STOP else go to Step 6.

The overall procedure applied for solving the considered optimization problem (1) can be summarized in the following steps:

Algorithm 5:

Step 1: Find all the minimal solutions by the Algorithm 1.
Step 2: Find all the sub-feasible regions considering all the minimal solutions.
Step 3: Solve the different optimization problems using Algorithm 4.
Step 4: Find single optimal solution out of the different optimal solutions obtained in Step 3.

5. Illustrative Example

Example 1: Consider the following fuzzy relation optimization problem:

$$\min\ f(x) = 5x_1^{-0.2}x_2^{-0.3}x_3^{2}x_4^{-1}x_5^{1} + 2x_1^{-0.2}x_2^{-1.5}x_3^{2}x_4^{-2}x_5^{1}$$
$$\text{s.t } A \circ x = b \text{ where } 0 \le x_j \le 1,\ i, j = 1,2,\ldots,5$$

$$A = \begin{bmatrix} 0.9 & 0.8 & 0.6 & 0.3 & 0.9 \\ 0.8 & 0.7 & 0.8 & 1 & 0.8 \\ 0.6 & 0.9 & 0.8 & 0.9 & 0.5 \\ 0.4 & 0.2 & 0.5 & 0.6 & 0.2 \\ 0.3 & 0.3 & 0.5 & 0.2 & 0.1 \\ 0.4 & 0.1 & 0.2 & 0.3 & 0.5 \end{bmatrix}$$
$$b = [0.8 \quad 0.8 \quad 0.8 \quad 0.5 \quad 0.5 \quad 0.4]^T$$

Maximal solution of $A \circ x = b$ can be easily computed by (4) as follows:

$$\hat{x} = (0.8 \quad 0.8 \quad 1 \quad 0.5 \quad 0.4)^T$$

Since $A \circ \hat{x} = b$, hence the system of FRE is consistent. The characteristic matrix $\hat{P}$ of system (2) is:

$$\hat{P} = \begin{bmatrix} 0.8 & 0.8 & \phi & \phi & \phi \\ 0.8 & \phi & [0.8,1] & \phi & \phi \\ \phi & 0.8 & [0.8,1] & \phi & \phi \\ \phi & \phi & [0.5,1] & 0.5 & \phi \\ \phi & \phi & [0.5,1] & \phi & \phi \\ 0.4 & \phi & \phi & \phi & 0.4 \end{bmatrix}$$

As row 4, dominates row 5, row 4 is redundant and can be removed from $\hat{P}$. After removing these rows, x_4 is semi-essential. After removing the redundant rows the simplified matrix P is:

$$P = \begin{bmatrix} 0.8 & 0.8 & 0 & 0 & 0 \\ 0.8 & 0 & [0.8,1] & 0 & 0 \\ 0 & 0.8 & [0.8,1] & 0 & 0 \\ 0 & 0 & [0.5,1] & 0 & 0 \\ 0.4 & 0 & 0 & 0 & 0.4 \end{bmatrix}$$

After removing the all zeroed columns we obtain the matrix P' as follows:

$$P' = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_5 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0 & 0 \\ 0.8 & 0 & [0.8,1] & 0 \\ 0 & 0.8 & [0.8,1] & 0 \\ 0 & 0 & [0.5,1] & 0 \\ 0.4 & 0 & 0 & 0.4 \end{bmatrix} \end{array}$$

Clearly x_3 is super essential. The associated truth function for the simplified characteristic matrix P' is:

$$F_{P'} = \left(\left\langle \frac{b_{i_1}}{x_{j_1}} \right\rangle \vee \left\langle \frac{b_{i_1}}{x_{j_2}} \right\rangle \right) \wedge \left(\left\langle \frac{b_{i_2}}{x_{j_1}} \right\rangle \vee \left\langle \frac{b_{i_2}}{x_{j_3}} \right\rangle \right) \wedge \left(\left\langle \frac{b_{i_3}}{x_{j_2}} \right\rangle \vee \left\langle \frac{b_{i_3}}{x_{j_3}} \right\rangle \right) \wedge \left\langle \frac{b_{i_5}}{x_{j_3}} \right\rangle \wedge \left(\left\langle \frac{b_{i_6}}{x_{j_1}} \right\rangle \vee \left\langle \frac{b_{i_6}}{x_{j_5}} \right\rangle \right)$$

After converting this fuzzy truth function in CNF to DNF using laws of conversion we have:

$$= \left(\left\langle \frac{0.8}{x_{j_1}} \right\rangle \left\langle \frac{0.8}{x_{j_2}} \right\rangle \left\langle \frac{0.5}{x_{j_3}} \right\rangle \right) \vee \left(\left\langle \frac{0.8}{x_{j_1}} \right\rangle \left\langle \frac{0.8}{x_{j_3}} \right\rangle \right) \vee \left(\left\langle \frac{0.8}{x_{j_2}} \right\rangle \left\langle \frac{0.8}{x_{j_3}} \right\rangle \left\langle \frac{0.4}{x_{j_5}} \right\rangle \right)$$
$$\vee \left(\left\langle \frac{0.4}{x_{j_1}} \right\rangle \left\langle \frac{0.8}{x_{j_2}} \right\rangle \left\langle \frac{0.4}{x_{j_3}} \right\rangle \right)$$

The corresponding minimal solutions of the system are:

$$\breve{x}^1 = (0.8 \quad 0.8 \quad 0.5 \quad 0 \quad 0)^T, \quad \breve{x}^2 = (0.8 \quad 0 \quad 0.8 \quad 0 \quad 0)^T$$
$$\breve{x}^3 = (0 \quad 0.8 \quad 0.8 \quad 0 \quad 0.4)^T, \quad \breve{x}^4 = (0.4 \quad 0.8 \quad 0.8 \quad 0 \quad 0)^T$$

Then the whole solution set of the fuzzy relation equation in (2) is:

$$X(A,b) = (0.8 \quad 0.8 \quad [0.5,1] \quad [0,0.5] \quad [0,0.4]) \cup (0.8 \quad [0,0.8] \quad [0.8,1] \quad [0,0.5] \quad [0,0.4])$$
$$\cup([0,0.8] \quad 0.8 \quad [0.8,1] \quad [0,0.5] \quad 0.4) \cup ([0.4,0.8] \quad 0.8 \quad [0.8,1] \quad [0,0.5] \quad [0,0.4])$$

It is clear that if $x_j = 0$ then $1/x_j = \infty$ .Keeping this fact in consideration and the four optimization problems formed by four minimal solutions can be written as:

$$\begin{aligned} P1 \quad & \min \ f(x) = 5.59x_3^2 x_4^{-1} x_5^1 + 2.92x_3^2 x_4^{-2} x_5^1 \\ & \text{s.t.} \quad 0.5 \le x_3 \le 1 \\ & \qquad 0 < x_4 \le 0.5 \\ & \qquad 0 < x_5 \le 0.4 \end{aligned}$$

$$\begin{aligned} P2 \quad & \min \ f(x) = 5.22x_2^{-0.3} x_3^2 x_4^{-1} x_5^1 + 2.09x_2^{-1.5} x_3^2 x_4^{-2} x_5^1 \\ & \text{s.t.} \quad 0 < x_2 \le 0.8 \\ & \qquad 0.8 \le x_3 \le 1 \\ & \qquad 0 < x_4 \le 0.5 \\ & \qquad 0 < x_5 \le 0.4 \end{aligned}$$

$$\begin{aligned} P3 \quad & \min \ f(x) = 2.14x_1^{-0.2} x_3^2 x_4^{-1} + 1.12x_1^{-0.2} x_3^2 x_4^{-2} \\ & \text{s.t.} \quad 0 < x_1 \le 0.8 \\ & \qquad 0.8 \le x_3 \le 1 \\ & \qquad 0 < x_4 \le 0.5 \end{aligned}$$

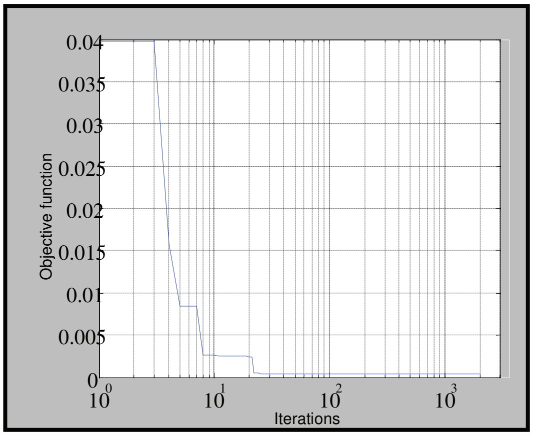
$$\begin{aligned} P4 \quad & \min \ f(x) = 5.35x_1^{-0.2} x_3^2 x_4^{-1} x_5^1 + 2.795x_1^{-0.2} x_3^2 x_4^{-2} x_5^1 \\ & \text{s.t.} \quad 0.4 \le x_1 \le 0.8 \\ & \qquad 0.8 \le x_3 \le 1 \\ & \qquad 0 < x_4 \le 0.5 \\ & \qquad 0 < x_5 \le 0.4 \end{aligned}$$

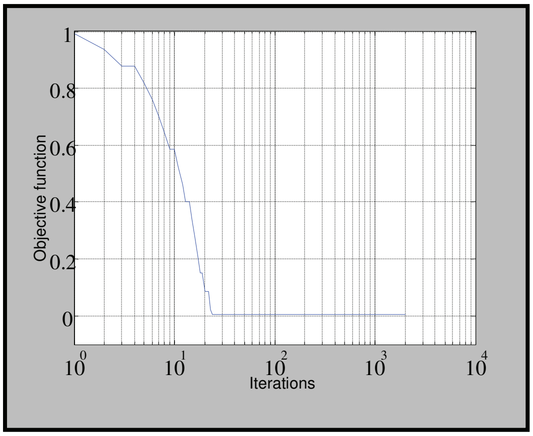
On solving these four optimization problems using *Algorithm 4* the four optimal solutions are listed in table 1 and the convergence of the corresponding GAs are shown in the figure 1.

Table 1. Optimal solutions for the four optimization problems

Problem	x_1	x_2	x_3	x_4	x_5	Optimal value f
$P1$	0.8000	0.8000	0.6751	0.4511	0.000032	0.00039451521903
$P2$	0.8000	0.7172	0.9420	0.3932	0.0001	0.00470617192855
$P3$	0.7933	0.8000	0.8002	0.4991	0.4000	5.89151635308652
$P4$	0.6826	0.8000	0.8131	0.2567	0.0000037	0.00016933802324

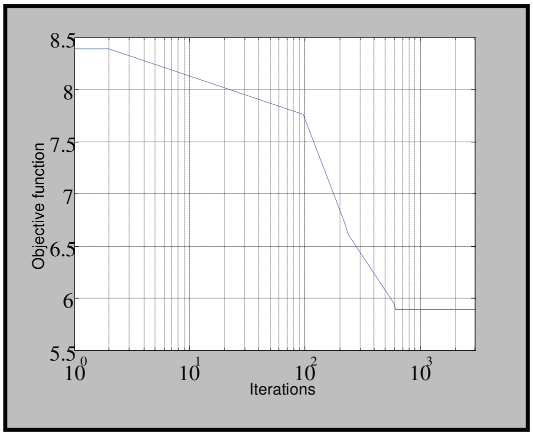
It is clear that $x^* = (0.6826 \quad 0.8000 \quad 0.8131 \quad 0.2567 \quad 0.0000037)$ with $f(x^*) = 0.00016933802324$ is the optimal solution of the problem (1).

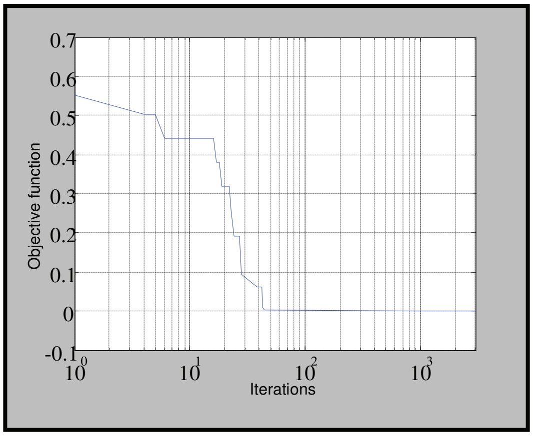




(a)

(b)





(c)

(d)

Fig. 1.(a) Convergence of GA for $P1$ (b) Convergence of GA for $P2$ (c) Convergence of GA for $P3$ (d) Convergence of GA for $P4$

6. Conclusion

In this paper, we have considered a fuzzy relation geometric programming problem with a posynomial geometric objective function subjected to max-min fuzzy relation equation constraints. The extensive nonlinear nature of the objective function and feasible domain hinders many nonlinear programming techniques to be applied directly. In the proposed method, firstly the fuzzy relational system is solved to determine the solution set. At the second level a genetic procedure is applied to determine the optimal solution. The solution procedure is based on solving the fuzzy relation system (2). Finding minimal solutions of (2) is an NP hard problem, so efficient determination of the solution set is an important aspect to be considered while applying the suggested procedure.

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