Closed geodesics in stationary manifolds with strictly convex boundary

A.M. Candela and A. Salvatore

Dipartimento di Matematica, via E. Orabona 4, 70125 Bari, Italy

Communicated by M. Willem
Received 20 May 1999

Abstract: Let $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ be a non-compact Lorentzian manifold. If the metric $\langle \cdot, \cdot \rangle_L$ is stationary and $\mathcal{M}$ has a strictly space-convex boundary, then variational tools allow to prove the existence of at least one closed spacelike geodesic in it.

Keywords: Stationary Lorentzian manifolds, closed geodesics, Ljusternik–Schnirelmann category, space-convex boundary.

MS classification: 53C50, 58E05, 58E10.

1. Introduction

The aim of this paper is to look for closed geodesics in a special class of Lorentzian manifolds with boundary. Let us recall that if $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ is a Lorentzian manifold, a smooth curve $z : [a, b] \rightarrow \mathcal{M}$ is a geodesic if

$$D_s \hat{z}(s) = 0 \quad \text{for all } s \in [a, b],$$

where $\hat{z}$ is the tangent field along $z$ and $D_s \hat{z}$ is the covariant derivative of $\hat{z}$ along $z$ induced by the Levi-Civita connection of $\langle \cdot, \cdot \rangle_L$.

It is well known that, if $z$ is a geodesic, then there exists a constant $E_z$, named energy of $z$, such that

$$E_z = \langle \hat{z}(s), \hat{z}(s) \rangle_L \quad \text{for all } s \in [a, b];$$

hence $z$ is called spacelike, respectively lightlike or timelike, if $E_z$ is positive, respectively null or negative. This classification is called causal character of geodesics and comes from General Relativity (cf. [12, 15]). We say that a geodesic $z : [a, b] \rightarrow \mathcal{M}$ is closed if

$$z(a) = z(b), \quad \hat{z}(a) = \hat{z}(b).$$

The research of closed geodesics with prescribed causal character is useful above all in order to have more information about the geometry of a Lorentzian manifold. Some existence results
A.M. Candela, A. Salvatore

for closed non-spacelike geodesics have been stated if \( M \) is compact (cf. [8, 9, 18]) while the existence of a closed spacelike geodesic holds, for example, if \( M = M_0 \times S^1 \) is compact and of splitting type (cf. [1]). Anyway in [9] an example of a compact Lorentzian manifold without spacelike closed geodesics makes interesting to investigate more about the existence of spacelike closed geodesics in non-compact Lorentzian manifolds. To this aim, let us introduce the following definition.

Definition 1.1. A Lorentzian manifold \((M, \langle \cdot, \cdot \rangle_L)\) is stationary if there exists a smooth connected finite-dimensional Riemannian manifold \( (M_0, \langle \cdot, \cdot \rangle) \) such that \( M = M_0 \times \mathbb{R} \) while \( \langle \cdot, \cdot \rangle_L \) has the following form:

\[
(\xi, \xi')_L = \langle \xi, \xi' \rangle + \langle \delta(x), \xi \rangle \tau' + \langle \delta(x), \xi' \rangle \tau - \beta(x) \tau \tau',
\]

for any \( z = (x, t) \in M = M_0 \times \mathbb{R} \) and \( \xi = (\xi, \tau), \xi' = (\xi', \tau') \in T_z M \equiv T_x M_0 \times \mathbb{R} \), where \( \delta : M_0 \rightarrow T M_0 \) and \( \beta : M_0 \rightarrow ]0, +\infty[ \) are smooth. In particular, the metric (1.1) is called static if \( \delta(x) \equiv 0 \).

It is easy to see that a closed geodesic on a stationary Lorentz manifold can never be timelike while it is lightlike only if it is reduced to a single point. On the other hand, the existence of closed spacelike geodesics has been proved in a stationary Lorentzian manifold if \( M_0 \) is compact (cf. [13]). Here, we want to extend this result to stationary manifolds with strictly space-convex boundary (cf. Definition 1.5). First of all let us recall the following definition (cf. [14]).

Definition 1.2. Let \( M \) be an open connected subset of a Lorentzian manifold \( M^* \). We say that \( M \) has a strictly space-convex boundary \( \partial M \) if any spacelike geodesic \( z : [a, b] \rightarrow M \cup \partial M \) is such that \( z([a, b]) \subset M \).

Remark 1.3. Let \( \partial M \) be strictly space-convex and \( z : [a, b] \rightarrow M \cup \partial M \) be a closed spacelike geodesic. It is easy to see that, by extending \( z \) to a bigger interval by periodicity, there results \( z([a, b]) \subset M \).

Remark 1.4. If \( M \) has a smooth boundary \( \partial M \), there exists a smooth function \( \Phi : M^* \rightarrow \mathbb{R} \) such that

\[
\Phi(z) = 0 \iff z \in \partial M,
\]

\[
\Phi(z) > 0 \iff z \in M,
\]

\[
\nabla_L \Phi(z) \neq 0 \quad \text{if} \ z \in \partial M
\]

(such a function can be defined by using the distance from the boundary). It is known that, if \( \partial M \) is strictly space-convex too, then there results

\[
H^\Phi_L(z)[\xi, \xi] \leq 0
\]

for any \( z \in \partial M \) and \( \xi \in T_z \partial M \) spacelike, where \( H^\Phi_L(z) : T_z M \times T_z M \rightarrow \mathbb{R} \) denotes the Hessian of \( \Phi \) at the point \( z \) (cf. [14, Proposition 4.1.3]). Vice versa if

\[
H^\Phi_L(z)[\xi, \xi] < 0 \quad \text{for all} \ z \in \partial M \text{ and} \ \xi \in T_z \partial M \text{ spacelike},
\]

(1.4)
then $\partial M$ is strictly space-convex. Indeed, suppose that $z : [a, b] \to M \cup \partial M$ is a spacelike geodesic such that there exists $s_0 \in ]a, b[$ with $z(s_0) \in \partial M$. Then $s_0$ is a minimum point of the $C^2$ function $\Phi \circ z$; hence,

$$0 \leq \frac{d^2}{ds^2} \Phi \circ z(s_0) = H^0_L(z(s_0))[\dot{z}(s_0), \ddot{z}(s_0)]$$

in contradiction with (1.4).

From now on, let $M$ be a stationary Lorentzian manifold.

**Definition 1.5.** We say that $(M, \langle \cdot, \cdot \rangle_L), M = M_0 \times \mathbb{R}$, is a stationary manifold with strictly space-convex boundary if there exists $(M^*, g^*)$ stationary Lorentzian manifold, $M^* = M^*_0 \times \mathbb{R}$, such that $M$ is an open connected subset of $M^*$ and $g^*_{\mid M_0} = \langle \cdot, \cdot \rangle_L$. Moreover

1. $\partial M_0$ is a $C^2$ submanifold of $M^*_0$;
2. $M_0 \cup \partial M_0$ is complete with respect to the Riemannian structure on $M^*_0$ (i.e., any geodesic $x : [a, b] \to M_0$ can be extended to a continuous curve $x : [a, b] \to M_0 \cup \partial M_0$);
3. $\partial M = \partial M_0 \times \mathbb{R}$ is strictly space-convex.

**Remark 1.6.** As $\partial M_0$ is a $C^2$ submanifold of $M^*_0$ then there exists a $C^2$ function $\phi : M^*_0 \to \mathbb{R}$ such that

$$M_0 = \{ x \in M^*_0 : \phi(x) > 0 \}, \quad \partial M_0 = \{ x \in M^*_0 : \phi(x) = 0 \}$$

and

$$\nabla \phi(x) \neq 0 \quad \text{for any } x \in \partial M_0,$$

where $\nabla \phi$ denotes the gradient of $\phi$ with respect to the Riemannian structure on $M^*_0$. If we set

$$\Phi(z) = \phi(x) \quad \text{for any } z = (x, t) \in M,$$

there results

$$\nabla_L \Phi(z) = (\nabla \phi(x), 0),$$

where $\nabla_L \Phi$ denotes the gradient of $\Phi$ with respect to the Lorentzian structure on $M^*$. It is easy to prove that $\Phi$ satisfies (1.2); moreover by $(H_3)$ it follows that $\Phi$ satisfies (1.3), too.

**Theorem 1.7.** Let $(M, \langle \cdot, \cdot \rangle_L), M = M_0 \times \mathbb{R}$, be a stationary manifold with strictly space-convex boundary such that $M_0$ is not contractible in itself and its fundamental group $\pi_1(M_0)$ is finite or it has infinitely many conjugacy classes. Suppose that there exist $\nu, N > 0$ such that

$$\nu \leq \beta(x) \leq N \quad \text{for all } x \in M_0.$$ 

Assume that there exist $x_0 \in M_0, U \in C^2(M_0, \mathbb{R}_+)$ and some positive constants $R, \rho, \lambda$, such that

$$x \in M_0, \quad d(x, x_0) \geq R \implies H^U_R(\xi, \xi) \geq \lambda(\xi, \xi) \quad \text{for all } \xi \in T_x M_0$$

and

$$x \in M_0, \quad d(x, x_0) \geq R \implies \phi(x) \geq \rho$$.
where \( d(\cdot, \cdot) \) is the distance in \( \mathcal{M}_0 \) and \( H^U_R(x)[\xi, \xi] \) denotes the Hessian of the function \( U \) at \( x \) in the direction \( \xi \) both induced by the Riemannian structure \( \langle \cdot, \cdot \rangle \) on \( \mathcal{M}_0 \). Here \( \phi \) defines the boundary \( \partial \mathcal{M}_0 \) as in (1.5). Furthermore, let us suppose that

\[
\lim_{d(x, x_0) \to +\infty} (\delta(x), \delta(x)) = 0,
\]

\[
\sup_{x \in \mathcal{M}_0} |\nabla U(x)| |\delta'(x)|_x < +\infty, \quad \sup_{x \in \mathcal{M}_0} |\nabla U(x)| |\nabla \beta(x)| < +\infty,
\]

where \( |\cdot|^2 = \langle \cdot, \cdot \rangle \) and \( |\delta'(x)|_x = \sup \{|\delta'(x)[v]| : v \in T_x \mathcal{M}_0, |v| = 1\} \). Then there exists at least one spacelike closed geodesic in \( \mathcal{M} \).

**Remark 1.8.** If \( \delta(x) = 0 \), Theorem 1.7 implies the existence of closed geodesics in static manifolds with strictly space-convex boundary. Let us point out that \( z = (x, t) \) is a closed geodesic if and only if \( t \) is constant and \( x \) is a closed geodesic in \( \mathcal{M}_0 \) (see, e.g., [13, Remark 2.9]). Consequently, the study of closed geodesics in static Lorentzian manifolds can be reduced to the research of closed geodesics in Riemannian manifolds; in particular, by [6, Theorem 1.2] it follows the existence of closed geodesics in static manifolds with non-smooth convex boundary.

**Remark 1.9.** It is easy to see that if \( z = (x, t) \) is a closed geodesic, then for any \( c \in \mathbb{R} \) \( z_c = (x, t + c) \) is a closed geodesic, too.

### 2. Variational setting

In this section we want to introduce a suitable functional whose critical points can be led back to closed geodesics.

From now on, let \( (\mathcal{M}, \langle \cdot, \cdot \rangle_L) \) be a stationary Lorentzian manifold such that \( \mathcal{M} = \mathcal{M}_0 \times \mathbb{R} \) while its metric \( \langle \cdot, \cdot \rangle_L \) is given in (1.1). Assume \([a, b] = [0, 1] = I\).

For all \( n \in \mathbb{N} \), define the Hilbert space

\[
H^1(S^1, \mathbb{R}^n) = \left\{ \gamma \in L^2(I, \mathbb{R}^n) : \gamma \text{ is absolutely continuous}, \quad \int_0^1 |\dot{\gamma}|^2 \, ds < +\infty, \quad \gamma(0) = \gamma(1) \right\}
\]

equipped with the norm

\[
\|\gamma\|_2^2 = |\gamma|^2 + |\dot{\gamma}|^2 = \int_0^1 |\gamma(s)|^2 \, ds + \int_0^1 |\dot{\gamma}(s)|^2 \, ds,
\]

where \( |\cdot|_2 \) is the usual norm in \( L^2(I, \mathbb{R}^n) \).

By Nash Embedding Theorem we can assume that \( \mathcal{M}_0 \) is a submanifold of an Euclidean space \( \mathbb{R}^N \) and its Riemannian metric \( \langle \cdot, \cdot \rangle \) is just the Euclidean metric on \( \mathbb{R}^N \); hence, the following definition can be stated:

\[
\Lambda^1 = \Lambda^1(\mathcal{M}_0) = \left\{ x \in H^1(S^1, \mathbb{R}^n) : x(s) \in \mathcal{M}_0 \text{ for all } s \in I \right\}.
\]
It is well known that \( \Lambda^1 \) is a submanifold of \( H^1(S^1, \mathbb{R}^N) \) and for any \( x \in \Lambda^1 \) the tangent space to \( \Lambda^1 \) at \( x \) is such that
\[
T_x \Lambda^1 \equiv \{ \xi \in H^1(S^1, \mathbb{R}^N) : \xi(s) \in T_x(\mathcal{M}_0) \text{ for all } s \in I \}.
\]

Assume \( Z = \Lambda^1 \times H^1(S^1, \mathbb{R}) \). Clearly \( Z \) is a Hilbert manifold such that at any \( z \in Z \) the tangent space is \( T_z Z \equiv T_x \Lambda^1 \times H^1(S^1, \mathbb{R}) \); moreover, on such a manifold the action integral
\[
f : z = (x, t) \in Z \mapsto f(z) \in \mathbb{R}
\]
is defined by setting
\[
f(z) = \frac{1}{2} \int_0^1 (\dot{\xi}, \dot{\xi}) \, ds = \frac{1}{2} \int_0^1 (\dot{\xi}, \dot{\xi}) + 2 \langle \delta(x), \dot{x} \rangle i - \beta(x)i^2 \rangle \, ds.
\]

Standard arguments allow to prove that \( f \) is a \( C^1 \) functional and for any \( z = (x, t) \in Z \) and \( \xi = (\xi, \tau) \in T_z Z \), there results
\[
f'(z)[\xi] = \int_0^1 \langle \dot{\xi}, \dot{\tau} \rangle \, ds + \int_0^1 \langle \delta'(x) \xi, \dot{\xi} \rangle i \, ds
\]
\[+ \int_0^1 \langle \delta(x), \dot{\xi} \rangle \, i \, ds + \int_0^1 \langle \delta(x), \dot{\tau} \rangle \, \dot{i} \, ds
\]
\[- \frac{1}{2} \int_0^1 \langle \nabla \beta(x), \dot{\xi} \rangle i^2 \, ds - \int_0^1 \beta(x) \dot{i} \, ds.
\]

It is easy to see that if \( z \) is a closed geodesic in \( \mathcal{M} \) then \( z \in Z \) is a critical point of \( f \); moreover, the vice versa can be proved.

**Proposition 2.1.** If \( z = (x, t) \) is a critical point of \( f \) in \( Z \) then it is a closed geodesic in \( \mathcal{M} \).

**Proof.** Let \( z = (x, t) \in Z \) be a critical point of \( f \). Then by (2.1) it is
\[
f'(z)[(0, \tau)] = \int_0^1 \langle \delta(x), \dot{x} \rangle \, \dot{i} \, ds - \int_0^1 \beta(x) \, \dot{i} \, ds = 0
\]
for all \( \tau \in C^\infty_0(I, \mathbb{R}) \); hence, a constant \( c \in \mathbb{R} \) exists such that
\[
\langle \delta(x), \dot{x} \rangle - \beta(x) \dot{i} = c \quad \text{almost everywhere in } I.
\]
By integrating, there results
\[
c = K(x) = \frac{\int_0^1 \langle \delta(x), \dot{x} \rangle / \beta(x) \rangle \, ds}{\int_0^1 (1/\beta(x)) \, ds} \in \mathbb{R};
\]
whence
\[
\dot{i} = \frac{\langle \delta(x), \dot{x} \rangle - K(x)}{\beta(x)} \quad \text{almost everywhere in } I.
\]
So, arguing as in [13], it can be proved that \( x \in C^2(I, \mathbb{R}^N) \), \( t \in C^2(I, \mathbb{R}) \) and \( z \) is a closed geodesic in \( \mathcal{M} \). \( \square \)

Now our problem is to find critical points of \( f \) in \( Z_0 = \Lambda^1 \times H^1_0 \), with \( H^1_0 = \{ t \in H^1(S^1, \mathbb{R}) : t(0) = \tau(1) = 0 \} \).
But, unlike the Riemannian case, the action functional \( f \) is unbounded from above and from below in \( Z_0 \), so to overcome this difficulty we introduce a new functional, bounded from below, whose critical points can be related to those ones of \( f \).

**Proposition 2.2.** Let \( \Theta : x \in \Lambda^1 \mapsto \Theta(x) \in H_0^1 \) be the \( C^1 \) function such that

\[
\Theta(x)(s) = \int_0^s \frac{\langle \delta(x), \dot{x} \rangle - K(x)}{\beta(x)} \, d\sigma \quad \text{for all } s \in I. \tag{2.5}
\]

The following assumptions are equivalent:

(i) \( z \in Z_0 \) is a critical point of \( f \);

(ii) \( z = (x, t) \) is such that \( x \in \Lambda^1 \) is a critical point of the functional

\[
J(x) = \frac{1}{2} \int_0^1 \left( \langle \dot{x}, \dot{x} \rangle + \frac{\langle \delta(x), \dot{x} \rangle^2}{\beta(x)} - \frac{K^2(x)}{\beta(x)} \right) \, ds \tag{2.6}
\]

and \( t = \Theta(x) \). Moreover, if (i) or (ii) holds, it is

\[
f(x, \Theta(x)) = J(x). \tag{2.7}
\]

**Proof.** Since \( f \) is a \( C^1 \) functional on \( Z_0 \), let us consider the partial derivatives of \( f \) in \( z = (x, t) \) given by

\[
\begin{align*}
f_x(z)[\xi] &= f'(z)[(\xi, 0)] \quad \text{for all } \xi \in T_x \Lambda^1, \\
f_t(z)[\tau] &= f'(z)((0, \tau)] \quad \text{for all } \tau \in H_0^1;
\end{align*}
\]

Clearly, the critical points of \( f \) in \( Z_0 \) have to be in the set

\[N = \{ z \in Z_0 : f_t(z) \equiv 0 \}.
\]

It is obvious that (2.2) and (2.4) imply

\[z = (x, t) \in N \iff t = \Theta(x); \tag{2.8}
\]

whence, \( N \) is the graph of the \( C^1 \) map \( \Theta \) and a smooth submanifold of \( Z_0 \). So the functional \( J \) in (2.6) is just the restriction of \( f \) to \( N \); hence, it is a \( C^1 \) functional such that there results

\[
J'(x)[\xi] = f_x(x, \Theta(x))[\xi] \quad \text{for every } x \in \Lambda^1, \xi \in T_x \Lambda^1. \tag{2.9}
\]

Let \( z = (x, t) \in Z_0 \). As for any \( (\xi, \tau) \in T_z Z_0 \) it is

\[
f'(z)[(\xi, \tau)] = f_x(z)[\xi] + f_t(z)[\tau],
\]

then (2.8) and (2.9) complete the proof. \( \square \)

**Lemma 2.3.** For any \( x \in \Lambda^1 \) there results

\[
J(x) \geq \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds. \tag{2.10}
\]

Hence, \( J(x) \geq 0 \) and

\[J(x) = 0 \iff x \text{ is constant.}\]
Proof. By (2.3) and the Hölder inequality it follows that
\[
K^2(x) \int_0^1 \frac{1}{\beta(x)} \, ds \leq \int_0^1 \frac{(\delta(x), \dot{x})^2}{\beta(x)} \, ds,
\]
then (2.10) holds. \(\square\)

As already remarked in Section 1, by Proposition 2.2 and Lemma 2.3 it follows that any non-trivial closed geodesic in \(\mathcal{M}\) has to be spacelike.

In order to prove the existence of critical points of \(J\) let us introduce some results of the Ljusternik–Schnirelman theory (for more details, see, e.g., [16, 17]).

Definition 2.4. Let \(X\) be a topological space. Given \(A \subseteq X\), the Ljusternik–Schnirelman category of \(A\) in \(X\), denoted by \(\text{cat}_X(A)\), is the least number of closed and contractible subsets of \(X\) covering \(A\). If it is not possible to cover \(A\) with a finite number of such sets it is \(\text{cat}_X(A) = +\infty\).

We assume \(\text{cat}(X) = \text{cat}_X(X)\).

In order to state the classical Ljusternik–Schnirelman Multiplicity Theorem we need the following definition.

Definition 2.5. Let \(\Lambda\) be a Riemannian manifold and \(g : \Lambda \to \mathbb{R}\) a \(C^1\) functional. We say that a sequence \((x_n)_{n \in \mathbb{N}} \subseteq \Lambda\) is a Palais–Smale sequence, briefly \((PS)\) sequence, if
\[
\sup_{n \in \mathbb{N}} |g(x_n)| < +\infty \quad \text{and} \quad \lim_{n \to +\infty} g'(x_n) = 0
\]
(here \(g'(x_n)\) goes to 0 in the norm induced on the cotangent bundle by the Riemannian metric on \(\Lambda\)). Moreover, \(g\) satisfies the Palais–Smale condition, briefly \((PS)\), if any \((PS)\) sequence has a convergent subsequence.

Theorem 2.6. Let \(\Lambda\) be a smooth Riemannian manifold and \(g : \Lambda \to \mathbb{R}\) a \(C^1\) functional which satisfies \((PS)\). Let us assume that \(\Lambda\) is complete or every sublevel of \(g\) in \(\Lambda\) is complete. If \(k \in \mathbb{N}, k > 0\), let us define
\[
\Gamma_k = \{ A \subseteq \Lambda : \text{cat}_A(A) \geq k \},
\]
\[
c_k = \inf_{A \in \Gamma_k} \sup_{x \in A} g(x).
\]
If \(\Gamma_k \neq \emptyset\) and \(c_k \in \mathbb{R}\), then \(c_k\) is a critical value of \(g\).

Remark 2.7. Let \(\Lambda\) and \(g\) be as in Theorem 2.6. If \(g\) is bounded from below, then for any \(c \in \mathbb{R}\) it is
\[
\text{cat}_\Lambda(g^c) < +\infty,
\]
where \(g^c = \{ x \in \Lambda : g(x) \leq c \}\).
Proposition 2.8. Let \( M_0 \) be a connected finite-dimensional manifold which is not contractible in itself. Suppose that its fundamental group \( \pi_1(M_0) \) is not infinite with finitely many conjugacy classes. Then \( \text{cat}(\Lambda^1) = +\infty \) and \( \Lambda^1 \) has compact subsets of arbitrarily high category.

3. Penalization arguments

Let \( (M, \langle \cdot, \cdot \rangle_L), M = M_0 \times \mathbb{R}, \) be a stationary manifold with strictly space-convex boundary. Since \( M_0 \) is not complete and, eventually, not bounded, the functional \( J \) does not satisfy the (PS) condition. Indeed we can consider a sequence \((x_n)_{n \in \mathbb{N}}\) of constant curves in \( \Lambda^1 \) such that \( x_n \to \tilde{x} \in \partial M_0 \) or \( d(x_n, x_0) \to +\infty \) as \( n \to +\infty \) for a certain \( x_0 \in M_0 \) (when \( M_0 \) is not bounded, too): \((x_n)_{n \in \mathbb{N}}\) is a (PS) sequence without subsequences converging in \( \Lambda^1 \). Thus we will introduce a family of penalized functionals \((f_\varepsilon)_{\varepsilon > 0}\) in such a way that every \( f_\varepsilon \), associated to \( f_\varepsilon \), satisfies the (PS) condition.

Here and in the following, we assume that \( M_0 \) is not bounded and there exists \( U \in C^2(M_0, \mathbb{R}_+) \) such that (1.8) holds (otherwise the proof of Theorem 1.7 is simpler).

Fixed \( \varepsilon > 0 \), let \( \psi_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+ \) be a \( C^2 \) “cut-function” defined as follows:

\[
\psi_\varepsilon(s) = \begin{cases}
0 & \text{if } 0 \leq s \leq \frac{1}{\varepsilon}, \\
\frac{\mu^n}{n!} \left( s - \frac{1}{\varepsilon} \right)^n & \text{if } s > \frac{1}{\varepsilon},
\end{cases}
\]

where \( \mu = \max\{1, \lambda\} \), \( \lambda \) given in (1.8). It is easy to prove that

\[
\psi_\varepsilon'(s) \geq \mu \psi_\varepsilon(s) \geq 0 \quad \text{for any } s \in \mathbb{R}_+, \\
\psi_\varepsilon(s) \leq \psi_\varepsilon'(s) \quad \text{for any } s \in \mathbb{R}_+.
\]

For any \( x \in M_0 \), define

\[
\phi_\varepsilon(x) = \psi_\varepsilon \left( \frac{1}{\Phi^2(x)} \right), \quad U_\varepsilon(x) = \psi_\varepsilon(U(x)),
\]

where \( \Phi \) is as in Remark 1.6.

Let us penalize the action functional \( f \) in the following way

\[
f_\varepsilon(z) = f(z) + \int_0^1 \phi_\varepsilon(x) \, ds + \int_0^1 U_\varepsilon(x) \, ds, \quad z = (x, t) \in Z_0.
\]

By standard arguments \( f_\varepsilon \) is of class \( C^1 \); moreover, arguing as in Proposition 2.1, any critical point \( z_\varepsilon \) of \( f_\varepsilon \) is \( C^2 \) and solves the following boundary problem

\[
-D_t \dot{z}_\varepsilon = 2\psi_\varepsilon \left( \frac{1}{\Phi^2(z_\varepsilon)} \right) \frac{\nabla L \Phi(z_\varepsilon)}{\Phi^4(z_\varepsilon)} - \psi_\varepsilon(U(z_\varepsilon)) \nabla_L U(z_\varepsilon),
\]

\[
\dot{z}_\varepsilon(0) = \dot{z}_\varepsilon(1),
\]

(3.5)
where $\Phi$ is as in (1.6) and $\mathcal{U}(z) = U(x)$ if $z = (x, t)$.

**Remark 3.1.** Since the penalization terms do not depend on the variable $t$, there results $f'(\tau(z)(0, \tau)] = f'(z)(0, \tau)]$ for all $\tau \in \mathcal{H}^1$. Then, Proposition 2.2 holds for the functionals $f_\varepsilon$ and $J_\varepsilon$ by using the same map $\Theta$, where

$$J_\varepsilon(x) = J(x) + \int_0^1 \phi_\varepsilon(x) \, ds + \int_0^1 U_\varepsilon(x) \, ds, \quad x \in \Lambda^1.$$  

Moreover, since $\psi_\varepsilon$ is positive, (2.10) implies

$$\frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds \quad \text{for all } x \in \Lambda^1.$$  

In order to relate the critical points of $J_\varepsilon$ to those ones of $J$ we need the following technical lemma.

**Lemma 3.2.** Let $U \in C^2(\mathcal{M}_0, \mathbb{R})$, $\lambda > 0$, $R > 0$ and $x_0 \in \mathcal{M}_0$ be such that (1.8) holds and $\{ x \in \mathcal{M}_0 : d(x, x_0) \geq R \}$ is complete. Then there exist some positive constants $c_1, c_2, c_3$ such that for any $x \in \mathcal{M}_0$ there results

$$\langle \nabla U(x), \nabla U(x) \rangle^{1/2} \geq \lambda d(x, x_0) - c_1,$$

$$U(x) \geq \frac{1}{2} \lambda d^2(x, x_0) - c_2 d(x, x_0) - c_3.$$  

**Proof.** See [4, Lemma 2.2].

**Remark 3.3.** The function $U$ satisfying (1.8) is introduced in order to give a control on $\mathcal{M}_0$ at infinity; hence, it is not restrictive to assume that $U$ is bounded on bounded sets and it is possible to choose

$$U_0 \geq \sup \left\{ U(x) : x \in \mathcal{M}_0, \ d(x, x_0) \leq R + 1 \right\}.$$  

Clearly, there results

$$U(x) > U_0 \implies d(x, x_0) \geq R + 1.$$  

**Proposition 3.4.** Let $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ be a stationary manifold with strictly space-convex boundary such that the hypotheses (1.7)–(1.11) hold. Taken $L, M > 0$, there exists $\varepsilon_0 = \varepsilon_0(L, M) > 0$ such that for every $\varepsilon \in ]0, \varepsilon_0[$ if $x_\varepsilon \in \Lambda^1$ satisfies

$$J_\varepsilon(x_\varepsilon) = 0, \quad L \leq J_\varepsilon(x_\varepsilon) \leq M,$$  

then

$$J(x_\varepsilon) = J_\varepsilon(x_\varepsilon), \quad J'(x_\varepsilon) = 0.$$  

**Proof.** It is enough to prove that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that any $x_\varepsilon \in \Lambda^1$ which satisfies (3.7) is such that

$$\sup_{x \in I} U(x_\varepsilon(s)) < \frac{1}{\varepsilon} \quad \text{if } \varepsilon \leq \varepsilon_1, \quad \inf_{x \in I} \phi^2(x_\varepsilon(s)) > \varepsilon \quad \text{if } \varepsilon \leq \varepsilon_2.$$  

(3.8)
First, let us prove (3.8). Arguing by contradiction, assume that there exist $\varepsilon_n \searrow 0$ and $(x_n)_{n\in\mathbb{N}} \subset \Lambda^1$ such that
\[
J_{\epsilon_0}'(x_n) = 0, \quad L \leq J_{\epsilon_0}(x_n) \leq M \quad \text{for all } n \in \mathbb{N}
\]
and
\[
\sup_{s \in I} U(x_n(s)) \geq \frac{1}{\varepsilon_n}
\]
As $U$ is bounded on bounded sets (see Remark 3.3) it follows that
\[
\sup \left\{ d(x_n(s), x_0) : s \in I, \ n \in \mathbb{N} \right\} = +\infty.
\]
On the other hand, (3.6) and (3.10) imply
\[
\left( \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, ds \right)_{n \in \mathbb{N}} \text{ is bounded} ;
\]
thus (3.11) and (3.12) give
\[
\inf_{s \in I} d(x_n(s), x_0) \longrightarrow +\infty \quad \text{as } n \to +\infty.
\]
Clearly, by (1.9) and (3.13) there exists $n_1 \in \mathbb{N}$ such that for $n \geq n_1$ it is $\inf_{s \in I} \phi(x_n(s)) \geq \rho > \sqrt{\varepsilon_n}$; whence, (3.1) and (3.4) imply
\[
\phi_{\epsilon_0}(x_n(s)) = \phi_{\epsilon_0}'(x_n(s)) = 0 \quad \text{for all } s \in I.
\]
Let us consider $n \geq n_1$. By (3.10) and Remark 3.1, defined $t_n = \Theta(x_n), z_n = (x_n, t_n)$ is a $C^2$
critical point of $f_{x_n}$, where by (3.14) it is $f_{x_n}(z) = f(z) + \int_0^1 U_{x_n}(x) \, ds$ for all $z = (x, t) \in Z_0$.
In particular, for any $\xi \in T_{x_n} \Lambda^1$ it results
\[
0 = f'_{x_n}(z_n)(\xi, 0) = \int_0^1 \langle \dot{x}_n, \dot{\xi} \rangle \, ds + \int_0^1 \langle \delta'(x_n)[\xi], \dot{x}_n \rangle \, ds
\]
\[
+ \int_0^1 \langle \delta(x_n), \dot{\xi} \rangle \, ds - \frac{1}{2} \int_0^1 \langle \nabla \beta(x_n), \dot{\xi} \rangle \dot{\xi}^2 \, ds + \int_0^1 \langle \nabla U_{x_n}(x_n), \dot{\xi} \rangle \, ds;
\]
whence, (2.5) and simple calculations prove that $x_n$ is a $C^2$ solution of the following equation
\[
D_x \dot{x}_n + \frac{\langle \delta(x_n), D_x \dot{x}_n \rangle}{\beta(x_n)} \delta(x_n) = - \frac{\langle \delta'(x_n)[\dot{x}_n], \dot{x}_n \rangle}{\beta(x_n)} \delta(x_n)
\]
\[
+ \left( \langle \delta(x_n), \dot{x}_n \rangle - K(x_n) \right) \frac{\langle \nabla \beta(x_n), \dot{x}_n \rangle}{\beta^2(x_n)} \delta(x_n)
\]
\[
+ \dot{t}_n \left( \delta''(x_n) - \delta'(x_n)[\dot{x}_n] \right)
\]
\[
- \frac{1}{2} \dot{t}_n^2 \nabla \beta(x_n) + \psi_{x_n}(U(x_n)) \nabla U(x_n),
\]
where $\delta''(x_n)$ is the adjoint of $\delta'(x_n)$. Taken $x \in \mathcal{M}_0$, let us define the linear operator
\[
A(x) : v \in T_x \mathcal{M}_0 \longmapsto A(x)[v] = \frac{\langle \delta(x), v \rangle}{\beta(x)} \delta(x) \in T_x \mathcal{M}_0.
\]
Clearly, it is
\[ |A(x)|_u \leq \frac{|\delta(x)|^2}{\beta(x)} \quad \text{for any } x \in M_0, \tag{3.16} \]
with \( |h(x)|_u = \sup \{|h(x)[v]| : v \in T_x M_0, |v| = 1 \} \) for any linear and continuous operator \( h(x) : T_x M_0 \to T_x M_0 \); hence, by (1.7), (1.10) and (3.16) it is
\[ \lim_{d(x, x_0) \to +\infty} |A(x)|_u = 0 \]
which implies the existence of \( R_1 \geq R \) such that, taken \( x \in M_0 \) verifying \( d(x, x_0) \geq R_1 \), the operator \( I + A(x) \) is invertible. Let \( B(x) = (I + A(x))^{-1} \) be its inverse. In particular, by (3.13) there exists \( n_2 \in \mathbb{N} \), \( n_2 \geq n_1 \), such that for \( n \geq n_2 \) and for all \( s \in I \) there results
\[ d(x_n(s), x_0) \geq R_1 \geq R \tag{3.17} \]
and \( B(x_n(s)) \) is well defined. Thus (3.15) becomes
\[ D_s \dot{x}_n = -\frac{\langle \delta'(x_n)[\dot{x}_n], \dot{x}_n \rangle}{\beta(x_n)} B(x_n)[\delta(x_n)] + \left( \langle \delta(x_n), \dot{x}_n \rangle - K(x_n) \right) \frac{\nabla \beta(x_n) \cdot \dot{x}_n}{\beta^2(x_n)} B(x_n)[\delta(x_n)] + \dot{t}_n B(x_n) \left( \delta'(x_n) - \delta(x_n) \right) \frac{\dot{x}_n}{\beta(x_n)} - \frac{1}{2} \dot{t}_n^2 B(x_n) \left[ \nabla \beta(x_n) + \psi'_{x_n}(U(x_n)) B(x_n) \left[ \nabla U(x_n) \right] \right]. \]
Let us define \( \bar{u}_n(s) = U(x_n(s)) \). By (3.5) it is \( \bar{u}_n(0) = \dot{u}_n(1) \), then by (1.8) and (3.17) the previous equation implies
\[ 0 = \int_0^1 \bar{u}_n(s) \, ds = \int_0^1 \left( H^U_{x_n}(x_n)[\dot{x}_n, \dot{x}_n] + \langle \nabla U(x_n), D_s \dot{x}_n \rangle \right) \, ds \]
\[ \geq \lambda \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, ds + \int_0^1 \langle \nabla U(x_n), D_s \dot{x}_n \rangle \, ds \]
\[ = \lambda \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, ds - \int_0^1 \frac{\langle \delta'(x_n)[\dot{x}_n], \dot{x}_n \rangle}{\beta(x_n)} \left( \langle \nabla U(x_n), B(x_n)[\delta(x_n)] \rangle \right) \, ds \]
\[ + \int_0^1 \left( \langle \delta(x_n), \dot{x}_n \rangle - K(x_n) \right) \frac{\langle \nabla \beta(x_n) \cdot \dot{x}_n \rangle}{\beta^2(x_n)} \left( \langle \nabla U(x_n), B(x_n)[\delta(x_n)] \rangle \right) \, ds \]
\[ + \int_0^1 \dot{t}_n \langle \nabla U(x_n), B(x_n) \left( \delta'(x_n) - \delta(x_n) \right) \frac{\dot{x}_n}{\beta(x_n)} \rangle \, ds \]
\[ - \frac{1}{2} \int_0^1 \dot{t}_n^2 \langle \nabla U(x_n), B(x_n) \left[ \nabla \beta(x_n) + \psi'_{x_n}(U(x_n)) B(x_n) \left[ \nabla U(x_n) \right] \right] \rangle \, ds \]
We claim that
\[ \lim_{n \to +\infty} \int_0^1 \dot{t}_n^2 \, ds = 0. \tag{3.18} \]
In fact, (1.7), (1.10) and (3.12) imply

$$\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle}{\beta(x_n)} \, ds = o(1), \quad \int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle^2}{\beta(x_n)} \, ds = o(1),$$

then $K(x_n) = o(1)$ by definition (2.3); hence, (3.18) follows by (2.5) (here $o(1)$ is any infinitesimal sequence). Then the definition of $J_n$ and (3.10) give

$$\int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, ds = 2J_n(x_n) - \int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle^2}{\beta(x_n)} \, ds + K^2(x_n) \int_0^1 \frac{1}{\beta(x_n)} \, ds - 2 \int_0^1 U_{x_n}(x_n) \, ds \geq 2L - 2 \int_0^1 U_{x_n}(x_n) \, ds + o(1).$$

Moreover, by using also the hypotheses (1.11) and arguing as in [5, Appendix], it can be proved that

$$\int_0^1 \frac{\langle \delta'(x_n), \dot{x}_n \rangle, \dot{x}_n \rangle}{\beta(x_n)} \{\nabla U(x_n), B(x_n) \{\delta(x_n)\} \} \, ds = o(1), \quad \int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle - K(x_n) \frac{\langle \nabla \beta(x_n), \dot{x}_n \rangle}{\beta^2(x_n)} \{\nabla U(x_n), B(x_n) \{\delta(x_n)\} \} \, ds = o(1), \quad \int_0^1 i_n^2 \{\nabla U(x_n), B(x_n) \{\delta(x_n)\} \} \, ds = o(1), \quad \int_0^1 \frac{\langle \nabla U(x_n), B(x_n) \{\delta(x_n)\} \} \, ds = o(1).$$

By (3.2) and the previous formulas there results

$$0 \geq \lambda \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, ds + 2 \int_0^1 \psi_{x_n}(U(x_n)) \, ds + o(1) \geq 2\lambda L + 2 \int_0^1 \left( \psi_{x_n}(U(x_n)) - \lambda \psi_{x_n}(U(x_n)) \right) \, ds + o(1) \geq 2\lambda L + o(1)$$

which gives a contradiction. Whence, (3.8) holds and for $n$ large enough it is

$$U_{x_n}(x_n(s)) = 0 \quad \text{for all } s \in I.$$ (3.19)

Now suppose that (3.9) does not hold, then there exist $\varepsilon_n \downarrow 0$ and $(x_n)_{n \in \mathbb{N}}$ in $L^1$ such that (3.10) is satisfied and

$$\inf_{s \in I} \phi(x_n(s)) \leq \sqrt{\varepsilon_n}.$$ (3.20)

By the first part of this proof, if $n$ is large enough (3.19) holds; hence, $z_n = (x_n, \Theta(x_n))$ is a $C^2$
critical point of $f_{x_n}$ satisfying the following equation

$$- D_s \dot{z}_n = 2\psi_{x_n}' \left( \frac{1}{\Phi^2(z_n)} \right) \nabla_L \Phi(z_n) \Phi^2(z_n).$$

Arguing as in [10, Lemma 4.7] it is possible to prove that $(x_n)_{n \in \mathbb{N}}$ converges in $H^1(S^1, \mathbb{R}^N)$ to a curve $x \in \Lambda^1(M_0 \cup \partial M_0)$ and $z = (x, \Theta(x))$ solves the equation

$$D_s \dot{z}(s) = \gamma(s) \nabla_L \Phi(z(s)),$$

where $\gamma \in L^2(I, \mathbb{R})$. Moreover, as in the proof of [10, Theorem 5.1] (see also [11, Lemma 3.8]), the condition (1.3) implies that $z$ is a closed geodesic in $M \cup \partial M$. Let us remark that by [10, Remark 4.5] it is

$$\lim_{n \to +\infty} \int_0^1 \psi_{x_n} \left( \frac{1}{\phi^2(x_n)} \right) ds = 0,$$

then $J_{\psi_{x_n}}(x_n) \to J(x)$ as $n \to +\infty$; hence, by (2.7) and (3.10), $z$ is a spacelike geodesic. Since (3.20) implies that $z$ touches the boundary $\partial M$, Definition 1.2 and Remark 1.3 give a contradiction. 

4. Proof of the main theorem

Let $M = M_0 \times \mathbb{R}$ be a stationary manifold with strictly space-convex boundary such that the hypotheses of Theorem 1.7 hold.

**Lemma 4.1.** Let $\phi$ be as in Remark 1.6 and assume that (1.9) holds. If $(x_n)_{n \in \mathbb{N}}$ is a sequence in $\Lambda^1$ such that

$$\left( \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle ds \right)_{n \in \mathbb{N}}$$

is bounded

and there exists $(s_n)_{n \in \mathbb{N}} \subset I$ such that

$$\lim_{n \to +\infty} \phi(x_n(s_n)) = 0,$$

then

$$\lim_{n \to +\infty} \int_0^1 \frac{1}{\phi^2(x_n)} ds = +\infty.$$

**Proof.** Cf. [2, Lemma 3.2].

**Lemma 4.2.** Let $\varepsilon > 0$ be fixed. For any $c \in \mathbb{R}$ the sublevel

$$J^\varepsilon_c = \{ x \in \Lambda^1 : J^\varepsilon(x) \leq c \}$$

is a complete metric space; moreover, $J^\varepsilon$ satisfies the $(PS)$ condition.
Proof. Let \( \varepsilon > 0 \) and \( c \in \mathbb{R} \) be fixed. Clearly, the sets
\[
\left\{ \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds : x \in J_e^c \right\}, \quad \left\{ \int_0^1 \phi(x) \, ds : x \in J_e^c \right\} \quad \text{and} \quad \left\{ \int_0^1 U(x) \, ds : x \in J_e^c \right\}
\]
are bounded, then by Lemmas 4.1 and 3.2 it can be easily deduced that there exist \( r, \mu > 0 \) such that
\[
J_e^c \subset \Lambda^1(B_{r,\mu}), \quad B_{r,\mu} = \left\{ x \in \mathcal{M}_0 : d(x, x_0) \leq r, \phi(x) \geq \mu \right\}.
\]
Since \( B_{r,\mu} \) is a compact subset of \( \mathcal{M}_0 \), then \( \Lambda^1(B_{r,\mu}) \) is complete which implies that the closed subset \( J_e^c \) is complete, too. Let us prove, now, that \( J_e^c \) verifies the \((PS)\) condition. Let \( (x_n)_{n \in \mathbb{N}} \subset \Lambda^1 \) be a \((PS)\) sequence, i.e.,
\[
\lim_{n \to +\infty} J_e'(x_n) = 0. \tag{4.2}
\]
By (4.1) and the previous remark there exist \( r, \mu > 0 \) such that
\[
x_n \in \Lambda^1(B_{r,\mu}) \quad \text{for any } n \in \mathbb{N}
\]
and
\[
\left( \int_0^1 \langle \dot{x}_n, \dot{x}_n \rangle \, ds \right)_{n \in \mathbb{N}} \quad \text{is bounded} ;
\]
whence,
\[
(x_n)_{n \in \mathbb{N}} \quad \text{is bounded in } H^1(S^1, \mathbb{R}^N) \tag{4.3}
\]
and there exists \( x \in \Lambda^1(B_{r,\mu}) \) such that \( x_n \rightharpoonup x \) weakly in \( H^1(I, \mathbb{R}^N) \) and uniformly in \( I \) up to subsequences. By [3, Lemma 2.1] it follows that there exist two bounded sequences \( (\xi_n)_{n \in \mathbb{N}} \) and \( (v_n)_{n \in \mathbb{N}} \) in \( H^1(I, \mathbb{R}^N) \) such that
\[
x_n - x = \xi_n + v_n, \quad \xi_n \in T_x \Lambda^1 \quad \text{for any } n \in \mathbb{N},
\]
\[
\xi_n \rightharpoonup 0 \quad \text{weakly in } H^1(I, \mathbb{R}^N) \quad \text{and} \quad v_n \rightharpoonup 0 \quad \text{strongly in } H^1(I, \mathbb{R}^N). \tag{4.4}
\]
Moreover, taken \( t_n = \Theta(x_n) \), by (2.5) and (4.3) the sequence \( (t_n)_{n \in \mathbb{N}} \) is bounded in \( H^1(I, \mathbb{R}) \), then, up to subsequences, \( t_n \rightharpoonup t \) weakly in \( H^1(I, \mathbb{R}) \). By (4.2), (2.8) and (2.9) imply
\[
o(1) = J_e'(x_n)[\xi_n] = J_e'(z_n)[(\xi_n, -\tau_n)], \tag{4.5}
\]
where \( z_n = (x_n, t_n) \) and \( \tau_n = t_n - t \). By the definition of \( f_e \), (4.5) becomes
\[
o(1) = \int_0^1 \langle \dot{x}_n, \dot{\xi}_n \rangle \, ds + \int_0^1 \langle \delta(x_n)[\xi_n], \dot{x}_n \rangle \, ds + \int_0^1 \langle \phi(x_n), \dot{\xi}_n \rangle \, ds \\
- \frac{1}{2} \int_0^1 \langle \beta(x_n), \dot{\tau}_n \rangle \, ds - \frac{1}{2} \int_0^1 \langle \beta'(x_n)[\xi_n], \dot{\tau}_n \rangle \, ds + \int_0^1 \beta(x_n) \dot{\xi}_n \, ds.
\]
Clearly, by (4.3) and (4.4) it follows that
\[ \int_0^1 \phi'_e(x_n)[\xi_n] \, ds = o(1), \quad \int_0^1 U'_e(x_n)[\xi_n] \, ds = o(1). \]
Then, arguing as in [13, Lemma 3.2], it is possible to prove that \( \xi_n \to 0 \) in \( H^1(I, \mathbb{R}^N) \); whence, \( x_n \to x \) strongly. \( \square \)

**Lemma 4.3.** For any \( c \in \mathbb{R} \) it results
\[ \text{cat}_{\Lambda^1}(J') < +\infty. \]

**Proof.** Setting
\[ F(x) = \int_0^1 \langle \dot{x}, \dot{x} \rangle \, ds, \]
by (2.10) it is \( J^c \subset \mathbb{R}^{2c} \); moreover, since \( \text{cat}_{\Lambda^1}(F^k) < +\infty \) for all \( k \in \mathbb{R} \) (see [6, Lemma 4.1]), the monotonicity property of the Lusternik–Schnirelmann category gives the conclusion. \( \square \)

**Proof of Theorem 1.7.** Let \( L > 0 \) be fixed. Lemma 4.3 implies that there exists \( \bar{k} \in \mathbb{N} \) such that \( B \cap J_L \neq \emptyset \) for all \( B \in \Gamma_\bar{k} \), where \( \Gamma_\bar{k} \) is defined as in (2.11) and \( J_L = \{ x \in \Lambda^1 \mid J(x) > L \} \). Since \( J_L \subset J_{e,L} \) for all \( \varepsilon > 0 \), it is \( B \cap J_{e,L} \neq \emptyset \) for all \( B \in \Gamma_\bar{k} \); hence,
\[ L \leq c_{e,\bar{k}}, \quad \text{where} \quad c_{e,\bar{k}} = \inf_{B \in \Gamma_\bar{k}} \sup_{x \in B} J_e(x). \quad (4.6) \]
By Proposition 2.8 there exists a compact set \( K \subset \Lambda^1 \) such that \( \text{cat}_{\Lambda^1}(K) \geq \bar{k} \). By (3.3) it is \( J_e(x) \leq J_1(x) \) for all \( x \in \Lambda^1 \) if \( \varepsilon \leq 1 \), then by (4.8) for all \( \varepsilon \leq 1 \) it follows
\[ L \leq c_{e,\bar{k}} \leq M, \quad \text{where} \quad M = \max_{x \in K} J_1(x). \quad (4.7) \]
Since for all \( \varepsilon \leq 1 \) Lemma 4.2 and Theorem 2.6 imply the existence of at least one critical point \( x_\varepsilon \) of \( J_e \) such that \( J_e(x_\varepsilon) = c_{e,\bar{k}} \) satisfies (4.9), Proposition 3.4 implies that, if \( \varepsilon \) is small enough, \( x_\varepsilon \) is a critical point of \( J \). Whence, \( z_\varepsilon = (x_\varepsilon, \Theta(x_\varepsilon)) \) is a closed geodesic in \( M \) such that \( f(z_\varepsilon) = J(z_\varepsilon) \geq L \). \( \square \)

**Remark 4.4.** Clearly, for any \( L > 0 \) if \( \varepsilon \) is small enough there exists a closed geodesic \( z_\varepsilon \) such that \( f(z_\varepsilon) \geq L \). In particular, there exists at least a closed spacelike geodesic. Unluckily, we have not a multiplicity result since the found geodesics may not be geometrically distinct.

**References**


