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Decomposition of Algebraic Functions

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Functional decomposition—whether a function $f(x)$ can be written as a composition of functions $g(h(x))$ in a non-trivial way—is an important primitive in symbolic computation systems. The problem of univariate polynomial decomposition was shown to have an efficient solution by Kozen and Landau (1989). Dickerson (1987) and von zur Gathen (1990a) gave algorithms for certain multivariate cases. Zippel (1991) showed how to decompose rational functions. In this paper, we address the issue of decomposition of algebraic functions. We show that the problem is related to univariate resultants in algebraic function fields, and in fact can be reformulated as a problem of *resultant decomposition*. We characterize all decompositions of a given algebraic function up to isomorphism, and give an exponential time algorithm for finding a non-trivial one if it exists. The algorithm involves genus calculations and constructing transcendental generators of fields of genus zero.

1. Introduction

Functional decomposition is the problem of representing a given function $f(x)$ as a composition of "smaller" functions $g(h(x))$. Decomposition of polynomials is useful in simplifying the representation of field extensions of high degree, and is provided as a primitive by many major symbolic algebra systems.

The first analysed algorithms for decomposition of polynomials were provided by Barton and Zippel (1976, 1985) and Alagar and Thanh (1985), who gave algorithms for the problem of decomposing univariate polynomials over fields of characteristic zero. Both solutions involved polynomial factorization and took exponential time. Kozen and Landau (1989) discovered a simple and efficient polynomial time solution that does not require factorization. It works over fields of characteristic zero and whenever the characteristic of the underlying field does not divide the degree of h. It also provides *NC* algorithms for irreducible polynomials over finite fields and all polynomials over fields of characteristic zero. Dickerson (1987) and von zur Gathen (1990a) gave algorithms for certain multivariate cases. In addition, von zur Gathen (1990b) also found algorithms for the case in which the characteristic of the field divides the degree of h. Zippel (1991) showed how to

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decompose rational functions efficiently over fields of any characteristic, thus resolving the polynomial problem for finite characteristic.

In this paper we address the decomposition problem for algebraic functions. We show that the problem bears an interesting and useful relationship to univariate resultants over algebraic function fields, and in fact can be reformulated as a certain resultant decomposition problem: whether some power of a given irreducible bivariate polynomial $f(x,z)$ can be expressed as the resultant with respect to y of two other bivariate polynomials $g(x,y)$, $h(y,z)$. We determine necessary and sufficient conditions for an algebraic function to have a non-trivial decomposition, and classify all such decompositions up to isomorphism. We give an exponential–time algorithm for finding a non-trivial decomposition of a given algebraic function if one exists. The algorithm involves calculating the genus of certain algebraic function fields and constructing transcendental generators of fields of genus zero.

2. Resultants and Algebraic Functions

2.1. THE UNIVARIATE RESULTANT

Here we review some basic facts about the univariate resultant; see Ierardi and Kozen (1993) and Zippel (1993) for a detailed introduction.

The *resultant* of two polynomials

$$g(y) = a\prod_{i=1}^{m}(y-\alpha_i) \qquad h(y) = b\prod_{j=1}^{\ell}(y-\beta_j)$$

with respect to y is the polynomial

$$\mathbf{res}_y(g,h) = a^\ell b^m \prod_{i,j}(\beta_j-\alpha_i) = b^m \prod_{h(\beta)=0} g(\beta)\,. \tag{2.1}$$

The resultant vanishes if and only if g and h have a common root. It can be calculated in a number of ways, including as the determinant of the *Sylvester matrix*, a certain $(m+\ell)\times(m+\ell)$ matrix containing the coefficients of g and h.

The following are some useful elementary properties, that follow immediately from (2.1).

$$\begin{aligned}
\mathbf{res}_y(g,h) &= (-1)^{m\ell}\,\mathbf{res}_y(h,g)\\
\mathbf{res}_y(g_1g_2,h) &= \mathbf{res}_y(g_1,h)\cdot\mathbf{res}_y(g_2,h)\\
\mathbf{res}_y(g,h_1h_2) &= \mathbf{res}_y(g,h_1)\cdot\mathbf{res}_y(g,h_2)\\
\mathbf{res}_y(c,h) &= c^\ell\\
\mathbf{res}_y(g,1) &= \mathbf{res}_y(1,h) = 1\\
\mathbf{res}_y(g,y-\beta) &= g(\beta)\\
\mathbf{res}_x(f(x),\mathbf{res}_y(g(x,y),h(y))) &= \mathbf{res}_y(\mathbf{res}_x(f(x),g(x,y)),h(y))\,.
\end{aligned} \tag{2.2}$$

Property (2.2) is an associativity property. Because of this property, we can write

$$\mathbf{res}_{x,y}(f(x),g(x,y),h(y))$$

unambiguously for the left- or right-hand side of (2.2).

We extend the definition to pairs of rational functions as follows. If neither g_1, h_2

nor g_2, h_1 have a common root, define

$$\mathbf{res}_y\left(\frac{g_1}{g_2}, \frac{h_1}{h_2}\right) = \frac{\mathbf{res}_y(g_1, h_1) \cdot \mathbf{res}_y(g_2, h_2)}{\mathbf{res}_y(g_1, h_2) \cdot \mathbf{res}_y(g_2, h_1)}.$$

This definition reduces to the previous one in the case of polynomials. All the properties listed above still hold, taking $m = \deg g_1 - \deg g_2$ and $n = \deg h_1 - \deg h_2$.

2.2. RESULTANTS AND DECOMPOSITION

Let K be an algebraically closed field, and let Ω be a *universal field* over K in the sense of van der Waerden (1970a); i.e., an algebraically closed field of infinite transcendence degree over K. Let $\mathbb{A}^2(\Omega)$ denote the affine plane over Ω.

Algebraic functions of γ are usually defined as elements of some finite extension of $K(\gamma)$, the field of rational functions of γ. We can also view algebraic functions more concretely as multivalued functions $\Omega \to 2^\Omega$ or as binary relations on Ω defined by their minimum polynomials. In the latter view, the decomposition problem is naturally defined in terms of ordinary composition of binary relations:

$$R \circ S = \{(u, w) \mid \exists v\ (u, v) \in R \wedge (v, w) \in S\}.$$

DEFINITION 2.1. *For $f(x, z) \in K[x, z]$, let*

$$V(f) = \{(\alpha, \gamma) \mid f(\alpha, \gamma) = 0\} \subseteq \mathbb{A}^2(\Omega)$$

be the affine variety generated by f. A decomposition *of f is a pair of polynomials $g(x, y) \in K[x, y]$ and $h(y, z) \in K[y, z]$ such that*

$$V(f) = \overline{V(g) \circ V(h)},$$

where the overbar denotes the Zariski closure in $\mathbb{A}^2(\Omega)$ (see Hartshorne, 1977).

The Zariski closure is taken in order to account for points at infinity in a composition. An alternative approach would be to consider f as a binary relation on the projective line.

This notion of decomposition is strongly related to the univariate resultant:

$$\begin{aligned} V(g) \circ V(h) &= \{(\alpha, \gamma) \mid \exists \beta\ g(\alpha, \beta) = h(\beta, \gamma) = 0\} \\ &= \{(\alpha, \gamma) \mid \mathbf{res}_y(g(\alpha, y), h(y, \gamma)) = 0\} \end{aligned}$$

by (2.1). The following results develop this relationship further.

LEMMA 2.2. *Let $g(x, y) \in K[x, y]$ and $h(y, z) \in K[y, z]$. Considering $g(x, y)$ and $h(y, z)$ as polynomials in y, let $g_m(x)$ and $h_\ell(z)$ be their respective leading coefficients. Then*

$$V(\mathbf{res}_y(g, h)) = (V(g) \circ V(h)) \cup V(g_m, h_\ell).$$

PROOF. Consider the two expressions

$$\mathbf{res}_y(g(\alpha, y), h(y, \gamma)) \tag{2.3}$$

$$\mathbf{res}_y(g(x, y), h(y, z))[x := \alpha, z := \gamma]. \tag{2.4}$$

The difference is whether α and γ are substituted for x and z before or after the resultant is taken. We claim that for any α, γ,

(i) if $g_m(\alpha) = h_\ell(\gamma) = 0$, then (2.4) vanishes;
(ii) if either $g_m(\alpha) \neq 0$ or $h_\ell(\gamma) \neq 0$, then (2.3) and (2.4) vanish or do not vanish simultaneously.

In case (i), we have

$$\mathbf{res}_y\,(g(x,y), h(y,z)) = \det S(x,z)\,,$$

where $S(x,z)$ is the Sylvester matrix of $g(x,y)$ and $h(y,z)$. Then

$$\mathbf{res}_y\,(g(x,y), h(y,z))\,[x := \alpha, z := \gamma] = \det S(\alpha,\gamma) = 0\,,$$

since the first row of $S(\alpha,\gamma)$ is the zero vector. In case (ii), say $h_\ell(\gamma) \neq 0$ (the other case is symmetric). Then

$$\begin{aligned}\mathbf{res}_y\,(g(x,y), h(y,z))\,[x := \alpha, z := \gamma] &= \mathbf{res}_y\,(g(x,y), h(y,\gamma))\,[x := \alpha] \\ &= h_\ell(\gamma)^{\deg_y g(x,y)} \prod_{h(\beta,\gamma)=0} g(\alpha,\beta) \\ \mathbf{res}_y\,(g(\alpha,y), h(y,\gamma)) &= h_\ell(\gamma)^{\deg_y g(\alpha,y)} \prod_{h(\beta,\gamma)=0} g(\alpha,\beta)\end{aligned}$$

thus both expressions are simultaneously zero or non-zero.

By (i) and (ii),

$$\begin{aligned}V(\mathbf{res}_y\,(g,h)) &= \{(\alpha,\gamma) \mid \mathbf{res}_y\,(g(x,y), h(y,z))\,[x := \alpha, z := \gamma] = 0\} \\ &= \{(\alpha,\gamma) \mid \mathbf{res}_y\,(g(\alpha,y), h(y,\gamma)) = 0 \vee g_m(\alpha) = h_\ell(\gamma) = 0\} \\ &= (V(g) \circ V(h)) \cup V(g_m, h_\ell)\,.\end{aligned}$$

□

THEOREM 2.3. *Let $g(x,y) \in K[x,y]$ and $h(y,z) \in K[y,z]$ be irreducible and non-degenerate (i.e., positive degree in each variable). Then*

$$V(\mathbf{res}_y\,(g,h)) = \overline{V(g) \circ V(h)}\,.$$

PROOF. We have $\overline{V(g) \circ V(h)} \subseteq V(\mathbf{res}_y\,(g,h))$ by Lemma 2.2 and the fact that $V(\mathbf{res}_y\,(g,h))$ is Zariski-closed.

Conversely, it follows from the assumption that $g(x,y)$ and $h(y,z)$ are irreducible and non-degenerate that for all α,β,γ such that $g(\alpha,\beta) = h(\beta,\gamma) = 0$, either all $\alpha,\beta,\gamma \in K$ or all are transcendental over K. We use this to show that $\mathbf{res}_y\,(g,h)$ has no factor of the form $u(x)$. Suppose it did. Let $a \in K$ be a root of u (recall that K is algebraically closed). Then $\mathbf{res}_y\,(g,h)\,[x := a] = 0$. Let γ be transcendental over K. We have

$$\begin{aligned}0 &= \mathbf{res}_y\,(g(x,y), h(y,z))\,[x := a, z := \gamma] \\ &= \mathbf{res}_y\,(g(x,y), h(y,\gamma))\,[x := a] \\ &= h_\ell(\gamma)^m \prod_{h(\beta,\gamma)=0} g(x,\beta)[x := a] \\ &= h_\ell(\gamma)^m \prod_{h(\beta,\gamma)=0} g(a,\beta)\,,\end{aligned}$$

thus $g(a,\beta) = h(\beta,\gamma) = 0$ for some β. But $a \in K$ and γ is transcendental over K, which contradicts our observation above.

By symmetry, $\mathbf{res}_y(g,h)$ has no factor $v(z)$.

Thus all irreducible factors of $\mathbf{res}_y(g,h)$ are non-degenerate. Let (α,γ) be a generic point of some irreducible component C of $V(\mathbf{res}_y(g,h))$. Then α and γ are transcendental over K. By Lemma 2.2, $(\alpha,\gamma) \in V(g) \circ V(h)$, so $C \subseteq \overline{V(g) \circ V(h)}$. Since C was arbitrary, $V(\mathbf{res}_y(g,h)) \subseteq \overline{V(g) \circ V(h)}$. □

COROLLARY 2.4. *Let $f(x,z)$, $g(x,y)$, and $h(y,z)$ be irreducible and non-degenerate. Then g,h give a decomposition of f if and only if $f^k = \mathbf{res}_y(g,h)$ for some $k > 0$.*

PROOF. If $f^k = \mathbf{res}_y(g,h)$, then by Theorem 2.3,

$$V(f) = V(f^k) = V(\mathbf{res}_y(g,h)) = \overline{V(g) \circ V(h)}\,.$$

Conversely, if $V(f) = \overline{V(g) \circ V(h)}$, then by Theorem 2.3, $V(f) = V(\mathbf{res}_y(g,h))$, and $f^k = \mathbf{res}_y(g,h)$ follows immediately from the Nullstellensatz and the assumption that f is irreducible. □

We use the term *functional decomposition*, despite the fact that the function $z \mapsto x$ specified by the bivariate polynomial $f(x,z)$ is in general multivalued, and would be more accurately termed a relation. However, at least in characteristic 0, these relations behave locally like functions; for example, consider the square root "function" $\sqrt{y}$, specified by the bivariate polynomial $x^2 - y$. It is this common intuition on which our terminology is based.

In light of Corollary 2.4 and the above discussion, we define the *decomposition problem* for algebraic functions as follows:

> Given an irreducible polynomial $f(x,z)$, find polynomials $g(x,y)$ and $h(y,z)$ and a positive integer k such that $f^k = \mathbf{res}_y(g,h)$.

This formulation directly generalizes the definition of functional decomposition for univariate polynomials and rational functions, considering univariate polynomials to be specified by bivariate polynomials $f(x,y)$ that are monic and linear in x, and rational functions to be specified by polynomials $f(x,y)$ that are linear in x; that is, the bivariate polynomial $h(y)x - g(y)$ specifies the rational function $g(y)/h(y)$. For example, to compose univariate polynomials $g(y)$ and $h(z)$, take the resultant of $y - h(z)$ and $x - g(y)$:

$$\mathbf{res}_y(x - g(y), y - h(z)) = x - g(h(z))\,.$$

Under this definition, every bivariate polynomial f is decomposable in infinitely many ways:

$$\mathbf{res}_y\left(f(x,y^k), y^k - z\right) = \prod_{\beta^k = z} f(x,\beta^k) = \prod_{\beta^k = z} f(x,z) = f^k\,. \tag{2.5}$$

However, these decompositions are not optimal in a sense to be made precise. In the next section we will define a notion of *minimality* for decompositions, and show that up to isomorphism there are only finitely many non-trivial minimal decompositions.

2.3. IRREDUCIBLE DECOMPOSITIONS

A decomposition $f = \mathbf{res}_y(g,h)$ is called *irreducible* if both g and h are irreducible as polynomials in $K[x,y]$ and $K[y,z]$, respectively. By the multiplicativity of the resultant, every decomposition factors into a product of irreducible decompositions.

2.4. MONIC DECOMPOSITIONS

A decomposition $f = \mathbf{res}_y(g,h)$ is called *monic* if $g \in K(y)[x]$ and $h \in K(z)[y]$ are monic. The next result says that we can restrict our attention to monic decompositions without loss of generality.

LEMMA 2.5. *Let $f \in K[x,z]$, $g \in K[x,y]$, $h \in K[y,z]$ be non-degenerate, g, h irreducible, f a power of an irreducible polynomial. Let $\widehat{f}$, $\widehat{g}$, and $\widehat{h}$ be the monic associates of f, g, h in $K(z)[x]$, $K(y)[x]$, and $K(z)[y]$ respectively. Then $f = \mathbf{res}_y(g,h)$ if and only if $\widehat{f} = \mathbf{res}_y\left(\widehat{g},\widehat{h}\right)$.*

PROOF. Let $f_n(z)$, $g_m(y)$, and $h_\ell(z)$ be the lead coefficients of f, g and h, respectively. Let

$$u(z) = \mathbf{res}_y\left(g_m(y), h(y,z)\right) \cdot h_\ell(z)^{\deg_y g - \deg_y g_m}.$$

Then

$$\mathbf{res}_y(g,h) = \mathbf{res}_y(g_m,h) \cdot \mathbf{res}_y\left(\widehat{g},h_\ell\right) \cdot \mathbf{res}_y\left(\widehat{g},\widehat{h}\right) = u \cdot \mathbf{res}_y\left(\widehat{g},\widehat{h}\right).$$

But since $\widehat{g}$ and $\widehat{h}$ are monic, so is $\mathbf{res}_y\left(\widehat{g},\widehat{h}\right)$, therefore if $f = \mathbf{res}_y(g,h) = u \cdot \mathbf{res}_y\left(\widehat{g},\widehat{h}\right)$, then $u = f_n$ and $\widehat{f} = \mathbf{res}_y\left(\widehat{g},\widehat{h}\right)$.

Conversely, if $\widehat{f} = \mathbf{res}_y\left(\widehat{g},\widehat{h}\right)$, then $uf = f_n \mathbf{res}_y(g,h)$. Remove common factors to get $vf = w \cdot \mathbf{res}_y(g,h)$, where $v, w \in K[z]$ are relatively prime. Now f has no factor in $K[z]$, so w is a unit. Likewise, as argued in the proof of Theorem 2.3, $\mathbf{res}_y(g,h)$ has no factor in $K[z]$, so v is a unit. □

2.5. INSEPARABLE DECOMPOSITIONS

In prime characteristic p, a decomposition $f(x,z)^k = \mathbf{res}_y\left(g(x,y), h(y,z)\right)$ is *separable* if f is separable as a polynomial in $K(z)[x]$, g is separable as a polynomial in $K(y)[x]$, and h is separable as a polynomial in $K(z)[y]$. The following argument shows that we can restrict our attention to separable decompositions without loss of generality.

Any inseparable polynomial $f(x^q,z)$, $q = p^n$, has a non-trivial decomposition

$$f(x^q,z) = \mathbf{res}_y\left(x^q - y, f(y,z)\right). \tag{2.6}$$

The polynomial $x^q - y$ decomposes into the composition of n copies of $x^p - y$. Also,

$$\begin{aligned}
\mathbf{res}_y\left(g(x,y), y^q - z\right) &= \mathbf{res}_y\left(g(x,y), (y - \sqrt[q]{z})^q\right) \\
&= \mathbf{res}_y\left(g(x,y), y - \sqrt[q]{z}\right)^q \\
&= g(x, \sqrt[q]{z})^q \\
&= g^{[q]}(x^q, z)
\end{aligned} \tag{2.7}$$

where $g^{[q]}(u,v)$ denotes the polynomial obtained from $g(u,v)$ by raising all the coefficients to the qth power.

Once we have decomposed $f(x^q,z)$ as in (2.6), we can attempt to decompose $f(y,z)$ further. The following results show that any decomposition of $f(x^q,y)$ gives an associated decomposition of $f(x,y)$, so we can take this step without loss of generality.

LEMMA 2.6. *If $f(x,z)^k = \mathbf{res}_y\left(g(x,y),h(y,z)\right)$ is a non-degenerate irreducible decomposition, g is separable in x, and h is separable in y, then f is separable in x.*

PROOF. Let γ be transcendental over K. Let β be a root of $h(y,\gamma)$ and let α be a root of $g(x,\beta)$. Then α is a root of $f(x,\gamma)$. Since h is separable in y, the extension $K(\beta,\gamma):K(\gamma)$ is separable. Since g is separable in x, the extension $K(\alpha,\beta,\gamma):K(\beta,\gamma)$ is separable. Combining these extensions, we have that the extension $K(\alpha,\beta,\gamma):K(\gamma)$ is separable, hence $f(x,\gamma)$ is separable. □

THEOREM 2.7. *Let q be a power of p and let $f(x^q,z)^k = \mathbf{res}_y\left(g(x,y),h(y,z)\right)$ be a monic non-degenerate irreducible decomposition, $f(x,z)$ separable. Then there exists a separable decomposition*

$$f(x,z)^k = \mathbf{res}_y\left(\widehat{g}^{[s]}(x,y),\widehat{h}(y,z)\right)$$

where $g(x,y)=\widehat{g}(x^r,y)$, $h(y,z)=\widehat{h}(y^s,z)$, and $q=rs$.

PROOF. Let r,s be powers of p such that g and h can be written $g(x,y)=\widehat{g}(x^r,y)$, $h(y,z)=\widehat{h}(y^s,z)$ with $\widehat{g}$, $\widehat{h}$ separable. Then $\widehat{g}$, $\widehat{h}$ are also irreducible, and so is $\widehat{g}^{[s]}(x,y)$.

$$\begin{aligned}
\mathbf{res}_y\left(x^q-y,f(y,z)^k\right) &= f(x^q,z)^k \\
&= \mathbf{res}_y\left(g(x,y),h(y,z)\right) \\
&= \mathbf{res}_y\left(\widehat{g}(x^r,y),\widehat{h}(y^s,z)\right) \\
&= \mathbf{res}_{y,w}\left(\widehat{g}(x^r,y),y^s-w,\widehat{h}(w,z)\right) \\
&= \mathbf{res}_w\left(\widehat{g}^{[s]}(x^{rs},w),\widehat{h}(w,z)\right) \qquad \text{by (2.7)} \\
&= \mathbf{res}_{y,w}\left(x^{rs}-y,\widehat{g}^{[s]}(y,w),\widehat{h}(w,z)\right)
\end{aligned}$$

and $\mathbf{res}_w\left(\widehat{g}^{[s]}(y,w),\widehat{h}(w,z)\right)$ is separable by Lemma 2.6. Thus $q=rs$ and

$$f(y,z)^k = \mathbf{res}_w\left(\widehat{g}^{[s]}(y,w),\widehat{h}(w,z)\right).$$

□

This argument shows that in any irreducible decomposition of f, any inseparability of f must stem from the inseparability of one of the composition factors, and this inseparability ultimately emerges as a composition factor of the form x^q-y.

By Theorem 2.7, we can henceforth assume without loss of generality that all decompositions are separable.

3. A Characterization of All Decompositions

In this section we give a characterization of all possible irreducible decompositions of an algebraic function that can arise. As above, we assume that K is algebraically closed and that Ω is a universal field over K.

Let γ be transcendental over K and let α be a non-constant algebraic function of γ with monic minimum polynomial $f(x,\gamma) \in K(\gamma)[x]$ of degree n. From the results of the previous section, the functional decomposition problem reduces to the problem of finding all monic irreducible decompositions of the form

$$f(x,\gamma)^k = \mathbf{res}_y\left(g(x,y), h(y,\gamma)\right) = \prod_{h(\beta,\gamma)=0} g(x,\beta)\,.$$

Moreover, we can assume without loss of generality that $f(x,\gamma)$ is separable.

Let A be the set of conjugates of α over $K(\gamma)$, $|A| = n$. Let $\mathbf{Sym}\,A$ denote the field of symmetric functions of A. This is the smallest field containing all the coefficients of $f(x,\gamma)$. Note that $\mathbf{Sym}\,A$ properly contains K, for otherwise $f(x,\gamma)$ would factor into linear factors since K is algebraically closed, contradicting the assumption that α is non-constant.

Now consider the following condition on algebraic functions β of γ:

CONDITION 3.1. *The monic minimum polynomial $g(x,\beta)$ of α over $K(\beta)$ divides $f(x,\gamma)$.*

If β is algebraic over $K(\gamma)$, then g exists, since α is algebraic over $K(\gamma)$ and γ is algebraic over $K(\beta)$. A subtle but important point to note is that Condition 3.1 does not imply that $f(x,\gamma)$ factors over $K(\beta)$. Indeed, $K(\beta)$ need not contain the coefficients of f or f/g. We give an example of this in Section 5. The polynomial $g(x,\beta)$ does divide $f(x,\gamma)$ in the field $K(\beta,\gamma)$, so $f(x,\gamma)$ does factor over this field.

The following theorem states that any β satisfying Condition 3.1 uniquely determines a monic irreducible decomposition of α; moreover, all monic irreducible decompositions of α arise in this way.

THEOREM 3.2. *Let α be an algebraic function of γ with monic minimum polynomial $f(x,\gamma) \in K(\gamma)[x]$ of degree n. Let β be algebraic over $K(\gamma)$ with monic minimum polynomial $h(y,\gamma) \in K(\gamma)[y]$ of degree ℓ. Let $g(x,\beta) \in K(\beta)[x]$ of degree m be the monic minimum polynomial of α over $K(\beta)$. If β satisfies Condition 3.1, i.e. if $g(x,\beta)$ divides $f(x,\gamma)$, then*

$$f(x,z)^{\frac{\ell m}{n}} = \mathbf{res}_y\left(g(x,y), h(y,z)\right)$$

is a monic irreducible decomposition of α. Moreover, all monic irreducible decompositions of α arise in this way.

PROOF. Let A be the set of roots of $f(x,\gamma)$ and let $B_\beta \subseteq A$ be the set of roots of $g(x,\beta)$. If η is a conjugate of β over $K(\gamma)$, let B_η be the set of roots of $g(x,\eta)$. The set B_η is the image of B_β under any Galois automorphism over $K(\gamma)$ mapping β to η. For any such conjugate η, $|B_\eta| = |B_\beta| = m$ and $B_\eta \subseteq A$, since the Galois group over $K(\gamma)$ preserves A setwise.

By the symmetry of the action of the Galois group on A, each $\delta \in A$ occurs in the same number of the B_η, say k. We determine k by counting in two ways the number of

pairs (δ, η) such that $\delta \in B_\eta$. First, it is the number of conjugates η of β times the size of each B_η, or ℓm. Second, it is the number of $\delta \in A$ times the number of B_η containing δ, or nk. Equating these two values gives $k = \ell m/n$, the exponent in the statement of the theorem. Moreover, it follows from the same argument that

$$\begin{aligned} f(x,\gamma)^k = \prod_{\delta \in A} (x-\delta)^k &= \prod_{h(\eta,\gamma)=0} \prod_{\delta \in B_\eta} (x-\delta) \\ &= \prod_{h(\eta,\gamma)=0} g(x,\eta) = \mathbf{res}_y \left(g(x,y), h(y,\gamma)\right) . \end{aligned}$$

Since γ is transcendental over K, we might as well replace it with the indeterminate z to get

$$f(x,z)^k = \mathbf{res}_y \left(g(x,y), h(y,z)\right) . \tag{3.1}$$

The decomposition is monic and irreducible by definition.

Now we show that every monic irreducible decomposition of α arises in this way. Suppose we have such a decomposition (3.1). Let β be a common root of $g(\alpha, y)$ and $h(y, \gamma)$. Such a β exists, since $f(\alpha, \gamma)$ vanishes, hence so does the resultant $\mathbf{res}_y \left(g(\alpha, y), h(y, \gamma)\right)$. Then β is algebraic over $K(\gamma)$ with minimum polynomial $h(y, \gamma)$; $g(x, \beta)$ is the minimum polynomial of α over $K(\beta)$; and

$$f(x,\gamma)^k = \mathbf{res}_y \left(g(x,y), h(y,\gamma)\right) = \prod_{h(\eta,\gamma)=0} g(x,\eta) .$$

Since $g(x, \beta)$ is one of the factors in the product, it divides $f(x, \gamma)$. □

At this juncture we make a few observations about minimal decompositions and uniqueness.

3.1. MINIMAL DECOMPOSITIONS

There may exist β of arbitrarily high degree over $K(\gamma)$ satisfying Condition 3.1. For example, for any k, $\beta = \sqrt[k]{\gamma}$ gives the decomposition

$$(x-z)^k = \mathbf{res}_y \left(x - y^k, y^k - z\right) .$$

This is also the situation with (2.5) above. However, we can bound the search for a suitable β as follows. Observe that if there exists a β satisfying Condition 3.1 with factor $g(x, \beta)$ of f, say with roots $B \subseteq A$, then α will have the same degree over any subfield of $K(\beta)$ containing the coefficients of g. Furthermore, any such subfield is again a purely transcendental extension of K by Lüroth's Theorem (see van der Waerden (1970b) and Zippel (1993)), so a transcendental generator of that subfield would give a decomposition with the same g and smaller degree h and smaller k. For a given g, the degree of h and exponent k are minimized by taking the smallest subfield containing the coefficients of g, namely $\mathbf{Sym}\, B$.

3.2. NON-TRIVIAL DECOMPOSITIONS

If the minimum polynomial $g(x, \beta)$ of α over $K(\beta)$ is f (as would occur in the case $\beta = \gamma$), then the minimal decomposition with this g occurs when β is a transcendental

generator of $\mathbf{Sym}\, A$. Since $\mathbf{Sym}\, A \subseteq K(\gamma)$, β would be a rational function of γ, and h would be linear of the form $y - u(\gamma)$, $u \in K(z)$, giving the decomposition

$$f(x,z) = \mathbf{res}_y\,(g(x,y), y - u(z)) = g(x, u(z))\,.$$

In this case α is the composition of an algebraic function and a rational function.

In case $g(x,\beta)$ is linear, say $g = x - v(\beta)$, the smallest field containing the coefficients of g is $K(v(\beta))$, so by using $v(\beta)$ instead of β we would obtain the trivial decomposition

$$f(x,z) = \mathbf{res}_y\,(x - y, h(y,z)) = h(x,z)\,.$$

To find a non-trivial decomposition, we must find a β such that $K(\beta)$ does not contain α.

3.3. uniqueness up to linear composition factors

The decomposition determined by β essentially depends only on the field $K(\beta)$, not on the choice of transcendental generator β. Any other transcendental generator of $K(\beta)$ is related to β by a non-singular fractional linear transformation

$$\beta \mapsto \frac{a\beta + b}{c\beta + d}\,, \quad ad - bc \neq 0\,,$$

which extends to an automorphism of $K(\beta)$. Any two decompositions defined with respect to two transcendental generators of the same field are equivalent up to invertible composition factors of the form $(cz + d)y - (az + b)$.

4. An Algorithm

As determined in the previous section, up to fractional linear transformations there are only finitely many minimal irreducible monic decompositions of f, at most one for each factor g of f. We have thus reduced the decomposition problem to the problem of finding a subset $B \subseteq A$ (the roots of g) such that the field $\mathbf{Sym}\, B$ (the field generated by the coefficients of g) is a purely transcendental extension of K, and then finding a transcendental generator β of $\mathbf{Sym}\, B$. Such a β is automatically algebraic over $K(\gamma)$, since $\mathbf{Sym}\, B \subseteq K(A)$, the splitting field of f over $K(\gamma)$.

We must first determine whether f has a factor g whose coefficients lie in a purely transcendental extension of K. Equivalently, we want to know when the field $\mathbf{Sym}\, B$ of symmetric functions in the roots B of g is isomorphic to a rational function field over K. This is true if and only if $\mathbf{Sym}\, B$ is of genus zero. Thus the problem reduces to the problem of determining the genus of an algebraic function field.

The following is a synopsis of our algorithm.

Algorithm 4.1.

1. Construct a splitting field of f and factor f over it. This can be done by repeatedly adjoining roots and factoring. Over $\mathbb{Q}$, the algorithm of Landau (1985) or Lenstra (1983) can be used here. Over finite fields, the computation is even simpler, since every extension is normal.
2. Let g be a non-trivial factor of f obtained by taking the product of some subset of the linear factors of f obtained in step 1. Then g can be written

$$g(x) = x^m + u_{m-1}x^{m-1} + \cdots + u_0\,,$$

where the u_i lie in some finite extension of $K(\gamma)$ that is a subfield of the splitting field. For each such g, perform steps 3 and 4.

3. The field $K(u_0, \ldots, u_{m-1})$ is the field $\mathbf{Sym}\, B$, where B is the set of roots of g. Pick one of the coefficients of g not in K, say u_0. We have two cases:
   (a) If $K(u_0, \ldots, u_{m-1}) = K(u_0)$, we are done: u_0 is a transcendental generator of $\mathbf{Sym}\, B$. This can be determined by asking whether $u_i \in K(u_0)$, $1 \le i \le m-1$. Membership in an algebraic extension can be tested by solving a linear system.
   (b) If $K(u_0, \ldots, u_{m-1}) \neq K(u_0)$, construct a primitive element θ of the extension such that $K(u_0, \ldots, u_{m-1}) = K(u_0, \theta)$. This can be done using Lang (1984), p. 290. Compute the genus of $K(u_0, \theta)$ by the Hurwitz genus formula or in some other fashion. An efficient algorithm is given in Kozen (1994). If the genus is non-zero, then no decomposition arises from this factor of f. If the genus is zero, compute a rational generator β of $K(u_0, \theta)$. Coates (1970), Trager (1984), Huang and Ierardi (1991), and Sendra and Winkler (1991) give efficient algorithms for computing rational generators. The coefficients of g can then be written as rational functions of β.
4. Let $h(y, \gamma)$ be the minimum polynomial of β over $K(\gamma)$. Return $g(x, y)$ and $h(y, z)$ as the decomposition factors.

Under suitable assumptions about the complexity of operations in K, the complexity of the algorithm as given above is exponential in the worst case, since there are exponentially many potential factors. For each such factor, the computation for that factor can be performed in polynomial time in the size of the representation of the algebraic numbers needed to express the result, or exponential time in the bit complexity model Huang and Ierardi (1991). We have not been too careful about the analysis here, because we are not optimistic about the practicality of the algorithm.

5. An Example

The following gives an example of a decomposition involving a β such that $g(x, \beta)$ divides $f(x, \gamma)$, but $f(x, \gamma)$ does not factor over $K(\beta)$. Consider the polynomial

$$f(x, z) = x^4 - zx^2(x+1) + z^3(x+1)^2 .$$

Let γ be transcendental over K, and let

$$\beta = \frac{\gamma(1+\sqrt{1-4\gamma})}{2} \qquad \eta = \frac{\gamma(1-\sqrt{1-4\gamma})}{2}$$
$$g(x, y) = x^2 - y(x+1) \qquad h(y, z) = y^2 - zy + z^3 .$$

Then β and η are conjugates over $K(\gamma)$ with minimum polynomial $h(y, \gamma)$, and

$$f(x, \gamma) = g(x, \beta) \cdot g(x, \eta) ,$$

thus Theorem 3.2 says that g and h should give a decomposition of f. Indeed,

$$\mathbf{res}_y\,(g(x, y), h(y, z)) = \begin{vmatrix} -(x+1) & 0 & 1 \\ x^2 & -(x+1) & -z \\ 0 & x^2 & z^3 \end{vmatrix} = f(x, z) .$$

To show $f(x, \gamma)$ does not factor over $K(\beta)$, it suffices to show that its trace γ is not in $K(\beta)$. But γ is a root of the irreducible polynomial $h(\beta, z)$, therefore is algebraic of degree three over $K(\beta)$.

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