MATHEMATICS

STAR EXTENSION OF PLANE CONVEX SETS

BY

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0. *Introduction. Notations*

Let R denote the euclidean plane. Its elements $a, b, p, q, x, y, \ldots$ will be called *points*. Subsets of R are written in capitals $K, S, T, \ldots$.

We introduce the following notations, S and T being arbitrary subsets of R:

$\bar{S}$: the topological closure of S,

S° : the set of all interior points of S,

S' : the set of all points $p \notin S$ (the complement of S),

$S \backslash T$: the set of all points $p \in S,\ p \notin T$; $S \backslash T = S \cap T'$,

$\eth S$: the boundary of S; $\eth S = \bar{S} \cap \bar{S}' = \bar{S} \backslash S^\circ$,

$[S]$: the convex hull of S,

$[S\,T]$: the convex hull of $S \cup T$.

If p and q are different points we define

$[p\,q]$: the closed line segment with end points p and q,

$(p\,q)$: the open line segment joining p and q,

$\{p\,q\}$: the straight line containing p and q.

We shall use combinations of these bracket symbols in the following sense:

$[p\,q\}$: the closed straight half line, containing q, which has end point p,

$p(q\}$: the open straight half line which is obtained by producing $[p\,q]$, in other words: the set of all points r, such that $q \in (p\,r)$.

Other combinations will require no explanation.

Straight lines in R will generally be denoted by Greek symbols $\lambda, \mu, \ldots$.

If $P(x)$ is a property, which holds for some points x of R, then the expression $\{x \in R \mid P(x)\}$ stands for "the set of all points $x \in R$ which have the property $P(x)$".

A non-empty subset S of R is said to be *star-shaped* with respect to p, if for all $q \in S$ the segment $[p\,q]$ is a subset of S.

The set K_S of all points $p \in S$, such that S is star-shaped with respect to p, is called the *kernel* of S. In formal notation we have

Definition 0.1. $K_S = \{p \in S \mid V_{q \in S}\, [p\,q] \subset S\}$.

H. BRUNN proved a theorem which may be modified as follows:

Theorem 0.1. K_S is either empty or a convex subset of S.

The original theorem of BRUNN occurs in Math. Ann. Vol. 73 (1913), pages 436–440. Another proof can be found in H. HADWIGER und H. DEBRUNNER, Kombinatorische Geometrie in der Ebene (Genève, 1959), page 30. Cf. G. PÓLYA und G. SZEGÖ, Aufgaben und Lehrsätze aus der Analysis I, Aufgabe 111.

In this paper we shall discuss a question which is strongly related to theorem 0.1., and was suggested by L. FEJES TÓTH (Budapest, 1961, private communication), viz.

Problem: Given a convex set K in R, does there exist a set $S \supset K$, such that $K_S = K$?

Remark 0.1. The trivial solution $S = K$ will be left out of consideration.

Remark 0.2. If a non-trivial solution of our problem exists, there will generally be more than one.

Definition 0.2. A set $S \neq K$ such that $K = K_S$ will be called a *K-star*. If, moreover, $\bar{S} \neq \bar{K}$, then S is named a *proper K-star*.

In section 1 we shall consider star extensions of subsets of a straight line, and some related cases.

In section 2 some lemmas will be formulated in order to prove a main theorem in section 3, viz.

Theorem: A strictly convex set K with a regular boundary admits of a proper K-star if and only if its boundary contains an arc whose intersection with K' is countable (theorem 3.1.).

Finally in section 4 we show that a convex closed set admits of a proper K-star, if and only if it is neither a half-plane nor a two-way infinite strip.

The author is very indebted to Prof. N. G. DE BRUIJN for his valuable contribution to the proof of theorem 3.1.

1. *Some special cases*

Let K be a convex subset of a straight line. Then we prove the following theorem:

Theorem 1.1. If K is a convex subset of a straight line λ in R, then a proper K-star exists if and only if $K \neq \lambda$.

Proof: If $K \neq \lambda$ then the union S of K and all points of R which do not belong to λ is obviously a proper K-star.

On the other hand let K be equal to λ and let S be a (necessarily proper) K-star. Suppose $p \in S \backslash K$. Then all interior points q of the convex hull $[p\, K]$ belong to S. In order to get a contradiction we shall prove that these points q must be elements of K_S: Let us assume that r is an arbitrary point of S. Then, if $\{r\, q\}$ and λ are parallel, $\{r\, q\}$ consists of interior points of $[p\, K]$ only, hence $[r\, q] \subset S$. If on the other hand $\{r\, q\} \cap \lambda = t$, definition 0.1. yields $[t\, r] \subset S$ and $[t\, q] \subset S$, so $[r\, q] \subset S$ because of the collinearity of t, r and q. In both cases we obtain $[r\, q] \subset S$ and therefore $q \in K_S$.

Remark 1.1. The second part of this proof also suffices to show that a convex, two-way infinite strip between two parallel lines, and that a convex half-plane does not admit of a *proper* star extension.

Remark 1.2. If K is an open half-plane, an improper K-star S can be built up from K and two of its boundary points. It is easily seen that a convex half-plane which is not open admits of no star extension at all.

Remark 1.3. If K is a two-way infinite convex strip with boundary $\partial K = \lambda \cup \mu$ then K admits of a non-proper star extension if and only if $\lambda \cap K = \phi$ or $\mu \cap K = \phi$.

Because of the foregoing arguments we may assume for the following sections that K contains interior points and that K is neither a half-plane nor a strip. Such a set K has a connected boundary ∂K, which is not a straight line.

2. *Supporting lines. Restricting lines of support. Restricting half-planes*

Let K be a convex set with a connected boundary ∂K, which is not a straight line. This means that we exclude half-planes and strips.

Definition 2.1. A *line of support* or a *supporting line* for K is a straight line λ, such that $\lambda \cap \overline{K} \neq \phi$ and $\lambda \cap K^\circ = \phi$.

Remark 2.1. The intersection of a supporting line λ with ∂K is a closed interval or a closed half-line of λ. The intersection of λ with K itself is empty or a convex subset of this interval.

Definition 2.2. A point $p \in \partial K$ is called *regular* if there exists exactly one line of support λ containing p. Otherwise p is defined to be a *singular* point.

Remark 2.2. The set of all singular points of ∂K is at most countable (cf. T. Bonnesen und W. Fenchel, Theorie der konvexen Körper, New York, 1948, page 15).

It will be useful to apply the following lemma:

Lemma 2.1. Suppose S is a K-star and p is a regular point of ∂K, such that $p \notin K$, $p \in S$. Then the supporting line λ_p through p contains a segment of ∂S.

Proof: Since $p \in S \backslash K$, K being the kernel K_S of S, there exists a point $q \in S$ such that $[p\, q] \not\subset S$. Because of the regularity of p we conclude that $q \in \lambda_p$ (if this were false then either $q(p\}\, K \neq \phi$ and hence $[p\, q] \subset S$, or $[p\,q] \cap K \neq \phi$ so that $[p\,q] \subset S$). As $[p\,q] \not\subset S$ there must be a third point $r \in (p\,q)$ such that $r \notin S$, and we shall show that $[q\,r] \subset \eth S$:

Let $t \in (q\,r)$ and Ω_t be an arbitrary neighbourhood of t. On the one hand Ω_t is the image of a neighbourhood Ω_p of p under the similarity transformation with centre q, which maps p onto t. As Ω_p contains points of $K = K_S$ and $q \in S$ we see that $\Omega_t \cap S \neq \phi$. On the other hand Ω_t may be considered the image of a neighbourhood $\Omega_p{}^*$ of p under the similarity mapping with centre r, which transforms p into t. Since $\Omega_p{}^* \cap K_S \neq \phi$ and $r \notin S$ we find $\Omega_t \cap S' \neq \phi$. Combining these results we conclude that $t \in \eth S$. This holds for all $t \in (q\,r)$, hence $(q\,r) \subset \eth S$, and therefore $[q\,r] \subset \eth S$, which proves our lemma.

Definition 2.3. A supporting line λ of K is defined to be *restricting* if it contains a point $p \in K$ and a regular point $q \in \eth K$ such that $q \notin K$. If λ is a restricting line of support then the closed half-plane $\bar{H}_\lambda$ containing K and having λ as its boundary is likewise called a *restricting half-plane.*

Lemma 2.2. If S is a K-star and λ is a restricting line of support, then S is a subset of the restricting half-plane $\bar{H}_\lambda$.

Proof: Assume that the assertion is false. Then from definition 2.3. it follows that there exists a point $p \in \lambda \cap K$ and a regular point $q \in \lambda \cap \eth K$, $q \notin K$. Moreover, there is a point $r \in S$, $r \notin \bar{H}_\lambda$. The regularity of q implies that $r(q\} \cap K \neq \phi$, whence we conclude that $q \in S$. On the other hand q is regular and $q \notin K$, so that λ must contain a point $t \in S$ which satisfies $[q\,t] \not\subset S$ (cf. the proof of lemma 2.1.). As $p \in \lambda \cap K$ we see that $[p\,q] \subset S$ and $[p\,t] \subset S$ and hence $[q\,t] \subset S$ because of the collinearity of p, q and t. This contradiction proves lemma 2.2.

Corollary: A K-star S is a subset of the intersection of all restricting half-planes $\bar{H}_\lambda$ of K.

Lemma 2.3. Suppose S is a K-star and λ is a line of support for K, containing three regular points p, q and r of $\eth K$ such that $p \in (q\,r)$, and $p \in K$, $q \notin K$, $r \notin K$. Then $S \cap \lambda = K \cap \lambda$.

Proof. Assume that there exists a point $t \in S \cap \lambda$, $t \notin K$. Now $[t\,p] \subset S$ and it is easily verified that $[t\,p]$ contains a regular point $u \notin K$, since $q \notin K$ and $r \notin K$. Hence we have another point $v \in \lambda \cap S$, such that $[u\,v] \not\subset S$. This, however, contradicts the fact that p, u and v are collinear and $p \in K$, $u \in S$, $v \in S$, which completes the proof.

Corollary to lemma 2.3. If K is the union of a convex open polygon $[p_1\,p_2\,\dots\,p_n]^\circ$ and n points $q_1 \in (p_1\,p_2)$, $q_2 \in (p_2\,p_3)$, $\dots$, $q_n \in (p_n p_1)$, then there exists no K-star.

3. *Star extension of a strictly convex set K with regular boundary*

Definition 3.1. A convex set K is called *strictly convex* if each line of support has exactly one point in common with $\eth K$.

Lemma 3.1. A star extension of a strictly convex set K is a proper K-star.

Proof: Since every set S satisfying $K \subset S \subset \bar{K}$ is convex an improper K-star S would be convex itself, hence $K_S = S \neq K$.

If K is strictly convex and ∂K has regular points only, the complete answer to our problem can be formulated as follows:

Theorem 3.1. For the existence of a star extension of a strictly convex set K with regular boundary it is necessary and sufficient that ∂K contains an arc A such that the set $A \backslash K$ is at most countable.

Proof: (Necessity)

Let S be a K-star. Then by lemma 3.1. S is a proper K-star. So we have a point $t \in S \backslash \bar{K}$. Consider the arc B defined by $B = \partial K \cap [t\, K]^\circ$. We shall show that the set $C = B \backslash K$ is countable.

Obviously C is a subset of S. For each point $p \in C$ lemma 2.1. applies, so that its supporting line λ_p contains a segment $[q_p\, r_p]$ of ∂S.

If x is a fixed point, chosen within K° we define the *shadow* W_p of $[q_p\, r_p]$ as follows:

$$W_p = \{y \in R \mid (x\, y) \cap (q_p\, r_p) \neq \phi\}.$$

Then W_p is an open set. We shall prove that the collection of all W_p ($p \in C$, x fixed) forms an at most double covering of R: A common point y of two shadows W_{p_1} and W_{p_2} gives rise to a point $s_1 \in (q_{p_1}\, r_{p_1}) \subset \partial S$ and a point $s_2 \in (q_{p_2}\, r_{p_2}) \subset \partial S$ such that $s_1 \in (x\, s_2\}$. If s_1 and s_2 were distinct (for example $s_2 \in (x\, s_1)$) then a suitable similarity mapping with a centre c sufficiently close to s_1 ($c \in S$) would transform a neighbourhood $\Omega_x \subset K$ of x into a neighbourhood of s_2 so that by definition 0.1. $s_2 \notin \partial S$. Hence $s_1 = s_2$. However, there are at most two points p of K, whose supporting line λ_p passes through s_1. Therefore, the collection of all W_p forms an at most double covering of R. It is a well-known fact that any collection of disjoint open subsets of R is at most countable. The same holds for an at most double covering of R. (In fact one can even show that all W_p are disjoint). Thus C is countable.

For proving the sufficiency condition in theorem 3.1. we shall use a coordinate system and an auxiliary function as described below sub (a) and (b).

(a) *Introduction of coordinates*

Let K be strictly convex and ∂K be regular. To every point p of ∂K corresponds exactly one line of support λ_p which can be oriented in such a sense that K lies left of λ_p. To every oriented line of support an argument is given with respect to a fixed direction (this argument is determined mod 2π). Every point t obtained by the intersection of two oriented supporting lines can be described by means of coordinates (φ_t, ψ_t), where

(φ_t, ψ_t) is the ordered pair of arguments of these lines, such that $0<\psi_t-\varphi_t<\pi$, when reduced mod 2π.

If t is a point of intersection of the respective oriented supporting lines λ_t and μ_t, which subtend an angle $\alpha<\pi$, we take all arguments with respect to the direction of λ_t. Then the coordinates of t will be $(0, \alpha)$ and the interior points u of $[t\ K]\backslash\bar{K}$ will have coordinates (φ_u, ψ_u) satisfying $0<\varphi_u<\psi_u<\alpha$. Obviously the mapping $P : u \rightarrow (\varphi_u, \psi_u)$ is one-to-one. It is even a topological mapping. For what follows it is useful to notice some other properties of P, viz.

(a1) Let $P(u)=(\varphi_u, \psi_u)$ be the image of u. The points v which are mapped into the interior of one of the regions A_i have the property that $\{u\ v\}$ separates t and $\bar{K}$. If v is mapped into the interior of one of the regions B_j then $\{u\ v\} \cap K^\circ \neq \phi$. The common boundaries of A_i and B_j correspond with the supporting lines through u (cf. fig. 1.).

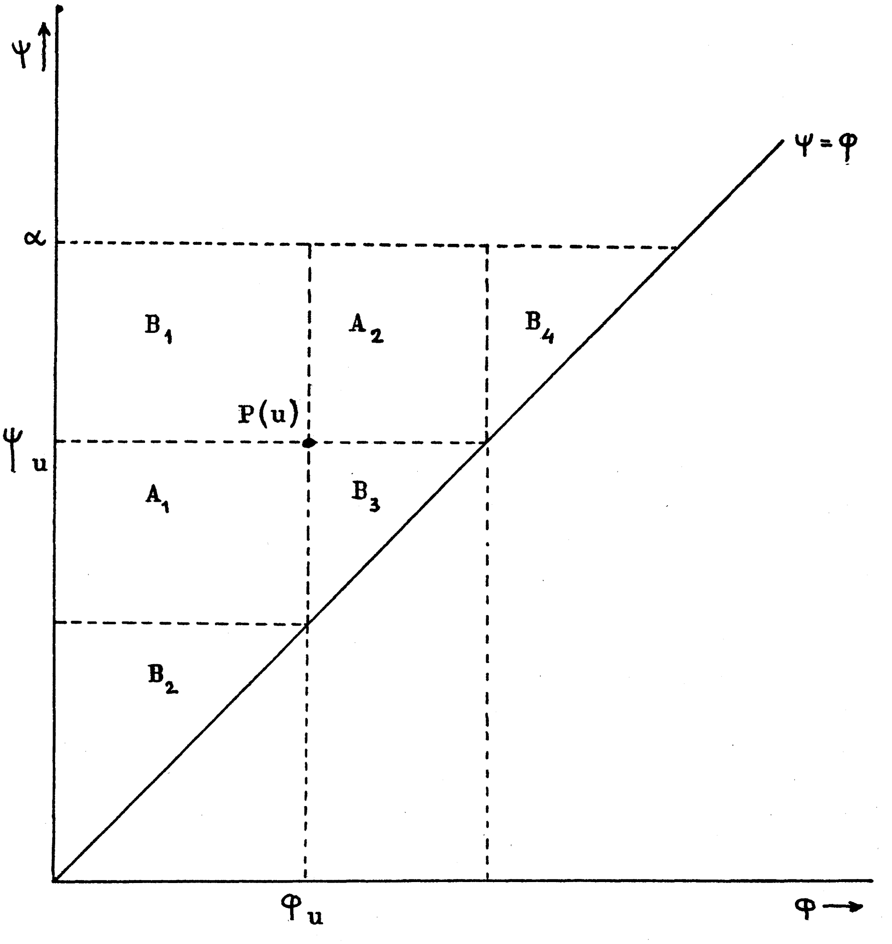

Fig. 1.

(a2) The interior of B_3 is the image of the interior points v of $[u\ K]\backslash\bar{K}$. Analytically we have in this case

$$\varphi_u < \varphi_v < \psi_v < \psi_u.$$

(a3) If the straight line v separates t and $\bar{K}$ its P-image in the (φ, ψ)-plane can be considered to be the graphical representation of a continuous, increasing, strictly monotonic function $\psi = \psi(\varphi)$.

(b) *Construction of a suitable real-valued auxiliary function*

Let there be given an infinite series of *positive* terms $a_1, a_2, \ldots$ such that $\sum_{k=1}^{\infty} a_k = 1$, and a countable, everywhere dense subset $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ of the real interval $(0, \alpha)$.

We define the functions f_k as follows:

$$f_k(\varphi) = \begin{cases} \varphi & (0 < \varphi < \gamma_k) \\ \alpha & (\gamma_k \leqslant \varphi < \alpha) \end{cases} \qquad (k = 1, 2, 3, \ldots).$$

Now let F be the function defined by the following linear expression in $f_1, f_2, \ldots$:

$$F(\varphi) = \sum_{k=1}^{\infty} a_k\, f_k(\varphi) \qquad (0 < \varphi < \alpha).$$

Then F has some remarkable properties, viz.

(b1) $\varphi < F(\varphi) < \alpha$ for all φ, $0 < \varphi < \alpha$.

(b2) F is a monotonic, strictly increasing function.

(b3) $\lim_{\varphi \downarrow \beta} F(\varphi) = F(\beta)$ for all β, $0 < \beta < \alpha$.

(b4) $\lim_{\varphi \uparrow \beta} F(\varphi) = F(\beta)$ for all $\beta \notin \Gamma$, $0 < \beta < \alpha$.

(b5) $\lim_{\varphi \uparrow \gamma_k} F(\varphi) = F(\gamma_k) - a_k(\alpha - \gamma_k) < F(\gamma_k) \qquad (k = 1, 2, 3, \ldots)$.

Proof of theorem 3.1. (Sufficiency)

Assume that ∂K contains an arc A such that $B = A\backslash K$ is at most countable. Without loss of generality we may suppose that A is so small that every pair of supporting lines in points of A intersect. In particular, let the supporting lines λ and μ in the end points of A intersect in the point t. Let the oriented lines λ and μ subtend an angle α. According to the coordinatisation defined sub (a) such that $P(t) = (0, \alpha)$ we see that to each point $p \in B = A\backslash K$ there corresponds exactly one oriented supporting line, and hence exactly one argument β $(0 < \beta < \alpha)$. Since B is countable the same holds for the collection of arguments β formed in this manner. We extend this collection to a countable, everywhere dense subset $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ of $(0, \alpha)$. Now by the construction sub (b) we introduce the auxiliary function F and consider its graphical representation $\tilde{F}$ in the (φ, ψ)-plane. Its inverse image according to the mapping P is a subset of R, that will be denoted by $P^{-1}(\tilde{F})$. Finally, for each point $r \in P^{-1}(\tilde{F})$ we take the convex hull $[r\ K]$ and we shall show that the set S, defined by

$$S = \cup_{r \in P^{-1}(\tilde{F})} [r\, K],$$

is a proper K-star.

Suppose that K_S is the kernel of S. It follows from (b1) that $S \neq K$. Since $[r\, K]$ is convex and K is a convex subset of $[r\, K]$ for all r we have $K \subset K_S$. So it remains to be proved that $K_S \backslash K = \phi$. If $K_S \backslash K$ contains a point u we must distinguish between two cases:

(1) u is an interior point of $[t\, K] \backslash \bar{K}$;

(2) $u \in B = A \backslash K$.

(1) Let u be an interior point of $[t\, K] \backslash \bar{K}$. Since $u \in S$ we conclude $u \in [r\, K]$ for some $r \in P^{-1}(\tilde{F})$ and that its image $P(u) = (\varphi_u, \psi_u)$ in the (φ, ψ)-plane satisfies

$$\varphi_r \leqslant \varphi_u < \psi_u \leqslant \psi_r = F(\varphi_r) \leqslant F(\varphi_u).$$

As Γ is everywhere dense in $(0, \alpha)$ there exists a number γ_k, such that $\varphi_u < \gamma_k < \psi_u$. Now we consider the point $w = P^{-1}((\gamma_k, F(\gamma_k))) \in S$. We observe the following inequality (cf. (b2)):

$$\varphi_u < \gamma_k < \psi_u \leqslant F(\varphi_u) < F(\gamma_k),$$

so that $\{u\, w\}$ separates t and $\bar{K}$ (cf. (a1)). In view of (a3) the P-image of $[u\, w]$ is the graphical representation of a continuous, monotonically increasing function $\psi = \psi(\varphi)$ on the interval $[\varphi_u, \gamma_k]$. Since $\psi(\gamma_k) = F(\gamma_k)$ we conclude by (b5) that there exists a point φ_0 of this interval with the property $\psi(\varphi_0) > F(\varphi_0)$. This means $[u\, w] \not\subset S$, in contradiction with our assumption that $u \in K_S \backslash K$.

(2) If u is a point of $B = A \backslash K$ we consider the corresponding argument γ_k of its supporting line λ. Let w be the point defined by $w = P^{-1}((\gamma_k, F(\gamma_k)))$, then $w \in S$ and

$$(u\, w) \cap S = \cup_{r \in P^{-1}(\tilde{F})} ((u\, w) \cap [r\, K]).$$

Therefore the P-image of $(u\, w) \cap S$ consists of all points (φ, ψ), where $\varphi = \gamma_k$, and $\gamma_k < \psi < \sup_{\tau < \gamma_k} F(\tau)$, which is less than $F(\gamma_k)$ (cf. (a2), (b5)). Hence $(u\, w) \notin S$ and so $[u\, w] \notin S$. This contradicts our hypothesis $u \in K_S$, and completes the proof of theorem 3.1.

Corollary to theorem 3.1.

A strictly convex closed set K with regular boundary admits of a proper K-star.

A strictly convex open set K with regular boundary admits of no K-star at all.

4. *The general case. Star extension of closed sets*

Though it is not difficult in any special case to determine whether a given convex set K admits of a (proper) K-star or not, the formulation of a general theorem seems to be rather complicated.

Obviously the proof of the existence of a K-star, sketched in section 3, also applies if K is a convex set, without being strictly convex, such that δK contains a strictly convex arc A with the property that $A \backslash K$ is countable.

The following example shows a difficulty which we have to consider when we formulate a general theorem concerning proper K-stars:

Example: We consider an open triangle $T = [a\, b\, c]^\circ$ and three points $p \in (a\, b)$, $q \in (b\, c)$, $r \in (a\, c)$. Now we define $K_1 = T \cup p \cup q \cup r$; $K_2 = T \cup p \cup q$; $K_3 = T \cup p$.

From the corollary to lemma 2.3. it follows that K_1 has no star extension at all. K_2 has an improper star extension, obtained by taking the union of K_2 and two distinct points of $(a\, c)$. On the other hand K_2 admits of no proper extension. This follows from lemma 2.1. and the corollary to lemma 2.2. Dealing with K_3, however, we see that K_3 has a proper star extension, viz. the union of K_3, $(a\, c)$, $(b\, c)$, and the sets S_1 and S_2, defined by $S_1 = \{x \in R \mid (x\, a) \cap (b\, c] \neq \phi\}$, $S_2 = \{y \in R \mid (y\, b) \cap (a\, c] \neq \phi\}$. Apparently the fact that for K_2 the lines $\{a\, b\}$ and $\{a\, c\}$ *both* satisfy the conditions of lemma 2.3. implies that the points of the intermediate open segment $(b\, c)$ cannot be interior points of a star extension. If $\{a\, b\}$ *only* satisfies the conditions of lemma 2.3. the points of $(b\, c)$ can actually be interior points of a K-star (for example, see K_3).

Finally we shall discuss our problem if K is a closed convex set. Then we are able to prove

Theorem 4.1. A closed convex set K has a star extension if and only if it is neither a half-plane nor a two-way infinite strip.

Sketch of proof: From remark 1.1. it follows that a closed half-plane and a closed strip have no star extension.

If on the other hand K is not a half-plane or a strip, there exist two different supporting lines λ and μ intersecting in a point p.

If $p \notin K$ then the open set $[p\, K]^\circ \backslash K$ can be coordinatised by oriented supporting lines analogously to the process sketched in section 3, and we can show the existence of a K-star in the same way.

If, on the contrary, $p \in K$ (which means that K is an angular region with vertex p) then the union of p and all points $q \neq p$, for which $q(p\} \cap K = \phi$, appears to be a K-star.

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