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Pseudoconvex Domains in Almost Complex Abstract Wiener Spaces

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The $\bar{\partial}$-operator on an almost complex abstract Wiener space (B, H, μ, J) is defined by making use of the Malliavin calculus. The authors then study pseudoconvex domains in B, domains where the $\bar{\partial}$-equations $\bar{\partial}u = f$ are solvable. As an application, they establish an approximation theorem of holomorphic forms and a Dolbeault type theorem. Examples of such domains, obtained through SDE, one also discussed.

0. INTRODUCTION

Let (B, H, μ, J) be an almost complex abstract Wiener space. The $\bar{\partial}$-operator is defined as the composition of the projection onto the space of (p, q)-forms and the exterior derivative, which is obtained by anti-symmetrizing the gradient operator D appearing in the Malliavin calculus. For details, see Section 1. In this paper, we study pseudoconvex domains in B, domains where the $\bar{\partial}$-equation $\bar{\partial}u = f$ possesses solutions.

To observe the solvability of $\bar{\partial}$-equations on a domain Ω in B, we extend $\bar{\partial}$ to a closed operator $T_{\sigma,\gamma}^{(p,q)}: L^2(\Omega; \bigwedge^{p,q}, e^{-\sigma}\, d\mu) \to L^2(\Omega; \bigwedge^{p,q}, e^{-\sigma-\gamma}\, d\mu)$, where σ, γ are suitable weight functions and $\bigwedge^{p,q}$ is a subspace of $\bigwedge^{p+q}(H^* \otimes \mathbf{C})$ consisting of p-complex and q-conjugate complex linear forms (Section 3). As in [2], the power $-\gamma$ is used to dominate the divergence of the derivative of an exhaustion function on Ω near the boundary and to avoid taking into account the boundary conditions. We

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show that $\bar{\partial}$-equations possess solutions in the sense that $\mathrm{Ker}(T_{\sigma,\gamma}^{(p,q+1)}) \cap L^2(\Omega; \bigwedge^{p,q+1}, e^{-\sigma} d\mu) \subset \mathrm{Image}(T_{\sigma,\gamma}^{(p,q)})$ provided that σ and γ enjoy the inequality

$$\varepsilon I + \partial\bar{\partial}(\sigma + 2\gamma) \geq \frac{1}{\varepsilon} \partial\gamma \otimes \bar{\partial}\gamma \tag{0.1}$$

(Theorem 3.2). To establish the existence theorem, we employ a functional analytic method similar to that used by Hörmander in [2]. Great pains are taken to establish the L^2-estimate on $T_{\sigma,\gamma}^{(p,q)}$ and its adjoint, which is stated in Lemma 3.9. The assumed inequality (0.1), which in a form with no εI-term has already appeared implicitly in [2], plays a key role in the estimation. The Gaussian measure, μ, we are dealing with a priori possesses a strictly plurisubharmonic function "$|z|^2$" as its exponent and it is this "$|z|^2$" that brings us the εI.

The solvability of $\bar{\partial}$-equations is taken advantage of to verify a Dolbeault-type theorem in Section 6. To be more precise, let E_0 be a complex Hilbert space and $F: \Omega \to E_0$ be a nice holomorphic mapping. For a family $\{U_i\}_{i \in \mathbf{I}}$ of open pseudoconvex sets in E_0 and $J \subset \mathbf{I}$ with $\# J < \infty$, another class of smooth (p, q)-forms on $F^{-1}(\bigcap_{i \in J} U_i)$, say $\mathscr{D}_{\bigcap_{i \in J} U_i}(\bigwedge^{p,q})$, is introduced and the $\bar{\partial}$-operator is extended to the class. We set $O_J^p = \{u \in \mathscr{D}_{\bigcap_{i \in J} U_i}(\bigwedge^{p,0}) : \bar{\partial} u = 0\}$. Then a quasi-sheaf $\mathcal{O}^p$ of holomorphic $(p, 0)$-forms is constructed algebraically from $\{O_J^p : J \subset \mathbf{I},\ \# J < \infty\}$ (Sections 5 and 6). Putting $\tilde{U} = \bigcup_{i \in \mathbf{I}} U_i$, we define

$$\bigwedge_{\tilde{U}}^{p,q} = \{f : F^{-1}(\tilde{U}) \to \bigwedge^{p,q} : f|_{F^{-1}(U_i)} \in \mathscr{D}_{U_i}(\bigwedge^{p,q}),\ i \in \mathbf{I}\}$$

$$(\bar{\partial}_{\bigwedge_{\tilde{U}}^{p,q}} f)|_{F^{-1}(U_i)} = \bar{\partial}(f|_{F^{-1}(U_i)}).$$

The following Dolbeault type theorem is verified,

$$H^q(\mathcal{O}^p) \simeq \mathrm{Ker}(\bar{\partial}_{\bigwedge_{\tilde{U}}^{p,q}}) / \mathrm{Image}(\bar{\partial}_{\bigwedge_{\tilde{U}}^{p,q-1}}),$$

where $H^q(\mathcal{O}^p)$ is the qth cohomology group with coefficients in $\mathcal{O}^p$ with respect to $\{U_i\}_{i \in \mathbf{I}}$ (Theorem 6.10).

As examples where the above assumptions are satisfied, we consider a domain obtained by using stochastic differential equations (SDE in abbreviation). Let $A_0, ..., A_{d'}: \mathbf{C}^{d'} \to \mathbf{C}^{d'}$ be holomorphic and $z(t, z)$ be the solution to the SDE:

$$dz(t) = \sum_{k=1}^{d'} A_k(z(t)) \circ dB^k(t) + A_0(z(t))\, dt,$$

$$z(0) = z,$$

where $(B^1(t), ..., B^{d'}(t))$ is a $\mathbf{C}^{d'}$-valued Brownian motion. The solution may explode. Choose $\alpha \in (0, \frac{1}{2})$, $m \in \mathbf{N}$ to satisfy that $2m\alpha > 1$ and $\alpha + (\frac{1}{2m}) < \frac{1}{2}$, and put

$$\phi_0 = \int_0^1 \int_0^1 \frac{|z(t, 0) - z(s, 0)|^{2m}}{|t-s|^{1+2m\alpha}} \, dt \, ds$$

$$\Omega = \{\phi_0 < \infty\}.$$

Then obviously ϕ_0 is an exhaustion function on Ω. To control the derivative $D\phi_0$, we consider $Y(t) = \partial z(t, z)/\partial z|_{z=0}$ and define

$$\phi_1 = \phi_0 + \int_0^1 \int_0^1 \frac{|Y(t) - Y(s)|^{2m}}{|t-s|^{1+2m\alpha}} \, dt \, ds + \int_0^1 \int_0^1 \frac{|Y(t)^{-1} - Y(s)^{-1}|^{2m}}{|t-s|^{1+2m\alpha}} \, dt \, ds.$$

We show that there are non-decreasing and convex $\chi, \tilde{\chi} \in C^\infty(\mathbf{R}; \mathbf{R})$ such that $\gamma = \chi(\phi_1)$ and $\sigma_0 = \tilde{\chi}(\phi_1)$ possess the desired properties (Section 7). The involvement of $Y(t)$ is due to that $Dz(t)$ is represented in terms of $Y(\cdot)$. It is also seen that $\Omega \equiv \{\phi_0 < r\}$ have σ_0 and γ possessing the desired properties. Moreover, regarding $z(\cdot)$ as a mapping of Ω to $L^2([0, 1]; \mathbf{C}^{d'})$, we see that $z(\cdot)$ is a nice holomorphic function such as that described above and hence a Dolbeault type theorem holds in this case.

Exhaustion functions and weight functions which we investigate are not smooth in the usual sense of the Malliavin calculus. For example, the above ϕ_j's are not integrable in general. In Section 1, after reviewing the Malliavin calculus briefly, we introduce some classes of differentiable functions assumed to be only either "locally integrable" or "locally bounded." In the classes our exhaustion and weight functions are contained. We study the $\bar{\partial}$-equation on the full space B in Section 2. Our aim in the section is to establish an estimation on $\bar{\partial}$ and its adjoint, which is used in Section 3 to obtain a similar estimation for $T_{\sigma,\gamma}^{(p,q)}$ and its adjoint. The solvability of $\bar{\partial}$-equations on B follows from the estimation. We remark that the L^2-$\bar{\partial}$-cohomology vanishing theorem in [8], which has been shown in a different manner from ours, also yields the solvability on B. As another application of the L^2-estimation established in Section 3, we show an approximation theorem of holomorphic functions in Section 4. In Section 5 the definition of quasi-sheaves is given and a Dolbeault type theorem is shown in Section 6. Section 7 is devoted to giving examples that take advantage of SDE.

1. Preliminaries

In this section, we give a brief review on an almost complex abstract Wiener space, differentiable forms on it, and $C_{1,p}$-capacities. Further, we define a linear operator $\bar{\partial}$ and give some new classes of differentiable forms.

A quadruple (B, H, μ, J) is called an almost complex abstract Wiener space if B is a real separable Banach space, H is a real separable Hilbert space imbedded in B continuously and densely, μ is a Borel probability measure on B such that

$$\int_B \exp[\sqrt{-1}\langle z, l\rangle]\, d\mu(z) = \exp[-\tfrac{1}{4}\|l\|_{H^*}^2] \tag{1.1}$$

for any $l \in B^*$ ($\equiv$ the dual space of B), and $J: B \to B$ is an isomorphism such that $J^2 = -1$ and $J|_H : H \to H$ is also isomorphic, where $\langle\, ,\rangle$ denotes the natural pairing of B and B^*. We denote by $H^{*\mathbf{C}}$ the complex Hilbert space of all continuous $\mathbf{R}$-linear operators of H into $\mathbf{C}$ and set

$$H^{*(1,0)} = \{\varphi \in H^{*\mathbf{C}} : J^*\varphi = \sqrt{-1}\,\varphi\}$$
$$H^{*(0,1)} = \{\varphi \in H^{*\mathbf{C}} : J^*\varphi = -\sqrt{-1}\,\varphi\},$$

where J^* is used to denote the natural extension of J^* on H^* to $H^{*\mathbf{C}}$.

Let $(H^{*\mathbf{C}})^{\otimes n}$ be the completion of the n-fold algebraic tensor $H^{*\mathbf{C}} \otimes \cdots \otimes H^{*\mathbf{C}}$ with respect to the Hilbert–Schmidt inner product,

$$\langle a, b\rangle = \sum_{i_1, \ldots, i_n = 1}^{\infty} a(h_{i_1}, \ldots, h_{i_n})\, \overline{b(h_{i_1}, \ldots, h_{i_n})},$$

$\{h_j\}$ being a CONS of H. Then $(H^{*\mathbf{C}})^{\otimes n}$ is a complex Hilbert space. The alternation operator $\mathscr{A}_n$ on $(H^{*\mathbf{C}})^{\otimes n}$ is given by

$$\mathscr{A}_n(h_1 \otimes \cdots \otimes h_n) = \frac{1}{n!}\sum_{\tau} \operatorname{sgn}(\tau)\, h_{\tau(1)} \otimes \cdots \otimes h_{\tau(n)}, \qquad h_1, \ldots, h_n \in H,$$

whee the summation is taken over all permutations τ of $\{1, \ldots, n\}$. We then define the closed subspace $\bigwedge^n H^{*\mathbf{C}}$ of $(H^{*\mathbf{C}})^{\otimes n}$ by

$$\bigwedge\nolimits^n H^{*\mathbf{C}} = \{A \in (H^{*\mathbf{C}})^{\otimes n} : \mathscr{A}_n(A) = A\},$$

and let $\bigwedge^{p,q}$ be the set of all $A \in \bigwedge^{p+q} H^{*\mathbf{C}}$ satisfying that

$$A((ah_1 + bJh_1), \ldots, (ah_n + bJh_n))$$
$$= (a + \sqrt{-1}\, b)^p\, (a - \sqrt{-1}\, b)^q\, A(h_1, \ldots, h_n)$$

for any $a, b \in \mathbf{R}$ and $h \in H$. $\bigwedge^n H^{*\mathbf{C}}$ is a complex Hilbert space equipped with an inner product $\langle a, b \rangle_{\mathrm{AHS}} = (1/n!)\langle a, b \rangle$ and $\bigwedge^{p,q}$ is a closed subspace of the complex Hilbert space $\bigwedge^{p+q} H^{*\mathbf{C}}$. We denote by $\pi_{p,q}$ the projection of $\bigwedge^{p+q} H^{*\mathbf{C}}$ onto $\bigwedge^{p,q}$ and set

$$l^{(1;0)} = 2\pi_{1,0} l = l - \sqrt{-1}\, l \qquad \text{and} \qquad l^{(0,1)} = 2\pi_{0,1} l = l + \sqrt{-1}\, l,$$

for $l \in H^{*\mathbf{C}}$.

A function $F: B \to \mathbf{C}$ is said to be smooth if it is of the form

$$F(z) = f(\langle z, l_1 \rangle, ..., \langle z, l_n \rangle),$$

where $f \in C_0^\infty(\mathbf{C}^n; \mathbf{C})$, $l_1, ..., l_n \in B^{*\mathbf{C}} (\equiv B^* \otimes \sqrt{-1}\, B^*)$. We denote the totality of smooth functions by $\mathscr{F}C_0^\infty(B; \mathbf{C})$. For a separable complex Hilbert space E, $\mathscr{F}C_0^\infty(B; E)$ consists of all linear combinations of finite number of elements of the form Fe, $F \in \mathscr{F}C_0^\infty(B; \mathbf{C})$ and $e \in E$. For $G \in \mathscr{F}C_0^\infty(B; E)$, $DG(z) \in H^{*\mathbf{C}} \otimes E$ is defined by

$$DG(z)[h] = \lim_{t \downarrow 0} \frac{1}{t} \{G(z + th) - G(z)\}.$$

Sobolev spaces $\mathbf{D}_p^r(E)$, $r \in \mathbf{R}$, $p \in (1, \infty)$, are completions of $\mathscr{F}C_0^\infty(B; E)$ with respect to the norms $\|\cdot\|_{r,p}$ defined by

$$\|G\|_{r,p} = \|(I - \mathscr{L})^{r/2} G\|_{L^p(B; E, d\mu)},$$

respectively, where $\mathscr{L}$ is the Ornstein–Uhlenbeck operator on B. We put

$$\mathbf{D}_{1+}^{-\infty}(E) = \bigcup_{r \in \mathbf{R}} \bigcup_{p \in (1,\infty)} \mathbf{D}_p^r(E)$$

$$\mathbf{D}_{\infty-}^{\infty}(E) = \bigcap_{r \in \mathbf{R}} \bigcap_{p \in (1,\infty)} \mathbf{D}_p^r(E).$$

For each $r \in \mathbf{R}$ and $p \in (1, \infty)$, it holds that

$$\sup\{\|DG\|_{r-1,p}/\|G\|_{r,p} : G \in \mathscr{F}C_0^\infty(B; E), G \neq 0\} < +\infty. \tag{1.2}$$

See [9]. Thus the operator D can be extended to a bounded operator of $\mathbf{D}_p^r(E)$ into $\mathbf{D}_p^{r-1}(H^{*\mathbf{C}} \otimes E)$, which is denoted by D again. Further, by virtue of (1.2), we can define a continuous linear operator

$$\bar{\partial} : \mathbf{D}_{1+}^{-\infty}(\bigwedge^{p,q}) \to \mathbf{D}_{1+}^{-\infty}(\bigwedge^{p,q})$$

so that

$$\bar{\partial} G = \pi_{p,q+1}((p+q+1)\mathscr{A}_{p+q+1}(DG)), \qquad G \in \mathscr{F}C_0^\infty(B; \bigwedge^{p,q}). \tag{1.3}$$

For the sake of later use, we introduce several more classes of differentiable functions. Let

$$\mathscr{K}_+ = \{K \subset B : K \text{ is compact and } \mu(K) > 0\}.$$

Take an arbitrary but fixed $K \in \mathscr{K}_+$ and define

$$\rho_K(x) = \begin{cases} \inf\{\|x-k\|_H : k \in K\} & \text{if } K \cap (x+H) \neq \varnothing \\ \infty & \text{otherwise.} \end{cases}$$

Then for any $g \in C_b^1(\mathbf{R}; \mathbf{R})$, $g(\rho_K) \in \mathbf{D}^1_{\infty-}(\mathbf{R})$ $(\equiv \bigcap_{p \in (1,\infty)} \mathbf{D}^1_p(\mathbf{R}))$ and $\|Dg(\rho_K)\|_{H^*} \leqslant \sup\{|g'(x)| : x \in \mathbf{R}\}$. See [4]. Moreover, the quasi-invariance of μ implies that

$$\mu(\rho_K < \infty) = 1. \tag{1.4}$$

For a separable Hilbert space E, we set

$$\mathbf{S}^1_K(E) = \{f : B \to E : g(\rho_K) f \in \mathbf{D}^1_{\infty-}(E) \text{ for every } g \in C_0^\infty(\mathbf{R}; \mathbf{R})\}$$

For $f \in \mathbf{S}^1_K(E)$, the derivative Df can be defined by

$$g_n(\rho_K)\, Df = g_n(\rho_K)\, D(g_{n+1}(\rho_K) f),$$

where $g_n \in C_0^\infty(\mathbf{R}; \mathbf{R})$ satisfies that $g_n(x) = 1$ if $|x| < n$ and $=0$ if $|x| > n+1$. Further, for such an f, $\bar{\partial} f$ can be also defined. We set

$$\mathbf{S}^{n+1}_K(E) = \{f \in \mathbf{S}^1_K(E) : Df \in \mathbf{S}^n_K(H^* \otimes E)\}$$

$$\mathbf{S}^n(E) = \bigcup_{K \in \mathscr{K}_+} \mathbf{S}^n_K(E).$$

For $f \in \mathbf{S}^n_K(E)$, $D^k f$, $k \leqslant n$, is defined so that

$$g_n(\rho_K)\, D^k f = g_n(\rho_K)\, D^{k-1}(g_{n+1}(\rho_K)\, Df).$$

We moreover define

$$\mathbf{S}^0_{K,b}(E) = \left\{ \begin{array}{r} f : B \to E : f \text{ is measurable and is } \mu\text{-essentially} \\ \text{bounded on each } \{\rho_K < r\},\ r < \infty \end{array} \right\}$$

and

$$\mathbf{S}^n_{K,b}(E) = \{f \in \mathbf{S}^n_K(E) : D^k f \in \mathbf{S}^0_{K,b}((H^*)^{\otimes k} \otimes E),\ 0 \leqslant k \leqslant n\}$$

$$\mathbf{S}^n_b(E) = \bigcup_{K \in \mathscr{K}_+} \mathbf{S}^n_{K,b}(E).$$

We have

LEMMA 1.1. *If* $f_j \in \mathbf{S}^{n_j}_b(E_j)$ *and* $g_j \in \mathbf{S}^{m_j}(E'_j)$, $j = 1, 2, \ldots$, *then there is a* $K \in \mathscr{K}_+$ *such that* $f_j \in \mathbf{S}^{n_j}_{K,b}(E_j)$ *and* $g_j \in \mathbf{S}^{m_j}_{K}(E'_j)$ *for every* j.

Proof. Let $K \in \mathscr{K}_+$ and $K_r = \{\rho_K \leqslant r\}$. Then $K_r \in \mathscr{K}_+$. See [4, Theorem 4.1]. Further, it is easily seen that $\mathbf{S}^n_{K,b}(E) = \mathbf{S}^n_{K_r,b}(E)$ and $\mathbf{S}^n_K(E) = \mathbf{S}^n_{K_r}(E)$. Hence, by (1.4), for each f_j and g_j, we can choose a $K_j, K'_j \in \mathscr{K}_+$ such that $f_j \in \mathbf{S}^{n_j}_{b,K_j}(E_j)$, $g_j \in \mathbf{S}^{m_j}_{K'_j}(E'_j)$, $\mu(K_j) > 1 - 2^{-j-2}$, and $\mu(K'_j) > 1 - 2^{-j-2}$. Then $K = \bigcap_j (K_j \cap K'_j)$ satisfies the desired properties. ∎

We close this section by recalling the capacities and the splitting property of μ. For $r \in (0, \infty)$ and $p \in (1, \infty)$, a capacity $C_{r,p}$ is a set function given by

$$C_{r,p}(O) = \inf\{\|u\|^p_{r,p} : u \in \mathbf{D}^r_p(\mathbf{R}), u(z) \geqslant 1, \mu\text{-a.e. } z \in O\} \tag{1.5}$$

for any open $O \subset B$ and

$$C_{r,p}(A) = \inf\{C_{r,p}(O) : O \text{ is open and contains } A\} \tag{1.6}$$

for any $A \subset B$. A capacity $C_{1,\infty}$ is defined by

$$C_{1,\infty}(A) = \sum_{n=1}^{\infty} 2^{-n} C_{1,n}(A), \qquad A \subset B. \tag{1.7}$$

We say that $A \subset B$ is $C_{1,\infty}$-quasiopen if there exists an increasing sequence $\{K_n\}$ of compact sets such that $K_n \subset B \backslash A$ and $C_{1,\infty}((B\backslash A)\backslash K_n) \downarrow 0$ $n \to \infty$.

Let $\imath : H^* \to H$ be a natural isomorphism. Choose a family $\{l_j\}_{j=1}^k \subset B^*$ such that $\{l_j, J^* l_j\}$ is an ONB in H^* and we set

$$H^0_{l_1, \ldots, l_k} = \text{the linear span of } \{\imath(l_j), \imath(J^* l_j) : 1 \leqslant j \leqslant k\},$$

$$H_{l_1, \ldots, l_k} = \text{the orthogonal complement of } H^0_{l_1, \ldots, l_k} \text{ in } H, \text{ and}$$

$$\overline{H_{l_1, \ldots, l_k}} = \text{the closure of } H_{l_1, \ldots, l_k} \text{ in } B.$$

Then the mapping defined by

$$\mathbf{C}^k \times \overline{H_{l_1, \ldots, l_k}} \ni (\xi + \sqrt{-1}\,\eta, w) \mapsto \sum_{j=1}^{k} (\xi_j \imath(l_j) + \eta_j \imath(J^* l_j)) + w \in B, \tag{1.8}$$

where $\xi = (\xi_1, \ldots, \xi_k)$, $\eta = (\eta_1, \ldots, \eta_k) \in \mathbf{R}^k$, is an isomorphism of $\mathbf{C}^k \times \overline{H_{l_1, \ldots, l_k}}$ onto B. Through this isomorphism, we have the splitting of the measure μ,

$$\mu = \mu_{\mathbf{C}^k} \times \mu_{l_1, \ldots, l_k}, \tag{1.9}$$

where $\mu_{\mathbf{C}^k}(d\zeta) = \pi^{-k} \exp(-|\zeta|^2)\, d\zeta$.

2. $\bar{\partial}$-EQUATION ON $L^2(B; \bigwedge^{p,q}, d\mu)$

In this section, we investigate $\bar{\partial}$-equations on $L^2(B; \bigwedge^{p,q}, d\mu)$. The observations made in this section play a fundamental role in the following sections and the notation introduced here is used throughout the paper.

We define

$$\mathrm{Dom}(T^{(p,q)}) = \{u \in L^2(B; \textstyle\bigwedge^{p,q}, d\mu) : \bar{\partial} u \in L^2(B; \bigwedge^{p,q+1}, d\mu)\} \qquad (2.1)$$

$$T^{(p,q)} u = \bar{\partial} u, \qquad u \in \mathrm{Dom}(T^{(p,q)}). \qquad (2.2)$$

The following lemma then holds.

LEMMA 2.1. (i) *The densely defined linear operator $T^{(p,q)}$ is a closed operator of $L^2(B; \bigwedge^{p,q}, d\mu)$ into $L^2(B; \bigwedge^{p,q+1}, d\mu)$.*

(ii) $\mathbf{D}_2^1(\bigwedge^{p,q}) \subset \mathrm{Dom}(T^{(p,q)})$.

(iii) $T^{(p,q)} P_t u = e^{-t} P_t(T^{(p,q)} u)$ *for every* $u \in \mathrm{Dom}(T^{(p,q)})$, *where* $\{P_t\}$ *is the Ornstein–Uhlenbeck semigroup.*

(iv) *If* $\xi \in \mathbf{D}_2^1(\mathbf{R})$, $|\xi| + \|D\xi\|_{H^*}$ *is bounded, and* $u \in \mathrm{Dom}(T^{(p,q)})$, *then* $\xi u \in \mathrm{Dom}(T^{(p,q)})$ *and* $T^{(p,q)}(\xi u) = \xi T^{(p,q)} u + \bar{\partial}\xi \wedge u$.

(v) $\mathrm{Image}(T^{(p,q)}) \subset \mathrm{Ker}(T^{(p,q+1)})$.

Proof. The first assertion follows from the continuity of

$$D : L^2(B; \textstyle\bigwedge^{p,q}, d\mu) = \mathbf{D}_2^0(\bigwedge^{p,q}) \to \mathbf{D}_2^{-1}(H^{*\mathbf{C}} \otimes \bigwedge^{p,q+1})$$

and the definition of $\bar{\partial}$. The second follows from the fact that D maps $\mathbf{D}_2^1(\bigwedge^{p,q})$ into $\mathbf{D}_2^0(H^{*\mathbf{C}} \otimes \bigwedge^{p,q})$. To see the third and fourth assertions, it suffices to recall the identities that

$$DP_t u = e^{-t} P_t(Du) \qquad \text{and} \qquad D(\xi u) = \xi\, Du + D\xi \otimes u$$

for $u \in \mathbf{D}_{1+}^{-\infty}(\bigwedge^{p,q})$ and $\xi \in \mathbf{D}_2^1(\mathbf{R})$. For these identities, see [9, 10].

To see the last assertion, let $u \in \mathscr{F}C_0^\infty(B; \bigwedge^{p,q})$. It is then easily seen that

$$\bar{\partial}(\bar{\partial} u) = 0. \qquad (2.3)$$

By the continuity of $D : \mathbf{D}_{1+}^{-\infty}(\bigwedge^{p,q}) \to \mathbf{D}_{1+}^{-\infty}(\bigwedge^{p,q+1})$, we see that (2.3) holds for $u \in \mathbf{D}_{1+}^{-\infty}(\bigwedge^{p,q})$ and for $u \in L^2(B; \bigwedge^{p,q}, d\mu)$. This implies that $\mathrm{Image}(T^{(p,q)}) \subset \mathrm{Ker}(T^{(p,q+1)})$. ∎

To study the adjoint operator $T^{(p,q)*}$ of the densely defined closed linear operator $T^{(p,q)} : L^2(B; \bigwedge^{p,q}, d\mu) \to L^2(B; \bigwedge^{p,q+1}, d\mu)$, we prepare some notation. For complex separable Hilbert spaces E_0, E_1, E_2, and densely defined closed operators $A_1 : E_0 \to E_1$ and $A_2 : E_0 \to E_2$, we simply say that

$u_n \to u$ in $\mathrm{Dom}(A_1)$ (resp., $\mathrm{Dom}(A_1) \cap \mathrm{Dom}(A_2)$) if $u_n \in \mathrm{Dom}(A_1)$ (resp., $\mathrm{Dom}(A_1) \cap \mathrm{Dom}(A_2)$) and u_n converges to u with respect to the graph norm

$$\|v\| = \|v\|_{E_0} + \|A_1 v\|_{E_1} \qquad (\text{resp., } \|v\| = \|v\|_{E_0} + \|A_1 v\|_{E_1} + \|A_2 v\|_{E_2}).$$

Note that if $\alpha \in H^*(1,0)$ then its complex conjugate $\bar{\alpha}$ is in $H^*(0,1)$. For $\alpha \in H^{*(1,0)}$ and $\eta \in \bigwedge^{p,q}$, we then define $i(\alpha)\eta \in \bigwedge^{p,q-1}$ by

$$\langle \bar{\alpha} \wedge \omega, \eta \rangle_{\mathrm{AHS}} = \langle \omega, i(\alpha)\eta \rangle_{\mathrm{AHS}}, \qquad \omega \in \textstyle\bigwedge^{p,q-1}.$$

A continuous operator $\partial : \mathbf{D}_{1+}^{-\infty}(\bigwedge^{p,q}) \to \mathbf{D}_{1+}^{-\infty}(\bigwedge^{p+1,q})$ can be defined so that

$$\partial G = \pi_{p+1,q}((p+q+1)\, \mathscr{A}_{p+q+1}(DG)), \qquad G \in \mathscr{F}C_0^\infty(\textstyle\bigwedge^{p,q}).$$

LEMMA 2.2. (i) $\mathbf{D}_2^1(\bigwedge^{p,q+1}) \subset \mathrm{Dom}(T^{(p,q)*})$ *and* $T^{(p,q)*}u = \pi_{p,q}(\mathscr{A}_{p+q}(D^*u))$, $u \in \mathbf{D}_2^1(\bigwedge^{p,q+1})$, D^* *being the adjoint operator of* $D : L^2(B; \bigwedge^{p,q}, d\mu) \to L^2(B; H^{*\mathbf{C}} \otimes \bigwedge^{p,q}, d\mu)$.

(ii) *For* $u \in \mathrm{Dom}(T^{(p,q)*})$, $P_t u \in \mathrm{Dom}(T^{(p,q)*})$ *and* $T^{(p,q)*}(P_t u) = e^t P_t(T^{(p,q)*}u)$.

(iii) *For any* $u \in \mathrm{Dom}(T^{(p,q+1)}) \cap \mathrm{Dom}(T^{(p,q)*})$, *there is a sequence* $\{u_n\} \subset \mathscr{F}C_0^\infty(B; \bigwedge^{p,q+1})$ *such that* $u_n \to u$ *in* $\mathrm{Dom}(T^{(p,q+1)}) \cap \mathrm{Dom}(T^{(p,q)*})$.

(iv) *If* $\xi \in \mathbf{D}_2^1(\mathbf{R})$, $|\xi| + \|D\xi\|_{H^*}$ *is bounded, and* $u \in \mathrm{Dom}(T^{(p,q)*})$, *then* $\xi u \in \mathrm{Dom}(T^{(p,q)*})$ *and* $T^{(p,q)*}(\xi u) = \xi T^{(p,q)*}u - i(\partial\xi)u$.

Proof. Let $u \in \mathbf{D}_2^1(\bigwedge^{p,q+1})$. By the definition $T^{(p,q)}$, we see that

$$\begin{aligned}
&\int_B \langle u, T^{(p,q)}v \rangle_{\mathrm{AHS}}\, d\mu \\
&= \int_B \frac{1}{(p+q+1)!} \langle u, \pi_{p,q+1}((p+q+1)\, \mathscr{A}_{p+q+1}(Dv)) \rangle\, d\mu \\
&= \int_B \frac{1}{(p+q)!} \langle u, Dv \rangle\, d\mu \\
&= \int_B \langle \pi_{p,q}(\mathscr{A}_{p+q}(D^*u)), v \rangle_{\mathrm{AHS}}\, d\mu
\end{aligned}$$

for every $v \in \mathscr{F}C_0^\infty(B; \bigwedge^{p,q})$, which means that assertion (i) holds.

Since $P_t u \in \mathbf{D}_2^\infty(\bigwedge^{p,q+1}) \equiv \bigcap_{r \in \mathbf{R}} \mathbf{D}_2^r(\bigwedge^{p,q+1})$ if $u \in L^2(B; \bigwedge^{p,q+1}, d\mu)$ [10], it follows from assertion (i) that $P_t u \in \mathrm{Dom}(T^{(p,q)*})$ provided that $u \in \mathrm{Dom}(T^{(p,q)*})$. By the symmetry of P_t and Lemma 2.1(iii), we can easily conclude that he second assertion holds.

Recall that $P_t v \to v$ in $L^2(B; \bigwedge^{p,q+1}, d\mu)$ as $t \downarrow 0$. See [10]. Thus, by (ii) and Lemma 2.1(ii), we see that $P_t u \to u$ in $\mathrm{Dom}(T^{(p,q+1)}) \cap \mathrm{Dom}(T^{(p,q)*})$ as $t \downarrow 0$. Hence it suffices to approximate $P_t u$ in $\mathrm{Dom}(T^{(p,q+1)}) \cap \mathrm{Dom}(T^{(p,q)*})$. Since $P_t u \in \mathbf{D}_2^\infty(\bigwedge^{p,q+1})$, there exists a sequence $\{v_n\} \subset \mathscr{F}C_0^\infty(B; \bigwedge^{p,q+1})$ converging to $P_t u$ in $\mathbf{D}_2^\infty(\bigwedge^{p,q+1})$. Then the continuity of $D: \mathbf{D}_2^1(\bigwedge^{p,q+1}) \to \mathbf{D}_2^0(H^{*\mathbf{C}} \otimes \bigwedge^{p,q+1})$ and $D^*: \mathbf{D}_2^1(H^{*\mathbf{C}} \otimes \bigwedge^{p,q}) \to \mathbf{D}_2^0(\bigwedge^{p,q})$, combined with (1.3) and (i), implies that $v_n \to P_t u$ in $\mathrm{Dom}(T^{(p,q+1)}) \cap \mathrm{Dom}(T^{(p,q)*})$.

Let $u \in \mathrm{Dom}(T^{(p,q)*})$. Then, by Lemma 2.1(iv),

$$\int_B \langle \xi u, T^{(p,q)} v \rangle_{\mathrm{AHS}}\, d\mu = \int_B \langle u, T^{(p,q)}(\xi v) - \bar{\partial}\xi \wedge v \rangle_{\mathrm{AHS}}\, d\mu$$

$$= \int_B \langle \xi T^{(p,q)*} u - i(\partial \xi) u, v \rangle_{\mathrm{AHS}}\, d\mu.$$

Thus assertion (iv) has been seen. ∎

We now establish a main estimation to show the existence theorems for $\bar{\partial}$-equations.

Lemma 2.3. *For every $u \in \mathrm{Dom}(T^{(p,q+1)}) \cap \mathrm{Dom}(T^{(p,q)*})$, it holds that*

$$\int_B \|u\|_{\mathrm{AHS}}^2\, d\mu \leq \int_B \|T^{(p,q+1)} u\|_{\mathrm{AHS}}^2\, d\mu + \int_B \|T^{(p,q)*} u\|_{\mathrm{AHS}}^2\, d\mu. \tag{2.4}$$

Proof. Consider a $(p, q+1)$-form $g(\zeta) = \sum'_{I,J} g_{I,J}(\zeta)\, d\zeta^I \wedge d\bar{\zeta}^J$ on $\mathbf{C}^k$, where $\sum'_{I,J}$ means that the summation is performed only over strictly increasing multi-indices $I = \{i_1 < \cdots < i_p\}$, $J = \{j_1 < \cdots < j_{q+1}\}$, $i \leq i_r$, $j_s \leq k$, and $d\zeta^I = d\zeta^{i_1} \wedge \cdots \wedge d\zeta^{i_p}$, $d\bar{\zeta}^J = d\bar{\zeta}^{j_1} \wedge \cdots \wedge d\bar{\zeta}^{j_{q+1}}$. The norm $|\!|\!| g |\!|\!|$ is given by

$$|\!|\!| g(\zeta) |\!|\!| = \sum_{I,J}{}' |g_{I,J}(\zeta)|^2.$$

We denote by $S^{(p,q)}$ the $L^2(\mu_{\mathbf{C}^k})$-closure of the usual $\bar{\partial}$-operator on $\mathbf{C}^k$ and by $S^{(p,q)*}$ its adjoint, where the inner product on the space of (p, q)-forms is determined by the norm $|\!|\!| \cdot |\!|\!|$. Applying the observation made in [2, pp. 82–84] with $\varphi(\zeta) = |\zeta|^2$ and $\psi = 0$, we have

$$\frac{1}{2}\int_{\mathbf{C}^k} |\!|\!| g |\!|\!|^2\, d\mu_{\mathbf{C}^k} \leq \int_{\mathbf{C}^k} |\!|\!| S^{(p,q+1)} g |\!|\!|^2\, d\mu_{\mathbf{C}^k} + \int_{\mathbf{C}^k} |\!|\!| S^{(p,q)*} g |\!|\!|^2\, d\mu_{\mathbf{C}^k} \tag{2.5}$$

for any C_b^∞-$(p, q+1)$-form g.

To show (2.4), by Lemma 2.2(iii), we may assume that $u \in \mathscr{F}C_0^{\infty}(\bigwedge^{p,q+1})$. Moreover, by virtue of the continuity of D and D^*, we may assume that u is of the form

$$u = \sum_{I,J}' f_{I,J}\, l_I^{(1,0)} \wedge l_J^{(0,1)},$$

where (i) $\{l_1, ..., l_k\} \subset B^*$ satisfies that $\{l_1, J^*l_1, ..., l_k, J^*l_k\}$ is an ONS in H^*, (ii) each $f_{I,J}$ is represented as

$$f_{I,J}(z) = g_{I,J}(\langle z, l_1^{(1,0)}\rangle, ..., \langle z, l_k^{(1,0)}\rangle)$$

for some $g_{I,J} \in C_b^{\infty}(\mathbf{C}^k; \mathbf{C})$, and (iii) $l_I^{(1,0)} = l_{i_1}^{(1,0)} \wedge \cdots \wedge l_{i_p}^{(1,0)}$, $l_J^{(0,1)} = l_{j_1}^{(0,1)} \wedge \cdots \wedge l_{j_{q+1}}^{(0,1)}$. Let

$$g = \sum_{I,J}' g_{I,J}\, d\zeta^I \wedge d\bar{\zeta}^J.$$

It is then easily seen that

$$\|u(z)\|_{\mathrm{AHS}}^2 = 2^{p+q+1} |||g(\langle z, l_1^{(1,0)}\rangle, ..., \langle z, l_k^{(1,0)}\rangle)|||^2$$

$$\|T^{(p,q+1)}u(z)\|_{\mathrm{AHS}}^2 = 2^{p+q+2} |||S^{(p,q+1)}g(\langle z, l_1^{(1,0)}\rangle, ..., \langle z, l_k^{(1,0)}\rangle)|||^2$$

$$T^{(p,q)*}u(z) = 2 \sum_{I,K}' (S^{(p,q)*}g)_{I,K}(\langle z, l_1^{(1,0)}\rangle, ..., \langle z, l_k^{(1,0)}\rangle)\, l_I^{(1,0)} \wedge l_K^{(0,1)}$$

$$\|T^{(p,q)*}u(z)\|_{\mathrm{AHS}}^2 = 2^{p+q+2} |||S^{(p,q)*}g(\langle z, l_1^{(1,0)}\rangle, ..., \langle z, l_k^{(1,0)}\rangle)|||^2,$$

where K runs over strictly increasing multi-indices of length q. Plugging these into (2.5), we obtain (2.4). ∎

We are now ready to establish the existence theorem for $\bar{\partial}$-equations on $L^2(B; \bigwedge^{p,q}, d\mu)$ and the related results.

THEOREM 2.4. (i) Image$(T^{(p,q)})$ = Ker$(T^{(p,q+1)})$.

(ii) Image$(T^{(p,q)})$ *and* Image$(T^{(p,q+1)*})$ *are both closed in* $L^2(B; \bigwedge^{p,q+1}, d\mu)$ *and*

$$L^2(B; \textstyle\bigwedge^{p,q+1}, d\mu) = \mathrm{Image}(T^{(p,q)}) \oplus \mathrm{Image}(T^{(p,q+1)*}) \qquad \text{(orthogonal)}.$$

Proof. By Lemmas 2.1 and 2.3, we have that

$$\int_B \|g\|_{\mathrm{AHS}}^2\, d\mu \leqslant \int_B \|T^{(p,q)*}g\|_{\mathrm{AHS}}^2\, d\mu$$

$$\text{for every} \quad g \in \mathrm{Ker}(T^{(p,q+1)}) \cap \mathrm{Dom}(T^{(p,q)*})$$

and that

$$\mathrm{Image}(T^{(p,q)}) \subset \mathrm{Ker}(T^{(p,q+1)}).$$

Applying Lemma 4.1.1 in [2], we obtain the first assertion.

Assertion (i) implies that $\mathrm{Image}(T^{(p,q)})$ is closed. Moreover, by Lemma 4.1.2 in [2], we see that $\mathrm{Image}(T^{(p,q+1)*})$ coincides with the orthogonal complement of $\mathrm{Ker}(T^{(p,q+1)})$. Hence $\mathrm{Image}(T^{(p,q+1)*})$ is closed and $L^2(B; \bigwedge^{p,q+1}, d\mu)$ is decomposed as stated in (ii). ∎

Remark 2.1. The second assertion of Theorem 2.4 means that the de Rham–Hodge–Kodaira decomposition of $L^2(B; \bigwedge^{p,q}, d\mu)$ holds and there is no harmonic form for the $\bar{\partial}$-laplacian. The decomposition has been also established by Nishimura [8] in a different manner from ours.

3. $\bar{\partial}$-Equation on $L^2(\Omega; \bigwedge^{p,q}, e^{-\sigma}\, d\mu)$—A General Observation

In this section we consider the existence theorems to $\bar{\partial}$-equations on $C_{1,\infty}$-quasi-open set Ω. We set

$$\Psi(\Omega) = \left\{ \begin{array}{l} \psi : \Omega \to [0, \infty) : \psi \text{ is measurable and} \\ \qquad\qquad \mathbf{1}_\Omega\, g(\psi) \in \mathbf{S}^1_b(\mathbf{R}) \cap \mathbf{S}^2(\mathbf{R}) \\ \qquad\qquad \text{for every } g \in C_0^\infty(\mathbf{R}; \mathbf{R}) \end{array} \right\}.$$

For $\psi \in \Psi(\Omega)$, we define

$$\mathscr{S}(\psi, \Omega) = \{\sigma : \Omega \to [0, \infty) : \mathbf{1}_\Omega\, g(\psi)\sigma \in \mathbf{S}^1_b(\mathbf{R}) \cap \mathbf{S}^2(\mathbf{R}) \text{ for every } g \in C_0^\infty(\mathbf{R}; \mathbf{R})\}$$

$$\Gamma(\psi, \Omega) = \{\gamma \in \mathscr{S}(\psi, \Omega) : \mathbf{1}_\Omega\, e^{-\gamma}\, \|D\psi\|^2_{H^*} \in \mathbf{S}^0_b(\mathbf{R})\}.$$

Note that $\log(1 + \|D\psi\|^2) \in \Gamma(\psi, \Omega)$ provided hat $\mathbf{1}_\Omega\, g(\psi) \in \mathbf{S}^3_b(\mathbf{R})$ for every $g \in C_0^\infty(\mathbf{R}; \mathbf{R})$. In what follows, we fix an arbitrary $\psi \in \Psi(\Omega)$.

Let

$$\mathbf{D}^1_{\infty-,0}(\Omega; E) = \{u \in \mathbf{D}^1_{\infty-}(E) : u = \mathbf{1}_\Omega u\}.$$

We have

Lemma 3.1. *Let $\sigma \in \mathscr{S}(\psi, \Omega)$ and $\gamma \in \Gamma(\psi, \Omega)$. Then the operator*

$$\bar{\partial} : L^2(\Omega; \textstyle\bigwedge^{p,q}, e^{-\sigma}\, d\mu) \supset \mathbf{D}^1_{\infty-,0}(\Omega; \bigwedge^{p,q}) \ni u \mapsto \bar{\partial} u \in L^2(\Omega; \bigwedge^{p,q}, e^{-\sigma-\gamma}\, d\mu)$$

is closable.

Proof. Suppose that $u_k \in \mathbf{D}^1_{\infty-,0}(\Omega; \bigwedge^{p,q})$ converges to 0 in $L^2(\Omega; \bigwedge^{p,q}, e^{-\sigma}\, d\mu)$ and $\bar{\partial} u_k \to v$ in $L^2(\Omega; \bigwedge^{p,q+1}, e^{-\sigma-\gamma}\, d\mu)$. Choose $f \in \mathbf{D}^\infty_{\infty-}(\bigwedge^{p,q+1})$, and $g, \tilde{g} \in C_0^\infty(\mathbf{R}; \mathbf{R})$ with $\tilde{g} = 1$ on $\operatorname{supp}[g]$. Then there is a $K \in \mathscr{K}_+$ such that $\mathbf{1}_\Omega g(\psi)$, $\mathbf{1}_\Omega \tilde{g}(\psi)\sigma$, and $\mathbf{1}_\Omega \tilde{g}(\psi)\gamma$ are all in $\mathbf{S}^1_{K,b}(\mathbf{R})$. Taking $\varphi \in C_b^\infty(\mathbf{R}; \mathbf{R})$ with $\varphi = 1$ on $(-\infty, 0]$ and $= 0$ on $[1, \infty)$, we define $\xi_n = \varphi(\rho_K - n)$. Then $e^{-\sigma-\gamma} g(\psi)\, \xi_n f = e^{-(\sigma+\gamma)\tilde{g}(\psi)\xi_{n+1}} g(\psi) \xi_n f \in \mathbf{D}^1_{\infty-}(\bigwedge^{p,q+1})$. Hence we obtain

$$
\begin{aligned}
&\int_\Omega \langle g(\psi)\, \xi_n f, v \rangle_{\mathrm{AHS}}\, e^{-\sigma-\gamma}\, d\mu \\
&= \lim_{k\to\infty} \int_B \langle e^{-\sigma-\gamma} g(\psi)\, \xi_n f, \bar{\partial} u_k \rangle_{\mathrm{AHS}}\, d\mu \\
&= \lim_{k\to\infty} \int_B \langle \pi_{p,q}(\mathscr{A}_{p+q}(D^*(e^{-\sigma-\gamma} g(\psi)\, \xi_n f))), u_k \rangle_{\mathrm{AHS}}\, d\mu \\
&= 0,
\end{aligned}
$$

where to see the last identity we have used that $d\mu \leqslant C e^{-\sigma}\, d\mu$ on $\operatorname{supp}[g(\psi)\xi_n]$ for some $C < +\infty$. Letting $g \uparrow 1$ and $n \uparrow \infty$, we see that $v = 0$ and hence the operator $\bar{\partial}$ is closable. ∎

We denote by $T^{(p,q)}_{\sigma,\gamma}$ the closed minimal extension of the above $\bar{\partial}$. We are now ready to state our results.

THEOREM 3.2. *Let $\sigma \in \mathscr{S}(\psi, \Omega)$ and $\gamma \in \Gamma(\psi, \Omega)$. Assume further that there exists an $\varepsilon > 0$ so that*

$$\varepsilon I + \partial\bar{\partial}(\sigma + 2\gamma) \geqslant \varepsilon^{-1}\, \partial\gamma \otimes \bar{\partial}\gamma.$$

If $f \in \operatorname{Ker}(T^{(p,q+1)}_{\sigma+\gamma,\gamma}) \cap L^2(\Omega; \bigwedge^{p,q+1}, e^{-\sigma}\, d\mu)$, then there is a $u \in \operatorname{Dom}(T^{(p,q)}_{\sigma,\gamma})$ such that $T^{(p,q)}_{\sigma,\gamma} u = f$ and $\|u\|_\sigma \leqslant (1-2\varepsilon)^{-1/2} \|f\|_\sigma$, where $\|f\|^2_\sigma = \int_\Omega \|f\|^2_{\mathrm{AHS}}\, e^{-\sigma}\, d\mu$.

As an application of this theorem, we have

THEOREM 3.3. *Suppose that there exist $\sigma_0 \in \mathscr{S}(\psi, \Omega)$, $\gamma \in \Gamma(\psi, \Omega)$, $\varepsilon \in (0, \frac{1}{2})$, and $n_0 \in \mathbf{N}$ such that*

$$\operatorname{ess.sup}\{\gamma(z) : \sigma_0(z) < n\} < \infty \qquad \textit{for any } n > n_0 \tag{3.1}$$

$$\varepsilon I + \partial\bar{\partial}(\sigma_0 + 2\gamma) \geqslant \varepsilon^{-1}\, \partial\gamma \otimes \bar{\partial}\gamma \qquad \mu\textit{-a.e. on } \{\sigma_0 > n_0\} \tag{3.2}$$

$$\partial\bar{\partial}\sigma_0 \geqslant 0. \tag{3.3}$$

Then, for any $\sigma \in \mathscr{S}(\psi, \Omega)$ with $\partial\bar{\partial}\sigma \geqslant 0$ μ-a.e. on Ω, it holds that

$$\mathrm{Ker}(T_{\sigma,\gamma}^{(p,q+1)}) \subset \mathrm{Image}(T_{\sigma,\gamma}^{(p,q)}).$$

For $\sigma \in \mathscr{S}(\psi, \Omega)$, we set

$$\bigwedge_{\sigma,\gamma}^{p,q}(\Omega) = \{u \in \mathrm{Dom}(T_{\sigma,\gamma}^{(p,q)}) : T_{\sigma,\gamma}^{(p,q)} u \in L^2(\Omega; \bigwedge^{p,q}, e^{-\sigma}\, d\mu)\}$$

$$\hat{T}_{\sigma,\gamma}^{(p,q)} = T_{\sigma,\gamma}^{(p,q)}|_{\bigwedge_{\sigma,\gamma}^{p,q}(\Omega)}.$$

Finally we show

THEOREM 3.4. *Assume the existence of σ_0, γ, and ε as in Theorem 3.3. If $\sigma \in \mathscr{S}(\psi, \Omega)$ satisfies that $\partial\bar{\partial}\sigma \geqslant 0$, then*

$$\mathrm{Image}(\hat{T}_{\sigma,\gamma}^{(p,q)}) = \mathrm{Ker}(\hat{T}_{\sigma,\gamma}^{(p,q+1)}).$$

In particular, the sequence

$$\bigwedge_{\sigma,\gamma}^{p,0}(\Omega) \xrightarrow{\hat{T}} \cdots \xrightarrow{\hat{T}} \bigwedge_{\sigma,\gamma}^{p,q}(\Omega) \xrightarrow{\hat{T}} \bigwedge_{\sigma,\gamma}^{p,q+1}(\Omega) \xrightarrow{\hat{T}} \bigwedge_{\sigma,\gamma}^{p,q+2}(\Omega) \xrightarrow{\hat{T}} \cdots$$

is exact, where $\hat{T} = \hat{T}_{\sigma,\gamma}^{(p,q)}$ on $\bigwedge_{\sigma,\gamma}^{p,q}(\Omega)$.

The proofs of the theorems are broken into several steps, each step being a lemma.

Taking $f_0 \in C^\infty(\mathbf{R}; \mathbf{R})$ such that $0 \leqslant f_0 \leqslant 1$, $f_0(t) = 1$, $t \leqslant 0$, and $f_0(t) = 0$, $t \geqslant 1$, we set

$$\eta_\nu = \mathbf{1}_\Omega f_0(\psi - \nu).$$

By Lemma 1.1, for every $\sigma \in \mathscr{S}(\psi, \Omega)$ and $\gamma \in \Gamma(\psi, \Omega)$, there is a $K \in \mathscr{K}_+$ such that $\eta_\nu\sigma$, $\eta_\nu\gamma$, $\eta_\nu \in \mathbf{S}_{K,b}^1(\mathbf{R}) \cap \mathbf{S}_K^2(\mathbf{R})$, and $e^{-\gamma} \|D\psi\|_{H^*}^2 \in \mathbf{S}_{K,b}^0(\mathbf{R})$, $\nu = 1, 2, \ldots$. We then put

$$\xi_n = f_0(\rho_K - n), \qquad n = 1, 2, \ldots.$$

It is satisfied that

$$\xi_n \in \mathbf{D}_{\infty-}^1(\mathbf{R}), \tag{3.4}$$

$$\|D\xi_n\|_{H^*} \leqslant \sup\{|f_0'(x)| : x \in \mathbf{R}\}, \tag{3.5}$$

$$D\xi_{n+1} = 0 \qquad \mu\text{-a.e. on } \mathrm{supp}[\xi_n], \tag{3.6}$$

$$\xi_n \|D\eta_\nu\|_{H^*}^2 \leqslant C_n e^\gamma \qquad \text{for some} \quad C_n < \infty \text{ independent of } \nu, \tag{3.7}$$

$$\|D(\eta_\nu \xi_n)\|_{H^*} \text{ is bounded.} \tag{3.8}$$

We first see several properties of $T^{(p,q)}_{\sigma,\gamma}$.

LEMMA 3.5. *Let $\sigma \in \mathscr{S}(\psi, \Omega)$ and $\gamma \in \Gamma(\psi, \Omega)$.*

(1) *If $u \in \mathrm{Dom}(T^{(p,q)})$, then $\eta_\nu \xi_n u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$ for any $n, \nu \in \mathbf{N}$ and $T^{(p,q)}_{\sigma,\gamma}(\eta_\nu \xi_n u) = T^{(p,q)}(\eta_\nu \xi_n u)$.*

(2) *If $\xi \in \mathbf{D}^1_{\infty-}(\mathbf{R})$, $\xi + \|D\xi\|_{H^*}$ is bounded, and $u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$, then $\xi u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$ and $T^{(p,q)}_{\sigma,\gamma}(\xi u) = \xi T^{(p,q)}_{\sigma,\gamma} u + \bar\partial \xi \wedge u$.*

(3) *For $u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$, $\xi_n u \to u$ in $\mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$ as $n \to \infty$, and $\eta_\nu \xi_n u \to \xi_n u$ in $\mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$ as $\nu \to \infty$.*

(4) *If $u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$, then $\eta_\nu \xi_n u \in \mathrm{Dom}(T^{(p,q)})$ and $T^{(p,q)}(\eta_\nu \xi_n u) = T^{(p,q)}_{\sigma,\gamma}(\eta_\nu \xi_n u)$.*

(5) *For $u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$, $\eta_{\nu+1} \xi_{n+1} P_t(\eta_\nu \xi_n u) \to \eta_\nu \xi_n u$ in $\mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$ as $t \downarrow 0$.*

(6) $\mathrm{Image}(T^{(p,q)}_{\sigma,\gamma}) \subset \mathrm{Ker}(T^{(p,q)}_{\sigma+\gamma,\gamma})$.

Proof. (1) Let $u \in \mathrm{Dom}(T^{(p,q)})$. As was seen in the proof of Lemma 2.2, there is a $\{u_m\} \subset \mathbf{D}^\infty_{\infty-}(\bigwedge^{p,q})$ converging to u in $\mathrm{Dom}(T^{(p,q)})$. Then $\eta_\nu \xi_n u_m \in \mathbf{D}^1_{\infty-,0}(\Omega; \bigwedge^{p,q})$. Since $\sigma \geqslant 0$ and η_ν, ξ_n are both bounded, we have that

$$\eta_\nu \xi_n u_m \to \eta_\nu \xi_n u \qquad \text{in} \quad L^2(\Omega; \textstyle\bigwedge^{p,q}, e^{-\sigma}\, d\mu) \text{ as } m \to \infty.$$

By virtue of Lemma 2.1(iv) and (3.8), we also obtain that

$$\begin{aligned}\bar\partial(\eta_\nu \xi_n u_m) &= \eta_\nu \xi_n \bar\partial u_m + \bar\partial(\eta_\nu \xi_n) \wedge u_m \\ &\to \eta_\nu \xi_n T^{(p,q)} u + \bar\partial(\eta_\nu \xi_n) \wedge u = T^{(p,q)}(\eta_\nu \xi_n u)\end{aligned}$$

in $L^2(\Omega; \bigwedge^{p,q+1}, e^{-\sigma-\gamma}\, d\mu)$ as $m \to \infty$, since $e^{-\sigma-\gamma}\, d\mu \leqslant d\mu$. This completes the proof of the first assertion.

(2) Take $\xi \in \mathbf{D}^1_{\infty-}(\mathbf{R})$ such that $\xi + \|D\xi\|_{H^*}$ is bounded and let $u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$. Choose $\{u_n\} \subset \mathbf{D}^1_{\infty-,0}(\Omega; \bigwedge^{p,q})$ with $u_n \to u$ in $\mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$. Then $\xi u_n \in \mathbf{D}^1_{\infty-,0}(\Omega; \bigwedge^{p,q})$. Since σ and γ are both non-negative, we have that

$$\bar\partial(\xi u_n) = \xi\, \bar\partial u_n + \bar\partial \xi \wedge u_n \to \xi T^{(p,q)}_{\sigma,\gamma} u + \bar\partial \xi \wedge u$$

in $L^2(\Omega; \bigwedge^{p,q+1}, e^{-\sigma-\gamma}\, d\mu)$ as $n \to \infty$. Thus, the second assertion has been seen.

(3) Let $u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$. Since $\xi_n \in \mathbf{D}^1_{\infty-}(\mathbf{R})$ and $\xi_n + \|D\xi_n\|$ is bounded, it follows from (2) that $\xi_n u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$. By (3.6), it holds that

$\xi_n \uparrow 1$ and $D\xi_n \to 0$ μ-a.e. as $n \to \infty$. Then, applying Assertion 2 and the dominated convergence theorem, we see that

$$\xi_n u \to u \qquad \text{in } L^2(\Omega; \textstyle\bigwedge^{p,q}, e^{-\sigma}\, d\mu) \qquad \text{as} \quad n \to \infty$$

$$T^{(p,q)}_{\sigma,\gamma}(\xi_n u) = \xi_n T^{(p,q)}_{\sigma,\gamma} u + \bar{\partial}\xi_n \wedge u \to T^{(p,q)}_{\sigma,\gamma} u$$
$$\text{in } L^2(\Omega; \textstyle\bigwedge^{p,q}, e^{-\sigma-\gamma}\, d\mu) \text{ as } n \to \infty$$

because $\gamma \geqslant 0$. Thus, the first half of Assertion (3) has been verified.

To see the second half, apply Assertion 2 with $\xi = \eta_\nu \xi_n$ and $\xi = \xi_n$, and note that

$$T^{(p,q)}_{\sigma,\gamma}(\eta_\nu \xi_n u) = \eta_\nu \xi_n T^{(p,q)}_{\sigma,\gamma} u + \bar{\partial}(\eta_\nu \xi_n) \wedge u$$
$$= \eta_\nu T^{(p,q)}_{\sigma,\gamma}(\xi_n u) + \xi_n \bar{\partial}\eta_\nu \wedge u.$$

Since $0 \leqslant \eta_\nu \leqslant 1$ and $\eta_\nu \uparrow 1$, we have

$$\eta_\nu T^{(p,q)}_{\sigma,\gamma}(\xi_n u) \to T^{(p,q)}_{\sigma,\gamma}(\xi_n u) \qquad \text{in } L^2(\Omega; \textstyle\bigwedge^{p,q+1}, e^{-\sigma-\gamma}\, d\mu) \text{ as } \nu \to \infty.$$

Moreover, by (3.7), we see that

$$\|\xi_n \bar{\partial}\eta_\nu \wedge u\|^2_{\mathrm{AHS}} \leqslant \xi_n^2 \|D\eta_\nu\|^2_{H^*} \|u\|^2_{\mathrm{AHS}} \leqslant C_n e^{\gamma} \|u\|^2_{\mathrm{AHS}}$$

for some $C_n < +\infty$. Then the dominated convergence theorem yields that

$$\xi_n \bar{\partial}\eta_\nu \wedge u \to 0 \qquad \text{in } L^2(\Omega; \textstyle\bigwedge^{p,q+1}, e^{-\sigma-\gamma}\, d\mu) \text{ as } \nu \to \infty$$

because $D\eta_\nu \to 0$ μ-a.e. as $\nu \to \infty$. Thus $\eta_\nu \xi_n u \to \xi_n u$ in $\mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$ as $\nu \to \infty$.

(4) Let $u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$ and choose $u_m \in \mathbf{D}^1_{\infty-,0}(\Omega; \bigwedge^{p,q})$ with $u_m \to u$ in $\mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$ as $m \to \infty$. As was seen in the proof of Assertion 2, $\eta_\nu \xi_n u_m \to \eta_\nu \xi_n u$ in $\mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$ as $m \to \infty$. The boundedness of σ and γ on $\mathrm{supp}[\eta_\nu \xi_n]$ then implies that

$$\eta_\nu \xi_n u_m \to \eta_\nu \xi_n u \qquad \text{in } L^2(B; \textstyle\bigwedge^{p,q}, d\mu)$$
$$T^{(p,q)}(\eta_\nu \xi_n u_m) = T^{(p,q)}_{\sigma,\gamma}(\eta_\nu \xi_n u_m) \to T^{(p,q)}_{\sigma,\gamma}(\eta_\nu \xi_n u) \qquad \text{in } L^2(B; \textstyle\bigwedge^{p,q}, d\mu),$$

as $m \to \infty$, which means that Assertion 4 holds.

(5) Let $u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$. By (4), $\eta_\nu \xi_n u \in L^2(B; \bigwedge^{p,q}, d\mu)$. Then the hypercontractivity of the Ornstein–Uhlenbeck semigroup $\{P_t\}$ implies that

$$P_t(\eta_\nu \xi_n u) \in \mathbf{D}^1_{2+\varepsilon}(\textstyle\bigwedge^{p,q}),$$

where $\varepsilon = \varepsilon_t \equiv e^t - 1$. By Assertion 1, we see that

$$\eta_{\nu+1} \xi_{n+1} P_t(\eta_\nu \xi_n u) \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma}).$$

Since $e^{-\sigma}\, d\mu \leqslant d\mu$, the strong continuity of $\{P_t\}$ implies that

$$\eta_{\nu+1}\xi_{n+1}P_t(\eta_\nu\xi_n u) \to \eta_\nu\xi_n u \qquad \text{in } L^2(\Omega; \textstyle\bigwedge^{p,q}, e^{-\sigma}\, d\mu) \text{ as } t \downarrow 0.$$

On account of Lemma 2.1(iii) and Assertions 1, 2, and 4 above, we obtain that

$$\begin{aligned} &T^{(p,q)}_{\sigma,\gamma}(\eta_{\nu+1}\xi_{n+1}P_t(\eta_\nu\xi_n u)) \\ &\quad = \eta_{\nu+1}\xi_{n+1}\, e^{-t}P_t(T^{(p,q)}(\eta_\nu\xi_n u)) + \bar{\partial}(\eta_{\nu+1}\xi_{n+1}) \wedge P_t(\eta_\nu\xi_n u) \\ &\quad \to \eta_{\nu+1}\xi_{n+1}T^{(p,q)}(\eta_\nu\xi_n u) = T^{(p,q)}_{\sigma,\gamma}(\eta_\nu\xi_n u) \end{aligned}$$

in $L^2(\Omega; \bigwedge^{p,q+1}, e^{-\sigma-\gamma}\, d\mu)$, where we have used the property that

$$\eta_\nu\xi_n\, D(\eta_{\nu+1}\xi_{n+1}) = 0, \tag{3.9}$$

which follows from (3.6) and the very definition of η_ν's.

(6) Let $u \in \mathbf{D}^1_{\infty-,0}(\Omega; \bigwedge^{p,q})$ and set $u_{\nu,n} = \eta_\nu\xi_n u$. Then, by Lemma 2.1(v),

$$T^{(p,q)}_{\sigma,\gamma}u_{\nu,n} = T^{(p,q)}u_{\nu,n} \in \operatorname{Ker}(T^{(p,q+1)}).$$

Since $\eta_{\nu+1}\xi_{n+1}T^{(p,q)}_{\sigma,\gamma}u_{\nu,n} = T^{(p,q)}_{\sigma,\gamma}u_{\nu,n}$, by the Assertion 1, we have that $T^{(p,q)}_{\sigma,\gamma}u_{\nu,n} \in \operatorname{Ker}(T^{(p,q+1)}_{\sigma+\gamma,\gamma})$. Letting $\nu, n \to \infty$, by (3), we can show that $T^{(p,q)}_{\sigma,\gamma}u \in \operatorname{Ker}(T^{(p,q+1)}_{\sigma+\gamma,\gamma})$. Thus we obtain

$$T^{(p,q)}_{\sigma,\gamma}(\mathbf{D}^1_{\infty-,0}(\Omega; \textstyle\bigwedge^{p,q})) \subset \operatorname{Ker}(T^{(p,q+1)}_{\sigma+\gamma,\gamma}).$$

For general $u \in \operatorname{Dom}(T^{(p,q)}_{\sigma,\gamma})$, choose $u_m \in \mathbf{D}^1_{\infty-,0}(\Omega; \bigwedge^{p,q})$ converging to u in $\operatorname{Dom}(T^{(p,q)}_{\sigma,\gamma})$. Then, $T^{(p,q)}_{\sigma,\gamma}u_m \to T^{(p,q)}_{\sigma,\gamma}u$ in $L^2(\Omega; \bigwedge^{p,q+1}, e^{-\sigma}\, d\mu)$ and $T^{(p,q+1)}_{\sigma+\gamma,\gamma}(T^{(p,q)}_{\sigma,\gamma}u_m) = 0$. Hence $T^{(p,q)}_{\sigma,\gamma}u \in \operatorname{Ker}(T^{(p,q+1)}_{\sigma+\gamma,\gamma})$ and $\operatorname{Image}(T^{(p,q)}_{\sigma,\gamma}) \subset \operatorname{Ker}(T^{(p,q)}_{\sigma+\gamma,\gamma})$. ∎

We next study $T^{(p,q)*}_{\sigma,\gamma}$. In what follows, we simply denote by $\langle\, ,\rangle_\sigma$ the inner product in $L^2(\Omega; \bigwedge^{p,q}, e^{-\sigma}\, d\mu)$ and do not mention which space $\bigwedge^{p,q}$ is considered.

LEMMA 3.6. *Let $\sigma \in \mathscr{S}(\psi, \Omega)$ and $\gamma \in \Gamma(\psi, \Omega)$.*

(1) *If $\xi \in \mathbf{D}^1_{\infty-}(\mathbf{R})$, $\xi + \|D\xi\|_{H^*}$ is bounded, and $v \in \operatorname{Dom}(T^{(p,q)*}_{\sigma,\gamma})$, then $\xi v \in \operatorname{Dom}(T^{(p,q)*}_{\sigma,\gamma})$ and $(T^{(p,q)*}_{\sigma,\gamma})(\xi v) = \xi T^{(p,q)*}_{\sigma,\gamma}v - e^{-\gamma}\, i(\partial\xi)v$.*

(2) *For every $v \in \operatorname{Dom}(T^{(p,q)*}_{\sigma,\gamma})$, $\xi_n v \to v$ in $\operatorname{Dom}(T^{(p,q)*}_{\sigma,\gamma})$ as $n \to \infty$ and $\eta_\nu\xi_n v \to \xi_n v$ in $\operatorname{Dom}(T^{(p,q)*}_{\sigma,\gamma})$ as $\nu \to \infty$.*

(3) *Let $v \in \operatorname{Dom}(T^{(p,q)*}_{\sigma,\gamma})$. Then $\eta_\nu\xi_n v \in \operatorname{Dom}(T^{(p,q)*})$ for every $\nu, n \in \mathbf{N}$ and*

$$T^{(p,q)*}_{\sigma,\gamma}(\eta_\nu\xi_n v) = e^{-\gamma}\{T^{(p,q)*}(\eta_\nu\xi_n v) + \eta_\nu\xi_n i(\partial(\sigma+\gamma))v\}.$$

(4) *If* $v \in \mathbf{D}^1_{2+\varepsilon}(\bigwedge^{p,q+1})$ *for some* $\varepsilon > 0$, *then* $\eta_\nu \xi_n v \in \operatorname{Dom}(T^{(p,q)*}_{\sigma,\gamma})$ *for every* $\nu, n \in \mathbf{N}$ *and*

$$T^{(p,q)*}_{\sigma,\gamma}(\eta_\nu \xi_n v) = e^{-\gamma}[\eta_\nu \xi_n \{T^{(p,q)*} v + i(\partial(\sigma+\gamma))v\} - i(\partial(\eta_\nu \xi_n))v].$$

(5) *For* $v \in \operatorname{Dom}(T^{(p,q)*}_{\sigma,\gamma})$ *and* $\nu, n \in \mathbf{N}$, $\eta_{\nu+1}\xi_{n+1}P_t(\eta_\nu \xi_n v) \to \eta_\nu \xi_n v$ *in* $\operatorname{Dom}(T^{(p,q)*}_{\sigma,\gamma})$ *as* $t \downarrow 0$.

Proof. (1) By Lemma 3.5(2), we obtain

$$\begin{aligned}\langle \xi v, T^{(p,q)}_{\sigma,\gamma} u\rangle_{\sigma+\gamma} &= \langle v, T^{(p,q)}_{\sigma,\gamma}(\xi u) - \bar{\partial}\xi \wedge u\rangle_{\sigma+\gamma}\\ &= \langle T^{(p,q)*}_{\sigma,\gamma} v - e^{-\gamma} i(\partial\xi)v, u\rangle_\sigma\end{aligned}$$

for any $u \in \operatorname{Dom}(T^{(p,q)}_{\sigma,\gamma})$, which means that Assertion 1 holds.

(2) Let $v \in \operatorname{Dom}(T^{(p,q)*}_{\sigma,\gamma})$. According to (3.5) and (3.6), and Assertion 1, the dominated convergence theorem implies that

$$\begin{aligned}\xi_n v \to v \qquad &\text{in } L^2(\Omega; \textstyle\bigwedge^{p,q+1}, e^{-\sigma-\gamma}\, d\mu)\\ T^{(p,q)*}_{\sigma,\gamma}(\xi_n v) = \xi_n T^{(p,q)*}_{\sigma,\gamma} v - e^{-\gamma} i(\partial p_n)v &\to T^{(p,q)*}_{\sigma,\gamma} v\\ &\text{in } L^2(\Omega; \textstyle\bigwedge^{p,q}, e^{-\sigma}\, d\mu).\end{aligned}$$

Thus $\xi_n v \to v$ in $\operatorname{Dom}(T^{(p,q)*}_{\sigma,\gamma})$.

Because of Assertion 1 and the boundedness of $\eta_\nu \xi_n + \|D(\eta_\nu \xi_n)\|_{H^*}$ and $\xi_n + \|D\xi_n\|_{H^*}$, we have

$$\begin{aligned}T^{(p,q)*}_{\sigma,\gamma}(\eta_\nu \xi_n v) &= \eta_\nu \xi_n T^{(p,q)*}_{\sigma,\gamma} v - e^{-\gamma} i(\partial(\eta_\nu \xi_n))v\\ &= \eta_\nu T^{(p,q)*}_{\sigma,\gamma}(\xi_n v) - e^{-\gamma}\xi_n i(\partial\eta_\nu)v.\end{aligned}$$

Applying the dominated convergence theorem, we see that

$$\begin{aligned}\eta_\nu \xi_n v \to \xi_n v \qquad &\text{in } L^2(\Omega; \textstyle\bigwedge^{p,q+1}, e^{-\sigma-\gamma}\, d\mu) \text{ as } \nu \to \infty\\ \eta_\nu T^{(p,q)*}_{\sigma,\gamma}(\xi_n v) \to T^{(p,q)*}_{\sigma,\gamma}(\xi_n v) \qquad &\text{in } L^2(\Omega; \textstyle\bigwedge^{p,q}, e^{-\sigma}\, d\mu) \text{ as } \nu \to \infty.\end{aligned}$$

On the other hand, by (3.7),

$$\|e^{-\gamma}\xi_n i(\partial\eta_\nu)v\|^2_{\mathrm{AHS}} \leqslant e^{-2\gamma}\xi_n^2 \|D\eta_\nu\|^2_{H^*} \|v\|^2_{\mathrm{AHS}} \leqslant C_n e^{-\gamma} \|v\|^2_{\mathrm{AHS}}$$

for some $C_n < +\infty$. Hence applying the dominated convergence theorem again, we obtain

$$e^{-\gamma}\xi_n i(\partial\eta_\nu)v \to 0 \qquad \text{in } L^2(\Omega; \textstyle\bigwedge^{p,q}, e^{-\sigma}\, d\mu) \text{ as } \nu \to \infty.$$

Thus, $\eta_\nu \xi_n v \to \xi_n v$ in $\operatorname{Dom}(T^{(p,q)*}_{\sigma,\gamma})$ as $\nu \to \infty$.

(3) Let $v \in \mathrm{Dom}(T_{\sigma,\gamma}^{(p,q)*})$. It follows from Assertion 1 that

$$\eta_{\nu+1}\xi_{n+1} T_{\sigma,\gamma}^{(p,q)*}(\eta_\nu \xi_n v) = T_{\sigma,\gamma}^{(p,q)*}(\eta_\nu \xi_n v). \tag{3.10}$$

Then, for any $w \in \mathbf{D}_{\infty-}^{\infty}(\wedge^{p,q})$, we have

$$\begin{aligned}\langle e^{-\sigma} T_{\sigma,\gamma}^{(p,q)*}(\eta_\nu \xi_n v), w\rangle_0 &= \langle T_{\sigma,\gamma}^{(p,q)*}(\eta_\nu \xi_n v), \eta_{\nu+1}\xi_{n+1} w\rangle_\sigma \\ &= \langle \eta_\nu \xi_n v, T^{(p,q)}(\eta_{\nu+1}\xi_{n+1} w)\rangle_{\sigma+\gamma} \\ &= \langle e^{-\sigma-\gamma}\gamma_\nu \xi_n v, T^{(p,q)} w\rangle_0,\end{aligned}$$

where we have used Lemma 3.5(1) to show the second equality and Lemma 2.1(iv) and the property (3.9) to see the last equality. This implies that

$$e^{-\sigma-\gamma}\eta_\nu \xi_n v \in \mathrm{Dom}(T^{(p,q)*})$$

and

$$T^{(p,q)*}(e^{-\sigma-\gamma}\eta_\nu \xi_n v) = e^{-\sigma} T_{\sigma,\gamma}^{(p,q)*}(\eta_\nu \xi_n v). \tag{3.11}$$

Since $g \equiv \eta_{\nu+1}\xi_{n+1} e^{\sigma+\gamma}$ is in $\mathbf{D}_{\infty-}^1(\mathbf{R})$ and $g + \|Dg\|_{H^*}$ is bounded, by Lemma 2.2(iv) and (3.9)–(3.11), we obtain that $\eta_\nu \xi_n v = g e^{-\sigma-\gamma}\eta_\nu \xi_n v \in \mathrm{Dom}(T^{(p,q)*})$ and

$$\begin{aligned}T^{(p,q)*}(\eta_\nu \xi_n v) &= g T^{(p,q)*}(e^{-\sigma-\gamma}\eta_\nu \xi_n v) - i(\partial g)(e^{-\sigma-\gamma}\eta_\nu \xi_n v) \\ &= \eta_{\nu+1}\xi_{n+1} e^{\gamma} T_{\sigma,\gamma}^{(p,q)*}(\eta_\nu \xi_n v) - i(\partial(\sigma+\gamma))(\eta_\nu \xi_n v) \\ &\quad - i(\partial(\eta_{\nu+1}\xi_{n+1}))(\eta_\nu \xi_n v) \\ &= e^{\gamma} T_{\sigma,\gamma}^{(p,q)*}(\eta_\nu \xi_n v) - \eta_\nu \xi_n i(\partial(\sigma+\gamma)) v.\end{aligned}$$

This completes the proof of Assertion 3.

(4) Let $v \in \mathbf{D}_{2+\varepsilon}^1(\wedge^{p,q+1})$, $\varepsilon > 0$. Then, by virtue of Assertion 3, Lemma 3.5(2)(4), (3.9), Lemma 2.2(iv), and that $\eta_{\nu+1}\xi_{n+1} e^{-\sigma-\gamma} \in \mathbf{D}_{-\infty}^1(\mathbf{R})$, we obtain that, for any $u \in \mathrm{Dom}(T_{\sigma,\gamma}^{(p,q)})$,

$$\begin{aligned}&\langle \eta_\nu \xi_n v, T_{\sigma,\gamma}^{(p,q)} u\rangle_{\sigma+\gamma} \\ &\quad = \langle v, T_{\sigma,\gamma}^{(p,q)}(\eta_\nu \xi_n u) - \bar{\partial}(\eta_\nu \xi_n) \wedge u\rangle_{\sigma+\gamma} \\ &\quad = \langle \eta_{\nu+1}\xi_{n+1} e^{-\sigma-\gamma} v, T^{(p,q)}(\eta_\nu \xi_n u)\rangle_0 - \langle e^{-\gamma} i(\partial(\eta_\nu \xi_n)) v, u\rangle_\sigma \\ &\quad = \langle T^{(p,q)*}(\eta_{\nu+1}\xi_{n+1} e^{-\sigma-\gamma} v), \eta_\nu \xi_n u\rangle_0 - \langle e^{-\gamma} i(\partial(\eta_\nu \xi_n)) v, u\rangle_\sigma \\ &\quad = \langle e^{-\sigma-\gamma}\{T^{(p,q)*} v + i(\partial(\sigma+\gamma)) v\}, \eta_\nu \xi_n u\rangle_0 - \langle e^{-\gamma} i(\partial(\eta_\nu \xi_n)) v, u\rangle_\sigma \\ &\quad = \langle e^{-\gamma}\{\eta_\nu \xi_n (T^{(p,q)*} v + i(\partial(\sigma+\gamma)) v)) - i(\partial(\eta_\nu \xi_n)) v\}, u\rangle_\sigma.\end{aligned}$$

Since $\eta_\nu \xi_n \|D(\sigma+\gamma)\|_{H^*}$ is bounded, this implies that $\eta_\nu \xi_n v \in \mathrm{Dom}(T_{\sigma,\gamma}^{(p,q)*})$ and that

$$T_{\sigma,\gamma}^{(p,q)*}(\eta_\nu \xi_n v) = e^{-\gamma}\{\eta_\nu \xi_n (T_{\sigma,\gamma}^{(p,q)*} v + i(\partial(\sigma+\gamma))v) - (\partial(\eta_\nu \xi_n))v\}.$$

(5) Let $v \in \mathrm{Dom}(T_{\sigma,\gamma}^{(p,q)*})$. Since $\sigma+\gamma$ is bounded on $\mathrm{supp}[\eta_\nu \xi_n]$, $\eta_\nu \xi_n v \in L^2(B; \bigwedge^{p,q}, d\mu)$. Then the hypercontractivity of the semigroup $\{P_t\}$ implies that

$$P_t(\eta_\nu \xi_n v) \in \mathbf{D}^1_{2+\varepsilon}(\textstyle\bigwedge^{p,q+1}), \qquad \varepsilon = \varepsilon_t = e^t - 1. \tag{3.12}$$

By Assertion 4 and Lemma 2.2, we see that $\eta_{\nu+1}\xi_{n+1} P_t(\eta_\nu \xi_n v) \in \mathrm{Dom}(T_{\sigma,\gamma}^{(p,q)*})$ and that

$$\begin{aligned} T_{\sigma,\gamma}^{(p,q)*}&(\eta_{\nu+1}\xi_{n+1} P_t(\eta_\nu \xi_n v)) \\ &= e^{-\gamma}[\eta_{\nu+1}\xi_{n+1}\{T^{(p,q)*}(P_t(\eta_\nu \xi_n v)) + i(\partial(\sigma+\gamma))\, P_t(\eta_\nu \xi_n v)\} \\ &\quad - i(\partial(\eta_{\nu+1}\xi_{n+1}))\, P_t(\eta_\nu \xi_n v)] \\ &= e^{-\gamma}[\eta_{\nu+1}\xi_{n+1}\{e^{-t} P_t(T^{(p,q)*}(\eta_\nu \xi_n v)) + i(\partial(\sigma+\gamma))\, P_t(\eta_\nu \xi_n v)\} \\ &\quad - i(\partial(\eta_{\nu+1}\xi_{n+1}))\, P_t(\eta_\nu \xi_n v)]. \end{aligned}$$

The nonnegativity of σ and γ an the strong continuity of $\{P_t\}$ on $L^2(B; \bigwedge^{p,q+1}, d\mu)$ then imply that Assertion 5 holds. ▮

This lemma implies another property of $T_{\sigma,\gamma}^{(p,q)}$.

LEMMA 3.7. *Let* $\sigma \in \mathscr{S}(\psi, \Omega)$ *and* $\gamma \in \Gamma(\psi, \Omega)$. *Consider* $\varphi : \Omega \to \mathbf{R}$ *satisfying that* $\mathbf{I}_\Omega\, g(\psi)\varphi \in \mathbf{S}^1(\mathbf{R})$ *for every* $g \in C_0^\infty(\mathbf{R}; \mathbf{R})$. *If* $u \in \mathrm{Dom}(T_{\sigma,\gamma}^{(p,q)})$, $\varphi u \in L^2(\Omega; \bigwedge^{p,q}, e^{-\sigma}\, d\mu)$, *and* $\varphi T_{\sigma,\gamma}^{(p,q)} u,\ \bar\partial\varphi \wedge u \in L^2(\Omega; \bigwedge^{p,q}, e^{-\sigma-\gamma}\, d\mu)$, *then* $\varphi u \in \mathrm{Dom}(T_{\sigma,\gamma}^{(p,q)})$ *and* $T_{\sigma,\gamma}^{(p,q)}(\varphi u) = \varphi T_{\sigma,\gamma}^{(p,q)} u + \bar\partial\varphi \wedge u$.

Proof. We first assume in addition that $\varphi \in \mathbf{D}^1_{\infty-}(\mathbf{R})$ and take $w \in \mathbf{D}^1_{\infty-,0}(\Omega; \bigwedge^{p,q+1})$ with $w = \eta_\nu \xi_n w$ for some ν and n. Then, by Lemma 3.6, we have

$$\begin{aligned} \langle \varphi u, T_{\sigma,\gamma}^{(p,q)*} w\rangle_\sigma &= \langle \varphi u, e^{-\gamma}\{T^{(p,q)*} w + \eta_{\nu+1}\xi_{n+1}\, i(\partial(\sigma+\gamma))w\}\rangle_\sigma \\ &= \langle u, \varphi T^{(p,q)*}(e^{-\sigma-\gamma} w)\rangle_0 \\ &= \langle u, T^{(p,q)*}(e^{-\sigma-\gamma}\varphi w) + i(\partial\varphi)\, e^{-\sigma-\gamma} w\rangle_0 \\ &= \langle u, T_{\sigma,\gamma}^{(p,q)*}(\varphi w)\rangle_\sigma + \langle u, i(\partial\varphi)\, e^{-\sigma-\gamma} w\rangle_0 \\ &= \langle \varphi T_{\sigma,\gamma}^{(p,q)} u + \bar\partial\varphi \wedge u, w\rangle_{\sigma+\gamma}. \end{aligned}$$

Since such w's are dense in $\mathrm{Dom}(T_{\sigma,\gamma}^{(p,q)*})$, we obtain that the assertion holds if $\varphi \in \mathbf{D}^1_{\infty-}(\mathbf{R})$.

For general φ, without loss of generality, we may assume that ξ_n also satisfies that $\eta_\nu \xi_n \varphi \in \mathbf{D}^1_{\infty-}(\mathbf{R})$, $\nu, n = 1, 2, \ldots$. By virtue of the above observation, we have

$$\eta_\nu \xi_n \varphi u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma}),$$
$$T^{(p,q)}_{\sigma,\gamma}(\eta_\nu \xi_n \varphi u) = \eta_\nu \xi_n \varphi T^{(p,q)}_{\sigma,\gamma} u + \bar{\partial}(\eta_\nu \xi_n \varphi) \wedge u.$$

As in the proof of Lemma 3.6, letting $\nu \to \infty$ and $n \to \infty$, we obtain the desired conclusion. ∎

We now show

LEMMA 3.8. *Let $\sigma \in \mathscr{S}(\psi, \Omega)$ and $\gamma \in \Gamma(\psi, \Omega)$. For each $u \in \mathrm{Dom}(T^{(p,q+1)}_{\sigma+\gamma,\gamma}) \cap \mathrm{Dom}(T^{(p,q)*}_{\sigma,\gamma})$, there is a sequence $\{u_n\} \in \mathbf{D}^1_{\infty-,0}(\Omega; \wedge^{p,q})$ converging to u in $\mathrm{Dom}(T^{(p,q+1)}_{\sigma+\gamma,\gamma}) \cap \mathrm{Dom}(T^{(p,q)*}_{\sigma,\gamma})$ as $n \to \infty$.*

Proof. Let $u \in \mathrm{Dom}(T^{(p,q+1)}_{\sigma+\gamma,\gamma}) \cap \mathrm{Dom}(T^{(p,q)*}_{\sigma,\gamma})$. By Lemmas 3.5 and 3.6, it suffices to approximate $\eta_{\nu+1}\xi_{n+1}P_t(\eta_\nu \xi_n u)$, $t > 0$. By (3.12), we can choose $\{w_m\} \subset \mathbf{D}^1_{\infty-}(\wedge^{p,q+1})$ converging to $P_t(\eta_\nu \xi_n u)$ in $\mathbf{D}^1_{2+\varepsilon_t}(\wedge^{p,q+1})$. On account of (3.8), we have that

$$\eta_{\nu+1}\xi_{n+1} w_m \to \eta_{\nu+1}\xi_{n+1} P_t(\eta_\nu \xi_n u) \qquad \text{in } \mathbf{D}^1_{2+\varepsilon_t}(\wedge^{p,q+1}).$$

Combining this with Lemmas 2.1 and 2.2, and the continuity of D and D^*, we see that

$$\eta_{\nu+1}\xi_{n+1} w_m \to \eta_{\nu+1}\xi_{n+1} P_t(\eta_\nu \xi_n u) \qquad \text{in } \mathrm{Dom}(T^{(p,q+1)}) \cap \mathrm{Dom}(T^{(p,q)*}).$$

By Lemma 3.5(1), Lemma 3.6(4), the nonnegativity of σ and γ, and the boundedness of $D(\sigma+\gamma)$ on $\mathrm{supp}[\eta_{\nu+1}\xi_{n+1}]$, we see that the convergence also takes place in $\mathrm{Dom}(T^{(p,q+1)}_{\sigma,\gamma}) \cap \mathrm{Dom}(T^{(p,q)*}_{\sigma,\gamma})$. ∎

We are now ready to show a key estimation analogous to Lemma 2.3.

LEMMA 3.9. *Let $\sigma \in \mathscr{S}(\psi, \Omega)$ and $\gamma \in \Gamma(\psi, \Omega)$. Suppose that $\varepsilon I + \partial\bar{\partial}(\sigma + 2\gamma) \geqslant \varepsilon^{-1} \partial\gamma \otimes \bar{\partial}\gamma$ holds for some $0 < < \frac{1}{2}$. Then, for every $u \in \mathrm{Dom}(T^{(p,q+1)}_{\sigma+\gamma,\gamma}) \cap \mathrm{Dom}(T^{(p,q)*}_{\sigma,\gamma})$, it holds that*

$$\| T^{(p,q)*}_{\sigma,\gamma} u \|^2_\sigma + \| T^{(p,q+1)}_{\sigma+\gamma,\gamma} u \|^2_{\sigma+2\gamma} \geqslant (1 - 2\varepsilon) \| u \|^2_{\sigma+2\gamma}. \tag{3.13}$$

Proof. On account of Lemmas 3.5, 3.6, and 3.8, it suffices to show (3.13) for $u \in \mathbf{D}^1_{\infty-,0}(\Omega; \wedge^{p,q+1})$ with $u = \eta_\nu \xi_n u$ for some ν, n. Define

$$\sigma' = \eta_{\nu+2}\xi_{n+2}\sigma, \qquad \gamma' = \eta_{\nu+2}\xi_{n+2}\gamma.$$

Since $\sigma, \gamma \in \mathscr{S}(\psi, \Omega)$, $\sigma', \gamma', e^{-\sigma'/2}$ are all in $\mathbf{D}^1_{\infty-}(\mathbf{R})$. According to

Lemmas 2.1, 2.2, 3.5, and 3.6, and the property that $i(\partial(\eta_{\nu+1}\xi_{n+1}))u=0$, we then have that

$$\begin{aligned}
&\|T^{(p,q)*}_{\sigma,\gamma}u\|^2_\sigma+\|T^{(p,q+1)}_{\sigma+\gamma,\gamma}u\|^2_{\sigma+2\gamma}\\
&\quad=\|e^{-\gamma'}\eta_{\nu+1}\xi_{n+1}\{T^{(p,q)*}u+i(\partial(\sigma'+\gamma'))u\}\|^2_{\sigma'}+\|T^{(p,q+1)}u\|^2_{\sigma'+2\gamma'}\\
&\quad=\|T^{(p,q)*}(e^{-\sigma'/2-\gamma'}u)+\tfrac{1}{2}i(\partial\sigma')(e^{-\sigma'/2-\gamma'}u)\|^2_0\\
&\qquad+\|T^{(p,q+1)}(e^{-\sigma'/2-\gamma'}u)+\tfrac{1}{2}\bar\partial(\sigma'+2\gamma')\wedge(e^{-\sigma'/2-\gamma'}u)\|^2_0\\
&\quad\geqslant\|T^{(p,q)*}\tilde u\|^2_0+\|T^{(p,q+1)}\tilde u\|^2_0\\
&\qquad+\mathrm{Re}\{\langle T^{(p,q)*}\tilde u,i(\partial\sigma')\tilde u\rangle_0+\langle T^{(p,q+1)}\tilde u,\bar\partial(\sigma'+2\gamma')\wedge\tilde u\rangle_0\},
\end{aligned}\tag{3.14}$$

where $\tilde u=e^{-(\sigma'/2)-\gamma'}u\in\mathbf{D}^1_{\infty-}(\wedge^{p,q+1})$. Note that

$$\xi_{n+1}\,D\sigma'\in\mathbf{D}^1_{\infty-}(H^*)\qquad\text{and}\qquad\xi_n\,D(\xi_{n+1}\,D\sigma')=\xi_n\,D^2\sigma,$$

and the same assertion holds for γ. Thus we obtain

$$\begin{aligned}
&\mathrm{Re}\langle T^{(p,q)*}\tilde u,i(\partial(\sigma'+2\gamma'))\tilde u\rangle_0\\
&\quad=-\mathrm{Re}\langle i(\partial(\sigma'+2\gamma'))\,T^{(p,q+1)}\tilde u,\tilde u\rangle_0+\mathrm{Re}\langle\mathrm{Der}(\partial\bar\partial(\sigma+2\gamma))\,\tilde u,\tilde u\rangle_0\\
&\quad=-\mathrm{Re}\langle T^{(p,q+1)}\tilde u,\bar\partial(\sigma'+2\gamma'))\wedge\tilde u\rangle_0+\mathrm{Re}\langle\mathrm{Der}(\partial\bar\partial(\sigma+2\gamma))\tilde u,\tilde u\rangle_0,
\end{aligned}$$

where for bounded linear $A:H^{*\mathbf{C}}\to H^{*\mathbf{C}}$, $\mathrm{Der}(A):(H^{*\mathbf{C}})^{\otimes n}\to(H^{*\mathbf{C}})^{\otimes n}$ is given by $\mathrm{Der}(A)(l_1\otimes\cdots\otimes l_n)=\sum_{j=1}^n l_1\otimes\cdots\otimes l_{j-1}fAl_j\otimes l_{j+1}\otimes\cdots\otimes l_n$. Thus, taking advantage of Lemma 2.3, we have

$$\begin{aligned}
&\|T^{(p,q)*}_{\sigma,\gamma}u\|^2_\sigma+\|T^{(p,q+1)}_{\sigma+\gamma,\gamma}u\|^2_{\sigma+2\gamma}\\
&\quad\geqslant\|T^{(p,q)*}\tilde u\|^2_0+\|T^{(p,q+1)}\tilde u\|^2_0\\
&\qquad+\mathrm{Re}\langle\mathrm{Der}(\partial\bar\partial(\sigma+2\gamma))\tilde u,\tilde u\rangle_0-2\,\mathrm{Re}\langle T^{(p,q)*}\tilde u,i(\partial\gamma')\tilde u\rangle_0\\
&\quad\geqslant(1-\varepsilon)\,\|T^{(p,q)*}\tilde u\|^2_0+\|T^{(p,q+1)}\tilde u\|^2_0\\
&\qquad+\mathrm{Re}\langle\mathrm{Der}(\partial\bar\partial(\sigma+2\gamma))\tilde u,\tilde u\rangle_0-\frac{1}{\varepsilon}\,\|i(\partial\gamma')\tilde u\|^2_0\\
&\quad\geqslant(1-\varepsilon)\,\|\tilde u\|^2_0+\mathrm{Re}\left\langle\mathrm{Der}\left(\partial\bar\partial(\sigma+2\gamma)-\frac{1}{\varepsilon}\,\partial\gamma' f\bar\partial\gamma'\right)\tilde u,\tilde u\right\rangle_0\\
&\quad\geqslant(1-2\varepsilon)\,\|u\|^2_{\sigma+2\gamma},
\end{aligned}$$

since $(\varepsilon/2)I+\partial\bar\partial(\sigma+2\gamma)-(1/\varepsilon)\,\partial\gamma'\otimes\bar\partial\gamma'\geqslant0$ on $\mathrm{supp}[\tilde u]$. The proof is completed. ∎

We now proceed to the proofs of the theorems.

Proof of Theorem 3.2. Fix an arbitrary $f \in \mathrm{Ker}(T^{(p,q+1)}_{\sigma+\gamma,\gamma}) \cap L^2(\Omega; \wedge^{p,q+1}, e^{-\sigma}\, d\mu)$. By virtue of Lemma 3.5(6), $(\mathrm{Ker}(T^{(p,q+1)}_{\sigma+\gamma,\gamma}))^{\perp}$, the orthogonal complement of $\mathrm{Ker}(T^{(p,q+1)}_{\sigma+\gamma,\gamma})$ in $L^2(\Omega; \wedge^{p,q+1}, e^{-\sigma-\gamma}\, d\mu)$, is contained in $\mathrm{Ker}(T^{(p,q)*}_{\sigma,\gamma})$. Let P be the orthogonal projection of $L^2(\Omega; \wedge^{p,q+1}, e^{-\sigma-\gamma}\, d\mu)$ onto $(\mathrm{Ker}(T^{(p,q+1)}_{\sigma+\gamma,\gamma}))^{\perp}$ and $v \in \mathrm{Dom}(T^{(p,q)*}_{\sigma,\gamma})$. Then

$$(I-P)v \in (\mathrm{Dom}(T^{(p,q)*}_{\sigma,\gamma}) \qquad \text{and} \qquad T^{(p,q)*}_{\sigma,\gamma}((I-P)v) = T^{(p,q)*}_{\sigma,\gamma} v.$$

By applying Lemma 3.9, we have

$$\begin{aligned} |\langle f, v\rangle_{\sigma+\gamma}| &= |\langle f, (I-P)v\rangle_{\sigma+\gamma}| \\ &\leqslant \| f\|_{\sigma}\, \|(I-P)v\|_{\sigma+2\gamma} \\ &\leqslant (1-2\varepsilon)^{-1/2}\, \|f\|_{\sigma}\, \|T^{(p,q)*}_{\sigma,\gamma} v\|_{\sigma}. \end{aligned}$$

Applying Hahn–Banach's theorem, we see the existence of $u \in L^2(\Omega; \wedge^{p,q}, e^{-\sigma}\, d\mu)$ such that $\|u\|_{\sigma} \leqslant (1-2\varepsilon)^{-1/2}\, \|f\|_{\sigma}$ and

$$\langle f, v\rangle_{\sigma+\gamma} = \langle u, T^{(p,q)*}_{\sigma,\gamma} v\rangle_{\sigma} \qquad \text{for any} \quad v \in \mathrm{Dom}(T^{(p,q)*}_{\sigma,\gamma}).$$

Hence $u \in \mathrm{Dom}(T^{(p,q)}_{\sigma,\gamma})$ and $T^{(p,q)}_{\sigma,\gamma} u = f$. ∎

Proof of Theorem 3.3. Choose a strictly increasing sequence $\{c_n\}$ such that

$$\text{ess.}\sup\{\gamma(z) : \sigma_0(z) < n\} \leqslant c_n.$$

Take $g_n, \varphi, \chi \in C^{\infty}(\mathbf{R}; \mathbf{R})$ so that

$$\begin{gathered} g_n'' \geqslant 0, \qquad 0 \leqslant g_n' \leqslant 1, \qquad g_n'(x) = 0, \\ x \in (-\infty, c_{n+1}), \qquad g_n(x) = x, \qquad x \in (c_{n+2}, \infty) \\ \varphi' \geqslant 0, \qquad \varphi(x) = 0, \qquad x \in (-\infty, 0], \\ \varphi(x) = 1, \qquad x \in (1, \infty) \qquad \chi' \geqslant 0, \qquad \chi(n_0) = 1. \end{gathered}$$

Define

$$\begin{aligned} G_n(t) &= \int_0^t \varphi(s-n+1)\,\chi(s)\, ds \\ \sigma_n(z) &= G_n(\sigma_n(z)) + \sigma(z) \\ \gamma_n(z) &= g_n(\gamma(z)) - g_n(c_{n+1}). \end{aligned}$$

Then $\sigma_n \in \mathscr{S}(\psi, \Omega)$, $\gamma_n \in \Gamma(\psi, \Omega)$, and

$$\varepsilon I + \partial\bar{\partial}(\sigma_n + 2\gamma_n) \geqslant \frac{1}{\varepsilon}\, \partial\gamma_n \otimes \bar{\partial}\gamma_n \qquad \mu\text{-a.e. on } \Omega \text{ for } n > n_0.$$

Indeed, on $\{\gamma < c_{n+1}\}$, the inequality is easily obtained by using that $\partial\bar{\partial}\sigma_0 \geqslant 0$ and $\partial\bar{\partial}\sigma \geqslant 0$. On $\{\gamma > c_{n+1}\}$, the inequality follows from assumption (3.3) and that $g_n' \leqslant 1$. Let $f \in \operatorname{Ker}(T_{\sigma,\gamma}^{(p,q+1)})$. Since $\sigma \leqslant \sigma_n$ and $\gamma \leqslant \gamma_n + g_n(c_{n+1})$,

$$f \in \operatorname{Ker}(T_{\sigma_n+\gamma_n,\gamma_n}^{(p,q+1)}) \cap L^2(\Omega; \textstyle\bigwedge^{p,q+1}, e^{-\sigma_n}\, d\mu).$$

As an application of Theorem 3.2, we see that there is a $u_n \in \operatorname{Dom}(T_{\sigma_n,\gamma_n}^{(p,q)})$ such that

$$\|u_n\|_{\sigma_n} \leqslant (1-2\varepsilon)^{-1/2}\, \|f\|_{\sigma_n} \leqslant (1-2\varepsilon)^{-1/2}\, \|f\|_{\sigma} \tag{3.15}$$

$$T_{\sigma_n,\gamma_n}^{(p,q)} u_n = f. \tag{3.16}$$

In particular, since $\sigma = \sigma_n$ on $\{\sigma_0 < n-1\}$,

$$\int_{\{\sigma_0 < n-1\}} \|u_n\|_{\mathrm{AHS}}^2\, e^{-\sigma}\, d\mu \leqslant (1-2\varepsilon)^{-1}\, \|f\|_\sigma^2.$$

Thus $\mathbf{1}_{\{\sigma_0 < n-1\}} u_n \to u$ weakly in $L^2(\Omega; \bigwedge^{p,q}, e^{-\sigma}\, d\mu)$ for some $u \in L^2(\Omega; \bigwedge^{p,q}, e^{-\sigma}\, d\mu)$.

Without loss of generality, we may and do assume that $\eta_\nu \sigma_0$ is also in $\mathbf{S}_K^2(\mathbf{R}) \cap \mathbf{S}_{K,b}^1(\mathbf{R})$. Let $v \in \mathbf{D}_{\infty-,0}^1(\Omega; \bigwedge^{p,q+1})$ and set $v_{\nu,m} = \eta_\nu \xi_m v$. Then, for $n \geqslant \operatorname{ess.sup}\{\sigma_0(z) : z \in \operatorname{supp}[\eta_\nu \xi_m]\}$, we have

$$\begin{aligned}
\langle \mathbf{1}_{\{\sigma_0 < n-1\}} u_n, T_{\sigma,\gamma}^{(p,q)*} v_{\nu,m} \rangle_\sigma &= \langle u_n, T_{\sigma,\gamma}^{(p,q)*} v_{\nu,m} \rangle_\sigma \\
&= \langle u_n, T_{\sigma_n,\gamma_n}^{(p,q)*} (e^{-\sigma-\gamma+\sigma_n+\gamma_n} v_{\nu,m}) \rangle_{\sigma_n} \\
&= \langle f, e^{-\sigma-\gamma+\sigma_n+\gamma_n} v_{\nu,m} \rangle_{\sigma_n+\gamma_n} \\
&= \langle f, v_{\nu,m} \rangle_{\sigma+\gamma}.
\end{aligned}$$

Letting $n \to \infty$, $\nu \to \infty$, and then $m \to \infty$, we have

$$\langle u, T_{\sigma,\gamma}^{(p,q)*} v \rangle_\sigma = \langle f, v \rangle_{\sigma+\gamma}, \qquad v \in \mathbf{D}_{\infty-,0}^1(\Omega, \textstyle\bigwedge^{p,q+1}).$$

We therefore have that $u \in \operatorname{Dom}(T_{\sigma,\gamma}^{(p,q)})$ and $T_{\sigma,\gamma}^{(p,q)} u = f$. ∎

Proof of Theorem 3.4. The inclusion that $\operatorname{Image}(\hat{T}_{\sigma,\gamma}^{(p,q)}) \supset \operatorname{Ker}(\hat{T}_{\sigma,\gamma}^{(p,q+1)})$ follows from Theorem 3.3.

Let $u \in \bigwedge_{\sigma,\gamma}^{p,q}(\Omega)$. Then $T_{\sigma,\gamma}^{(p,q)} u \in L^2(\Omega; \bigwedge^{p,q+1}, e^{-\sigma}\, d\mu)$. Moreover, by Lemma 3.5, there exist $u_n \in \mathbf{D}_{\infty-,0}^1(\Omega; \bigwedge^{p,q})$, $\nu(n)$, and $m(n)$ such that

$$\begin{aligned}
u_n &= \eta_{\nu(n)} \xi_{m(n)} u_n \\
u_n &\to u && \text{in } L^2(\Omega; \textstyle\bigwedge^{p,q}, e^{-\sigma}\, d\mu) \\
T_{\sigma,\gamma}^{(p,q)} u_n &\to T_{\sigma,\gamma}^{(p,q)} u && \text{in } L^2(\Omega; \textstyle\bigwedge^{p,q+1}, e^{-\sigma-\gamma}\, d\mu).
\end{aligned}$$

Note that $T^{(p,q)}_{\sigma,\gamma}u_n \in \operatorname{Ker}(T^{(p,q+1)}_{\sigma,\gamma})$. Indeed, it follows from Lemma 2.1 that $T^{(p,q)}_{\sigma,\gamma}u_n = T^{(p,q)}u_n \in \operatorname{Ker}(T^{(p,q+1)})$. Lemma 3.5 then implies that $T^{(p,q)}_{\sigma,\gamma}u_n = \eta_{v(n)+1}\xi_{m(n)+1}T^{(p,q)}u_n \in \operatorname{Ker}(T^{(p,q+1)}_{\sigma,\gamma})$.

Choose $w \in \mathbf{D}^1_{\infty-,0}(\Omega; \wedge^{p,q+2})$ with $w = \eta_v \xi_m w$ for some v and m. Note that $(T^{(p,q)*}_{\sigma,\gamma}w)\,e^{\gamma} \in L^2(\Omega; \wedge^{p,q}, e^{-\sigma-\gamma}\,d\mu)$. We then have

$$\begin{aligned}\langle T^{(p,q+1)*}_{\sigma,\gamma}w, T^{(p,q)}_{\sigma,\gamma}u\rangle_\sigma &= \langle (T^{(p,q+1)*}_{\sigma,\gamma}w)\,e^{\gamma}, T^{(p,q)}_{\sigma,\gamma}u\rangle_{\sigma+\gamma} \\ &= \lim_{n\to\infty} \langle (T^{(p,q+1)*}_{\sigma,\gamma}w)\,e^{\gamma}, T^{(p,q)}_{\sigma,\gamma}u_n\rangle_{\sigma+\gamma} \\ &= 0.\end{aligned}$$

Since $\mathbf{D}^1_{\infty-,0}(\Omega; \wedge^{p,q+2})$ is dense in $\operatorname{Dom}(T^{(p,q+1)*}_{\sigma,\gamma})$, we obtain that $T^{(p,q)}_{\sigma,\gamma}u \in \operatorname{Ker}(T^{(p,q+1)}_{\sigma,\gamma})$. Thus $\hat{T}^{(p,q)}_{\sigma,\gamma}u \in \operatorname{Ker}(\hat{T}^{(p,q+1)}_{\sigma,\gamma})$. ∎

4. An Approximation Theorem

We shall establish an approximation theorem as an application of Lemma 3.9. We set

$$\mathscr{P}\mathscr{S}(\psi, \Omega) = \{\sigma \in \mathscr{S}(\psi, \Omega) : \partial\bar{\partial}\sigma \geqslant 0\ \mu\text{-a.e. on } \Omega\}.$$

Throughout this section, we consider $\sigma_0 \in \mathscr{P}\mathscr{S}(\psi, \Omega)$, $\sigma \in \mathscr{S}(\psi, \Omega)$, and $\gamma \in \Gamma(\psi, \Omega)$ such that for some $\sigma_1 \in \mathscr{S}(\psi, \Omega)$ and $\varepsilon \in (0, \frac{1}{2})$,

$$\operatorname{ess.sup}\{\sigma_1(z) : \sigma_0(z) < 1\} < \infty \tag{4.1}$$

$$\varepsilon I + \partial\bar{\partial}(\sigma + \sigma_1 + 2\gamma) \geqslant \frac{1}{\varepsilon}\,\partial\gamma \otimes \bar{\partial}\gamma \qquad \mu\text{-a.e. on } \Omega. \tag{4.2}$$

Set

$$\mathscr{A}(\sigma) = \bigcup \{\operatorname{Ker}(T^{(p,0)}_{\sigma+\alpha,\gamma}) : \alpha \in \mathscr{S}(\psi, \Omega), \operatorname{ess.sup}\{\alpha(z) : \sigma_0(z) < 1\} < \infty\}$$

$$\Omega_{1-\delta} = \{z \in \Omega : \sigma_0(z) < 1 - \delta\}, \qquad \delta \in (0, 1)$$

Theorem 4.1. *For each $u \in \wedge^{p,0}_{\sigma,\gamma}(\Omega)$ with $T^{(p,0)}_{\sigma,\gamma}u|_{\Omega_1} = 0$, there is a sequence $\{u_n\} \subset \mathscr{A}(\sigma)$ such that*

$$\|u|_{\Omega_{1-\delta}} - u_n|_{\Omega_{1-\delta}}\|_\sigma \to 0 \qquad \textit{as} \quad n \to \infty.$$

Proof. Let $\mathscr{H} = \{u \in \wedge^{(p,0)}_{\sigma,\gamma}(\Omega) : T^{(p,0)}_{\sigma,\gamma}u|_{\Omega_1} = 0\}$. Choose an arbitrary but fixed $v \in L^2(\Omega; \wedge^{p,0}, e^{-\sigma}\,d\mu)$ such that $v = 0$ on $\Omega \setminus \Omega_{1-\delta}$ and $v \perp \mathscr{A}(\sigma)$ in $L^2(\Omega; \wedge^{p,0}, e^{-\sigma}\,d\mu)$. It then suffices to show that v is also orthogonal to $\mathscr{H}$.

We first claim that if $\tilde{\sigma} \in \mathscr{S}(\psi, \Omega)$ satisfies that

$$\tilde{\varepsilon} I + \partial\bar{\partial}(\sigma + \tilde{\sigma} + 2\gamma) \geqslant \tilde{\varepsilon}^{-1}\,\partial\gamma \otimes \bar{\partial}\gamma \tag{4.3}$$

$$\operatorname{ess.sup}\{\tilde{\sigma}(z) : \sigma_0(z) < 1\} \tag{4.4}$$

for some $0<\tilde{\varepsilon}<\frac{1}{2}$, then there exists a measurable $g\in L^2(\Omega;\bigwedge^{p,1},e^{\tilde{\sigma}-\sigma}\,d\mu)$ such that

$$\int_\Omega \|g\|^2_{\mathrm{AHS}}\,e^{\tilde{\sigma}-\sigma}\,d\mu\leq(1-2\tilde{\varepsilon})^{-1}\int_\Omega \|v\|^2\,e^{\tilde{\sigma}-\sigma}\,d\mu \tag{4.5}$$

$$\int_\Omega \langle v,w\rangle_{\mathrm{AHS}}\,e^{-\sigma}\,d\mu=\int_\Omega\langle g,T^{(p,0)}_{\sigma,\gamma}w\rangle_{\mathrm{AHS}}\,e^{-\sigma}\,d\mu\qquad\text{for}\quad w\in\textstyle\bigwedge^{(p,0)}_{\sigma,\gamma}(\Omega). \tag{4.6}$$

To see this, note that

$$\begin{aligned}ve^{\tilde{\sigma}}&\in\mathrm{Ker}(T^{(p,0)}_{\tilde{\sigma}+\sigma,\gamma})^{\perp}\ (\text{in } L^2(\Omega;\textstyle\bigwedge^{p,0},e^{-\tilde{\sigma}-\sigma}\,d\mu))\\&=\overline{\mathrm{Image}(T^{(p,0)*}_{\tilde{\sigma}+\sigma,\gamma})}^{\,L^2(\Omega;\bigwedge^{p,0},e^{-\tilde{\sigma}-\sigma}d\mu)}\\&=\overline{T^{(p,0)*}_{\tilde{\sigma}+\sigma,\gamma}[\mathrm{Dom}(T^{(p,0)*}_{\tilde{\sigma}+\sigma,\gamma})\cap\mathrm{Ker}(T^{(p,1)}_{\tilde{\sigma}+\sigma+\gamma,\gamma})]}^{\,L^2(\Omega;\bigwedge^{p,0},e^{-\tilde{\sigma}-\sigma}d\mu)}.\end{aligned} \tag{4.7}$$

It follows from Lemma 3.9 that

$$\begin{aligned}\|f\|_{\tilde{\sigma}+\sigma+2\gamma}&\leq(1-2\tilde{\varepsilon})^{-1/2}\,\|T^{(p,0)*}_{\tilde{\sigma}+\sigma,\gamma}f\|_{\tilde{\sigma}+\sigma},\\ &\qquad f\in\mathrm{Dom}(T^{(p,0)*}_{\tilde{\sigma}+\sigma,\gamma})\cap\mathrm{Ker}(T^{(p,1)}_{\tilde{\sigma}+\sigma+\gamma,\gamma}).\end{aligned} \tag{4.8}$$

Hence there are $\{f_n\}\in\mathrm{Dom}(T^{(p,0)*}_{\tilde{\sigma}+\sigma,\gamma})\cap\mathrm{Ker}(T^{(p,1)}_{\tilde{\sigma}+\sigma+\gamma,\gamma})$ and $f\in L^2(\Omega;\bigwedge^{p,1},e^{-\tilde{\sigma}-\sigma-2\gamma}\,d\mu)$, such that

$$\begin{aligned}f_n&\to f &&\text{in } L^2(\Omega;\textstyle\bigwedge^{p,1},e^{-\tilde{\sigma}-\sigma-2\gamma}\,d\mu),\\ T^{(p,0)*}_{\tilde{\sigma}+\sigma,\gamma}f_n&\to ve^{\tilde{\sigma}} &&\text{in } L^2(\Omega;\textstyle\bigwedge^{p,0},e^{-\tilde{\sigma}-\sigma}\,d\mu).\end{aligned}$$

Let $g=fe^{-\tilde{\sigma}-\gamma}$. Equation (4.8) then implies (4.5). To see (4.6), let $w\in\bigwedge^{p,0}_{\sigma,\gamma}(\Omega)$. Then we have

$$\begin{aligned}&\int_\Omega\langle v,w\rangle_{\mathrm{AHS}}\,e^{-\sigma}\,d\mu\\&=\lim_{n\to\infty}\int_\Omega\langle T^{(p,0)*}_{\tilde{\sigma}+\sigma,\gamma}f_n,w\rangle_{\mathrm{AHS}}\,e^{-\tilde{\sigma}-\sigma}\,d\mu\\&=\lim_{n\to\infty}\int_\Omega\langle f_n,T^{(p,0)}_{\sigma,\gamma}w\rangle_{\mathrm{AHS}}\,e^{-\tilde{\sigma}-\sigma-\gamma}\,d\mu\\&=\lim_{n\to\infty}\int_\Omega\langle f_n e^{-(\tilde{\sigma}+\sigma+2\gamma)/2},(T^{(p,0)}_{\sigma,\gamma}w)\,e^{-(\tilde{\sigma}+\sigma)/2}\rangle_{\mathrm{AHS}}\,d\mu\\&=\int_\Omega\langle f,T^{(p,0)}_{\sigma,\gamma}w\rangle_{\mathrm{AHS}}\,e^{-\tilde{\sigma}-\sigma-\gamma}\,d\mu,\end{aligned}$$

where to see the second identity we have used the fact that $\mathrm{Dom}(T_{\sigma,\gamma}^{(p,0)}) \subset \mathrm{Dom}(T_{\tilde{\sigma}+\sigma,\gamma}^{(p,0)})$ and $T_{\sigma,-}^{(p,0)} = T_{\tilde{\sigma}+\sigma,\gamma}^{(p,0)}$ on $\mathrm{Dom}(T_{\sigma,\gamma}^{(p,0)})$. Hence (4.6) has been verified.

Choose a non-decreasing, convex $\chi \in C^{\infty}(\mathbf{R};\mathbf{R})$ such that $\chi(x)=0$ on $(-\infty, 1-2\delta]$ and $\chi(1-\delta) \geqslant 1$. If we put

$$\tilde{\sigma}_n = n\chi(\sigma_0) + \sigma_1,$$

then it is easily seen that $\tilde{\sigma}_n \in \mathscr{S}(\psi, \Omega)$ and satisfies (4.3) and (4.7). We therefore obtain $g_n \in L^2(\Omega; \bigwedge^{p,1}, e^{\tilde{\sigma}_n - \sigma}\, d\mu)$ such that

$$\sup_n \int_{\Omega} \|g_n\|_{\mathrm{AHS}}^2\, e^{\tilde{\sigma}_n - \sigma}\, d\mu < \infty \tag{4.9}$$

$$\int_{\Omega} \langle v, w\rangle_{\mathrm{AHS}}\, e^{-\sigma}\, d\mu = \int_{\Omega} \langle g_n, T_{\sigma,\gamma}^{(p,0)} w\rangle_{\mathrm{AHS}}\, e^{-\sigma}\, d\mu, \qquad w \in \textstyle\bigwedge_{\sigma,\gamma}^{p,q}(\Omega). \tag{4.10}$$

Since $\tilde{\sigma}_n \leqslant \tilde{\sigma}_{n+1}$, choosing a subsequence if necessary, we may assume that $\{g_n\}$ converges weakly in $L^2(\Omega; \bigwedge^{p,1}, e^{\sigma_1 - \sigma}\, d\mu)$ to a limit g. Moreover, we have

$$\int_{\{\sigma_0 \geqslant 1-\delta\}} \|g_n\|_{\mathrm{AHS}}^2\, e^{\sigma_1 - \sigma}\, d\mu \leqslant e^{-n} \sup_k \int_{\Omega} \|g_k\|_{\mathrm{AHS}}^2\, e^{\tilde{\sigma}_k - \sigma}\, d\mu \to 0 \qquad \text{as} \quad n \to \infty. \tag{4.11}$$

Since σ_1 is essentially bounded on Ω_1 and v vanishes outside of $\Omega_{1-\delta}$, it follows from (4.10) and (4.11) that

$$\int_{\Omega} \langle v, w\rangle_{\mathrm{AHS}}\, e^{-\sigma}\, d\mu = \int_{\Omega_{1-\delta}} \langle g, T_{\sigma,\gamma}^{(p,0)} w\rangle_{\mathrm{AHS}}\, e^{-\sigma}\, d\mu, \qquad \text{for any} \quad w \in \textstyle\bigwedge_{\sigma,\gamma}^{p,0}(\Omega).$$

Hence $v \perp \mathscr{H}$. ∎

5. Quasi-Sheaves and Cohomology

We first introduce the notion of quasi-sheaves. Let $\mathbf{I}$ be an arbitrary set and

$$\mathbf{I}_0^n = \{I = (i_1, ..., i_n) \in \mathbf{I}^n : i_j \neq i_k \text{ if } j \neq k\},$$

$$\mathbf{I}_{00}^n = \{J \subset \mathbf{I} : \# J = n\},$$

$$\mathbf{I}_{00} = \bigcup_{n=1}^{\infty} \mathbf{I}_{00}^n.$$

For $I \in \mathbf{I}_{00}$, $J(I)$ denotes the subset of $\mathbf{I}$ consisting of the components of I.

We call a pair $\mathscr{V} = (\{V_J : J \in \mathbf{I}_{00}\}, \{\pi_{JJ'} : J, J' \in \mathbf{I}_{00}, J \subset J'\})$ a quasi-sheaf (indexed by $\mathbf{I}$) if

(i) each V_J is a vector space,

(ii) $\pi_{JJ'} : V_J \to V_{J'}$ is linear,

(iii) $\pi_{KJ'} \circ \pi_{JK} = \pi_{K'J'} \circ \pi_{JK'}$, if $K, K' \subset J'$, $J \subset K, K'$.

For quasi-sheaves $\mathscr{V}^{(j)} = (\{V_J^{(j)}\}, \{\pi_{JJ'}^{(j)}\})$, $j = 1, 2$, a morphism $\varphi = \{\varphi_J : J \in \mathbf{I}_{00}\}$ from $\mathscr{V}^{(1)}$ to $\mathscr{V}^{(2)}$ is a family of linear mappings $\varphi_J : V_J^{(1)} \to V_J^{(2)}$ satisfying that $\pi_{JJ'}^{(2)} \circ \varphi_J = \varphi_{J'} \circ \pi_{JJ'}^{(1)}$. In this case, $\varphi\mathscr{V}^{(1)} \equiv (\{\varphi_J^{(1)} V_J^{(1)}\}, \{\pi_{JJ'}^{(2)}\})$, $\mathrm{Ker}(\varphi) \equiv (\{\mathrm{Ker}(\varphi_J)\}, \{\pi_{JJ'}^{(1)}\})$ are also quasi-sheaves.

Each permutation τ of $\{1, 2, ..., n+1\}$ acts on $I = (i_1, ..., i_{n+1}) \in \mathbf{I}_0^{n+1}$ as $\tau I = (i_{\tau(1)}, ..., i_{\tau(n+1)})$. We then set

$$\mathscr{C}_n(\mathscr{V}) = \{\{v_I : I \in \mathbf{I}_0^{n+1}\} : v_{\tau I} = \mathrm{sgn}(\tau) v_I \text{ for any permutation } \tau \text{ of } \{1, ..., n+1\}\}.$$

For $v_n = \{v_I\} \in \mathscr{C}_n(\mathscr{V})$, $\delta_n v_n = \{w_{I'} : I' \in \mathbf{I}_0^{n+2}\} \in \mathscr{C}_{n+1}(\mathscr{V})$ is given by

$$w_{I'} = \sum_{k=1}^{n+2} (-1)^{k+1} \pi_{J(\underline{I'}_k) J(I')} v_{\underline{I'}_k},$$

where $\underline{I'}_k = (i_1, ..., i_{k-1}, i_{k+1}, ..., i_{n+2})$. It is easily seen that $\delta_{n+1} \circ \delta_n = 0$ and hence we can define the nth cohomology group with coefficients in the quasi-sheaf $\mathscr{V}$ by

$$H^n(\mathscr{V}) = Z^n(\mathscr{V}) / \delta_{n-1} \mathscr{C}_{n-1}(\mathscr{V}), \qquad n \geqslant 0,$$

where $Z^n(\mathscr{V}) = \{v_n \in \mathscr{C}_n(\mathscr{V}) : \delta_n v_n = 0\}$ and $\delta_{-1} \mathscr{C}_{-1}(\mathscr{V}) = \{0\}$. We say that a quasi-sheaf $\mathscr{V}$ is fine if $H^n(\mathscr{V}) = 0$ for every $n \geqslant 1$.

Every morphism $\varphi = \{\varphi_J\}$ of $\mathscr{V}^{(1)}$ to $\mathscr{V}^{(2)}$ determines a linear mapping $\varphi_n : \mathscr{C}_n(\mathscr{V}^{(1)}) \to \mathscr{C}_n(\mathscr{V}^{(2)})$ so that $\varphi_n(\{v_I\}) = \{\varphi_{J(I)} v_I\}$. It is straightforward to show that $\delta_n \varphi_n = \varphi_{n+1} \delta_n$. We can therefore define a linear mapping $\varphi_{n*} : H^n(\mathscr{V}^{(1)}) \to H^n(\mathscr{V}^{(2)})$ by $\varphi_{n*}[v_n] = [\varphi_n v_n]$, $v_n \in Z^n(\mathscr{V})$, where $[\cdot]$ denotes the cosets.

For quasi-sheaves $\mathscr{V}^{(j)}$ and morphisms $\varphi^{(j)} = \{\varphi_J^{(j)}\}$ of $\mathscr{V}^{(j)}$ to $\mathscr{V}^{(j+1)}$, a sequence

$$\mathscr{V}^{(1)} \xrightarrow{\varphi^{(1)}} \mathscr{V}^{(2)} \xrightarrow{\varphi^{(2)}} \mathscr{V}^{(3)} \xrightarrow{\varphi^{(3)}} \cdots \xrightarrow{\varphi^{(n-1)}} \mathscr{V}^{(n)} \xrightarrow{\varphi^{(n)}} \mathscr{V}^{(n+1)} \xrightarrow{\varphi^{(n+1)}} \cdots$$

is said to be exact if $\operatorname{Ker}(\varphi_J^{(j+1)}) = \operatorname{Image}(\varphi_J^{(j)})$, $j = 1, 2, \ldots, J \in \mathbf{I}_{00}$. We have

PROPOSITION 5.1. *Assume that*

$$0 \xrightarrow{0} \mathscr{V} \xrightarrow{\iota} \mathscr{V}^{(0)} \xrightarrow{\varphi^{(0)}} \mathscr{V}^{(1)} \xrightarrow{\varphi^{(1)}} \cdots \xrightarrow{\varphi^{(n-1)}} \mathscr{V}^{(n)} \xrightarrow{\varphi^{(n)}} \mathscr{V}^{(n+1)} \xrightarrow{\varphi^{(n+1)}} \cdots \tag{5.1}$$

is an exact sequence of quasi-sheaves and that every $\mathscr{V}^{(j)}$ *is fine. Then*

$$H^n(\mathscr{V}) \simeq H^0(\varphi^{(n-1)}\mathscr{V}^{(n-1)})/\varphi_{0*}^{(n-1)} H^0(\mathscr{V}^{(n-1)}), \qquad n \geq 1. \tag{5.2}$$

Proof. Suppose that $0 \xrightarrow{0} \mathscr{V} \xrightarrow{\varphi} \mathscr{V}' \xrightarrow{\psi} \mathscr{V}'' \xrightarrow{0} 0$ is exact. By a standard argument, we see the existence of linear maps $\delta_{n*} : H^n(\mathscr{V}'') \to H^{n+1}(\mathscr{V})$ such that the sequence

$$0 \xrightarrow{0} H^0(\mathscr{V}) \xrightarrow{\varphi_{0*}} H^0(\mathscr{V}') \xrightarrow{\psi_{0*}} H^0(\mathscr{V}'') \xrightarrow{\delta_{0*}} H^1(\mathscr{V}) \xrightarrow{\varphi_{1*}} \cdots$$
$$\xrightarrow{\delta_{n-1*}} H^n(\mathscr{V}) \xrightarrow{\varphi_{n*}} H^n(\mathscr{V}') \xrightarrow{\psi_{n*}} H^n(\mathscr{V}'') \xrightarrow{\delta_{n*}} H^{n+1}(\mathscr{V}) \xrightarrow{\varphi_{n+1*}} \cdots$$

is exact. See [3, Chap. 3].

Note that $0 \xrightarrow{0} \varphi^{(n-1)}\mathscr{V}^{(n-1)} \subsetneq \mathscr{V}^{(n)} \xrightarrow{\varphi^{(n)}} \varphi^{(n)}\mathscr{V}^{(n)} \xrightarrow{0} 0$ is exact. Combining this with the above observation, we can conclude the desired equivalence. For details, see [3]. ∎

By the exactness of (5.1) and (5.2), in the same situation as that in Proposition 5.1, it holds that

$$H^n(\mathscr{V}) \simeq H^0(\operatorname{Ker}(\varphi^{(n)}))/\varphi_{0*}^{(n-1)} H^0(\mathscr{V}^{(n-1)}), \qquad n \geq 1. \tag{5.3}$$

6. A DOLBEAULT-TYPE THEOREM

Let $A \subset B$ be a measurable set and M be a Polish space. For measurable $f : A \to [0, \infty)$ and $g : A \to M$, we say that g is compactly dominated by f ($g \ll f$ in notation) if for each n there is a compact $L_n \subset M$ such that $\mu(g \notin L_n, f \leq n) = 0$. We set

$$\mathscr{X} = \{\chi : [0, \infty) \to [0, \infty) : \chi \text{ is } C^2, \text{ non-decreasing, and convex}\}.$$

We remark that every $f \in C([0, \infty); [0, \infty))$ possesses $\chi \in \mathscr{X}$ so that $f \leqslant \chi$. We have

PROPOSITION 6.1. *Let $\sigma, \gamma \in \mathscr{S}(\psi, \Omega)$ and assume that*

$$\partial\bar{\partial}\sigma \geqslant A \qquad \text{for some symmetric Hilbert Schmidt } A > 0 \tag{6.1}$$

$$(D\gamma, D^2\gamma) \ll \sigma. \tag{6.2}$$

Then there is a $\chi \in \mathscr{X}$ such that

$$\partial\bar{\partial}(\chi(\sigma) + 2\gamma) \geqslant 4\, \partial\gamma \otimes \bar{\partial}\gamma. \tag{6.3}$$

Moreover, for every $\chi_0 \in \mathscr{X}$, there is a $\chi_1 \in \mathscr{X}$ such that (6.3) *holds with $\chi = \chi_1$ and $\gamma = \chi_0(\sigma) + y$.*

Proof. It follows from (6.2) that there is an increasing function $g : [0, \infty) \to [0, \infty)$ such that

$$g(n) A \geqslant -2\, \partial\bar{\partial}\gamma + 4\, \partial\gamma \otimes \bar{\partial}\gamma \qquad \text{on} \quad \{\sigma \leqslant n\}.$$

Choosing $\chi \in \mathscr{X}$ such that $\chi' \geqslant g$, we can conclude from this and (6.1) that (6.3) holds.

Since

$$\partial(\chi_0(\sigma) + \gamma) \otimes \bar{\partial}(\chi_0(\sigma) + \gamma) \leqslant 2\{(\chi_0'(\sigma))^2\, \partial\sigma \otimes \bar{\partial}\sigma + \partial\gamma \otimes \bar{\partial}\gamma\},$$

to see the last assertion it suffices to choose $\chi_2 \in \mathscr{X}$ such that $\chi_2'(\sigma) A \geqslant -2\, \partial\bar{\partial}\gamma + 8\, \partial\gamma \otimes \bar{\partial}\gamma$ and set

$$\chi_1(x) = 2\chi_2(x) + 8 \int_0^x \int_0^y (\chi_0'(z))^2\, dz\, dy. \quad \blacksquare$$

For $\sigma \in \mathscr{S}(\psi, \Omega)$, we set

$$\begin{aligned} W_\sigma(\Omega; E) = \{u \in \mathbf{D}^1_{\infty-,0}(\Omega; E) : u = 0 \\ \text{and } Du = 0 \ \mu\text{-a.e. on } \{\sigma > n\} \text{ for some } n\} \end{aligned}$$

$$\begin{aligned} \mathscr{L}^2_\sigma(\Omega; E) &= \bigcup \{L^2(\Omega; E, e^{-f(\sigma)}\, d\mu) : f : [0, \infty) \to [0, \infty) \text{ is continuous}\} \\ &= \bigcup \{L^2(\Omega; E, e^{-\chi(\sigma)}\, d\mu) : \chi \in \mathscr{X}\}. \end{aligned}$$

To introduce a new class of differentiable (p, q)-forms on Ω, we prepare

LEMMA 6.2. (i) *If* $w \in W_\sigma(\Omega; \wedge^{p,q})$ *then* $T^{(p,q)*}w=0$ μ*-a.e. on* $\{\sigma > m\}$ *for some* m. *In particular, for any* $u \in \mathscr{L}^2_\sigma(\Omega; \wedge^{p,q})$ *and* $w \in W_\sigma(\Omega; \wedge^{p,q})$,

$$\int_\Omega |\langle u, T^{(p,q)*}w\rangle_{\mathrm{AHS}}|\, d\mu < \infty.$$

(ii) *If* $v \in \mathscr{L}^2_\sigma(\Omega; \wedge^{p,q+1})$ *satisfies that*

$$\int_\Omega \langle v, w\rangle_{\mathrm{AHS}}\, d\mu = 0, \qquad w \in W_\sigma(\Omega; \wedge^{p,q+1}),$$

then $v=0$ μ*-a.e. on* Ω.

Proof. Choose $g \in C^\infty(\mathbf{R}; \mathbf{R})$ with $g' \leqslant 0$, $g=1$ on $(-\infty, 0]$ and $=0$ on $[1, \infty)$. Fix an m with $w=0$ and $Dw=0$ on $\{\sigma > m\}$. Then, $w = g(\sigma - m)w$ and it holds that

$$\begin{aligned} T^{(p,q)*}(\eta_\nu \xi_n w) &= T^{(p,q)*}(\eta_\nu \xi_n g(\sigma - m)w) \\ &= T^{(p,q)*}(\eta_\nu \xi_n g(\eta_{\nu+2}\xi_{n+2}\sigma - m)w) \\ &= g(\eta_{\nu+2}\xi_{n+2}\sigma - m)\, T^{(p,q)*}(\eta_\nu \xi_n w) \\ &\quad - i(\partial(g(\eta_{\nu+2}\xi_{n+2}\sigma - m)))(\eta_\nu \xi_n w). \end{aligned}$$

Hence we have

$$T^{(p,q)*}(\eta_\nu \xi_n w) = 0 \qquad \mu\text{-a.e. on } \{\sigma > m+1\}.$$

Since

$$T^{(p,q)*}(\eta_\nu \xi_n w) = \eta_\nu \xi_n T^{(p,q)*}w - i(\partial(\eta_\nu \xi_n))w, \tag{6.4}$$

we see that $T^{(p,q)*}w=0$ μ-a.e. on $\{\sigma > m+1,\ \eta_\nu \xi_n = 1\}$ for every ν, n. Letting $\nu \to \infty$ and then $n \to \infty$, we obtain that $T^{(p,q)*}w=0$ μ-a.e. on $\{\sigma > m+1\}$. Thus Assertion (i) has been verified.

To see the second assertion, let g be as above. Then $\eta_\nu \xi_n g(\sigma - m)f \in W_\sigma(\Omega; \wedge^{p,q+1})$ for every $f \in \mathbf{D}^1_{\infty -}(\wedge^{p,q+1})$. By the assumption, we have

$$\int_\Omega \langle \eta_\nu \xi_n g(\sigma - m)v, f\rangle_{\mathrm{AHS}}\, d\mu = 0, \qquad f \in \mathbf{D}^1_{\infty -}(\wedge^{p,q+1}).$$

We therefore have

$$\eta_\nu \xi_n g(\sigma - m)v = 0 \qquad \mu\text{-a.e. on } \Omega.$$

Letting $m \to \infty$, $\nu \to \infty$, and $n \to \infty$, we obtain that $v=0$ μ-a.e. on Ω. ∎

We can now define

$$\mathscr{D}_\sigma(\Omega;\wedge^{p,q}) = \left\{ \begin{array}{l} u\in\mathscr{L}^2_\sigma(\Omega;\wedge^{p,q})\text{: there is a } v\in\mathscr{L}^2_\sigma(\Omega;\wedge^{p,q+1}) \text{ such that} \\ \qquad\qquad \int_\Omega \langle u, T^{(p,q)*}w\rangle_{\mathrm{AHS}}\,d\mu = \int_\Omega \langle v, w\rangle_{\mathrm{AHS}}\,d\mu \\ \qquad\qquad \text{for every } w\in W_\sigma(\Omega;\wedge^{p,q+1}) \end{array} \right\}$$

$$\bar\partial^{(\sigma)}_{p,q}u = v \qquad \text{for} \quad u\in\mathscr{D}_\sigma(\Omega;\wedge^{p,q}).$$

We then have

LEMMA 6.3. *Assume that there is a $\gamma\in\Gamma(\psi,\Omega)$ with $\gamma\ll\sigma$. Then it holds that*

$$\mathscr{D}_\sigma(\Omega;\wedge^{p,q}) = \bigcup_{\chi\in\mathscr{X}} \mathrm{Dom}(T^{(p,q)}_{\chi(\sigma),\gamma}).$$

Furthermore, if $u\in\mathrm{Dom}(T^{(p,q)}_{\chi(\sigma),\gamma})$, then $\bar\partial^{(\sigma)}_{p,q}u = T^{(p,q)}_{\chi(\sigma),\gamma}u$.

Proof. It follows from Lemma 3.6 that for any $u\in\mathscr{L}^2_\sigma(\Omega;\wedge^{p,q})$, $w\in W_\sigma(\Omega;\wedge^{p,q+1})$, and $\chi\in\mathscr{X}$,

$$\begin{aligned} &\langle u, T^{(p,q)*}(\eta_\nu\xi_n w)\rangle_0 \\ &\quad= \langle u, e^\gamma T^{(p,q)*}_{\chi(\sigma),\gamma}(\eta_\nu\xi_n w) - \eta_\nu\xi_n i(\partial(\chi(\sigma)+\gamma))w\rangle_0 \\ &\quad= \langle u, e^{-\chi(\sigma)} T^{(p,q)*}_{\chi(\sigma),\gamma}(e^{\chi(\sigma)+\gamma}\eta_\nu\xi_n w)\rangle_0 \\ &\quad= \langle u, T^{(p,q)*}_{\chi(\sigma),\gamma}(e^{\chi(\sigma)+\gamma}\eta_\nu\xi_n w)\rangle_{\chi(\sigma)}. \end{aligned} \tag{6.5}$$

Suppose first that $u\in\mathrm{Dom}(T^{(p,q)}_{\chi(\sigma),\gamma})$. By (6.5), we have

$$\langle u, T^{(p,q)*}(\eta_\nu\xi_n w)\rangle_0 = \langle T^{(p,q)}_{\chi(\sigma),\gamma}u, \eta_\nu\xi_n w\rangle_0.$$

Since $\gamma\ll\sigma$,

$$\gamma\leqslant\chi_0(\sigma) \qquad \text{for some} \quad \chi_0\in\mathscr{X}. \tag{6.6}$$

Hence we see that $T^{(p,q)}_{\chi(\sigma),\gamma}u\in\mathscr{L}^2_\sigma(\Omega;\wedge^{p,q})$ and

$$\lim_{n\to\infty}\lim_{\nu\to\infty}\langle T^{(p,q)}_{\chi(\sigma),\gamma}u, \eta_\nu\xi_n w\rangle_0 = \langle T^{(p,q)}_{\chi(\sigma),\gamma}u, w\rangle_0.$$

On the other hand, it follows from (3.7), (6.4), (6.6), and Lemma 6.2(i) that

$$\lim_{n\to\infty}\lim_{\nu\to\infty}\langle u, T^{(p,q)*}(\eta_\nu\xi_n w)\rangle_0 = \langle u, T^{(p,q)*}w\rangle_0.$$

Thus, $u\in\mathscr{D}_\sigma(\Omega;\wedge^{p,q})$.

We next assume that $u \in \mathscr{D}_\sigma(\Omega; \bigwedge^{p,q})$. Let $f \in \mathbf{D}^1_{\infty-,0}(\Omega; \bigwedge^{p,q})$. For each ν, n, if $m > \text{ess. sup}\{\eta_{\nu+1}\xi_{n+1}\sigma(z) : z \in \Omega\}$, then

$$\eta_\nu \xi_n f = \eta_\nu \xi_n g(\sigma - m) f \in W_\sigma(\Omega; \textstyle\bigwedge^{p,q}),$$

where g is the function appearing in the proof of Lemma 6.2. By (6.5), we obtain

$$\begin{aligned}\langle \bar{\partial}^{(\sigma)}_{p,q} u, \eta_\nu \xi_n f\rangle_0 &= \langle u, T^{(p,q)*}(\eta_\nu \xi_n f)\rangle_0 \\ &= \langle u, T^{(p,q)*}_{\chi(\sigma),\gamma}(e^{\chi(\sigma)+\gamma}\eta_\nu \xi_n f)\rangle_{\chi(\sigma)},\end{aligned}$$

where $\chi \in \mathscr{X}$ is chosen so that $\bar{\partial}^{(\sigma)}_{p,q} u \in L^2(\Omega; \bigwedge^{p,q+1}, e^{-\chi(\sigma)}\, d\mu)$. It then follows that

$$\langle \bar{\partial}^{(\sigma)}_{p,q} u, \eta_\nu \xi_n f\rangle_{\chi(\sigma)+\gamma} = \langle u, T^{(p,q)*}_{\chi(\sigma),\gamma}(\eta_\nu \xi_n f)\rangle_{\chi(\sigma)},$$

which means that $u \in \text{Dom}(T^{(p,q)}_{\chi(\sigma),\gamma})$ and $T^{(p,q)}_{\chi(\sigma),\gamma} u = \bar{\partial}^{(\sigma)}_{p,q} u$. ∎

The lemma yields

COROLLARY 6.4. *If there is a $\gamma \in \Gamma(\psi, \Omega)$ with $\gamma \ll \sigma$, then it holds that*

$$\bar{\partial}^{(\sigma)}_{p,q}(\mathscr{D}_\sigma(\Omega; \textstyle\bigwedge^{p,q})) \subset \mathscr{D}_\sigma(\Omega; \textstyle\bigwedge^{p,q+1}) \quad \textit{and} \quad \bar{\partial}^{(\sigma)}_{p,q+1} \circ \bar{\partial}^{(\sigma)}_{p,q} = 0.$$

Proof. Let $u \in \mathscr{D}_\sigma(\Omega; \bigwedge^{p,q})$. By Lemma 6.3, $u \in \text{Dom}(T^{(p,q)}_{\chi(\sigma),\gamma})$ for some $\chi \in \mathscr{X}$. Applying Lemma 3.5(6), we see that

$$T^{(p,q)}_{\chi(\sigma),\gamma} u \in \text{Dom}(T^{(p,q+1)}_{\chi(\sigma)+\gamma,\gamma}) \quad \text{and} \quad T^{(p,q+1)}_{\chi(\sigma)+\gamma,\gamma}(T^{(p,q)}_{\chi(\sigma),\gamma} u) = 0.$$

Since $\gamma \ll \sigma$, there is a $\tilde{\chi} \in \mathscr{X}$ such that $\gamma \leqslant \tilde{\chi}(\sigma)$. We then obtain that $T^{(p,q)}_{\chi(\sigma),\gamma} u \in \text{Dom}(T^{(p,q+1)}_{(\chi+\tilde{\chi})(\sigma),\gamma})$ and that $T^{(p,q+1)}_{(\chi+\tilde{\chi})(\sigma),\gamma}(T^{(p,q)}_{\chi(\sigma),\gamma} u) = 0$. The desired assertion follows by applying Lemma 6.3 again. ∎

We now show the exactness of $\bar{\partial}^{(\sigma)}_{p,q}$:

PROPOSITION 6.5. *Assume that there is a $\gamma \in \Gamma(\psi, \Omega)$ with $\gamma \ll \sigma$, $\partial\bar{\partial}\sigma \geqslant 0$ μ-a.e. on Ω; and* (6.3) *is fulfilled for some $\chi \in \mathscr{X}$. Then it holds that*

$$\text{Image}(\bar{\partial}^{(\sigma)}_{p,q}) = \text{Ker}(\bar{\partial}^{(\sigma)}_{p,q+1}).$$

Proof. By Corollary 6.4, it suffices to show that $\text{Ker}(\bar{\partial}^{(\sigma)}_{p,q+1}) \subset \text{Image}(\bar{\partial}^{(\sigma)}_{p,q})$. Let $u \in \text{Ker}(\bar{\partial}^{(\sigma)}_{p,q+1})$. By Lemma 6.3, $u \in \text{Ker}(T^{(p,q+1)}_{\tilde{\chi}(\sigma),\gamma})$ for some $\tilde{\chi} \in \mathscr{X}$. We then see that $u \in \text{Ker}(T^{(p,q+1)}_{(\tilde{\chi}+\chi)(\sigma)+\gamma,\gamma}) \cap L^2(\Omega; \bigwedge^{p,q}, e^{-(\tilde{\chi}+\chi)(\sigma)}\, d\mu)$ and that

$$\partial\bar{\partial}((\tilde{\chi}+\chi)(\sigma) + 2\gamma) \geqslant 4\, \partial\gamma \otimes \bar{\partial}\gamma.$$

Applying Theorem 3.2, we obtain $v \in \mathrm{Dom}(T^{(p,q)}_{(\tilde{\chi}+\chi)(\sigma),\gamma})$ so that $T^{(p,q)}_{(\tilde{\chi}+\chi)(\sigma),\gamma} v = u$, which implies that $u \in \mathrm{Image}(\bar{\partial}^{(\sigma)}_{p,q})$ by virtue of Lemma 6.3. ∎

Let E_0 be a complex Hilbert space and U be an open subset of E_0. We define

$$\mathrm{Exh}(U) = \{f : U \to [0, \infty) : f \text{ is } C^2 \text{ and each } \{f \leqslant n\} \text{ is closed in } U\}.$$

By virtue of [6, Chap. II, Theorem 2], we see that

(a) $\mathrm{Exh}(U) \neq \varnothing$,

(b) for every closed $A \subset U$, there is an $f \in \mathrm{Exh}(U)$ such that $A \subset \{f \leqslant m\}$ for some m. Fix arbitrary $\sigma_0 \in \mathscr{S}(\psi, \Omega)$ and $F : \Omega \to E_0$ satisfying that

(1) $\mathbf{1}_\Omega g(\psi) F \in \mathbf{S}^2(E_0)$ for every $g \in C_0^\infty(\mathbf{R}; \mathbf{R})$,

(2) $\bar{\partial} F = 0$ μ-a.e. on Ω,

(3) $(F, DF) \ll \sigma_0$.

We then have

Lemma 6.6. *Let $f \in \mathrm{Exh}(U)$ and $\gamma \in \Gamma(\psi, \Omega)$. Then*

(i) $\tilde{\psi} = \psi + f(F) \in \Psi(F^{-1}(U))$.

(ii) $\tilde{\sigma} = \sigma_0 + f(F) \in \mathscr{S}(\tilde{\psi}, F^{-1}(U))$.

(iii) *There is a $\chi \in \mathscr{X}$ such that $\tilde{\gamma} = \gamma + \chi(\tilde{\sigma}) \in \Gamma(\tilde{\psi}, F^{-1}(U))$.*

Proof. Let $g \in C_0^\infty(\mathbf{R}; \mathbf{R})$. It is easily seen that $\mathbf{1}_{F^{-1}(U)} g(\tilde{\psi}) \in \mathbf{S}^2(\mathbf{R})$ and that $\mathbf{1}_{F^{-1}(U)} g(\tilde{\psi}) \tilde{\sigma} \in \mathbf{S}^2(\mathbf{R})$. To see the boundedness of them and their derivatives, take a $K \in \mathscr{K}_+$ such that $\eta_v \sigma_0 \in \mathbf{S}^1_{K,b}(\mathbf{R})$, $v = 1, 2, \ldots$. On $\{\rho_K \leqslant n, \psi \leqslant m\}$, especially on $\{\rho_K \leqslant n, \tilde{\psi} \leqslant m\}$, ${}_0$ and $D\sigma_0$ are bounded and (F, DF) is contained in a compact set. Moreover, since $f \in \mathrm{Exh}(U)$, $f(F)$ and $f'(F)$ are both bounded on $\{\rho_K \leqslant n, \tilde{\psi} \leqslant m\}$. Hence $\mathbf{1}_{F^{-1}(U)} g(\tilde{\psi})$, $\mathbf{1}_{F^{-1}(U)} g(\tilde{\psi}) \tilde{\sigma} \in \mathbf{S}^1_b(\mathbf{R})$. Thus Assertions (i) and (ii) have been proved.

Note that $(F, DF) \in L_n \times L'_n$ on $\{\tilde{\sigma} \leqslant n\}$ for some compact $L_n \subset E_0$ and $L'_n \subset H^* \otimes E_0$. Since $f \in \mathrm{Exh}(U)$, we may think of L_n as a compact set contained in U. We can therefore find a $\chi \in \mathscr{X}$ such that $\|D(f(F))\|_{H^* \otimes E_0} \leqslant e^{\chi(\tilde{\sigma})}$. The last assertion then follows. ∎

Lemma 6.7. *Let $f, \varphi \in \mathrm{Exh}(U)$. Then it holds that*

(i) $W_{\sigma_0 + f(F)}(F^{-1}(U); E) = W_{\sigma_0 + \varphi(F)}(F^{-1}(U); E)$.

(ii) $\mathscr{L}^2_{\sigma_0 + f(F)}(F^{-1}(U); E) = \mathscr{L}^2_{\sigma_0 + \varphi(F)}(F^{-1}(U); E)$.

(iii) $\mathscr{D}_{\sigma_0 + f(F)}(F^{-1}(U); \bigwedge^{p,q}) = \mathscr{D}_{\sigma_0 + \varphi(F)}(F^{-1}(U); \bigwedge^{p,q})$.

Proof. Since $F \ll \sigma_0$, for each n, there is an m such that

$$\{\sigma_0 + f(F) \leqslant n\} \subset \{\sigma_0 + \varphi(F) \leqslant m\}$$

and (6.7)

$$\{\sigma_0 + \varphi(F) \leqslant n\} \subset \{\sigma_0 + f(F) \leqslant m\}.$$

This implies the first assertion.

Equation (6.7) also yields that

$$\sigma_0 + \varphi(F) \leqslant \chi(\sigma_0 + f(F)) \qquad \text{and} \qquad \sigma_0 + f(F) \leqslant \chi(\sigma_0 + \varphi(F))$$

for some $\chi \in \mathscr{X}$, which implies the second assertion.

The third assertion follows from the first two assertions. ∎

We can now define

$$\mathscr{D}_U(\wedge^{p,q}) = \mathscr{D}_{\sigma_0 + f(F)}(F^{-1}(U); \wedge^{p,q}),$$
$$\bar{\partial}_{p,q} = \bar{\partial}^{(\sigma_0 + f(F))}_{p,q},$$

where $f \in \operatorname{Exh}(U)$. We say that U is pseudoconvex if there is an $f \in \operatorname{Exh}(U)$ with $\partial\bar{\partial} f \geqslant 0$.

PROPOSITION 6.8. *Assume that* (a) *there exists a* $\gamma \in \Gamma(\psi, \Omega)$ *with* $(\gamma, D\gamma, D^2\gamma) \ll \sigma_0$, (b) *Eq.* (6.1) *is fulfilled with* $\sigma = \sigma_0$, *and* (c) U *is pseudoconvex. Then the sequence*

$$\mathscr{D}_U(\wedge^{p,0}) \xrightarrow{\bar{\partial}_{p,0}} \cdots \xrightarrow{\bar{\partial}_{p,q-1}} \mathscr{D}_U(\wedge^{p,q}) \xrightarrow{\bar{\partial}_{p,q}} \mathscr{D}_U(\wedge^{p,q+1}) \xrightarrow{\bar{\partial}_{p,q+1}} \cdots$$

is exact.

Proof. Choose $f \in \operatorname{Exh}(U)$ with $\partial\bar{\partial} f \geqslant 0$ and set $\tilde{\psi} = \psi + f(F)$ and $\tilde{\sigma} = \sigma_0 f(F)$. As in Lemma 6.6, take a $\chi_0 \in \mathscr{X}$ so that $\tilde{\gamma} = \gamma + \chi_0(\tilde{\sigma}) \in \Gamma(\tilde{\psi}, F^{-1}(U))$. It is easily seen that $\gamma \in \mathscr{S}(\tilde{\psi}, F^{-1}(U))$, $\partial\bar{\partial}\tilde{\sigma} \geqslant A$, and $(\gamma, D\gamma, D^2\gamma) \ll \tilde{\sigma}$. As an application of Proposition 6.1, we obtain a $\chi_1 \in \mathscr{X}$ such that

$$\partial\bar{\partial}(\chi_1(\tilde{\sigma}) + 2\tilde{\gamma}) \geqslant 4\, \partial\tilde{\gamma} \otimes \bar{\partial}\tilde{\gamma} \qquad \text{on } F^{-1}(U).$$

Then the assertion follows from Proposition 6.5. ∎

Let $\{U_i\}_{i \in I}$ be a locally family of open sets in E_0 and set $\tilde{U} = \bigcup_{i \in \mathbf{I}} U_i$; i.e., every $x \in \tilde{U}$ has a neighbourhood V such that $\#\{i : U_i \cap V \neq \varnothing\} < \infty$. We denote by $\{\varphi_i\}_{i \in \mathbf{I}}$ the corresponding partition of unity (for details, see [6]). For $u_i : F^{-1}(U_i) \to \wedge^{p,q}$, we define

$$T(\{u_i\}_{i \in \mathbf{I}}) = \sum_{i \in \mathbf{I}} \varphi_i(F) u_i.$$

LEMMA 6.9. *Suppose that there is a $\gamma \in \Gamma(\psi, \Omega)$ with $\gamma \ll \sigma_0$. Then the mapping*

$$T : \bigoplus_{i \in \mathbf{I}} \mathscr{D}_{U_i \cap U}(\textstyle\bigwedge^{p,q}) \to \mathscr{D}_{\tilde{U} \cap U}(\textstyle\bigwedge^{p,q})$$

is a well-defined linear mapping.

Proof. Choose $f \in \mathrm{Exh}(U)$ and $f_i \in \mathrm{Exh}(U_i)$ such that $\{\varphi_i > 0\} \subset \{f_i \leqslant m_i\}$ for some m_i. We set $\tilde{\psi}_i = \psi + f(F) + f_i(F)$, $\tilde{\sigma}_i = \sigma_0 + f(F) + f_i(F)$, and take $\tilde{\chi}_i \in \mathscr{X}$ such that $\tilde{\gamma}_i = \gamma + \tilde{\chi}_i(\tilde{\sigma}_i) \in \Gamma(\tilde{\psi}_i, F^{-1}(U_i \cap U))$.

Let $u_i \in \mathscr{D}_{U_i \cap U}(\bigwedge^{p,q})$. Since

$$\int_{F^{-1}(U_i \cap U)} |u_i|^2_{\mathrm{AHS}}\, e^{-\chi_1(\tilde{\sigma}_i)}\, d\mu < \infty,$$

for some $\chi_1 \in \mathscr{X}$ and $\tilde{\sigma}_i \leqslant \sigma_0 + f(F) + m_i$ on $\{\varphi_i(F) > 0\}$, we have

$$\int_{F^{-1}(\tilde{U} \cap U)} \varphi_i^2(F)\, |u_i|^2_{\mathrm{AHS}}\, e^{-\chi_1(\sigma_0 + f(F) + m_i)}\, d\mu$$
$$\leqslant \int_{F^{-1}(U_i \cap U)} |u_i|^2_{\mathrm{AHS}}\, e^{-\chi_1(\tilde{\sigma}_i)}\, d\mu < \infty.$$

Thus

$$\varphi_i(F)\, u_i \in \mathscr{L}^2_{\sigma_0 + f(F)}(F^{-1}(\tilde{U} \cap U); \textstyle\bigwedge^{p,q}).$$

Since $(\tilde{\gamma}_i, F, DF) \ll \tilde{\sigma}_i$, Lemmas 3.7 and 6.3 imply that

$$\varphi_i(F)\, u_i \in \mathrm{Dom}(T^{(p,q)}_{\chi_i(\tilde{\sigma}_i), \tilde{\gamma}_i})$$
$$T^{(p,q)}_{\chi_i(\tilde{\sigma}_i), \tilde{\gamma}_i}(\varphi_i(F)\, u_i) = \varphi_i(F)\, T^{(p,q)}_{\chi_i(\tilde{\sigma}_i), \tilde{\gamma}_i} u_i + \bar{\partial}(\varphi_i(F)) \wedge u_i,$$

for some $\chi_i \in \mathscr{X}$. Noting that $\tilde{\gamma}_i \leqslant \gamma + \tilde{\chi}_i(\sigma_0 + f(F) + m_i)$ on $\{f_i(F) \leqslant m_i\}$, we can conclude that

$$T^{(p,q)}_{\chi_i(\tilde{\sigma}_i), \tilde{\gamma}_i}(\varphi_i(F)\, u_i) \in \mathscr{L}^2_{\sigma_0 + f(F)}(F^{-1}(\tilde{U} \cap U); \textstyle\bigwedge^{p,q}).$$

As in the proof of Lemma 6.3, we can further show that

$$\int_{F^{-1}(\tilde{U} \cap U)} \langle \varphi_i(F)\, u_i, T^{(p,q)*} w \rangle_{\mathrm{AHS}}\, d\mu$$
$$= \int_{F^{-1}(\tilde{U} \cap U)} \langle T^{(p,q)}_{\chi_i(\tilde{\sigma}_i), \tilde{\gamma}_i}(\varphi_i(F)\, u_i), w \rangle_{\mathrm{AHS}}\, d\mu$$

for every $w \in W_{\sigma_0 + f(F)}(F^{-1}(\tilde{U} \cap U); \bigwedge^{p,q+1})$. Since $F \ll \sigma_0$, we then have that

$$\sum_{i \in \mathbf{I}} \varphi_i(F) u_i \in \mathscr{L}^2_{\sigma_0 + f(F)}(F^{-1}(\tilde{U} \cap U); \bigwedge^{p,q}),$$

$$\sum_{i \in \mathbf{I}} T^{(p,q)}_{\chi_i(\tilde{\sigma}_i), \tilde{\gamma}_i}(\varphi_i(F) u_i) \in \mathscr{L}^2_{\sigma_0 + f(F)}(F^{-1}(\tilde{U} \cap U); \bigwedge^{p,q+1}),$$

and

$$\int_{F^{-1}(\tilde{U} \cap U)} \left\langle \sum_{i \in \mathbf{I}} \varphi_i(F) u_i, T^{(p,q)*} w \right\rangle_{\mathrm{AHS}} d\mu$$
$$= \int_{F^{-1}(\tilde{U} \cap U)} \left\langle \sum_{i \in \mathbf{I}} T^{(p,q)}_{\chi_i(\tilde{\sigma}_i), \tilde{\gamma}_i}(\varphi_i(F) u_i), w \right\rangle_{\mathrm{AHS}} d\mu,$$

for any $w \in W_{\sigma_0 + f(F)}(F^{-1}(\tilde{U} \cap U); \bigwedge^{p,q+1})$. Thus $\sum_{i \in \mathbf{I}} \varphi_i(F) u_i \in \mathscr{D}_{\tilde{U} \cap U}(\bigwedge^{p,q})$. ∎

In what follows we consider an example of quasi-sheaves and show a Dolbeault type theorem. For $J, J' \in \mathbf{I}_{00}$, $J \subset J'$, we define

$$V_J^{p,q} = \begin{cases} \mathscr{D}_{\bigcap_{i \in J} U_i}(\bigwedge^{p,q}) & \text{if } \bigcap_{i \in J} U_i \neq \varnothing, \\ \{0\} & \text{otherwise} \end{cases}$$

$$\pi_{JJ'}^{p,q} u = \begin{cases} u|_{\bigcap_{i \in J'} U_i} & \text{if } \bigcap_{i \in J'} U_i \neq \varnothing \text{ and } u \in V_J^{p,q}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\bar{\partial}_{p,q,J} = \begin{cases} \bar{\partial}_{p,q} \text{ on } \mathscr{D}_{\bigcap_{i \in J} U_i}(\bigwedge^{p,q}) & \text{if } \bigcap_{i \in J} U_i \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

We have the following Dolbeault type theorem.

THEOREM 6.10. *Assume that* (a) *there exists a* $\gamma \in \Gamma(\psi, \Omega)$ *with* $(\gamma, D\gamma, D^2\gamma) \ll \sigma_0$, (b) *Eq.* (6.1) *is satisfied with* $\sigma = \sigma_0$, *and* (c) *every* U_i *is pseudoconvex. Then the following assertions holds.*

(i) $\mathscr{A}^{p,q} \equiv (\{V_J^{p,q}\}, \{\pi_{JJ'}^{p,q}\})$ *is a quasi-sheaf.*

(ii) $\bar{\partial}_{p,q} \equiv \{\bar{\partial}_{p,q,J}\}$ *is a morphism of* $\mathscr{A}^{p,q}$ *to* $\mathscr{A}^{p,q+1}$.

(iii) *Let* $\mathcal{O}^p = \mathrm{Ker}(\bar{\partial}_{p,0})$. *Then the sequence*

$$0 \xrightarrow{0} \mathcal{O}^p \hookrightarrow \mathscr{A}^{p,0} \xrightarrow{\bar{\partial}_{p,0}} \mathscr{A}^{p,1} \xrightarrow{\bar{\partial}_{p,1}} \cdots \xrightarrow{\bar{\partial}_{p,1}} \mathscr{A}^{p,q} \xrightarrow{\bar{\partial}_{p,q}} \mathscr{A}^{p,q+1} \xrightarrow{\bar{\partial}_{p,q+1}} \cdots \tag{6.8}$$

is exact.

(iv) *Every $\mathscr{A}^{p,q}$ is fine. Moreover, it holds that*

$$H^q(\mathscr{O}^p) \simeq \operatorname{Ker}(\bar{\partial}_{\wedge_{\tilde{U}}^{p,q}})/\operatorname{Image}(\bar{\partial}_{\wedge_{\tilde{U}}^{p,q-1}}),$$

where

$$\wedge_{\tilde{U}}^{p,q} = \{u : F^{-1}(\tilde{U}) \to \wedge^{p,q} : u|_{F^{-1}(U_i)} \in \mathscr{D}_{U_i}(\wedge^{p,q}),\ i \in \mathbf{I}\}$$

$$\bar{\partial}_{\wedge_{\tilde{U}}^{p,q}} u|_{F^{-1}(U_i)} = \bar{\partial}_{p,q,\{i\}}(u|_{F^{-1}(U_i)}).$$

Proof. Let $\bigcap_{i \in J} U_i \neq \varnothing$ and $f_i \in \operatorname{Exh}(U_i)$. Then $\sum_{i \in J} f_i \in \operatorname{Exh}(\bigcap_{i \in J} U_i)$. Hence, if $J \subset J'$, then

$$W_{\sigma_0 + \sum_{i \in J'} f_i(F)}\left(F^{-1}\left(\bigcap_{i \in J'} U_i\right); \wedge^{p,q}\right)$$

$$\subset W_{\sigma_0 + \sum_{i \in J} f_i(F)}\left(F^{-1}\left(\bigcap_{i \in J} U_i\right); \wedge^{p,q}\right)$$

$$\mathscr{L}^2_{\sigma_0 + \sum_{i \in J'} f_i(F)}\left(F^{-1}\left(\bigcap_{i \in J'} U_i\right); \wedge^{p,q}\right)$$

$$\supset \pi_{JJ'}^{p,q}\left(\mathscr{L}^2_{\sigma_0 + \sum_{i \in J} f_i(F)}\left(F^{-1}\left(\bigcap_{i \in J} U_i\right); \wedge^{p,q}\right)\right).$$

This implies that

$$\pi_{JJ'}^{pq}(\mathscr{D}_{\bigcap_{i \in J} U_i}(\wedge^{p,q})) \subset \mathscr{D}_{\bigcap_{i \in J'} U_i}(\wedge^{p,q})$$

$$\pi_{JJ'}^{p,q} \circ \bar{\partial}_{p,q,J} = \bar{\partial}_{p,q,J'} \circ \pi_{JJ'}^{p,q}.$$

Thus the first and second assertions have been verified.

Note that $\bigcap_{i \in J} U_i$ is also pseudoconvex. Hence the third assertion follows from Proposition 6.8.

To see the fourth assertion, take $\omega_{n+1} = \{\omega_{I'}\}_{I' \in \mathbf{I}_0^{n+2}} \in \mathscr{C}_{n+1}(\mathscr{A}^{p,q})$ with $\delta_{n+1}\omega_{n+1} = 0$. If we define $v_n = \{v_I\}_{I \in \mathbf{I}_0^{n+1}}$ by $v_I = \sum_{i \notin J(I)} \varphi_i(F)\, \omega_{(i,I)}$, then, by Lemma 6.9, we see that $v_n \in \mathscr{C}_n(\mathscr{A}^{p,q})$. It is straightforward to see that $\delta_n v_n = \omega_{n+1}$ (for example, see [3]). Thus $\mathscr{A}^{p,q}$ is fine. By virtue of (5.3), we then obtain that

$$H^q(\mathscr{O}^p) \simeq H^0(\operatorname{Ker}(\bar{\partial}_{p,q}))/(\bar{\partial}_{p,q-1})_{0*}\, H^0(\mathscr{A}^{p,q-1}).$$

To complete the proof, it suffices to note that

$$H^0(\operatorname{Ker}(\bar{\partial}_{p,q})) = \operatorname{Ker}(\bar{\partial}_{\wedge_{\tilde{U}}^{p,q}})$$

and

$$(\bar{\partial}_{p,q-1})_{0*}\, H^0(\mathscr{A}^{p,q-1}) = \operatorname{Image}(\bar{\partial}_{\wedge_{\tilde{U}}^{p,q-1}}). \quad \blacksquare$$

7. Pseudoconvex Domains Given through SDE

In this section, we consider the case where B is a space of continuous functions on $[0, 1]$ starting at 0 with values in $\mathbf{R}^d$ $(d=2d')$, H is its Cameron–Martin space equipped with the norm $\|h\|_H^2 = 2\int_0^1 |\dot{h}(s)|^2\, ds$, and μ is the standard Wiener measure on B. For $w \in B$, we denote by $w(t)$ its position at time t and by $x^k(t, w)$ (resp. $y^k(t, w)$) the $(2k-1)$th (resp., $2k$th) component of $w(t)$, $k = 1, 2, ..., d'$. If we set $B^k(t, w) + \sqrt{-1}\, y^k(t, w)$, then an almost complex structure $J: B \to B$ is given by $B^k(t, Jw) = \sqrt{-1}\, B^k(t, w)$.

Our first aim of this section is to give some examples of functions in $\mathbf{S}^n(\mathbf{R}^m)$ and $\mathbf{S}_h^n(\mathbf{R}^m)$. For $W \in C^\infty(\mathbf{R}^m; \mathbf{R}^m)$, we define $W^\# \in C^\infty(\mathbf{R}^m; \mathbf{R}^m \otimes \mathbf{R}^m)$ and $W^\natural \in C^\infty(\mathbf{R}^m \times (\mathbf{R}^m \otimes \mathbf{R}^m); \mathbf{R}^m \otimes \mathbf{R}^m))$ by

$$W^\#(y) = \left(\frac{\partial W^i}{\partial y^j}(y)\right)_{1 \leq i, j \leq m}$$

and

$$W^\natural(y, a) = \left(W(y), \left(\sum_{k=1}^m (W^\#)^i_k (y)\, a^k_j \right)_{1 \leq i \leq, j \leq m} \right),$$

where $y \in \mathbf{R}^m$, $a \in \mathbf{R}^m \otimes \mathbf{R}^m$. Take an arbitrary but fixed $D \in N$ and put $q_1 = D$ and $q_{n+1} = q_n + q_n^2$. Note that $(x, e_1, ..., e_n)$ $(x \in \mathbf{R}^D,\ e_j \in \mathbf{R}^{q_j} \otimes \mathbf{R}^{q_j})$ is a coordinate system on $\mathbf{R}^{q_{n+1}}$, since $D + q_1^2 + \cdots + q_n^2 = q_{n+1}$. For $V \in C^\infty(\mathbf{R}^D; \mathbf{R}^D)$, we define $V^{\natural(n)} \in C^\infty(\mathbf{R}^{q_{n+1}}; \mathbf{R}^{q_{n+1}})$ and $V^{\#(n)} \in C^\infty(\mathbf{R}^{q_n}; \mathbf{R}^{q_n} \otimes \mathbf{R}^{q_n})$ successively by

$$V^{\natural(1)}(x, e_1) = V^\natural(x, e_1) \qquad \text{and} \qquad V^{\#(1)}(x) = V^\#(x)$$

$$V^{\natural(n)}(x, e_1, ..., e_n) = (V^{\natural(n-1)})^\natural\, ((x, e_1, ..., e_{n-1}), e_n)$$

$$V^{\#(n)}(x, e_1, ..., e_{n-1}) = (V^{\natural(n-1)})^\#\, (x, e_1, ..., e_{n-1}).$$

Let $V_k \in C_0^\infty(\mathbf{R}^D; \mathbf{R}^D)$ and consider a system of SDEs

$$dX(t) = \sum_{k=1}^d V_k(X(t)) \circ dw^k(t) + V_0(X(t))\, dt \tag{7.1}$$

$$dY^{(1)}(t) = \sum_{k=1}^d V_k^{\#(1)}(X(t))\, Y^{(1)}(t) \circ dw^k(t) + V_0^{\#(1)}(X(t))\, Y^{(1)}(t)\, dt \tag{7.2}$$

$$\vdots$$

$$\begin{aligned} dY^{(n)}(t) &= \sum_{k=1}^{d} V_k^{\#(n)}(X(t), Y^{(1)}(t), \ldots, Y^{(n-1)}(t))\, Y^{(n)}(t) \circ dw^k(t) \\ &\quad + V_0^{\#(n)}(X(t), Y^{(1)}, \ldots, Y^{(n-1)}(t))\, Y^{(n)}(t)\, dt \qquad (7.3) \\ X(0) &= x_0 \\ Y^{(j)}(0) &= I_{\mathbf{R}^{q_j} \otimes \mathbf{R}^{q_j}}, \end{aligned}$$

where $w(t) = (w^1(t), \ldots, w^d(t))$, $w \in B$. Since each $V_k^{\#(n)}(\cdot)$ depends only on the coordinate $(x, e_1, \ldots, e_{n-1})$, we see inductively that there exists a unique strong solution $(X(t, x_0; w), Y^{(1)}(t, x_0; w), Y^{(2)}(t, x_0; w), \ldots)$ to the system of SDEs. By the definition, we note that

$$V_k^{\#(n)}(x, e_1, \ldots, e_{n-1}) = \begin{pmatrix} V_k^{\#(n-1)}(x, e_1, \ldots, e_{n-2}) & 0 \\ * & * \end{pmatrix}$$

and hence we have

$$Y^{(n)}(t, x_0) = \begin{pmatrix} Y^{(n-1)}(t, x_0) & 0 \\ * & * \end{pmatrix}. \qquad (7.4)$$

Moreover, every $Y^{(n)}(t, x_0) \in \mathbf{R}^{q_n} \otimes \mathbf{R}^{q_n}$ is invertible and the inverse $Y^{(n)-1}(t, x_0)$ enjoys

$$\begin{aligned} dY^{(n)-1}(t) &= -\sum_{k=1}^{d} Y^{(n)-1}(t)\, V_k^{\#(n)} \\ &\quad \times (X(t, x_0), Y^{(1)}(t, x_0), \ldots, Y^{(n-1)}(t, x_0)) \circ dw^k(t) \\ &\quad - Y^{(n)-1}(t)\, V_0^{\#(n)}(X(t, x_0), Y^{(1)}(t, x_0), \ldots, Y^{(n-1)}(t, x_0)\, dt \\ Y^{(n)}(0) &= I_{\mathbf{R}^{q_n} \otimes \mathbf{R}^{q_n}}. \end{aligned}$$

For arbitrarily fixed $\alpha \in (0, \frac{1}{2})$ and $m \in \mathbf{N}$ with $2m\alpha > 1$ and $\alpha + (\frac{1}{2m}) < \frac{1}{2}$, we put

$$\begin{aligned} \Phi_0^{(\alpha, m; x_0)}(w) &= \int_0^1 \int_0^1 \frac{|X(t, x_0; w) - X(s, x_0; w)|^{2m}}{|t-s|^{1+2m\alpha}}\, dt\, ds \\ \Phi_n^{(\alpha, m; x_0)}(w) &= \Phi_{n-1}^{(\alpha, m; x_0)}(w) \\ &\quad + \int_0^1 \int_0^1 \frac{|Y^{(n)}(t, x_0; w) - Y^{(n)}(s, x_0; w)|^{2m}}{|t-s|^{1+2m\alpha}}\, dt\, ds \\ &\quad + \int_0^1 \int_0^1 \frac{|Y^{(n)-1}(t, x_0; w) - Y^{(n)-1}(s, x_0; w)|^{2m}}{|t-s|^{1+2m\alpha}}\, dt\, ds. \end{aligned}$$

Our first goal will be

THEOREM 7.1. (i) $X(t, x_0) \in \mathbf{S}_b^\infty(\mathbf{R}^D) \equiv \bigcap_n \mathbf{S}_b^n(\mathbf{R}^D)$, $Y^{(n)}(t, x_0) \in \mathbf{S}_b^\infty(\mathbf{R}^{q_n} \otimes \mathbf{R}^{q_n})$. (ii) $\Phi_n^{(\alpha,m;x_0)} \in \mathbf{S}_b^\infty(\mathbf{R})$ *and there exist* $C(n, l) < \infty$ *and* $0 \leqslant p(n, l) < \infty$, *which depend only on* D, α, m, n, l, *and the bounds of* V_k's *and their derivatives of order* $\leqslant n + l$, *such that*

$$\|D^l \Phi_n^{(\alpha,m;x_0)}\|_{(H^*)\otimes l} \leqslant C(n, l)(1 + \Phi_{n+l}^{(\alpha,m;x_0)})^{p(n,l)}. \tag{7.5}$$

Furthermore, it holds that

$$C(0, 1) = \left(\frac{16D(1 + 2m\alpha)(1 + \sqrt{D})}{2m\alpha - 1}\right)^2 \times \left(1 + \frac{2}{(m - 2m\alpha)(m - 2m\alpha + 1)}\right) b_{V_1, \dots, V_d} \tag{7.6}$$

$$p(0, 1) = 1 + \frac{1}{m}, \tag{7.7}$$

where $b_{V_1,\dots,V_d} = \sup\{|V_k(x)| : 1 \leqslant k \leqslant d,\ x \in \mathbf{R}^D\}$.

The proof is broken into several steps. We first see

LEMMA 7.2. $\Phi_n^{(\alpha,m;x_0)} < \infty$ *μ-a.e. on B.*

Proof. Since V_k's are bounded, by Doob's inequality, we have

$$E[\Phi_0^{(\alpha,m;x_0)}] \leqslant C_m \int_0^1 \int_0^1 \frac{|t-s|^m}{|t-s|^{1+2m\alpha}}\, dt\, ds < \infty$$

for some $C_m < \infty$. Thus, the assertion holds for $n = 0$.

Note that $V_k^{\#(1)}(x) = V_k^{\#}(x)$ are all bounded and hence

$$\sup_{0 \leqslant t \leqslant 1} |Y^{(1)}(t, x_0)| + |Y^{(1)-1}(t, x_0)| \in \bigcap_{1 < p < \infty} L^p(B; \mathbf{R}, d\mu).$$

Then applying Doob's inequality we see that the assertion holds for $n = 1$.

We shall show the assertion by induction. Hence suppose that $\Phi_n^{(\alpha,m;x_0)} < \infty$ μ-a.e. By [7, Theorem 2.1.3], we have

$$|\phi(t) - \phi(s)| \leqslant 16 \frac{1 + 2m\alpha}{2m\alpha - 1} |t - s|^{\alpha - 1/2m} \times \left(\int_0^1 \int_0^1 \frac{|\phi(u) - \phi(v)|^{2m}}{|u - v|^{1+2m\alpha}}\, du\, dv\right)^{1/2m} \tag{7.8}$$

for every $\phi \in C([0,1];\mathbf{R})$. Hence we have

$$\sup_{0\leqslant t\leqslant 1} |X(t,x_0)| \leqslant \frac{16\sqrt{D}(1+2m\alpha)(1+|x_0|)}{2m\alpha-1}(1+\Phi_0^{(\alpha,m;x_0)})^{1/2m} \tag{7.9}$$

$$\sup_{\substack{0\leqslant t\leqslant 1\\ 1\leqslant j\leqslant n}} |Y^{(j)}(t,x_0)| \vee |Y^{(j)-1}(t,x_0)| \leqslant \frac{16q_n(1+2m\alpha)(1+\sqrt{q_n})}{2m\alpha-1}(1+\Phi_n^{(\alpha,m;x_0)})^{1/2m}. \tag{7.10}$$

Remember that $q_1+\sum_{j=1}^n q_j^2=q_{n+1}$ and hence $(x, e_1, ..., e_n)$ ($x\in\mathbf{R}^D$, $e_j\in\mathbf{R}^{q_j}\otimes\mathbf{R}^{q_j}$) is a coordinate system on $\mathbf{R}^{q_{n+1}}$. Choose $\varphi_L\in C_0^\infty(\mathbf{R}^{q_{n+1}};\mathbf{R})$ so that $\varphi_L(\xi)=1$ for $|\xi|\leqslant 16q_n(1+2m\alpha)(1+|x_0|+\sqrt{q_n})(1+L)^{1/2m}/(2m\alpha-1)$. If we set

$$J_{L,k}(x, e_1, ..., e_n)=\varphi_L(x, e_1, ..., e_n)\, V_k^{\#(n)}(x, e_1, ..., e_n)$$

and denote by $\tilde{Y}^{(n+1)}(t,x_0)$ the solution to the SDE

$$\begin{aligned} d\tilde{Y}^{(n+1)}(t) &= \sum_{k=1}^d J_{L,k}(X(t,x_0), Y^{(1)}(t,x_0), ..., Y^{(n)}(t,x_0)) \\ &\quad\times \tilde{Y}^{(n+1)}(t)\circ dw^k(t) \\ &\quad + J_{L,0}(X(t,x_0), Y^{(1)}(t,x_0), ..., Y^{(n)}(t,x_0))\, \tilde{Y}^{(n+1)}(t)\,dt \\ \tilde{Y}^{(n+1)}(0) &= I_{\mathbf{R}^{q_{n+1}}\otimes\mathbf{R}^{q_{n+1}}}, \end{aligned}$$

then, by (7.10), it holds that

$$Y^{(n+1)}(t,x_0)=\tilde{Y}^{(n+1)}(t,x_0), \qquad 0\leqslant t\leqslant 1, \quad \mu\text{-a.e. on } \{\Phi_n^{(\alpha,m;x_0)}\leqslant L\}. \tag{7.11}$$

As in the case that $n=1$, we can conclude from this identity that $\Phi_{n+1}^{(\alpha,m;x_0)}<\infty$ μ-a.e. on $\{\Phi_n^{(\alpha,m;x_0)}\leqslant L\}$. Letting $L\to\infty$, we see that the assertion holds for $n+1$. ∎

LEMMA 7.3. *For each n, there is a $K\in\mathscr{K}_+$ such that $X(t,x_0)\in\mathbf{S}_K^\infty(\mathbf{R}^D)\cap\mathbf{S}_{K,b}^0(\mathbf{R}^D)$, $Y^{(j)}(t,x_0)$, $Y^{(j)-1}(t,x_0)\in\mathbf{S}_K^\infty(\mathbf{R}^{q_j}\otimes\mathbf{R}^{q_j})\cap\mathbf{S}_{K,b}^0(\mathbf{R}^{q_j}\otimes\mathbf{R}^{q_j})$, $0\leqslant t\leqslant 1$, $1\leqslant j\leqslant n$, and $\Phi_n^{(\alpha,m;x_0)}\in\mathbf{S}_{K,b}^0(\mathbf{R})$.*

Proof. Set $N=D+2\sum_{j=1}^n q_j^2$. Let $C^\varepsilon([0,1];\mathbf{R}^N)$ be the space of ε-Hölder continuous functions defined on [0, 1] with values in $\mathbf{R}^N$, and $W^\varepsilon((0,1);\mathbf{R}^N)$ be the L^2-Sobolev space or order ε over (0, 1) as in [1]. We denote by $\tilde{C}^{\varepsilon}([0,1];\mathbf{R}^N)$ the closure of $C^1([0,1];\mathbf{R}^N)$ in $C^{\varepsilon}([0,1];\mathbf{R}^N)$ and by H_1^{ε} the closed subspace of $W^{\varepsilon}((0,1);\mathbf{R}^N)$ consisting of h's with $h(0)=0$, respectively. We consider

$$G(w)=(X(t,x_0;w),\ Y^{(1)}(t,x_0;w),\ Y^{(1)-1}(t,x_0;w),\ \ldots,\ Y^{(n)}(t,x_0;w),\ Y^{(n)-1}(t,x_0;w)).$$

By virtue of Theorem (0.2) in [5], for every $w\in B$, the mapping

$$G(w+\cdot):H_1^{\delta}\ni h\mapsto G(w+h)\in\tilde{C}^{\beta}([0,1];\mathbf{R}^N) \tag{7.12}$$

is continuous, where $\beta=\alpha+(\frac{1}{2}m)$ and $\delta=(\frac{1}{2})+\beta\vee(\frac{1}{3})$. Indeed, the boundedness assumption on vector fields governing SDE made in [5] is needed only to see the unique existence of a strong solution to the SDE. Since our system of SDE's possesses a unique strong solution, the result in [5] is applicable to our system of SDE's. In what follows, for the sake of simplicity we write E for $\tilde{C}^{\beta}([0,1];\mathbf{R}^N)$.

We set

$$B_H(r)=\{h\in H:\|h\|_H\leqslant r\}.$$

For $A\subset H_1^{\delta}$, we denote by $C_{H_1^{\delta}}(A;E)$ the space of continuous functions of A into E, where the subscript H_1^{δ} is used to emphasize that the continuity is considered with respect to the topology inherited from H_1^{δ}. By the above observation,

$$G(w+\cdot)|_{B_H(r)}\in C_{H_1^{\delta}}(B_H(r);E)\qquad\text{for every}\quad r>0\text{ and }w\in B.$$

Since the mapping

$$B\ni w\mapsto (G(w+\cdot)|_{B_H(k)})_{k\in\mathbf{N}}\in\prod_{k=1}^{\infty}C_{H_1^{\delta}}(B_H(k);E)$$

is measurable [5, Theorem (0.2)], by virtue of Lusin's theorem and the inner regularity of the Wiener measure μ, there is a $K\in\mathscr{K}_+$ such that the mapping

$$K\ni w\mapsto G(w+\cdot)|_{B_H(k)}\in C_{H_1^{\delta}}(B_H(k);E)$$

is continuous for any $k\in\mathbf{N}$. By virtue of the compactness of K in B and that of $B_H(k)$ in H_1^{δ}, this implies that the mapping $G:K+B_H(k)\to E$ is continuous for every k. In particular, we have

$$\sup\left\{\frac{|G(w)(t)-G(w)(s)|}{|t-s|^{\beta}}:0\leqslant s<t\leqslant 1,\ w\in K+B_H(k)\right\}<\infty,$$
$$k=1,2,\ldots. \tag{7.13}$$

Recall an obvious inequality

$$|\Phi_n^{(\alpha,m;x_0)}(w)| \leqslant (2n+1)\left(\sup_{0\leqslant s<t\leqslant 1} \frac{|G(w)(t)-G(w)(s)|}{|t-s|^{\beta}}\right)^{2m}. \tag{7.14}$$

Combining this with (7.13) we obtain that

$$\Phi_n^{(\alpha,m;x_0)} \in \mathbf{S}^0_{K,b}(\mathbf{R}).$$

Now plugged into (7.9) and (7.10), (7.14) also yields that

$$X(t, x_0) \in \mathbf{S}^0_{K,b}(\mathbf{R}^D),$$
$$Y^{(j)}(t, x_0),\ Y^{(j)-1}(t, x_0) \in \mathbf{S}^0_{K,b}(\mathbf{R}^{q_n} \otimes \mathbf{R}^{q_n}),$$

for $0 \leqslant t \leqslant 1$ and $1 \leqslant j \leqslant n$. Further, it also follows from (7.10), (7.13), and (7.14) that, on each $\{\rho_K < r\}$, $X(t, x_0)$, $Y^{(1)}(t, x_0)$, $Y^{(1)-1}(t, x_0)$, ..., $Y^{(n)}(t, x_0)$, and $Y^{(n)-1}(t, x_0)$ coincide with solutions to SDE's governed by C_0^{∞}-vector fields. Hence

$$X(t, x_0) \in \mathbf{S}^{\infty}_{K}(\mathbf{R}^D),$$
$$Y^{(j)}(t, x_0),\ Y^{(j)-1}(t, x_0) \in \mathbf{S}^{\infty}_{K}(\mathbf{R}^{q_n} \otimes \mathbf{R}^{q_n}), \qquad 0 \leqslant t \leqslant 1,\ 1 \leqslant j \leqslant n.$$

The proof is completed. ∎

LEMMA 7.4. (i) *If* $f_j \in \mathbf{S}^1_b(E)$, $f_j \to f$ *μ-a.e.*, $Df_j \to F$ *μ-a.e., and* $\|f_j\|_E + \|Df_j\|_{H^* \otimes E} \leqslant G$, $j = 1, 2, \ldots,$ *for some* $G \in \mathbf{S}^0_b(\mathbf{R})$, *then* $f \in \mathbf{S}^1_b(E)$, $Df = F$, *and* $\|Df\|_{H^* \otimes E} \leqslant G$.

(ii) *Let T be a compact metric space, m be a probability measure on T, and* $f(t, \cdot) \in \mathbf{S}^1_b(E)$. *Assume that* $t \mapsto f(t, w)$, $t \mapsto Df(t, w)$ *are both continuous and that there is a* $G \in \mathbf{S}^0_b(\mathbf{R})$ *such that* $\|f(t, w)\|_E + \|Df(t, w)\|_{H^* \otimes E} \leqslant G$ *for every* $t \in T$. *Then* $\int_T f(t)\, m(dt) \in \mathbf{S}^1_b(E)$ *and* $D(\int_T f(t)\, m(dt)) = \int_T Df(t)\, m(dt)$.

Proof. The second assertion follows from the first one by using the Riemann sum approximation and the bounded convergence theorem. To see the first assertion, choose $K \in \mathscr{K}_+$ such that $f_j \in \mathbf{S}^1_{K,b}(E)$ and $G \in \mathbf{S}^0_{K,b}(\mathbf{R})$. For any $g \in C_0^{\infty}(\mathbf{R}; \mathbf{R})$,

$$g(\rho_K) f_j \in \mathbf{D}^1_{\infty-}(E)$$

and

$$D(g(\rho_K) f_j) = D(g(\rho_K)) \otimes f_j + g(\rho_K)\, Df_j.$$

Since $g(\rho_K)G$ is bounded, the bounded convergence theorem implies that

$$g(\rho_K)f_j \to g(\rho_K)f \qquad \text{in } L^p(B; E, d\mu),$$
$$D(g(\rho_K)f_j) \to D(g(\rho_K))\otimes f + g(\rho_K)F \qquad \text{in } L^p(B; H^*\otimes E, d\mu)$$

for any $p\in(1,\infty)$. Thus $g(\rho_K)f\in \mathbf{D}^1_{\infty -}(E)$ and $D(g(\rho_K)f)=D(g(\rho_K))\otimes f + g(\rho_K)F$. This implies the desired conclusion. ∎

For the sake of simplicity, we set

$$Y^{(0)}(s,t)=X(t,x_0)-X(s,x_0),$$
$$Y^{(n)}(s,t)=Y^{(n)}(t,x_0)-Y^{(n)}(s,x_0),$$
$$Z^{(0)}(s,t)=0,$$
$$Z^{(n)}(s,t)=Y^{(n)-1}(t,x_0)-Y^{(n)-1}(s,x_0),$$

and write Φ_n for $\Phi_n^{(\alpha,m;x_0)}$.

We then have

LEMMA 7.5. *For each $l\in\mathbf{N}$, $n\geqslant 0$, there exist $C_1(n,l)$, $C_2(n,l)<\infty$, and $0\leqslant p_1(n,l)$, $p_2(n,l)<\infty$, depending only on D, α, m, n, l, and the bounds of V_k's and their derivatives of order up to $n+l$, such that*

$$\|D^lY^{(n)}(s,t)\|_{H^*}\leqslant C_1(n,l)(|t-s|^{1/2}+|Y^{(n+l)}(s,t)|)(1+\Phi_{n+l})^{p_1(n,l)} \tag{7.15}$$
$$\|D^lZ^{(n)}(s,t)\|_{H^*}\leqslant C_2(n,l)(|t-s|^{1/2}+|Y^{(n+l)}(s,t)| + |Z^{(n+l)}(s,t)|)(1+\Phi_{n+l})^{p_2(n,l)} \tag{7.16}$$

for any $s,t\in[0,1]$. In particular, for every n and l, there is a $K\in\mathscr{K}_+$ such that

$$X(t,x_0)\in \mathbf{S}^l_{K,b}(\mathbf{R}^D),$$
$$Y^{(n)}(t,x_0),\ Y^{(n)-1}(t,x_0)\in \mathbf{S}^l_{K,b}(\mathbf{R}^{q_n}\otimes\mathbf{R}^{q_n}), \qquad 0\leqslant t\leqslant 1.$$

Proof. We show the assertion by induction on l.

By (7.11), we can show that

$$D\begin{pmatrix} X(t,x_0)\\ Y^{(1)}(t,x_0)\\ \vdots \\ Y^{(n)}(t,x_0)\end{pmatrix}[h]$$
$$= Y^{(n+1)}(t,x_0)\sum_{k=1}^{d}\int_0^t Y^{(n+1)-1}(u,x_0) \times V_k^{\natural(n)}(X(u,x_0),Y^{(1)}(u,x_0),\ldots,Y^{(n)}(u,x_0))\,\dot h^k(u)\,du. \tag{7.17}$$

Combining this with (7.10) and the elementary inequality that

$$\int_a^b |\dot{h}|\,(s)\,ds \leqslant |b-a|^{1/2}\,\|h\|_H,$$

we see that (7.15) holds for $l=1$.

Since

$$DY^{(n)-1}(t,x_0) = -Y^{(n)-1}(t,x_0)\,DY^{(n)}(t,x_0)\,Y^{(n)-1}(t,x_0), \tag{7.18}$$

and (7.4) yields that

$$|Y^{(n-1)}(s,t)| \leqslant |Y^{(n)}(s,t)| \qquad \text{and} \qquad |Z^{(n-1)}(s,t)| \leqslant |Z^{(n)}(s,t)|, \tag{7.19}$$

on account of (7.10), we see that (7.16) for $l=1$ follows from (7.15) for $l=1$.

Suppose now that (7.15) and (7.16) are fulfilled for every $k \leqslant l$ and n. By applying Lemma 7.4, we can conclude from (7.17) that

$$\begin{aligned} D^{l+1}\begin{pmatrix} X(t,x_0) \\ Y^{(1)}(t,x_0) \\ \vdots \\ Y^{(n)}(t,x_0) \end{pmatrix}[h] &= \sum_{j=0}^{l} [D^j Y^{(n+1)}(t,x_0)] \\ &\quad \times \sum_{k=1}^{d} \int_0^t D^{l-j}(Y^{(n+1)-1}(u,x_0)\,V_k^{\natural(n)}(\cdots))\,\dot{h}^k(u)\,du. \end{aligned} \tag{7.20}$$

By virtue of the assumption of induction, (7.10), (7.19), and (7.20), we can easily show that (7.15) holds for $l+1$. Applying (7.10), (7.18), and (7.19) again, we see that (7.16) also holds for $l+1$. ∎

Proof of Theorem 7.1. We have already established the first assertion. To see the second assertion, for the sake of simplicity, in this proof all constants of the form $C_*(n,l)$ or $p_*(n,l)$ are assumed to depend only on D, α, m, n, l, and the bounds of V_k's and their derivatives of order $\leqslant n+l$, and we will not mention it each time when constants appear.

Let

$$\Phi_{n,k} = \iint_{\substack{|t-s|\geqslant 1/k \\ 0\leqslant s,t\leqslant 1}} \frac{\sum_{j=0}^{n} \{(Y^{(j)}(s,t))^{2m} + (Z^{(j)}(s,t))^{2m}\}}{|t-s|^{1+2m\alpha}}\,dt\,ds.$$

Then, by Lemmas 7.4 and 7.5 and (7.8), $\Phi_{n,k} \in \mathbf{S}_b^{\infty}(\mathbf{R})$ and

$$D^l \Phi_{n,k} = \iint_{\substack{|t-s| \geq 1/k \\ 0 \leq s,t \leq 1}} \frac{\sum_{j=0}^n \{D^l[(Y^{(j)}(s,t))^{2m}] + D^l[(Z^{(j)}(s,t))^{2m}]\}}{|t-s|^{1+2m\alpha}} \, dt \, ds.$$

Recall that there exist universal constants $a_{k_1,\ldots,k_p}^{l,m}$ such that

$$D^l(F^{2m}) = \sum_{p=1}^{2m \wedge l} F^{2m-p} \sum_{\substack{k_1+\cdots+k_p=l \\ k_i \geq 1}} a_{k_1,\ldots,k_p}^{l,m} D^{k_1}F \otimes \cdots \otimes D^{k_p}F.$$

Hence, by (7.15), (7.16), and (7.19), there are $C_3(n,l) < \infty$ and $0 \leq p_3(n,l) < \infty$ such that

$$\begin{aligned}
&\int_0^1 \int_0^1 \frac{\sum_{j=0}^n \{D^l[(Y^{(j)}(s,t))^{2m}] + D^l[(Z^{(j)}(s,t))^{2m}]\}}{|t-s|^{1+2m\alpha}} \, dt \, ds \\
&\quad \leq C_3(n,l)(1+\Phi_{n+l})^{p_3(n,l)} \\
&\qquad \times \sum_{j=0}^n \sum_{p=1}^{2m \wedge l} \left\{ \int_0^1 \int_0^1 \right. \\
&\qquad \times \frac{(|t-s|^{1/2} + |Y^{(j+l)}(s,t)|)^p \, |Y^{(j)}(s,t)|^{2m-p}}{|t-s|^{1+2m\alpha}} \, dt \, ds + \int_0^1 \int_0^1 \\
&\qquad \times \left. \frac{(|t-s|^{1/2} + Y^{(j+l)}(s,t)| + |Z^{(j+l)}(s,t)|)^p \, |Z^{(j)}(s,t)|s^{m-p}}{|t-s|^{1+2m\alpha}} \, dt \, ds \right\}.
\end{aligned}$$

Applying Hölder's inequality, we obtain constants $C_4(n,l) < \infty$ and $0 \leq p_4(n,l) < \infty$ such that

$$\begin{aligned}
&\int_0^1 \int_0^1 \frac{\sum_{j=0}^n [D^l(Y^{(j)}(s,t))^{2m} + D^l(Z^{(j)}(s,t))^{2m}]}{|t-s|^{1+2m\alpha}} \, dt \, ds \\
&\quad \leq C_4(n,l)(1+\Phi_{n+l})^{p_4(n,l)}.
\end{aligned}$$

By Lemma 7.4, we can conclude that $\Phi_n = \lim_{k \to \infty} \Phi_{n,k} \in \mathbf{S}_b^{\infty}(\mathbf{R})$ and (7.5) holds.

The precise expression of $C(0,1)$ and $p(0,1)$ can be shown by repeating the above argument and taking into account the identity

$$DX(t,x_0)[h] = Y^{(1)}(t,x_0) \sum_{k=1}^d \int_0^1 Y^{(1)-1}(s,x_0) \, V_k(X(s,x_0)) \dot{h}^k(s) \, ds.$$

We omit the details. ∎

We now give examples of compact domination $g \ll f$.

THEOREM 7.6. (i) $(X(t, x_0), DX(t, x_0), D^2X(t, x_0)) \ll \Phi_2^{(\alpha, m; x_0)}$.

(ii) *If we define* $F: B \to L^2([0, 1]; \mathbf{R}^d)$ *by* $F = X(\cdot, x_0)$, *then* $(F, DF, D^2F) \ll \Phi_2^{(\alpha, m; x_0)}$.

(iii) *Let* $0 < \alpha < \frac{1}{2}$ *and* $n \in \mathbf{N}$. *Then there is an* $m_0 \in \mathbf{N}$ *so that, for each* $m \geq m_0$, $2m\alpha > 1$, $\alpha + (\frac{1}{2m}) < \frac{1}{2}$, *and there are* $0 < \beta < \frac{1}{2}$ *and* m' *satisfying that* $2m'\beta > 1$, $\beta + (\frac{1}{2m'}) < \frac{1}{2}$, *and* $(\Phi_n^{(\alpha, m; x_0)}, D\Phi_n^{(\alpha, m; x_0)}, D^2\Phi_n^{(\alpha, m; x_0)}) \ll \Phi_{n+2}^{(\beta, m'; x_0)}$.

Proof. (i) Fix an arbitrary i and define $k_l(u)$ by

$$DX^i(t, x_0)[h] = \sum_{l=1}^{d} \int_0^1 k_l(u) \frac{dh^l}{du}(u)\, du.$$

It holds that

$$h_l(u) = \mathbf{1}_{[0,t]}(u) \sum_{j,k=1}^{d} (Y^{(1)})^i_j(t)(Y^{(1)-1})^j_k(u)\, V^k_l(X(u, x_0)).$$

By (7.8)–(7.10), we see that $[0, t] \ni u \mapsto h_l(u)$ is Hölder continuous and the Hölder constant is dominated by $\Phi_2^{(\alpha, m; x_0)}$. Thus, by Ascoli–Arzelà's theorem, for each n, there exists a compact $L_n \subset C([0, t]; \mathbf{R})$ such that $\mu(h_l|_{[0,t]} \notin L_n, \Phi_2^{(\alpha, m; x_0)} \leq n) = 0$. Then through the natural imbedding $C([0, t]; \mathbf{R}^d) \subsetneq L^2([0, t]; \mathbf{R}^d) \subsetneq L^2([0, 1]; \mathbf{R}^d)$, we see that $DX(t, x_0) \ll \Phi_2^{(\alpha, m; x_0)}$.

Define $k_{l,l'}(u, v)$ by

$$D^2X^i(t, x_0)[h_1, h_2] = \sum_{l,l'=1}^{d} \int_0^1 \int_0^1 k_{l,l'}(u, v) \frac{dh_1^l}{du}(u) \frac{dh_2^{l'}}{du}(v)\, du\, dv.$$

It then holds that $k_{l,l'}(u, v) = k_{l',l}(v, u)$ and $k_{l,l'}(u, v) = (d/dv)[Dk_l(u)](v)$. Using expression (7.17), we can conclude that the mapping $\Delta_t \equiv \{(u, v) \in [0, t]^2 . v \leq u\} \ni (u, v) \mapsto k_{l,l'}(u, v)$ is Hölder continuous and the Hölder constant is dominated by $\Phi_0^{(\alpha, m; x_0)}$. Then through the natural imbedding $C(\Delta_t; \mathbf{R}^d) \subsetneq L^2(\Delta_t; \mathbf{R}^d) \subsetneq L^2(\Delta_1; \mathbf{R}^d)$, we see that $D^2X(t, x_0) \ll \Phi_2^{(\alpha, m; x_0)}$.

(ii) By virtue of Lemma 7.5, $t \mapsto X(t, x_0)$, $t \mapsto DX(t, x_0)$, and $t \mapsto D^2X(t, x_0)$ are all Hölder continuous and the Hölder constants are dominated by $\Phi_2^{(\alpha, m; x_0)}$. By the above observation and Ascoli's theorem, for each n, there are compact $L_n \subset C([0, 1]; H^*)$ and $L'_n \subset C([0, 1]; (H^*)^{\otimes 2})$ such that $\mu(DX(\cdot, x_0 \notin L_n(L_n, \Phi_2^{(\alpha, m; x_0)} \leq n) = 0$ and $\mu(D^2X(\cdot, x_0) \notin L'_n, \Phi_2^{(\alpha, m; x_0)} \leq n) = 0$. Then, through the natural continuous imbedding

$$C([0, 1]; (H^*)^{\otimes n}) \subsetneq L^2([0, 1]; (H^*)^{\otimes n}) \subsetneq (H^*)^{\otimes n} \otimes L^2([0, 1]; \mathbf{R}),$$

we see that $(F, DF, D^2F) \ll \Phi_2^{(\alpha, m; x_0)}$.

(iii) For the sake of simplicity, we fix x_0 and consider only the case where $n = 0$ (the assertion in the other cases $n \geq 1$ can be seen in the same way). Hence we write $\Phi^{(\alpha,m)}$, $X(t)$, $Y^{(j)}(t)$ for $\Phi_0^{(\alpha,m;x_0)}$, $X(t,x_0)$, $Y^{(j)}(t,x_0)$. Define $g_l(u)$ by

$$D\Phi^{(\alpha,m)}[h]=\sum_{l=1}^{d}\int_0^1 g_l(u)\frac{dh^l}{du}(u)\,du.$$

We have

$$\begin{aligned} g_l(u)=8m\sum_{i,j,k=1}^{d}\Bigg[&\left\{\int_0^u dt\int_0^t ds\,\frac{(X^i(t)-X^i(s))\,|X(t)-X(s)|^{2m-2}}{|t-s|^{1+2m\alpha}}\right.\\ &\left.\times(Y^i_j(t)-Y^i_j(s))\right\}\\ &+\left\{\int_u^1 dt\int_0^u ds\,\frac{(X^i(t)-X^i(s))\,|X(t)-X(s)|^{2m-2}}{|t-s|^{1+2m\alpha}}\,Y^i_j(s)\right\}\Bigg]\\ &\times(Y^{(1)-1}(u))^j_k V^k_l(X(u)). \end{aligned}$$

For $u\leqslant u'$, it holds that

$$\begin{aligned} &\left|\int_0^u dt\int_0^t ds\,\frac{(X^i(t)-X^i(s))\,|X(t)-X(s)|^{2m-2}}{|t-s|^{1+2m\alpha}}(Y^i_j(t)-Y^i_j(s))\right.\\ &\qquad\left.-\int_0^{u'} dt\int_0^t ds\,\frac{(X^i(t)-X^i(s))\,|X(t)-X(s)|^{2m-2}}{|t-s|^{1+2m\alpha}}(Y^i_j(t)-Y^i_j(s))\right|\\ &\leqslant\int_u^{u'} dt\int_0^t ds\,\frac{|X(t)-X(s)|^{2m-1}}{|t-s|^{1+2m\alpha}}\,|Y(t)-Y(s)|\\ &\leqslant\left(\int_u^{u'} dt\int_0^t ds\,|t-s|^{-1/2}\right)^{1/2}\\ &\qquad\times\left(\int_0^1\int_0^1\frac{|X(t)-X(s)|^{4m}}{|t-s|^{1+(1/2)+4m\alpha}}\,dt\,ds\right)^{(2m-1)/4m}\\ &\qquad\times\left(\int_0^1\int_0^1\frac{|Y(t)-Y(s)|^{4m}}{|t-s|^{1+(1/2)+4m\alpha}}\,dt\,ds\right)^{1/4m}. \end{aligned}$$

Moreover, by applying the Hölder inequality, we obtain

$$\begin{aligned} &\left|\int_u^1 dt\int_0^u ds\,\frac{(X^i(t)-X^i(s))\,|X(t)-X(s)|^{2m-2}}{|t-s|^{1+2m\alpha}}\,Y^i_j(s)\right.\\ &\qquad\left.-\int_{u'}^1 dt\int_0^{u'} ds\,\frac{(X^i(t)-X^i(s))\,|X(t)-X(s)|^{2m-2}}{|t-s|^{1+2m\alpha}}\,Y^i_j(s)\right|\\ &\leqslant C(m,\alpha)\,|u-u'|^{1/2}\,(1+\Phi_1^{(\alpha,m;x_0)})^{1/4m}\,(\Phi^{(\beta,4m-2)})^{1/2}, \end{aligned}$$

where $\beta=(4m\alpha+1)/(4m-2)$. Since $\Phi^{(\alpha,m)}\leqslant\Phi^{(\alpha',m)}$ for $\alpha\leqslant\alpha'$ and $(\Phi^{(\alpha,m)})^p\leqslant\Phi^{(\alpha+(1/4mp),mp)}$ for $p\geqslant 2$, we therefore see that $u\mapsto g_l(u)$ is Hölder continuous and the Hölder constant is dominated by $\Phi_2^{(\beta',m';x_0)}$ for some β' and m' with $2\beta'm'>1$ and $\beta'+(\frac{1}{2m'})<\frac{1}{2}$ provided that m is sufficiently large. Thus $D\Phi^{(\alpha,m)}\ll\Phi_2^{(\beta',m';x_0)}$.

If we denote by $g_{l,l'}$ the kernel of $D^2\Phi^{(\alpha,m)}$, then in exactly the same manner we can show that $\Delta\ni u\mapsto g_{l,l'}(u,v)$ is Hölder continuous and that the Hölder constant is dominated by some $\Phi_2^{(\beta'',m'';x_0)}$. This implies that $D^2\Phi^{(\alpha,m)}\ll\Phi_2^{(\beta'',m'';x_0)}$. ∎

We next give an example of Ω,σ, and γ satisfying assumptions in Theorems 3.2–3.4. Let $A_0, A_1, ..., A_{d'}$ be holomorphic mappings of $\mathbf{C}^{d'}$ to $\mathbf{C}^{d'}$, and $z(t)$ and $Y(t)$ be the solutions to the following SDEs on $\mathbf{C}^{d'}$ and $\mathbf{C}^{d'}\otimes\mathbf{C}^{d'}$, respectively,

$$dz(t)=\sum_{k=1}^{d'} A_k(z(t))\circ dB^k(t)+A_0(z(t))\,dt \tag{7.21}$$

$$dY(t)=\sum_{k=1}^{d'} \partial A_k(z(t))\,Y(t)\circ dB^k(t)+\partial A_0(z(t))\,Y(t)\,dt \tag{7.22}$$

$$z(0)=0$$

$$Y(0)=\mathrm{id},$$

where $\partial A_k=(\partial A_k^i/\partial z^j)_{1\leqslant i,j\leqslant d}$ and $\partial/\partial z^j$ is the complex derivative. The solutions may explode but $Y(t)$ is invertible up to time of explosion. We set

$$\phi_0=\int_0^1\int_0^1\frac{|z(t)-z(s)|^{2m}}{|t-s|^{1+2m\alpha}}\,dt\,ds \tag{7.23}$$

$$\phi_1=\phi_0+\int_0^1\int_0^1\frac{|Y(t)-Y(s)|^{2m}+|Y(t)^{-1}-Y(s)^{-1}|^{2m}}{|t-s|^{1+2m\alpha}}\,dt\,ds \tag{7.24}$$

$$\Omega_r=\{\phi_0<r\},\qquad 0<r\leqslant\infty.$$

We have

LEMMA 7.7. (i) *$\phi_0\in\Psi(\Omega_\infty)$ and $\phi_1\in\mathscr{S}(\phi_0,\Omega_\infty)$.*

(ii) *There is a non-decreasing and convex $\chi\in C^\infty(\mathbf{R};\mathbf{R})$ such that $\gamma_\infty\equiv\chi(\phi_1)$ satisfies that*

$$\operatorname*{ess.sup}_{\Omega_\infty}\|D\phi_0\|_{H^*}^2\,e^{-\gamma_\infty}<\infty.$$

In particular, $\gamma_\infty\in\Gamma(\phi_0,\Omega_\infty)$.

(iii) *$\partial\bar\partial\phi_0\geqslant 0$, and $\partial\bar\partial\phi_1\geqslant 0$ μ-a.e. on Ω_∞.*

Proof. By (7.8), we have

$$|z(t)| \leqslant \frac{16\sqrt{d}(1+2m\alpha)}{2m\alpha-1}\,\phi_0^{1/2m}.$$

Let $R>0$ and take $\varphi_R \in C_0^\infty(\mathbf{C}^{d'};\mathbf{R})$ such that $\varphi_R(z)=1$ if $|z| \leqslant 16\sqrt{d}(2m\alpha-1)^{-1}(1+2m\alpha)(R+1)^{1/2m}$. Denote by $z_R(t)$ and $Y_R(t)$ the solutions to the SDEs

$$\begin{aligned}
dz_R(t) &= \sum_{k=1}^{d'} (\varphi_R A_k)(z_R(t)) \circ dB^k(t) + (\varphi_R A_0)(z_R(t))\,dt\\
dY_R(t) &= \sum_{k=1}^{d'} \partial(\varphi_R A_k)(z_R(t))\,Y_R(t) \circ dB^k(t)\\
&\quad + \partial(\varphi_R A_0)(z_R(t))\,Y_R(t)\,dt\\
z(0) &= 0\\
Y(0) &= \mathrm{id},
\end{aligned}$$

and define $\phi_{0,R}$ and $\phi_{1,R}$ by (7.23) and (7.24) with $z=z_R$ and $Y=Y_R$, respectively. If $g \in C_0^\infty(\mathbf{R};\mathbf{R})$ enjoys $\mathrm{supp}[g] \subset (-R-1, R+1)$, then we have that

$$g(\phi_0)=g(\phi_{0,R}) \qquad \text{and} \qquad g(\phi_0)\phi_1 = g(\phi_{0,R})\,\phi_{1,R}.$$

Thinking of $(z,\bar{z})$ as a real coordinate system on $\mathbf{C}^{d'}$ and set $X_R(t) = (z_R(t), \overline{z_R(t)})$. If we define $Y_R^{(1)}(t)$ by (7.2) with $X(t)=X_R(t)$, then

$$Y_R^{(1)}(t) = \begin{pmatrix} Y_R(t) & 0 \\ 0 & \overline{Y_R(t)} \end{pmatrix} \qquad \text{on } \{\phi_0 < R\}. \tag{7.25}$$

Applying Theorem 7.1, we see that $\phi_0 \in \Psi(\Omega_\infty)$ and $\phi_1 \in \mathscr{S}(\phi_0, \Omega_\infty)$.

To see the second assertion, note that

$$\phi_0 = \phi_{0,R}, \qquad \phi_1 = \phi_{1,R}, \qquad \text{and} \qquad D\phi_0 = D\phi_{0,R} \qquad \text{on } \{\phi_0 < R\}. \tag{7.26}$$

Let

$$b(R) = \sup\left\{ |A_k(z)| : k=0,1,\ldots,d',\ |z| \leqslant \frac{16\sqrt{d}(1+2m\alpha)}{2m\alpha-1}(R+1)^{1/2m} \right\}.$$

By Theorem 7.1 and (7.26), we obtain that

$$\|D\phi_0\|_{H^*} \leqslant C_0 b(R)(1+\phi_1)^{1+1/m} \qquad \text{on } \{\phi_0 < R\},$$

where $C_0 = (16d(1+2m\alpha)(1+\sqrt{d})/2m\alpha - 1)^2 (1 + (2/(m-2m\alpha)(m-2m\alpha+1)))$. Thus we see

$$\|D\phi_0\|_{H^*} \leqslant C_0 b(\phi_1)(1+\phi_1)^{1+1/m}.$$

It therefore suffices to choose a non-decreasing and convex $\chi \in C^\infty(\mathbf{R};\mathbf{R})$ so that

$$\chi(x) \geqslant 2\log[b(x)(1+x)^{1+1/m}] \qquad \text{for} \quad x \geqslant 0.$$

We now show the last assertion. If $\varphi \in C^\infty(\mathbf{C}^n;\mathbf{C})$ and $F \in \mathbf{S}^1(\mathbf{C}^n)$, then

$$D\varphi(F) = \sum_{j=1}^n \frac{\partial\varphi}{\partial z^j}(F)\,DF^j + \sum_{j=1}^n \frac{\partial\varphi}{\partial \bar{z}^j}(F)\,D\overline{F^j},$$

and hence

$$\partial\varphi(F) = \sum_{j=1}^n \frac{\partial\varphi}{\partial z^j}(F)\,\partial F^j + \sum_{j=1}^n \frac{\partial\varphi}{\partial \bar{z}^j}(F)\,\partial\overline{F^j}. \tag{7.27}$$

$$\bar{\partial}\varphi(F) = \sum_{j=1}^n \frac{\partial\varphi}{\partial z^j}(F)\,\bar{\partial} F^j + \sum_{j=1}^n \frac{\partial\varphi}{\partial \bar{z}^j}(F)\,\overline{\partial F^j}. \tag{7.28}$$

We further recall that

$$\begin{aligned} &D\left(\int_0^T f(t)\circ dB^k(t)\right)[h] \\ &\quad = \int_0^T Df(t)[h]\circ dB^k(t) + \int_0^T f(t)\frac{d}{dt}B^k(t,h)\,dt. \end{aligned} \tag{7.29}$$

Define $A^{\#(2)}_{k,R}(z,y) \in \mathbf{C}^{d'+(d')^2} \otimes \mathbf{C}^{d'+(d')^2} \otimes \mathbf{C}^{d'+(d')^2}$ to be the Jacobian matrix with respect to the holomorphic differentiation of the mapping

$$\mathbf{C}^{d'} \times (\mathbf{C}^{d'} \otimes \mathbf{C}^{d'}) \ni (z,y) \mapsto A^{\natural}_{k,R}(z,y) = (\varphi_R A_k(z), (\partial(\varphi_R A_k))y).$$

We set

$$A^{\natural(2)}_{k,R}(z,y,y^{(2)}) = (\varphi_R A_k(z), (\partial(\varphi_R A_k))\,y, A^{\#(2)}_{k,R}(z,y)\,y^{(2)}),$$

where $y^{(2)} \in \mathbf{C}^{d'+(d')^2} \otimes \mathbf{C}^{d'+(d')^2}$. Let $Y^{(2)}_R(t)$ be the solution to the SDE:

$$\begin{aligned} dY^{(2)}_R(t) &= \sum_{k=1}^{d'} A^{\#(2)}_{k,R}(z_R(t), Y_R(t))\,Y^{(2)}_R(t)\circ dB^k(t) \\ &\qquad + A^{\#(2)}_{0,R}(z_R(t), Y_R(t))\,Y^{(2)}_R(t)\,dt \\ Y^{(2)}_R(0) &= I_{\mathbf{C}^{d'+(d')^2}\otimes\mathbf{C}^{d'+(d')^2}}. \end{aligned}$$

Since $\varphi_R A_k$, $A^{\natural}_{k,R}$, $A^{\natural(2)}_{k,R}$ are holomorphic on $\{\varphi_R = 1\}$ and $B(t, \pi_{1,0} h) = B(t, h)$, we have that

$$d\bar{\partial}\begin{pmatrix} z_R(t) \\ Y_R(t) \\ Y_R^{(2)}(t) \end{pmatrix} = \sum_{k=1}^{d'} (\partial A^{\natural(2)}_{k,R})(z_R(t), Y_R(t), Y_R^{(2)}(t))\, \bar{\partial}\begin{pmatrix} z_R(t) \\ Y_R(t) \\ Y_R^{(2)}(t) \end{pmatrix} \circ dB^k(t) + (\partial A^{\natural(2)}_{0,R})(z_R(t), Y_R(t), Y_R^{(2)}(t))\, \bar{\partial}\begin{pmatrix} z_R(t) \\ Y_R(t) \\ Y_R^{(2)}(t) \end{pmatrix} dt$$

$$d\partial\begin{pmatrix} z_R(t) \\ Y_R(t) \end{pmatrix}[h] = \sum_{k=1}^{d'} A^{\#(2)}_{k,R}(z_R(t), Y_R(t))\partial\begin{pmatrix} z_R(t) \\ Y_R(t) \end{pmatrix}[h] \circ dB^k(t) + A^{\#(2)}_{0,R}(z_R(t), Y_R(t))\partial\begin{pmatrix} z_R(t) \\ Y_R(t) \end{pmatrix}[h]\, dt + \sum_{k=1}^{d'} A^{\natural}_{k,R}(z_R(t), Y_R(t))\frac{d}{dt} B^k(t, h)\, dt$$

on $\{\phi_0 < R\}$ for every $R > 0$. Thus we see that

$$\bar{\partial}\begin{pmatrix} z_R(t) \\ Y_R(t) \\ Y_R^{(2)}(t) \end{pmatrix} = 0$$

$$\partial\begin{pmatrix} z_R(t) \\ Y_R(t) \end{pmatrix}[h] = \sum_{k=1}^{d'} Y_R^{(2)}(t)\int_0^t Y_R^{(2)-1}(s)\, A^{\natural}_{k,R}(z_R(s), Y_R(s))\frac{d}{ds} B^k(s, h)\, ds$$

for $0 \leqslant t \leqslant 1$ on $\{\phi_0 < R\}$. We can therefore conclude that

$$\bar{\partial} z_R(t) = 0, \qquad \bar{\partial} Y_R(t) = 0,$$
$$\bar{\partial}\partial z_R(t) = 0, \qquad \text{and} \qquad \bar{\partial}\partial Y_R(t) = 0 \qquad \text{on } \Omega_R.$$

The third assertion follows from these. ▮

LEMMA 7.8. *There exists a non-negative, non-decreasing, and convex* $\tilde{\chi} \in C^{\infty}(\mathbf{R}; \mathbf{R})$ *such that* $\sigma_{\infty} \equiv \tilde{\chi}(\phi_1) \in \mathscr{S}(\phi_0, \Omega_{\infty})$ *and fulfills conditions* (3.1)–(3.3) *in Theorem* 3.3 *with* $\sigma_0 = \sigma_{\infty}$, $\gamma = \gamma_{\infty}$, *and* $\varepsilon = \frac{1}{4}$.

Proof. Since

$$\partial\bar{\partial}\phi_1 \geqslant 0,$$

$$\partial\bar{\partial}\gamma_{\infty} \geqslant 0, \qquad \text{and} \qquad \partial\gamma_{\infty} \otimes \bar{\partial}\gamma_{\infty} = (\chi')^2(\phi_1)\, \partial\phi_1 \otimes \bar{\partial}\phi_1,$$

it suffices to put $\tilde{\chi}(x) = 4\int_0^x \int_0^v (\chi')^2(u)\, du\, dv$. ∎

Thus, as an application of Theorems 3.3 and 3.4, we have established

THEOREM 7.9. *Let* Ω_{∞} *and* γ_{∞} *be as above. Then, for every* $\sigma \in \mathscr{S}(\phi_0, \Omega_{\infty})$ *with* $\partial\bar{\partial}\sigma \geqslant 0$ $\bar{\mu}$ *a.e. on* Ω_{∞}, *it holds that*

$$\operatorname{Ker}(T^{(p,q+1)}_{\sigma,\gamma_{\infty}}) \subset \operatorname{Image}(T^{(p,q)}_{\sigma,\gamma_{\infty}}).$$

Moreover, the sequence

$$\textstyle\bigwedge^{p,0}_{\sigma,\gamma_{\infty}}(\Omega_{\infty}) \xrightarrow{\hat{T}} \cdots \xrightarrow{\hat{T}} \bigwedge^{p,q}_{\sigma,\gamma_{\infty}}(\Omega_{\infty})$$

$$\xrightarrow{\hat{T}} \textstyle\bigwedge^{p,q+1}_{\sigma,\gamma_{\infty}}(\Omega_{\infty}) \xrightarrow{\hat{T}} \bigwedge^{p,q+2}_{\sigma,\gamma_{\infty}}(\Omega_{\infty}) \xrightarrow{\hat{T}} \cdots$$

is exact, where $\hat{T} = \hat{T}^{(p,q)}_{\sigma,\gamma_{\infty}}$ *on* $\bigwedge^{p,q}_{\sigma,\gamma_{\infty}}(\Omega_{\infty})$.

We next consider Ω_r, $r < \infty$. If we put

$$\psi_r(w) = -\log(r - \phi_0(w)) + \log r, \qquad w \in \Omega_r,$$

then, as in the proof of Lemma 7.7, we see that

$$\psi_r \in \Psi(\Omega_r) \qquad \text{and} \qquad \phi_1 \in \mathscr{S}(\psi_r, \Omega_r).$$

Let $\gamma_r = 2\psi_r + \gamma_{\infty}$. Then, by Lemma 7.7,

$$\operatorname*{ess.sup}_{\Omega_r} \|D\psi_r\|^2_{H^*} e^{-\gamma_r} < \infty.$$

Hence $\gamma_r \in \Gamma(\psi_r, \Omega_r)$. Further, since

$$\partial\gamma_r \otimes \bar{\partial}\gamma_r \leqslant \frac{8}{(r-\phi_0)^2}\, \partial\phi_0 \otimes \bar{\partial}\phi_0 + 2(\chi')^2(\phi_1)\, \partial\phi_1 \otimes \bar{\partial}\phi_1,$$

the function

$$\sigma_r \equiv 32\psi_r + 8\int_0^{\phi_1} \int_0^v (\chi')^2(u)\, du\, dv$$

satisfies (3.1)–(3.3) in Theorem 3.3 with $\sigma_0 = \sigma_r$, $\gamma = \gamma_r$, and $\varepsilon = \frac{1}{4}$. We therefore have

THEOREM 7.10. *Let Ω_r and γ_r be as above. Then, for every $\sigma \in \mathcal{S}(\psi_r, \Omega_r)$ with $\partial\bar{\partial}\sigma \geq 0$ μ-a.e. on Ω_r, it holds that*

$$\operatorname{Ker}(T^{(p,q+1)}_{\sigma,\gamma_r}) \subset \operatorname{Image}(T^{(p,q)}_{\sigma,\gamma_r}).$$

Moreover, the sequence

$$\bigwedge\nolimits^{p,0}_{\sigma,\gamma_r}(\Omega_r) \xrightarrow{\hat{T}} \cdots \xrightarrow{\hat{T}} \bigwedge\nolimits^{p,q}_{\sigma,\gamma_r}(\Omega_r)$$
$$\xrightarrow{\hat{T}} \bigwedge\nolimits^{p,q+1}_{\sigma,\gamma_r}(\Omega_r) \xrightarrow{\hat{T}} \bigwedge\nolimits^{p,q+2}_{\sigma,\gamma_r}(\Omega_r) \xrightarrow{\hat{T}} \cdots$$

is exact, where $\hat{T} = \hat{T}^{(p,q)}_{\sigma,\gamma_r}$ on $\bigwedge^{p,q}_{\sigma,\gamma_r}(\Omega_r)$.

We next consider an example of σ_0 and $F: \Omega \to E_0$ which appeared in the Dolbeault type theorem. We construct $A_k^{\#(j)}$ from A_k in the same way as we did $V_k^{\#(j)}$ from V_k by using the complex coordinate instead of the real one. Let $Y^{(j)}(t)$ be the solution to the SDEs

$$dY^{(j)}(t) = \sum_{k=1}^{d'} A_k^{\#(j)}(z(t), Y^{(1)}(t), \ldots, Y^{(j-1)}(t))\, Y^{(j)}(t) \circ dB^k(t)$$
$$+ A_0^{\#(j)}(z(t), Y^{(1)}(t), \ldots, Y^{(j-1)}(t))\, Y^{(j)}(t)\, dt,$$

with the initial condition $Y^{(j)}(0) = \mathrm{id}$. Obviously $Y^{(1)}(t) = Y(t)$. For $n \geq 0$, we define $\phi_n^{(\beta,k)}$ by

$$\phi_0^{(\beta,k)} = \int_0^1 \int_0^1 \frac{|z(t) - z(s)|^{2k}}{|t-s|^{1+2k\beta}}\, dt\, ds.$$

$$\phi_n^{(\beta,k)} = \phi_{n-1}^{(\beta,k)} + \int_0^1 \int_0^1 \frac{|Y^{(n)}(t) - Y^{(n)}(s)|^{2k} + |Y^{(n)-1}(t) - Y^{(n)-1}(s)|^{2k}}{|t-s|^{1+2k\beta}}\, dt\, ds.$$

As in the proof of Lemma 7.2, we see that $\phi_n^{(\beta,k)} < \infty$ on Ω_∞. Suppose further that $m \geq m_0$, m_0 being the constant obtained in Theorem 7.6(iii). By Theorem 7.6, we see that $(\gamma_\infty, D\gamma_\infty, D^2\gamma_\infty) \ll \phi_3^{(\beta,k)}$ for some $0 < \beta < \frac{1}{2}$ and $k \in \mathbf{N}$ with $2k\beta > 1$ and $\beta + (\frac{1}{2k}) < \frac{1}{2}$. As before, we can show that $\partial\bar{\partial}\phi_n^{(\beta,k)} \geq 0$ on Ω_∞. Thus we can construct a $\sigma_0 \in \mathcal{S}(\phi_0, \Omega_\infty)$ satisfying $\sigma_0 \geq \phi_3^{(\beta,k)}$ and

$$\sigma_0 \geq A \qquad \text{for some symmetric Hilbert–Schmidt } A > 0.$$

In particular,

$$(\gamma_\infty, D\gamma_\infty, D^2\gamma_\infty) \ll \sigma_0.$$

Then from Theorems 6.10 and 7.6 follows

THEOREM 7.11. *Assume either that* $E_0 = \mathbf{C}^{nd'}$ *and* $F = (z(t_1), ..., z(t_n))$ *or that* $E_0 = L^2([0, 1]; \mathbf{C}^{d'})$ *and* $F = z(\cdot)$. *If* $\{U_i\}_{i\in I}$ *is a locally finite family of pseudoconvex open sets in* E_0, *then*

$$H^q(\mathcal{O}^p) \simeq \mathrm{Ker}(\bar{\partial}_{\wedge_U^{p,q}})/\mathrm{Image}(\bar{\partial}_{\wedge_U^{p,q-1}}).$$

We finally consider an example of approximation theorems studied in Section 4. We continue to work with the above σ_0 and γ_∞. If $\sigma \in \mathcal{S}(\phi_0, \Omega_\infty)$ is decomposed as $\sigma = \sigma_p + \sigma_c$ with $\sigma_p \in \mathcal{PS}(\phi_0, \Omega_\infty)$, and $\sigma_c \in \mathcal{S}(\phi_0, \Omega_\infty)$ such that $D^2\sigma_c \ll \sigma_0$, then we can find a $\chi \in \mathcal{X}$ such that (4.2) is satisfied with $\sigma_1 = \chi(\sigma_0)$. Indeed, we can find $\chi \in \mathcal{X}$ satisfying $\partial\bar{\partial}(\chi(\sigma_0) + \sigma_c) \geqslant 0$ because $D^2\sigma_c \ll \sigma_0$. Obviously, σ_1 fulfills (4.1). Thus we obtain

THEOREM 7.12. *Suppose that* $\sigma \in \mathcal{S}(\phi_0, \Omega_\infty)$ *is decomposed as above. Then, for each* $u \in \bigwedge_{\sigma,\gamma_\chi}^{p,0}(\Omega)$ *with* $T_{\sigma,\gamma_\chi}^{(p,0)} u|_{\Omega_1} = 0$, *there is a sequence* $\{u_n\} \subset \mathcal{A}(\sigma)$ *such that*

$$\|u|_{\Omega_{1-\delta}} - u_n|_{\Omega_{1-\delta}}\|_\sigma \to 0 \qquad \text{as} \quad n \to \infty.$$

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