Central and Local Limit Theorems Applied to Asymptotic Enumeration. III. Matrix Recursions

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Let a multivariate sequence \( a_n(k) \) be obtained by a matrix recursion. It is shown that it is usually easy to establish central and local limit theorems for \( a_n(k) \). The proof requires a lemma on multisection of multivariate series which appears to be new. The applications of the limit theorems include covering by polyominoes, enumeration of plane animals, occupancy problems, 0–1 matrices, and nonexistence of critical phenomena.

I. INTRODUCTION

We continue the study of central and local limit theorems applied to asymptotic enumeration, concentrating on those problems solved by transfer matrices. Let \( \mathbf{x} \) and \( \mathbf{k} \) be \( d \)-dimensional vectors and set \( \mathbf{x}^k = x_1^{k_1} \cdots x_d^{k_d} \). We study \( a_n(k) \) such that

\[
\sum_k a_n(k) \mathbf{x}^k = \sum_{i,j} C_{ij}(\mathbf{x}) T_{ij}^{(n)}(\mathbf{x}),
\]

\((1.1)\)

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where $C$ and $T$ are matrices whose entries are Laurent series with nonnegative coefficients converging for all values of $x$ under consideration. Here $T_{ij}^{(n)}$ is the $(i, j)$ entry of $T^n$. We define the normalized sequence

$$p_n(k) = a_n(k) \frac{1}{\sum_i a_n(i)}$$

and say that $a_n(k)$ satisfies a central limit theorem with mean $m_n$, covariance matrix $B_n$, and $n \equiv n_0 \mod p$ if

$$\sup_{x} \left| \sum_{i \leq u < m} p_n(i) - \frac{1}{\sqrt{(2\pi)^d |B_n|}} \int_{x \leq u} \exp(-\frac{1}{2}xB_n^{-1}x') \, dx \right| = o(1),$$

(1.2)

where $n \equiv n_0 \mod p$ and $x'$ denotes the transpose of $x$. Let $A \subseteq \mathbb{Z}^d$ be a lattice. We say that $a_n(k)$ satisfies a local limit theorem modulo $A$ for $n \equiv n_0 \mod p$ if

$$\sup_k \left| \sqrt{(2\pi)^d |B_n|} p_n(k) - h(k + A) \exp(-\frac{1}{2}xB_n^{-1}x') \right| = o(1),$$

(1.3)

where $n \equiv n_0 \mod p$, $h$ is some function of the cosets of $A$ and $x = k - m_n$. We use the local limit theorem to obtain estimates for $a_n(k)$ over a wide range. When $A = \mathbb{Z}^d$ there is only one coset and it turns out that $h = 1$ by Lemma 4.2. See [1] for additional terminology not defined here.

In Section 2 we state Theorem 1 which ensures that $a_n(k)$ satisfies central and local limit theorems. The applications show that the hypotheses of the theorem are usually very easily verified. Section 4 contains the proof of the theorem and, in (4.1) and (4.2), asymptotic estimates for $m_n$, $B_n$, and $a_n(k)$. Lemma 1 of this section is an apparently new observation on multisection of multivariate series.

Applications of Theorem 1 are given in Section 3. Our first example deals with covering $n \times w$ rectangular arrays with polyominoes where $w$ is fixed. We then consider enumeration of plane animals of fixed width, various occupancy problems with labeled boxes, and enumeration of $q \times n$ matrices of zeroes and ones where we keep track of the number of changes. In Section 6 we indicate briefly how the ideas of this paper relate to the phase transition problem of statistical mechanics. Limit theorems have been established when $d = 1$ and $T(1)$ is a transition matrix for a Markov chain. See Montroll [8] and Romanovsky [11]. Some applications have been made to correlated random walks. The simplicity and greater generality of Theorem 1 allow us to extend these applications [2].
1. 2. An Existence Theorem

Let $\mathcal{D}(T)$ be a directed graph with edge $(i, j)$ if and only if $T_{ij} \neq 0$. Define an equivalence relation on the vertices by $i \equiv j$ if and only if there is a directed path from $i$ to $j$ and one from $j$ to $i$. If $\beta$ is an equivalence class, $T_\beta$ denotes the submatrix of $T$ whose indices are restricted to $\beta$. By putting $T$ in block triangular form, the blocks corresponding to equivalence classes, one sees that the eigenvalues of $T$ are the eigenvalues of $T_\beta$'s.

**Definition 1.** Suppose $r_l > 0$ for all $l$. We say that $a_n(k)$ is admissible at $r$ for $n \equiv n_0 \mod p$ if there is a block $\alpha$ and indices $i$ and $j$ such that

(i) the entries in $C$ and $T$ have Laurent series expansions about 0 having nonnegative coefficients and converging in a neighborhood of $r$;

(ii) the greatest common divisor of the lengths of the directed cycles of $\mathcal{D}(T_\alpha)$ is $p$;

(iii) $C_{ij}(x) \neq 0$;

(iv) there is a directed path in $\mathcal{D}(T)$ from $i$ to $\alpha$ and one from $\alpha$ to $j$ such that the sum of the two path lengths is $n_0$ modulo $p$;

(v) for $\beta \neq \alpha$ every eigenvalue of $T_\beta(r)$ has modules less than the maximum modules of the eigenvalues of $T(r)$.

In many applications all vertices of $\mathcal{D}(T)$ fall into a single equivalence class. In this case conditions (iii)-(v) are trivially satisfied. Also, the entries in $C$ and $T$ are often polynomials. If these conditions both hold, admissibility does not depend on $r$ nor on the pattern of zeroes in $C$. Note that if $T_\alpha$ has a nonzero diagonal entry, then $p = 1$.

**Definition 2.** Suppose $a_n(k)$ is admissible and let $\alpha$ be as in Definition 1. If $i, j \in \alpha$, let $\mathcal{O}_{ij}^{(s)}$ be the additive Abelian group generated by vectors of the form $k - l$, where $k$ and $l$ are exponents of terms in the $(i, j)$ entry of $T_\alpha(x)^s$. If the entry is zero, set $\mathcal{O}_{ij}^{(s)} = \phi$. Let $\Lambda$ denote the lattice $\bigcup_{s,i,j} \mathcal{O}_{ij}^{(s)}$.

**Theorem 1.** Suppose $a_n(k)$ is admissible at 1 for $n \equiv n_0 \mod p$ and that $\Lambda$ is $d$-dimensional. Then $a_n(k)$ satisfies a central limit theorem for $n \equiv n_0 \mod p$ with means and covariance matrix asymptotically proportional to $n$. Let $q$ be such that $qc \in \Lambda$ for all $c \in \mathbb{Z}^d$. Then $a_n(k)$ satisfies a local limit theorem modulo $\Lambda$ for $n \equiv n_0$ modulo $pq$.

**Remarks.** We can replace $\Lambda$ by any sublattice of $\Lambda$ in the theorem. As remarked after Definition 1, admissibility is often easily checked and we
often have $p = 1$. The condition on the dimensionality of $A$ simply asserts that it is as large as possible and so can usually be verified by examining a few terms of $T_{ij}^{(s)}$ for some small $s$. In a similar vein, one can often show that $A = \mathbb{Z}^d$ fairly easily. We shall see this in the next section.

3. Applications

Polyominoes

Our first example concerns the problem of covering rectangular arrays with dimers (dominoes) or, more generally, polyominoes. For connections with physics, see Section 5 and Read [10].

To illustrate the concepts we first consider the number $a_n(k)$ of ways of arranging $k_1$ horizontal dimers and $k_2$ vertical dimers in an $n \times 2$ array where empty squares (monomers) are allowed. We can imagine slicing between rows $m$ and $m + 1$ and describing what happens at each cell boundary on the cut; i.e., whether a dimer is sliced or not. Using $s$ to denote sliced and $u$ to denote unsliced we have four states: $1 = uu$, $2 = us$, $3 = su$, $4 = ss$. When we move from one row to the next we keep track of the number of dimers completed in that step. This leads to the matrix

$$T = \begin{bmatrix}
1 + x_1 & 1 & 1 \\
0 & x_2 & 0 \\
x_2 & x_2 & 0 \\
x_2^2 & 0 & 0
\end{bmatrix},$$

where $T_{ij}$ is associated with going from state $i$ to $j$. Here $T_{11} = 1 + x_1$, for example, because we can go from $1 = uu$ to $uu$ and place either none or one horizontal dimer. On the other hand $T_{24} = 0$ because it is impossible to complete the sliced dimer in $2 = us$ and end up in $ss$. Our desired generating function in $r', s'$, so $C_{ij}(x) = 1$ if $i = j = 1$ and zero otherwise.

Since we can get from any state to any other, the vertices of $\mathcal{D}(T)$ form a single equivalence class and $a_n(k)$ is admissible with $p = 1$. Note that $T_{11}^{(2)} = 1 + 2x_1 + x_1^2 + 2x_2 + x_2^2$. Thus $(1, 0), (0, 1) \in \mathcal{O}_{11}^{(2)}$ and so $\mathcal{O}_{11}^{(2)} = \mathbb{Z}^2$. Hence we have a central and local limit theorem by Theorem 1. (Saying we have a central limit theorem is redundant since local implies central.)

This argument gives a local limit theorem for $n \times w$ arrays: Let $1$ denote the totally uncut state. Then $T_{11}^{(2)} = 1 + x_1 + x_2 + \cdots$, and so $(1, 0), (0, 1) \in \mathcal{O}_{11}^{(2)}$.

This idea can be extended to placing any sort of polyominoes in any number of dimensions and keeping track of the number of occurrences of
any event. If we wish to place polyominoes whose maximum length is $q$ and keep track of all types of polyominoes except monomers, it suffices to look at $T_{11}^{(q)} = 1 + x_1 + \cdots + x_d$, where $d$ is the number of different polyomino types. We can also keep track of such things as isolated polyominoes and correlation between neighboring polyominoes. The only constraint is that a bounded amount of information suffices to determine all possible transitions from line $m$ to line $m+1$. To illustrate, consider keeping track of dimers with $x_1$ and isolated monomers with $x_2$. Let state 1 mean the row contains only monomers. The following picture helps to show that $T_{11}^{(5)} = 1 + x_1 + x_1^2 x_2 + \cdots$. We easily obtain a local limit theorem with $A = \mathbb{Z}^2$.

\[
\begin{array}{cccc}
M & M & \cdots & M \\
D & D & M & \cdots & M \\
M & D & M & \cdots & M \\
D & D & M & \cdots & M' \\
D & M & \cdots & M \\
M & M
\end{array}
\]

**Animals**

Our next set of examples concerns the enumeration of plane animals. A plane animal is a connected union of unit squares whose vertices have integer coordinates with two squares connected if and only if they share an edge. Read [9] treats enumeration of $n \times w$ plane animals for fixed $w$. To illustrate the ideas, we consider $w = 3$. We then obtain local limit theorems for all $w > 1$, and consider variations.

An animal can be built up by adding one row of squares at a time. One of the eight patterns of open (0) and used (1) squares, 000, is forbidden because the animal is to be connected. Connectivity also rules out certain transitions such as 100 to 001. There is another problem: If we are at 101 we need to know if the previous rows connect the two end squares or not. Read handles this by introducing the states $u0u$ for connected and $u0v$ for not connected. This gives eight states:

\[
\begin{align*}
1 &= u00 & 3 &= 00u & 5 &= u0u & 7 &= uu0 \\
2 &= 0u0 & 4 &= 0uu & 6 &= u0v & 8 &= uu.
\end{align*}
\]

We may start in any state but $u0u$ and end in any state but $u0v$. Let $T_{ij}$ keep track of the number of squares gained when state $j$ follows state $i$. Then
and $\sum a_{n+1}(k) x^4 = \sum C_{ij}(x) T_{ij}^{(4)}(x)$. Now consider the $n \times w$ case and let $f$ be the index of the filled state $uu \ldots u$. There is a path from every state to every other state and $T_{jj} \neq 0$. Hence $a_n(k)$ is admissible with $p = 1$. Since $T_{jj}^{(2)} = x^{2w} + x^{2w-1} + \cdots$, we see that $T_{jj}^{(2)} = T$ and so we have a local limit theorem with $A = \mathbb{Z}$.

There is a problem with this approach: The actual width of the animal may be less than $w$. We could handle this by Read's method, but a computationally poor method is better for proving existence. We need to know whether both the left and right columns, just the left, just the right, or neither contain squares of the animal at this point. Let $B, L, R, N$ denote these conditions. Then a vertex of $\mathcal{D}(T)$ may be denoted by $wA$, where $w$ ranges over the $w$-dimensional vectors introduced above and $A = B, L, R, N$. Then $wA \equiv zD$ implies $A = D$. Let $T_A$ be the corresponding diagonal block. It is easily seen that $T_B$ is the $T$ introduced in the previous paragraph and $T_A$ is like $T$ but with additional zeroes when $A \neq B$; e.g., at all entries involving the filled state. Using Marcus [7, 5.7.5, p. 126] one easily shows that $a_n(k)$ is admissible with $a = B$. The other hypotheses of Theorem 1 hold by the previous paragraph. This gives a local limit theorem.

We can make other modifications. For example, we may count the number of enclosed regions as well as the number of cells. If $x_2$ keeps track of enclosed regions, $T_{jj}^{(2)} = x_1^{2w} + x_1^{2w-1} + \cdots$. Since the connectivity information is enough to keep track of how many closed regions are formed when we seal off openings, we can get a local limit theorem. For example, $u0u0u \rightarrow uuuuuu gives two regions and $u0u0u \rightarrow uuuuuu$ gives one. We cannot keep track of the number of enclosed squares in this way as $u0u \rightarrow u0u \rightarrow \cdots \rightarrow u0u \rightarrow uuu$ versus $u0u \rightarrow u0u \rightarrow \cdots \rightarrow u0u \rightarrow u00$ shows.
Still another modification is to show that the length of animals of width \( w \) satisfies a local limit theorem as the number of cells goes to infinity; i.e., 
\( a_n(k) \) equals the number of \( n \)-celled, \( k \)-long, \( w \)-wide animals whereas it was previously the number of \( n \)-long, \( k \)-celled, \( w \)-wide animals. We now require a transition matrix in which the number of cells in the animal, instead of the length, increases by one at each step. We can imagine an \( \infty \times w \) array being read left to right and top to bottom (like English). Stop every time a cell in the animal is encountered. For \( w = 2 \) the states are

\[
\begin{array}{cccc}
uu & 0u & uu & 0u & u0 \\
0 & u & u & u & u \\
\end{array}
\]

Using this idea, one can proceed as before to establish local limit theorems. If state \( S \) is \( u \cdots u \), then 
\[ T_{ij}^{(w+2)} = x_2 + x_3 + \cdots \cdot \]

**Occupancy**

Our next class of examples consists of occupancy problems. To begin with, consider distributing identical balls into \( n \) boxes in a row so that each pair of adjacent boxes contains at least \( p \) balls and no box contains more than \( q > p/2 \) balls. Let state \( i \) \( (0 < i < q) \) be the state in which the present box contains \( i \) balls. Thus 
\[ T_{ij} = 0 \text{ if } i + j < p \text{ and } T_{ij} = x^i \text{ if } i + j \geq p. \]
If \( q > p \), we can limit ourselves to states \( 0, 1, \ldots, p \) where state \( p \) means at least \( p \) balls are in the box. Then 
\[ T_{ij} = (x^p - x_{q+i}^p)/(1 - x). \]
Since we may start and end with any number of balls,

\[ C = (1, x, x^2, \ldots, y)'(1, l, \ldots, 1), \]

where \( y = x^q \) or \((x^p - x_{q+i}^p)/(1 - x). \) A local limit theorem is easily established. (Note that in this case we use \( T^{n-1} \) for \( n \) boxes.)

Suppose the first and last boxes are adjacent; i.e., we have a circular array. It is easily seen that the generating function is \( trace(T^n) \).

Of course, a large number of variations are possible, we consider one. A total of \( n \) red and green balls are placed in adjacent boxes so that we have a string of nonempty boxes followed by a string of empty boxes. Thus \( n \) now keeps track of balls, not boxes, and each power of \( T \) adds another ball. We count the number of occupied boxes with \( x_1 \), the number of such boxes without red balls by \( x_2 \), and the number of runs of boxes containing green balls by \( x_3 \). The content of a box may be described by specifying whether the number of green (red) balls is zero or not. Thus gives three states:

1 = green absent, red present,
2 = green present, red absent,
3 = green present, red present.
When balls are being placed in a box, green balls are put in before red ones. We count the contribution of a box when we first move to a new one. The matrices are

\[
T = \begin{bmatrix}
1 + x_1 & x_1 & 0 \\
x_1 x_3 & 1 + x_1 x_2 & 1 \\
x_1 x_3 & x_1 & 1
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
x_1 \\
x_1 x_3 \\
x_1 x_3
\end{bmatrix}
\]

For example, \( T_{22} \) is obtained by noting that we can add a green ball to the present box or start a new box with a green ball. The latter ends a box which has no red balls, giving a term of \( x_1 x_2 \). One easily sees that \( T^{(3)}_{22} = 1 + x_1 x_2 + x_1^2 x_2 x_3 + x_1^3 x_2 + \cdots \). Thus \( O^{(3)}_{22} = \mathbb{Z}^3 \), and Theorem 1 applies.

**Arrays**

Our last example concerns the enumeration of rectangular arrays having \( q \) rows and \( n \) columns, with \( m_i \) ones and \( r_i \) changes in the \( i \)th row, and with \( s_i \) vertical changes between the \( i \)th and \((i + 1)\)th rows. (See Carlitz [3] and Hodel [6].) For example,

\[
\begin{pmatrix}
1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix}
\]

has \( m = (3, 2, 2), r = (2, 1, 3), s = (1, 2) \).

A state consists of a \( q \)-tuple of zeroes and ones. If \( x_i \) keeps track of \( m_i \), \( y_i \) of \( r_i \), and \( u_i \) of \( s_i \), then

\[
T_y = x_i y_i \oplus u_i^j,
\]

where \( \oplus \) denotes addition modulo 2 and \( (dj)_k = j_k \oplus j_{k+1} \). It is easily seen that \( a_n(k) \) is admissible with \( p = 1 \) and \( \alpha \) the set of all indices. We now consider \( \mathcal{A}^{(3)}_{00} \). From \( 0 \rightarrow i \rightarrow j \rightarrow 0 \) we get a term with exponent

\[
e(i, j) = (i + j, i + j + (i \oplus j), Ai + dj).
\]

Since \( e(0, 0) = (0, 0, 0), e(i, j) \in \mathcal{A}^{(3)}_{00} \). Thus we get the following in \( \mathcal{A}^{(3)}_{00} \):

\[
e(i, 1 - i) - e(0, 1) = (0, 0, 2di),
\]

\[
f = e(i, 1) - e(0, 1) = (i, 0, di),
\]

\[
e(i, i) - 2f = (0, 2i, 0).
\]

It follows that \( \mathcal{A}^{(3)}_{00} \supseteq (2\mathbb{Z})^{3q-1} \). We get a local limit theorem. Unfortunately \( \mathcal{A} \) has a large number of equivalence classes. It seems probable that, in fact,
a local limit theorem holds modulo the lattice of vectors of the form \((i, j, 2k + A)\) where \(i, j \in \mathbb{Z}^q\) and \(k \in \mathbb{Z}^{q-1}\).

4. PROOF OF THEOREM 1

Let \(A\) be a lattice in \(\mathbb{Z}^d\) with the rows of the \(d \times d\) matrix \(L\) as an integral basis. We require the following easily proved lemma which does not seem to be in the literature although it is a natural extension of multisection of power series in one variable.

**Lemma 1.** Let

\[
\psi(x) = \sum_{k \in \mathbb{Z}^d} a_n(k) x^k,
\]

and let \(\psi_c(x)\) denote the sum restricted to \(k \in c + A\). Define \(e(t) = e^{2\pi it}\) and \((xs)_i = x_is_i\). Then

\[
\psi_c(x) = \frac{1}{(\mathbb{Z}^d : A)} \sum_{b} \psi(xe(L^{-1}b')) e(-cL^{-1}b'),
\]

where \(b\) ranges over a set of coset representatives for \(A'\), the lattice generated by the rows of \(L'\).

Let \(R\) be a compact subset of \((0, \infty)^d\). Suppose that \(a_n(k)\) is admissible at all \(x \in R\) for \(n \equiv k \mod p\) and that \(a\) is the same for all \(x \in R\). By the Perron–Frobenius theorem [7, p. 124], \(T_a(x)\) has a simple positive real eigenvalue \(\lambda(x)\). Except for the eigenvalues \(\lambda(x) e^{2\pi i/p}\), the remaining eigenvalues have smaller modulus. This can be extended to a neighborhood of \(R\) in \(C^d\) by continuity (though, of course, \(\lambda(x)\) is not real). By standard arguments using Jordan canonical form, every entry in \(T^n\) has the form \(\sum p_i(n) \lambda_i^n\), where \(p_i\) is a polynomial whose degree is less than the multiplicity of the eigenvalue \(\lambda_i\). Some entry in \(T^n\) grows as fast as \(\lambda(x)^n\) when \(p \nmid n\). By the previous observations on powers of \(T_a\) one can easily see by using paths in \(\mathcal{D}(T_a)\) that either \(t_i^n\) is zero for \(n \equiv n_0 \mod p\) or it grows like \(\lambda(x)^n\). By Definition 1(iii) and (v) and the above,

\[
\sum_k a_n(k) x^k \sim g(x) \lambda(x)^n
\]

for \(n \equiv n_0 \mod p\) uniformly in a neighborhood of \(R\). Also note that \(\lambda(x)\) is
analytic in \( R \) since it is a simple root of a polynomial with analytic coefficient and nonvanishing leading terms. Define
\[
m_i = \frac{\partial \log \lambda(x)}{\partial \log x_i}, \quad B_{ij} = \frac{\partial^2 \log \lambda(x)}{\partial \log x_i \partial \log x_j}.
\]

**Lemma 2.** If \( \alpha_n(k) \) is admissible at \( 1 \) for \( n \equiv n_0 \mod p \) and \( B(1) \) is nonsingular, then \( \alpha_n(k) \) satisfies a central limit theorem for \( n \equiv n_0 \mod p \) with mean and covariance asymptotic to \( n\mu(1) \) and \( nB(1) \).

**Proof.** Apply Theorem 1 of [1].

**Lemma 3.** Let \( q \) be such that \( qc \in A \) for all \( c \in \mathbb{Z}^d \). Then the eigenvalues of \( T_\alpha(x)^p \) are the eigenvalues of a matrix in which the exponent of each term of each entry is in \( A \).

**Proof.** By Definition 2, each entry of \( T_\alpha(x)^p \) has exponents in a single coset \( z_{ij}(s) + A \) of \( A \). By Definition 1(ii) it follows that for some \( s_0 \), \( T_\alpha^{s_0} \) consists of \( p \) diagonal blocks having no zero entries when \( s \geq s_0 \). Since \( T_\alpha^{s_0}(x) = (T_\alpha^p)^q \), \( z_{ij}(qs) \equiv qz_{ij}(sp) \equiv 0 \) modulo \( A \) for \( s \geq s_0 \). By similar consideration of matrix multiplication when \( z_{ij}(qp) \) is defined
\[
z_{ij}(qs) = z_{ij}(q(s - 1)p) + z_{ij}(qp) = z_{ij}(qp),
\]
and
\[
z_{ij}(qs) + z_{ij}(qs) \equiv z_{ij}(qs) + z_{ij}(qs) \equiv z_{ij}(qs).
\]
Hence \( z_{ij}(qp) \equiv z_{ij}(qs) - z_{ij}(qs) \). Let \( D \) be a diagonal matrix with \( d_{ii} = x^{z_{ij}(qs)} \). Then \( DT_\alpha^{s_0}D^{-1} \) is a matrix in which the exponent of each term of each entry is in \( A \).

**Lemma 4.** Let \( R \) be a compact subset of \( (0, \infty)^d \). Suppose that for all \( r \in R \)

(i) \( \alpha_n(k) \) is admissible at \( r \) for \( n \equiv n_0 \mod p \) with \( n_0 \) and \( p \) independent of \( r \);

(ii) \( B(r) \) is nonsingular;

(iii) if \( z_j = r_j \exp(2\pi ic_j) \) and the largest eigenvalue of \( T(z) \) has modulus \( \lambda(r) \), then \( L_\epsilon^\dagger \in \mathbb{Z}^d \).

Let \( q \) be as in Lemma 3. Then \( \alpha_n(k) r^k \) satisfies a local limit theorem modulo \( A \) for \( n \equiv n_0 \mod pq \) with \( m_n = n\mu(r) \) and \( B_n = nB(r) \). Furthermore, the bound in (1.3) is uniform over \( R \) and the average of \( h(k + A) \) over the cosets of \( A \) is 1. Also,
\[
\sup_k |\alpha_n(k) r^k(2\pi n)^d |B(r)| \lambda(r)^{-n} - g(r; k + A)| = o(1),
\]
(4.3)
where the supremum is over all \( k \in \mathbb{Z}^d \) such that \( r \in R \), where \( m(r) = k/n \) and \( g \) is analytic in \( r \). The bound in \( o(1) \) is uniform when \( r \) is in a compact subset of \( (0, \alpha)^d \). If there exists \( i, j \) and a path \( \pi \) from \( i \) to \( j \) through \( \alpha \) of length \( n_0 \) modulo \( p \) such that \( C_{ij} \) times the product of \( T_{uv} \) over all edges \( (u, v) \) of \( \pi \) contains a term in \( x^k \), then \( h(k + A) \neq 0 \) and \( g(r; k + A) \neq 0 \).

**Proof.** We convert to the basis \( L \) by defining

\[
\varphi_n(x) = \sum_{k \in l + \Lambda} a_n(k) x^{(k - h)L^{-1}}.
\]

Since \( k - l \in \Lambda \), \( k - l = uL \) for some \( u \in \mathbb{Z}^d \) and so the powers of \( x \) are integral. Define a map from \( x \) space to \( v \) space \( x_i = \Pi_{j}^{i} u_i \), then

\[
\varphi_n(x) = \psi(v) = v^{-1} \sum_{k \in l + \Lambda} a_n(k) v^k.
\]

Set \( D_{ij} = \sum_{k} C_{kj} T_{kl}^{(n_0)} \), \( U = T_{pq} \), \( m = (n - n_0)/pq \). Then \( \sum_{i,j} T_{ij}^{(n)} = \sum_{i,j} D_{ij} U_{ij}^{(m)} \). By Lemma 3, the eigenvalues of \( (T_{pq})^{\alpha} \) are the same as the eigenvalues of \( S \), a matrix whose terms have exponents in \( \Lambda \). Every element of \( \Lambda \) has the form \( uL \) with \( u \in \mathbb{Z}^d \). For \( b \in \mathbb{Z}^d \),

\[
e(L^{-1}b')^uL = e(uLL^{-1}b') = 1.
\]

Thus \( S(xe(L^{-1}b')) = S(x) \). It follows that when \( U_{ij}^{(m)} \) is expanded by eigenvalues, the large terms (due to block \( \alpha \)) are the same at \( x \) and \( xe(L^{-1}b') \). It follows that for \( v \in N(R, \epsilon) \), an \( \epsilon \)-neighborhood of \( R \), there is a function \( g(b) \) and a function \( h(v) \in (0, 1) \) such that \( \psi(ve(L^{-1}b')) = g(b) \psi(v) + O(h(v)^\alpha) \).

Use (4.1) to compute \( v'\psi(fk) \). Set \( g(r, l + A) = \sum g(b) \). Condition (iii) states that \( |\lambda(v)| = \lambda(|v|) \) implies \( v = |v| \), since \( \log v = L^{-1} \log x \), and hence

\[
(2\pi i)^{-1} \text{Im}(\log v) = (2\pi i)^{-1} \text{Im}(L^{-1} \log x) \in \mathbb{Z}^d
\]

for \( x \) such that \( |\lambda(x)| = \lambda(|x|) \). Hence we may apply Theorem 2 of [1] to obtain (1.3) and (4.3).

Define \( S = \{ x : |x_i - k_i| \leq \sqrt{\epsilon n}/2 \text{ for } i = 1, \ldots, d \} \). By (1.2)

\[
\sqrt{(2\pi)^d |B_n|} \sum_{i \in S \cap \mathbb{Z}^d} p_n(i) \sim (\epsilon n)^{d/2} (\exp(-\frac{1}{2} xB_n^{-1}x') + O(\epsilon))
\]

and by (1.3)

\[
\sqrt{(2\pi)^d |B_n|} \sum_{i \in S \cap \mathbb{Z}^d} p_n(i) \sim \sum_{i \in S \cap \mathbb{Z}^d} h(i + A)(\exp(-\frac{1}{2} xB_n^{-1}x') + O(\epsilon)).
\]

By comparing these, we see that the average value of \( h(i + A) \) is asymptotic to \( 1 + O(\epsilon) \).
We prove the last statement in the lemma. By definition, $D_{ij}$ contains a term in $x^k$. All diagonal entries in $U^m_{\alpha}$ grow at the same rate as $\lambda^{\alpha m}$. By Lemma 3, $\sum D_{ij} U^{(\alpha m)}_{ij}$ grows at this rate. 

We now show how the conditions in Theorem 1 imply the conditions in the lemmas. Before beginning we need a probabilistic result.

**Lemma 5.** Let $X$ and $Y$ be random variables taking integer values. Let their generating functions be $q(u)$ and $r(u)$; i.e.,

$$q(u) = \sum_i \Pr\{X = i\} u^i.$$ 

Define generating functions for random variables $P$ and $S$ by $q(u)r(u)$ and $(aq(u) + br(u))/(a + b)$, respectively, where $a, b \geq 0$. Then

$$\text{var}(P) = \text{var}(X) + \text{var}(Y) \quad \text{and} \quad \text{var}(S) \geq \min(\text{var}(X), \text{var}(Y)).$$

**Proof.** The result for $\text{var}(P)$ is well known since $P = X + Y$. In the second we can assume $a + b = 1$. Let $s(u) = aq(u) + br(u)$. Then

$$\text{var}(S) = s''(1) + s'(1)^2$$

$$= (aq'' + br'') + (aq' + br') - (aq' + br')^2$$

$$= a \text{var}(X) + b \text{var}(Y) + a(q')^2 + b(r')^2 - (aq' + br')^2$$

$$\geq a \text{var}(X) + b \text{var}(Y)$$

since $a + b = 1$ and $(a + b)(a(q')^2 + b(r')^2) - (aq' + br')^2 = ab(q' - r')^2$. 

When we speak of the covariance matrix of $x$ in $g(x)$ we mean the covariance matrix of the random variables with generating function $g(x)/g(1)$.

**Lemma 6.** Suppose $T$ is admissible at $r$. If $\mathcal{A}^{(i)}_{ij}$ is $d$-dimensional for some $i, j \in \alpha$, then $B(r)$ is nonsingular.

**Proof.** Suppose $B(r)$ is singular. By the basic theory of quadratic forms, there is a nonsingular $M$ such that $(M'B(r)M)_{ij} = 0$. Define new variables $z$ by $x_i = r_i\Pi z^{m_{ij}}$. By the assumption on $\mathcal{A}^{(i)}_{ij}$, $z_1$ must appear in the transformed $T^{(i)}_{ij}$ with two distinct powers. Partition $T^{(i)}_{\alpha}$ into a $p \times p$ block matrix as guaranteed by Definition 1(ii). Again by Definition 1(ii), $T_{\alpha}^{k}$ will have $p$ blocks all of which have no zero entries for all sufficiently large $k$, say $k \geq m$. By looking at

$$T^{m+i}_{\alpha}T_{\alpha}^{m+p-i}$$

for $i = 0, 1, \ldots, p - 1$,
we see that $z_i$ appears with two distinct powers in every nonzero entry of $T^{a+2m+p}_a$. Let $k = s + 2m + p$ and let $v$ be the minimum over all $i, j \in \alpha$ of the variance of $z_i$ in $T^{(a)}_{ij}$. Clearly, $v > 0$. By Lemma 5, the variance of $z_i$ in nonzero $T^{(n)}_{ij}$ for $i, j \in \alpha$ is at least $v[n/k] \sim (v/k)n$. This contradicts the proof of Theorem 1 [1] which shows that the covariance matrix of $z$ in $T^{(n)}_{ij}$ differs from $nMB(r)M$ by $o(n)$ and is easily extended to singular $B(r)$.

By Lemmas 2 and 6, the central limit part of Theorem 1 is proved.

**Lemma 7.** If $C_{ij} = 1$ for some $i, j \in \alpha$, then Lemma 4(iii) holds.

**Proof.** By Marcus [7, 5.7.5, p. 126], the conclusion follows if for some $i, j \in \alpha$ and some $s$

$$|T^{(s)}_{ij}(z)| < T^{(s)}_{ij}(r)$$

(4.4)

when $z_k = r_k \exp(2\pi ic_k)$ and $Lc' \in \mathbb{Z}^d$. Suppose equality holds in (4.4). Then every term in $T^{(s)}_{ij}(z)$ has the same argument (angle in the complex plane). Hence $kc'$ is constant modulo 1 as $k$ runs through exponents of terms in $T^{(s)}_{ij}(x)$. Hence $Lc' \in \mathbb{Z}^d$.  

The proof of Theorem 1 is a straightforward application of Lemmas 2, 4, 6, and 7.

### 5. Calculations

In this section we make some remarks concerning the calculation of the mean, covariance matrix, and asymptotic formulas. Several examples have been given in [1] illustrating the calculations if the generating function for $a_n(k)$ is known, so we will not deal with this problem here.

Since

$$\sum a_n(k)x^k y^n = \sum C_{ij}T^{(n)}_{ij}y^n = \sum C_{ij}S_{ij},$$

(5.1)

where $S = (I - Ty)^{-1}$, the generating functions for $a_n(k)$ can be obtained in a straightforward way. Read gives more efficient methods for animals [9] and dimer problems [10]. We point out another shortcut. We concluded our discussion of animals in Section 3 by proving normality of the length of $n$-celled animals of width $w$. The required generating function is more easily obtained by interchanging $x$ and $y$: look at $k$-celled, $n$-long animals of width $w$. For example, when $w = 2$, the generating function was obtained by Read from $2 \times 2$ matrices in [9], while the direct approach in Section 3 would involve a $5 \times 5$ matrix.
If we only wish to compute the mean and the covariance matrix, it suffices to obtain $\lambda(x)$ and use (4.2). We remark that, by (5.1), $\lambda(x)$ is the reciprocal of the smallest zero $y$ of the denominator of $\sum a_n(k) x^n y$. An explicit expression for $\lambda(x)$ or even for $|\lambda(x)I - T(x)|$ is not needed to compute mean and covariance. We illustrate with 2-wide animals for which Read [9] obtains

$$\sum a_n(k) x^n y^n = y(2, 1)(I - yT(x))^{-1} \left( \begin{array}{c} x \\ x^2 \end{array} \right),$$

where

$$T(x) = \left( \begin{array}{cc} x & x \\ 2x^2 & 2x^2 \end{array} \right).$$

Thus

$$\begin{vmatrix} x - \lambda & x \\ 2x^2 & 2x^2 - \lambda \end{vmatrix} = 0.$$  \hspace{1cm} (5.2)

With $x = 1$ we obtain $\lambda(1) = 1 + \sqrt{2}$. Differentiating (5.2) a row at a time gives

$$\begin{vmatrix} 1 - \lambda' & 1 \\ 2x^2 & 2x^2 - \lambda \end{vmatrix} + \begin{vmatrix} x - \lambda & x \\ 4x & 4x - \lambda' \end{vmatrix} = 0$$

and so $\lambda'(1) = 4\lambda/(2\lambda - 3)$. By (4.2),

$$m = \frac{\partial \log \lambda}{\partial \log x} = \frac{x\lambda'}{\lambda} = \frac{4}{2\lambda - 3} = \frac{4}{7} (2\sqrt{2} + 1).$$

6. **Statistical Mechanics**

We now apply our results to the following problem in statistical mechanics (closely related to that of DiMarzio [4]). Consider a cubic lattice of volume $m_1 m_2 n$; that is, a rectangular box of sides $m_1, m_2, n$. A site is a unit cube, hence this box has $m_1 m_2 n$ sites. Suppose that we place rods of dimensions $1 \times 1 \times x$ in the lattice so that they are parallel to the sides of the box. The rods may be thought of as linear polymers and the empty sites as solute molecules. Let $k_1$ denote the number of solute molecules and $k_i, i = 2, 3, 4,$ the number of rods in the three directions of the lattice. There are then 16 distinguishable ways in which a pair of nearest neighbor sites, represented by $ij$, can be occupied, namely, $11, 1i, 1i, ii, ij$ ($i = 2, 3, 4; j = 2, 3, 4; i \neq j$). (The description of a solute molecule next to a segment of
the rod in direction $i$ is $l_i$, for example). Of the 16 variables $k_{ij}$ denoting the number of pairs of sites $i, j$, only 12 are independent since

$$2k_{ii} + \sum_{i \neq j} k_{ij} = 6k_i.$$ 

The grand partition function for the lattice is [5, p. 246]

$$Z_n(k_1, \ldots, k_{ij}, \ldots) = \sum_{k_i} \sum_{k_{ij}} g_n(k_i, k_{ij}, \ldots) e^{-\sum k_{ij}w_{ij}/kT(\lambda_1, \varphi_1)^{k_1} \ldots (\lambda_4, \varphi_4)^{k_4}},$$

where $g(k_1, \ldots, k_{ij}, \ldots)$ denotes the number of distinguishable configurations for the specified values of $k_i$ and $k_{ij}$. Here $w_{ij}$ denotes the contribution of each $ij$ pair of neighbors to the interaction energy and is often treated as a constant although we can just as easily consider it a function of temperature. The $\varphi_i$ represent the partition function of each molecule (including rotational and internal vibrational degrees of freedom), and $\lambda_i$ the absolute activity. If we suppose that the molecules are noninteresting (as does DiMarzio), i.e., that $w_{ij} = 0$, this expression simplifies since $\sum k_{ij} g(k_1, \ldots, k_{ij}, \ldots)$ is simply the number of ways of filling the lattice with $k_i$ solute molecules and $k_i$ polymers in direction $i$ and

$$Z_n(k_1) = \sum_{k_i} g(k_1, \ldots, k_i) (\lambda_1, \varphi_1)^{k_1} \ldots (\lambda_4, \varphi_4)^{k_4}. $$

Note that if we do not make this assumption we merely have more variables in the transfer matrix entries to count the number of sites $ij$. If we let $x_i = \lambda_i \varphi_i$ (and $x_{ij} = e^{-w_{ij}/kT}$ in the more general case), we have

$$Z_n(x) \sim g(x) \lambda^n(x),$$

where $\lambda(x)$ is the largest eigenvalue of the transfer matrix. Hence the free energy per lattice site approaches the limit

$$-\frac{F}{kT} = \lim_{n \to \infty} \frac{1}{m_1 m_2 n} \log Z_n(x) = \frac{1}{m_1 m_2} \log \lambda(x).$$

We have seen at the beginning of Section 4 that $\lambda$, and so, in the limit, the free energy per lattice site, is an analytic function of $x$ on any compact subset of $(0, \infty)^d$. Thus there can be no phase change unless some $x_i$ is zero.

Clearly the same conclusion follows for interacting molecules, indeed for polymers of any shape and any set of allowed orientations. Note also that the same conclusion follows when the interactions of any finite set of neighbors is taken into account, not only nearest neighbors.
REFERENCES


