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JOURNAL OF ALGEBRA 44, 319–338 (1977)

Maximal Orders and Localization. I

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Communicated by A. W. Goldie

Received February 10, 1975

During the past 40 years, the classical and nonclassical theories of maximal orders (in simple algebras) have experienced parallel development with a reasonably complete structure theory available only in global dimension 1 For the classical theory, pertinent references are [2, 6, 15–18]; for the nonclassical theory, [1, 7, 11–14, 19]. Maximal orders having global dimension $\geqslant 2$ have been studied primarily in the classical setting by Ramras and Riley. Some results are available but a complete classification awaits discovery.

In this paper, we initiate a program similar in approach to that of Asano, Robson, and Michler for Asano orders, but broader in scope, which will include many important orders, both classical and nonclassical. In a different vein, we have tried to develop noncommutative techniques which are also useful in the classical context, where heretofore, commutative methods have been employed to prove noncommutative results.

Section 1 is a potpourri of facts about reflexivity, a concept ubiquitous to both the classical and nonclassical theories. Lemma 1 3 provides a useful module–theoretic criterion for reflexivity of an essential right ideal which avoids any hint of localization, and which has such consequences as: in a quasi-local maximal order with global dimension $\leqslant 2$, the intersection of projective right ideals is principal.

In Section 2, we begin a study of the arithmetic (à la Asano and Robson) of what we call *RI*-orders, culminating in an axiomization of their arithmetic. The model for such orders is *any* Noetherian maximal order with global dimension $\leqslant 2$. These orders provide a rich source of examples, more general than Asano orders, and at the same time, provide a natural departure from the rich theory of Dedekind prime rings.

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ISSN 0021–8693

Section 3 elucidates the basic properties of Asano's overring S and the intersection T of all the Goldie localizations at invertible primes. When the underlying ring R is a Noetherian RI-order, we show that S is an RI-order but "unbounded" in the sense that it possesses no nontrivial reflexive S-ideals. Using techniques of Hajarnavis and Lenagan, T is shown to be a bounded RI-order and that the globalization theorem of Kuzmanovich holds, namely, $R = S \cap T$. This result suggests that in the bounded case, R is actually the noncommutative analog of a Krull domain since $R = \bigcap_{P \in \mathscr{P}} R_P$, where $\mathscr{P}$ is the set of *all* height one primes and each R_P, the Goldie localization at P, is almost a noncommutative DVR.

Section 4 examines a more general localization due to Asano and generalizes many of the results which hold for Asano orders.

Finally, Section 5 applies some results of the preceding sections to the case of quasi-local Noetherian orders having global dimension $\leqslant 2$. Here we prove a structure theorem for these orders (Theorem 5.2) in the spirit of the Auslander–Goldman, Ramras, and Michler theorems. In particular, we show as a consequence of purely noncommutative methods, that several important results of Ramras deduced by classical techniques, hold in a more general setting.

1. Preliminaries

Throughout this paper, Q will denote a simple Artin ring. A subring R of Q is called an *order* in Q if Q is a classical two-sided quotient ring of R. For brevity a ring R will be called an order if it is an order in a simple Artin ring

Two orders R and S in Q are *equivalent*, $R \sim S$, if there exist regular (unit) elements $a, b, c, d \in Q$ such that $aRb \subseteq S$ and $cSd \subseteq R$. Two orders R and S in Q are *left* (*right*) *equivalent* $R \overset{\ell}{\sim} S$ ($R \overset{r}{\sim} S$) if there exist regular elements $a, b \in Q$ such that $Ra \subseteq S$ ($aR \subseteq S$) and $Sb \subseteq R$ ($bS \subseteq R$).

An order, R in Q is a *maximal left equivalent*, *maximal right equivalent*, respectively, *maximal equivalent* order, if whenever S is an order in Q with $R \subseteq S$ and $R \overset{\ell}{\sim} S$, $R \overset{r}{\sim} S$, respectively, $R \sim S$, then $R = S$ A maximal equivalent order will be called a maximal order

Let R be an order in Q. A right (left) R-submodule I of Q is called a *fractional right* (*left*) *R-ideal* if $aR \supseteq I \supseteq bR$ ($Ra \supseteq I \supseteq Rb$) for units a, b of Q. If $I \subseteq R$, then R is called an *integral right* (*left*) *R-ideal* Of course, these are just the essential right (left) ideals of R A left and right fractional (integral) R-ideal is called a *fractional* (integral) *R-ideal* In fact, since an integral R-ideal is an ordinary ideal of R, we shall refer to these as *ideals*; fractional R-ideals will be called *R-ideals*.

Let I_R be a fractional right R-ideal. It is easily verified that the set $\{q \in Q \mid qI \subseteq I\}$ is an order in Q, equivalent to R. This order is called the *left order* of I and denoted $O_\ell(I)$. Similarly, we define the *right order* of I and denote it by $O_r(I)$ Once again, this is an order of Q, equivalent to R. Since I_R is an essential submodule of Q_R, $O_\ell(I)$ can be identified with End I_R via the map taking $q \in O_\ell(I)$ to q_ℓ, left multiplication by q.

For a fractional right R-ideal I, its *inverse*, I^{-1}, is defined by

$$\begin{aligned} I^{-1} &= \{q \in Q \mid IqI \subseteq I\} \\ &= \{q \in Q \mid Iq \subseteq O_\ell(I)\} \\ &= \{q \in Q \mid qI \subseteq O_r(I)\}. \end{aligned}$$

The following proposition due to Robson [19] will be needed later.

PROPOSITION 1.1. *Let I be a fractional right R-ideal. Then $II^{-1} = O_l(I)$ if and only if I is a projective right $O_r(I)$-ideal; equivalently, I is a projective right R-ideal* ∎

Closely related to I^{-1} is the R-dual of I, $I^* = \hom_R(I, R)$. As before, since I_R is essential, I^* can be identified with the set $\{q \in Q \mid qI \subseteq R\}$. Note that when R is a maximal order and I is an R-ideal, $O_\ell(I) = R$ and hence, $I^{-1} = I^*$. We shall call an R-ideal I *invertible* if $I^*I = II^* = R$ There is a canonical map φ: $I \to I^{**}$ defined by evaluation, namely, $(f)\,\varphi(x) = f(x)$, $\forall\, x \in I, f \in I^*$ If φ is a monomorphism (isomorphism), I is called *torsionless* (*reflexive*).

For any module M_R, the biendomorphism ring of M, Biend M_R is defined as follows. If $S = \text{End } M_R$, Biend $M_R = \text{End}_S M$, where ${}_S M$ is the canonical left S-module. M_R is called *balanced* if Biend $M_R = R$

LEMMA 1 2. *For R an order in Q the following statements are equivalent.*

(1) *R is a maximal left equivalent order*

(2) *$O_\ell(A) = R$ for each nonzero ideal A of R*

(3) *Every torsionless faithful left R-module is balanced*

Proof (1) ⇒ (2) If $S = O_\ell(A)$, then clearly $R \subseteq S$ Since A is a nonzero ideal of the prime ring R, A is an essential right ideal of R so A contains a regular element d, hence $Sd \subseteq SA \subseteq A \subseteq R$ so $R \overset{\ell}{\sim} S$, and therefore, $R = S$

(2) ⇒ (3). If $S = \text{Biend}\,({}_R M)$, then the canonical map $R \to S$ is a one-to-one since ${}_R M$ is faithful. Identifying R as a subring of S, $ST \subseteq R$, where T is the trace ideal of ${}_R M$ (see [5, Proposition 1.1]) Cozzens [5, Proposition 1.1] also has shown that for ${}_R M$ torsionless faithful, $(sm)f = s(mf)$ whenever $s \in S$, $m \in M$ and $f \in M^* = \text{Hom}({}_R M, {}_R R)$. Hence if $s \in S$ and $sT = 0$, $0 = s(MM^*) = (sM)\,M^*$. Since M is torsionless $sM = 0$, so $s = 0$. It now follows that left multiplication by $s \in S$ induces a unique R-endomorphism of the ideal ST of R regarded as a right R-module so by (2), $S = R$.

(3) ⇒ (1). If $R \subseteq S \subseteq Q$ and $R \overset{\ell}{\sim} S$, $Sd \subseteq R$ for some regular element d of R. Consequently, $A = SdR$ is a nonzero ideal of R, so A is a faithful torsionless left R-module. Clearly $S \subseteq \text{Biend}\,({}_RA) = R$ and (1) follows. ∎

Remark. If R is an order in Q then R is a maximal order if and only if R is a maximal left equivalent and maximal right equivalent order (see [10, p 284]). Thus, R is a maximal order if and only if $O_l(A) = R = O_r(A)$ for all ideals A of R.

The next lemma has a number of interesting consequences Among these is the fact that in a two-sided Noetherian ring with glb $R \leqslant 2$, the intersection of *any* collection of essential projective ideals is projective Later, we shall show that this implies that in a quasi-local maximal order having global dimension $\leqslant 2$, the intersection of projective right ideals is principal. These generalize results of Ramras [15] obtained for a classical maximal R-order Λ, where glb $\Lambda = 2$ and R is a two-dimensional regular local ring.

LEMMA 1 3 *Let R be an order in Q.*

If I_R is essential right ideal of R, then I_R is reflexive if and only if $(R/I)_R$ embeds monomorphically in a product of copies of the right R-module $(Q/R)_R$.

Proof. If $I = I^{**}$ and $a \in R\text{–}I$ then there is an element $q \in I^*$ such that $qa \notin R$, hence q induces by left multiplication an R-homomorphism $R/I \to Q/R$ which distinguishes $a + I \in R/I$ from 0, thus R/I embeds monomorphically in a product of copies of Q/R.

Conversely, if there is an embedding of R/I in a product of copies of Q/R, $R/I \xrightarrow{\alpha} \Pi Q/R$, then α followed by each projection $\Pi Q/R \to Q/R$ is given by some left multiplication by $q \in I^*$ Therefore, there is a subset $B \subseteq I^*$ such that if $a \in R$ and $Ba \subseteq R$, then $a \in I$ Clearly then, if $a \in I^{**}$ $Ba \subseteq I^*a \subseteq R$, so $a \in I$ and the lemma follows ∎

COROLLARY 1 4. *If I_1, I_2 are essential reflexive right ideals of R, then so is $I_1 \cap I_2$*

Proof. $R/I_1 \cap I_2$ embeds monomorphically in $R/I_1 \times R/I_2$. ∎

COROLLARY 1 5. *If I is an essential right ideal of R, then I^{**} is a reflexive right ideal of R and $I \subseteq I^{**} \subseteq R$*

Proof Clearly $I \subseteq I^{**} \subseteq R$ If $r \in R\text{–}I^{**}$ then there is a $q \in I^*$ such that $qr \notin R$, so each element of R/I^{**} can be distinguished from zero by a map $R/I^{**} \to Q/R$ and, therefore, R/I^{**} embeds monomorphically in a product of copies of the right R-module Q/R. ∎

Riley [17] has shown that if Λ is a (classical) maximal R-order, R an integrally closed Noetherian domain, then a prime P is minimal if and only if P is reflexive.

The next theorem generalizes the sufficiency to any maximal order (later, we shall show that the necessity is also valid for any bounded maximal order). This theorem will prove basic to the results of Sections 2–4 where maximal reflexives turn out to be in a very precise sense, the building blocks of the reflexive R-ideals.

THEOREM 1.6. *Let R be a maximal order If P is a nonzero maximal reflexive proper ideal of R. Then P is a height* 1 (*minimal nonzero*) *prime ideal of R.*

Proof. Suppose $AB \subseteq P$, where A, B are ideals of R properly containing P It may be assumed without loss of generality that $A^* = R$, otherwise A^{**} would be a proper reflexive ideal of R properly containing P, an impossiblity. Similarly, $B^* = R$ Now, if $q \in P^*$, then $qAB \subseteq qP \subseteq R$ so $qA \subseteq B^* = R$ hence $q \in A^* = R$, so $P^* \subseteq R$ which is untenable. Thus, P is a prime ideal.

Now if $B \subsetneqq P$ and B is a nonzero prime ideal of R, then $BP^* \subseteq PP^* \subseteq R$ and $(BP^*) P \subseteq B$. Since $P \nsubseteq B$, $BP^* \subseteq B$ as B is prime However, R is a maximal order so $P^* \subseteq R$ an impossibility and the theorem follows ∎

The next result gives some basic characterizations of a reflexive prime ideal in a maximal order which are useful.

PROPOSITION 1.7. *Let R be a maximal order and P a nonzero prime ideal of R, then the following statements are equivalent:*

(1) $P = P^{**}$,

(2) $P^* \supsetneqq R$,

(3) $PP^* \supsetneqq P$.

Moreover, whenever P is reflexive, P is minimal.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). If $PP^* = P$ then $P^* \subseteq R$ by the maximality of R contradicting (2)

(3) $\Rightarrow$ (1) If $r \in P^{**}$ then $(PP^*) r \subseteq P$. Since PP^* is an ideal of R and $PP^* \nsubseteq P$ by (3) $r \in P$ and (1) follows.

Suppose $P' \subsetneqq P$ and P' is a prime ideal, then $(P'P^*) P \subseteq P'$ hence $P'P^* \subseteq P'$ and by the maximality of R, $P^* \subseteq R$ a contradiction. ∎

2. *RI*-ORDERS

For a bounded order R, Asano [19] showed that the R-ideals form a group under multiplication if and only if R is a maximal order satisfying the a.c.c. on integral R-ideals and such that prime integral R-ideals are maximal. Robson [19] removed the boundedness assumption, characterizing right orders R whose

R-ideals form a group under multiplication. These he called *Asano right orders*. Important examples of maximal orders exist in global dimension 2 and more generally, which are not Asano. For example, any two dimensional regular local ring; and if R is any simple Goldie ring, M_R any finite (Goldie)-dimensional reflexive, nonprojective generator, $k = \text{End } M_R$ is a maximal order which is not Asano (see [5]).

Noetherian maximal orders with global dimension $\leqslant 2$ have an important property which can be thought of as the natural generalization of the defining property of an Asano order, namely, each reflexive R-ideal (right or left) is invertible This assertion follows from Bass' characterization of Noetherian rings with global dimension $\leqslant 2$ as rings over which duals of finitely generated modules are projective [3, Proposition 5.2]. For, if I is a reflexive R-ideal, say I_R is reflexive, $II^* = O_\ell(I) = R$ by projectivity of I_R and maximality of R. Once again, by maximality of R, $I^* = \{q \in Q \mid Iq \subseteq R\} \Rightarrow_R I$ is reflexive $\Rightarrow I^*I = R \Rightarrow I$ is invertible. Since orders other than those with global dimension $\leqslant 2$ satisfy this property (both of the aforementioned examples do), we shall call orders satisfying the condition that R-ideals, left or right reflexive, are invertible, *RI-orders* Of course, each Asano order is an *RI*-order.

The first theorem characterizes these orders in eactly the same way as Robson characterizes Asano orders [68, Theorem 2.1] In fact, the proof is essentially his adapted to our more general setting. First, a useful lemma.

LEMMA 2.1. *Let A and B be R-ideals with B invertible and A_R reflexive. Then $(BA)_R$ is reflexive.*

Proof. First, observe that $(BA)^* = A^*B^*$. For, clearly $(BA)^* \supseteq A^*B^*$ and if $qBA \subseteq R$, $qB \subseteq A^* \Rightarrow qBB^* = qR \subseteq A^*B^*$

If $(A^*B^*)\, q \subseteq R \Rightarrow B^*q \subseteq A^{**} = A \Rightarrow q \in BA \Rightarrow (BA)^{**} = BA$. ∎

By a *reflexive* R-ideal we mean an R-ideal which is R-reflexive both as a right and as a left R-module.

THEOREM 2.2. *The following statements are equivalent for a prime ring R:*

(1) *If A is an R-ideal which is reflexive as a right R-module, then $AX = R$ for some R-ideal X*

(1*) *If B is an R-ideal which is reflexive as a left R-module, then $YB = R$ for some R-ideal Y*

(2) *R is a maximal order and reflexive ideals of R are invertible.*

(3) *R is a maximal order and the reflexive R-ideals form a group under multiplication*

(4) *Each R-ideal which is reflexive as a left or right R-module is invertible.*

Proof. (1) $\Rightarrow$ (1*). If B is an R-ideal which is reflexive as a left R-module, then B^* is an R-ideal which is reflexive as a right R-module, so by 1, $B^*X = R$ for some R-ideal X. It now follows that $B_R{}^*$ is finitely generated and projective and $O_\ell(B^*) = R$. Since ${}_RB$ is reflexive and $B_R{}^*$ is finitely generated and projective, ${}_RB$ is finitely generated and projective. Clearly $O_r(B) = O_\ell(B^*) = R$, so let $Y = B^*$

(1*) $\Rightarrow$ (2) If B is an ideal of R, then $O_r(B) \subseteq O_r(B^{**})$, B^{**} the bidual of ${}_RB$ By 1*, $YB^{**} = R$ for some R-ideal Y, hence if $q \in O_r(B)$, then $Rq = YB^{**}q \subseteq YB^{**} = R$ so $q \in R$, thus by Lemma 1 2, R is a maximal right equivalent order and by symmetry a maximal left equivalent order and, therefore, a maximal order

Now, if B is a reflexive ideal of R by (1*) (and (1)), B is invertible.

(2) $\Rightarrow$ (3). If A is any R-ideal, then set $B = \{r \in R \mid Ar \subseteq R\}$. Since $Ad \subseteq R$ for some regular element d of R, B is a nontrivial ideal of R Now if $r \in R$–B, then $Ar \not\subseteq R$ so R/B is embeddable monomorphically in a product of copies of $(Q/R)_R$ via left multiplications by elements of A and, therefore, B_R is reflexive by Lemma 1.3. By (2), B is invertible. If A is reflexive, then by Lemma 2.1, AB is reflexive so by Lemma 2.1, A has a right inverse and by symmetry A has a left inverse and (3) is shown.

(3) $\Rightarrow$ (4). This is trivial, as an R-ideal which is either left or right reflexive is reflexive by the maximality of R.

(4) $\Rightarrow$ (1). Trivial. ▮

The next results give an explicit description of invertible ideals and hence, the group of invertible R-ideals in the spirit of the Asano–Robson characterization. However, in the absence of the Asano axioms, some chain conditions must be assumed.

PROPOSITION 2.3. *Let R be a maximal order. Then each maximal invertible ideal is prime If R satisfies the* a.c.c. *on invertible ideals, each invertible ideal is a product of maximal invertible ideals. Finally, if R is an RI-order, the group of invertible R-ideals is a free abelian group on the maximal invertible ideals.*

Proof. The first assertion is a special case of Theorem 1 6.

Let A be an invertible ideal maximal with respect to the property that A is not a product of maximal invertible ideals. Then $A \subsetneqq P$ with P a maximal invertible ideal $A \subseteq AP^{-1} \subseteq R$, and if $A = AP^{-1} \Rightarrow P^{-1} \subseteq O_r(A) = R$, a contradiction. Thus, $A \subsetneqq AP^{-1}$. Since AP^{-1} is an invertible ideal properly containing A, AP^{-1} is a product of maximal invertibles and hence, so is $(AP^{-1})\, P = A$ a contradiction.

To establish the last assertion, it suffices to show that the maximal invertibles commute. To that end, suppose P and P' are distinct maximal invertibles. $P \cap P' \subseteq P \Rightarrow P \cap P' = PA$ for some invertible ideal ($P^{-1}\,(P \cap P')$ in fact).

Now $PP' \subseteq P \cap P' \subseteq PA$ and hence $P' \subseteq A$ by invertibility of $P \Rightarrow P' = A$. Similarly, $P'P = P \cap P'$ by symmetry. Hence, $PP' = P'P$ as claimed ▌

Our final result in this section is an attempt to say something about the height 1 (minimal) primes in an RI-order, paralleling the Asano–Robson description of primes in an Asano order. That our description turns out to be a characterization of RI-orders should come as no surprise in view of Proposition 2.3. Recall that an order is *bounded* if each fractional right or left R-ideal contains a nonzero R-ideal.

PROPOSITION 2.4 *Let R be a maximal order satisfying the* a.c.c. *on reflexive ideals Then R is an RI-order if and only if each reflexive minimal prime is invertible. In this case, each minimal prime is invertible whenever R is bounded.*

Proof The necessity is clear For the sufficiency, observe that each maximal reflexive is a minimal prime by Theorem 1.6 and hence, invertible by assumption. Choose, if possible, a reflexive ideal A, maximal with respect to the property of being noninvertible. $A \subsetneq P$ a maximal reflexive which is necessarily invertible. As in Proposition 2.3, $A \neq AP^{-1}$. By Lemma 2.1, AP^{-1} is reflexive and hence invertible by maximality of A Thus, A is invertible.

To establish the last assertion, it suffices to show that when R is a bounded RI-order, each minimal prime P is reflexive (equivalently, invertible). To that end, choose a regular element $x \in P$. By boundedness, xR contains an nonzero ideal A which we can clearly assume to be reflexive. By assumption, A can be expressed as a product of invertible primes $\Rightarrow$ P coincides with (one of) these by minimality of P. ▌

As a consequence of Proposition 2 4, we are now in a position to exhibit a broad class of classical, not necessarily Asano, RI-orders, namely, *any* maximal R-order Λ, where R is a Noetherian integrally closed domain. For, if P is a minimal prime of Λ, choose $0 \neq x \in P \cap R$. Λx is clearly an invertible ideal of Λ contained in P, and hence, a product of maximal invertible ideals of Λ by Proposition 2 3. Since these are necessarily prime, P coincides with one of them by minimality of P. Thus each minimal prime is invertible and Λ is an RI-order by Proposition 2.4. Riley [17] has shown that whenever Λ is a maximal R-order in Σ, a full matrix algebra over K, the quotient field of R, R Noetherian integrally closed, the minimal primes of Λ are projective whenever the minimal primes of R are projective

3. NOETHERIAN RI-ORDERS

For R a prime maximal order denote $S = S(R) = \bigcup B^{-1}$, where the union is taken over all nonzero ideals of R. Since $B^{-1} = B^{-1-1-1}$ and B^{-1-1} is a reflexive ideal, $S = \bigcup B^{-1}$, where the union is taken over all nonzero reflexive ideals B.

If $B_1, B_2, .., B_n$ are nonzero reflexive ideals of R, then for $B = (B_1B_2 \cdot\ B_n)^{-1-1}$, $B_i^{-1} \subseteq B_n^{-1} \cdots B_2^{-1}B_1^{-1} \subseteq (B_1B_2\ \cdot\ B_n)^{-1} = B^{-1}$ and so it follows that the union $\bigcup B^{-1}$ taken over all nonzero reflexive ideals B of R is directed and S is a subring of Q containing R. S is called the *Asano overring of R*

THEOREM 3.1 *Let R be a Noetherian RI-order and S the Asano overring of R, then*

(1) *S is a left and right flat epimorphic ring extension of R.*

(2) *For each right ideal Π of S, $(\Pi \cap R)\, S = \Pi$*

(3) *S is a Noetherian maximal order with no nontrivial reflexive ideals.*

Proof. (1) To show that S is an epimorphic ring extension of R, it is sufficient to show that for $t \in S$, $t \otimes 1 = 1 \otimes t$ in $S \otimes_R S$ (e.g., see [20]). Let B be an invertible ideal of R such that $tB \subseteq R$, then for $b \in B$ $[(t \otimes 1) - (1 \otimes t)]\, b = 0$, hence $[(t \otimes 1) - (1 \otimes t)]\, BB^{-1} = [(t \otimes 1) - (1 \otimes t)]\, R$ and, therefore, $t \otimes 1 = 1 \otimes t$

S is R-flat as a left and right R-module since it is a directed union of invertible, hence, projective R-ideals of Q

Statement (2) is an easy consequence of (1) and will be omitted.

(3) Let A be an ideal of S and $q \in Q$ such that $qA \subseteq A$, hence $q(A \cap R) \subseteq A$. Since $q(A \cap R)$ is isomorphic to the image of an ideal of R and R is Noetherian $q(A \cap R)$ is finitely generated as a right R-module, hence there is an invertible ideal B of R such that $Bq(A \cap R) \subseteq R$. Now it follows that $Bq \subseteq (A \cap R)^{-1} \subseteq S$ and therefore, $q \in Rq = B^{-1}Bq \subseteq B^{-1}S = S$ Similarly, if $Ap \subseteq A$ for some $p \in Q$, then $p \in S$. By Lemma 1.2, S is a maximal order. If Π is a nontrivial reflexive (with respect to S) ideal of S and $q \in \Pi^{-1}$, then $q(\Pi \cap R) \subseteq S$. Since $q(\Pi \cap R)$ is a finitely generated right R-module, there is an invertible ideal B of R such that $Bq(\Pi \cap R) \subseteq R$, hence $q \in Rq = B^{-1}Bq \subseteq B^{-1} (\Pi \cap R)^{-1} \subseteq S$, thus $\Pi^{-1} = S$ and, therefore, $\Pi = S$ ▮

COROLLARY 3 2. *If R is a Noetherian RI-order, then S the Asano overring of R is a Noetherian RI-order.*

Proof. S is Noetherian in view of part (1) of the theorem and the remainder of the corollary follows from the theorem. ▮

COROLLARY 3.3. *If R is a Noetherian RI-order and S the Asano overring of R, then*

$$\mathrm{glb}(S) \leqslant \mathrm{glb}(R)$$

Proof This corollary follows from the fact that S is a flat epimorphic ring extension of R. ▮

Goldie has defined the localization of a Noetherian ring R at a prime ideal P utilizing $C(P) = \{c \in R \mid cx \in P \Rightarrow x \in P\}$ and has shown that under certain conditions on P, $C(P)$ is a right Ore set of regular elements and that the localization of R at P is the classical ring of right quotients of R with respect to the right Ore set $C(P)$ One of the above conditions is that the intersection of the symbolic powers of P (see [69]) is zero In the event that P is an invertible ideal of R, Michler [14] has shown that the nth symbolic power of P is P^n Michler [14] has also shown that if P is an invertible prime ideal of a Noetherian prime ring R, then $\bigcap_{n=1}^{\infty} P^n = 0$.

Goldie's other condition on P assuring that the localization of R at P is classical is that P satisfies an Artin–Rees type condition. Chatters and Ginn [74] have shown that if P is an invertible prime ideal of R then P satisfies the second of Goldie's conditions guaranteeing that the localization of R at P is classical

Summarizing the above, the localization of a Noetherian prime ring R at an invertible prime ideal P of R is the classical ring of quotients of R, necessarily two-sided in view of the symmetric hypotheses, with respect to the left and right Ore set $C(P)$ of regular elements of R

Goldie has called a ring Λ a local ring if Λ has a unique maximal ideal $M =$ rad Λ, Λ/M is an Artin ring and $\bigcap_{n=1}^{\infty} M^n = 0$ and has shown that R_P is a local ring with unique maximal ideal $PR_P = R_P P$. In what follows R_P denotes the localization of R at P for an invertible prime ideal P of a Noetherian prime ring R.

The following theorem lists some properties of R_P.

THEOREM 3 4. *Let R be Noetherian RI-order and P an invertible prime (maximal invertible) ideal of R, then the following hold·*

(1) *R_P is a left and right R-flat epimorphic ring extension of R and $(A \cap R) R_P = A$ for each right ideal A of R_P.*

(2) *R_P is a Noetherian local prime ring with Jacobson radical $J(R_P) = PR_P = R_P P$.*

(3) *R_P is a hereditary principal right and left ideal ring and a bounded Asano order.*

Proof Property (1) for a classical quotient ring with respect to an Ore set of regular elements is easily verified

The fact that R_P is a local ring (in Goldie's sense) follows from earlier remarks. Since R_P is a left and right R-flat epimorphic ring extension of R and R is Noetherian it follows that R_P is Noetherian. Since P is invertible so is $PR_P = R_P P$ and (3) now follows from Hajarnavis and Lenagan [11, Proposition 1.3] ▌

The proof of the following lemma is a modification of the proof of [11, Lemma 3 4] adapted to the weaker hypotheses of a Noetherian *RI*-order.

LEMMA 3 5 *If c is a regular element of R, a Noetherian prime RI-order, then*

$c \in C(P)$ for all but a finite number of invertible prime (maximal reflexive) ideals P of R.

Proof Since R is a Noetherian ring $\sum_{\mathscr{P}} [(P^{-1}c) \cap R] = \sum_{i=1}^{n} [(P_i^{-1}c) \cap R]$, where the left-hand summation is taken over all $P \in \mathscr{P}$. If $P \in \mathscr{P}$, then

$$P^{-1}c \cap R \subseteq \sum_{i=1}^{n} P_i^{-1}c \cap P^{-1}c = \left[\left(\sum P_i^{-1}\right) \cap P^{-1}\right] c$$

$$\subseteq \left[\left(\prod_{i=1}^{n} P_i\right)^{-1} \cap P^{-1}\right] c \subseteq \left[\left(\prod_{i=1}^{n} P_i\right) + P\right]^{-1} c.$$

Now if $P \neq P_i$ for $i = 1, 2,..., n$ then $\prod_{i=1}^{n} P_i \not\subseteq P$ and by the maximal reflexivity of P, $[(\Pi P_i) + P]^{-1} = R$ hence $P^{-1}c \cap R \subseteq Rc$ and, therefore, by the invertibility of P, $Rc \cap P \subseteq Pc$. Now if $xc \in P$ then $xc \in Rc \cap P \subseteq Pc$ so $x \in P$ and, therefore, $c \in C(P)$ and the lemma follows. ∎

We shall now investigate some properties of the subring $T(R) = \bigcap_{\mathscr{P}} R_P$ of Q, where R is a Noetherian RI-order.

THEOREM 3 6. *Let R be a Noetherian RI-order and $T = T(R)$, then T is a directed union, $T = \bigcup K^* = (\bigcup L^*)$, where the union is taken over reflexive right (left) ideals of R such that $KR_P = R_P(R_PL = R_P)$ for all $P \in \mathscr{P}$*

Proof. If K_R is a reflexive right ideal of R such that $KR_P = R_P$ for every invertible prime ideal P of R, then for $q \in K^*$, $q \in qKR_P \subseteq RR_P = R_P$ so $q \in R_P$. Since P was an arbitrary invertible prime ideal of R, $K^* \subseteq \bigcap_{\mathscr{P}} R_P = T$. Now if $x \in T$, then $K = \{r \in R \mid xr \in R\}$ is a reflexive right ideal of R by Lemma 1 3, since R/K is embeddable in Q/R by left multiplication by x. Since $x \in R_P$ there is a $c \in C(P)$ such that $xc \in R$, so $K \cap C(P) \neq \varnothing$ for every invertible prime ideal P of R, hence $KR_P = R_P$ for each invertible prime ideal P of R. Thus, we have shown that $T = \bigcup K^*$. Now if $K_1 ,.., K_n$ are reflexive right ideals of R such that $K_iR_P = R_P$ for all P, $i = 1,..., n$, then by Lemma 1.3, $K_1 \cap K_2 \cap \cdots \cap K_n = K$ is a reflexive right ideal of R with $K_i^* \subseteq K^*$, $KR_P = R_P$ as R_P is flat, and the theorem follows. The left-sided analog for the theorem is shown similarly. ∎

COROLLARY 3.7. *If R is a Noetherian RI-order with* glb $R \leqslant 2$, *then T is a left and right R-flat epimorphic extension ring of R.*

Proof By Bass [63], reflexive right (left) ideals of R are projective For $T = \bigcup K^*$ (as above), ${}_RK^*$ is an R-projective left R-module, so T is a directed union, hence a direct limit, of projective left R-modules so ${}_RT$ is R-flat Similarly, T_R is R-flat.

Now if $t \in T$, $tK \subseteq R$ for some reflexive right ideal K of R such that $K^* \subseteq T$. In $T \otimes_R T$ $((t \otimes 1) - (1 \otimes t))K = 0$, so $((t \otimes 1) - (1 \otimes t))KK^* = 0$.

Since K_R is projective $1 \in KK^*$, hence $t \otimes 1 = 1 \otimes t$ and it follows that T is a ring epimorphic extension of R ∎

Some further properties of T will be established in the following.

PROPOSITION 3.8. *Let R be a Noetherian RI-order and I a right ideal of T, then the following hold*

(1) $IR_P = (I \cap R) R_P$.

(2) $(IR_P)^* = R_P(I \cap R)^*$.

(3) $I^* = \bigcap_{\mathscr{P}} R_P(I \cap R)^*$.

(4) *If I_T is reflexive, then* $(I \cap R)^{**} \subseteq I$.

(5) *If I_T is reflexive, then $I \cap R$ is reflexive.*

(6) *If I_T is reflexive, then* $\bigcap_{\mathscr{P}} (IR_P) = I$.

Proof of (1). Clearly $(I \cap R) R_P \subseteq IR_P$. If $x \in IR_P$ then $xc \in I$ for some $c \in C(P)$. Since $xc \in T$ there is a reflexive right ideal K of R such that $xcK \subseteq R$ and $KR_P = R_P$ Now $xcK \subseteq I \cap R$ and $xR_P = xcKR_P \subseteq (I \cap R) R_P$

Proof of (2). Clearly $R_P(I \cap R)^* \subseteq (IR_P)^*$ by 1 Now, if $qIR_P \subseteq R_P$, then $q(I \cap R) \subseteq R_P$. Since $I \cap R$ is a finitely generated right ideal of R, $q(I \cap R)$ is a finitely generated right R-submodule of T, so by Theorem 3 6 there is a reflexive left ideal L of R with $R_PL = R_P$ such that $Lq(I \cap R) \subseteq R$. Thus, $Lq \subseteq (I \cap R)^*$ and $R_PLq \subseteq R_P(I \cap R)^*$.

Proof of (3). If $qI \subseteq T$, then $qIR_P \subseteq R_P$ so by (2) $q \in (IR_P)^* = R_P(I \cap R)^*$, hence $I^* \subseteq \bigcap_{\mathscr{P}} (R_P(I \cap R)^*)$ Now, if $q \in \bigcap_{\mathscr{P}} (R_P(I \cap R)^*)$ then $qI \subseteq R_P$ for all $P \in \mathscr{P}$, hence $qI \subseteq \bigcap_{\mathscr{P}} R_P = T$ so $q \in I^*$.

Proof of (4). If $q \in (I \cap R)^{**}$ then $(I \cap R)^* q \subseteq R$, hence $R_P(I \cap R)^* q \subseteq R_P$ for all $P \in \mathscr{P}$. Since $R_P(I \cap R)^* = (IR_P)^* \supseteq I^*$, $I^*q \subseteq R_P$ for all $P \in \mathscr{P}$, hence $I^*q \subseteq \bigcap_{\mathscr{P}} R_P = T$, so $q \in I^{**} = I$.

Proof of (5). Since $(I \cap R)^{**} \subseteq I$ by 4, then $(I \cap R)^{**} \subseteq I \cap R$, so $I \cap R$ is reflexive

Proof of (6). Clearly $I \subseteq \bigcap_{\mathscr{P}} (IR_P)$. If $q \in \bigcap_{\mathscr{P}}(IR_P)$ then $I^*q \subseteq I^*IR_P \subseteq R_P$ for all $P \in \mathscr{P}$, hence $I^*q \subseteq T$, so $q \in I^{**} = I$. ∎

LEMMA 3 9. *Let R be a Noetherian RI-order and I an ideal of R, then $IR_P = R_PI$ for every $P \in \mathscr{P}$.*

Proof Let I be maximal among those ideals of R for which $IR_P \neq R_PI$, with P fixed Clearly, $I \subseteq P$, otherwise $I \cap C(P) \neq \varnothing$ and then $IR_P = R_P = R_PI$. Now $IP^{-1} \subseteq R$ and $I \subsetneqq IP^{-1}$ since if $I = IP$ then $I \subseteq \bigcap_{n=1}^{\infty} P^n = 0$. Since

$R_P P = PR_P$, $P^{-1}R_P = R_P P^{-1}$, hence $IP^{-1}R_P = IR_P P^{-1} = R_P I P^{-1}$ and so $IR_P = R_P I$ a contradiction. ▮

COROLLARY 3.10. *If R is a Noetherian RI-order and I is an ideal of T, then $IR_P = R_P I$.*

Proof. By Proposition 3.8, part (1), $IR_P = (I \cap R)R_P = R_P(I \cap R) = R_P I$. ▮

THEOREM 3 11. *Let R be a Noetherian RI order, then $T = \bigcap_{\mathscr{P}} R_P$ is a bounded RI-order.*

Proof. Let I be a nonzero ideal of T and $q \in O_\ell(I)$, that is, $qI \subseteq I$ then $qIR_P \subseteq IR_P$ for each $P \in \mathscr{P}$. Since IR_P is an ideal of R_P by Corollary 3.10, and R_P is a maximal order, $q \in R_P$ for each $P \in \mathscr{P}$, hence $q \in \bigcap_{\mathscr{P}} R_P = T$ and the maximality of T now follows from Lemma 1.2.

If I is a reflexive ideal of T, then by Proposition 3.8, $I \cap R$ is a reflexive ideal of R. Clearly, $(I \cap R)^* \subseteq I^*$, hence

$$II^* \supseteq (I \cap R)(I \cap R)^* = R \qquad \text{so} \qquad 1 \in II^*,$$

and by symmetry and maximality of T, I is an invertible ideal of T.

The boundedness of T uses the argument of Hajarnavis and Lenagan [11, Theorem 3 5] with Lemma 3.5, and, therefore, the proof will be omitted. ▮

THEOREM 3.12. *If R is a Noetherian RI-order then $R = S \cap T$, where $S = S(R)$, $T = T(R)$.*

Proof. The proof of the corresponding result for Asano orders found in Hajarnavis and Lenagan [11, Theorem 3.1] can be used without change to provide a proof of this result. ▮

PROPOSITION 3.13 *Let R be a Noetherian RI-order and I a nonzero prime ideal of R, then*

(1) *If I contains an invertible ideal, then $IS = S$.*

(2) *If I does not contain an invertible ideal, then $IS \cap R = I$*

Proof. The proof of (1) is obvious.

Suppose I does not contain an invertible ideal. If $x \in IS \cap R$, then $xB \subseteq I$ for some invertible ideal B of R. Since $B \not\subseteq I$ and I is prime $x \in I$ and (2) follows. ▮

COROLLARY 3.14. *If I is a prime ideal of R, then* Ker $(R/I \to R/I \otimes_R S)$ *is either* 0 *or* R/I.

Proof. Since Ker $(R/I \to R/I \otimes_R S) = IS \cap R/I$ the corollary follows from the proposition. ▮

LEMMA 3.15. *Let R be a Noetherian RI-order and I a prime ideal of R not containing any invertible, then $IB = BI$ for every invertible ideal B of R.*

Proof. Since $(BIB^{-1})B \subseteq I$ and $B \not\subseteq I$, $BIB^{-1} \subseteq I$ so $BI \subseteq IB$ and by symmetry $BI = IB$. ∎

COROLLARY 3.16. *If R is a Noetherian RI-order and I prime ideal of R not containing any invertible of R, then $IS = SI$.*

Proof. $IS = \bigcup IB^{-1}$, where the union is taken over all invertible ideals B of R. By the lemma, $IB^{-1} = B^{-1}I$ hence $IS = \bigcup IB^{-1} = \bigcup B^{-1}I = SI$. ∎

THEOREM 3.17. *Let R be a Noetherian prime RI-order and $S = S(R)$, then*

(1) *If I is a prime ideal of R not containing an invertible ideal, then $IS = SI$ is a prime ideal of S.*

(2) *If Π is a prime ideal of S, then $\Pi = IS$ for a unique prime ideal $I = \Pi \cap R$ of R, thus $I \to IS$, $\Pi \to \Pi \cap R$ are one–one correspondences, mutually inverse, between the class of prime ideals of R which do not contain any invertible and the prime ideals of S.*

Proof. (1) By Lemma 3, $SI = IS$ and by Silver [20, Corollary 1 10], $SI = IS$ is a prime ideal of S.

(2) Let $I = \Pi \cap R$. Then since R is Noetherian I contains a product of prime ideals each of which can be chosen to contain I with the product irredundant, hence there are ideals A, B such that $AB \subseteq I$, where A is a prime ideal containing I and $B \not\subseteq I$.

Case 1. If A contains an invertible ideal, then $B \subseteq SB = SAB \subseteq SI = \Pi$ hence $B \subseteq \Pi \cap R = I$ a contradiction.

Case 2. If A does not contain an invertible then using 1, $SAB = ASB$ so $SAB = SASB \subseteq \Pi$ and by the primality of Π, $SA \subseteq \Pi$ or $SB \subseteq \Pi$, hence $A \subseteq I$ or $B \subseteq I$. Since $B \not\subseteq I$ is impossible, $A \subseteq I$ and, therefore, $A = I$. ∎

COROLLARY 3.18 *For R a Noetherian RI-order $S(R)$ is a simple ring if and only if every nonzero prime ideal of R contains an invertible ideal of R.* ∎

Let Λ be a (classical) maximal R-order in a simple algebra where R is a Noetherian integrally closed domain. By remarks at the end of Section 2, Λ is an RI-order and since Λ is bounded, $\Lambda = \bigcap_{P \in \mathscr{P}} \Lambda_P$ by Theorem 3 12. Here, $\mathscr{P}$ is the set of all minimal (invertible) primes of Λ. On the other hand, $\Lambda = \bigcap_{p \in ht_1(R)} \Lambda_p$, where $\Lambda_p = \Lambda \otimes_R R_p$ and $ht_1(R)$ is the collection of all minimal primes of R (e.g , see [15]) The natural question to ask is, What is the relationship between these two decompositions of Λ? The next proposition supplies the anticipated answer

PROPOSITION 3.19. *Let Λ be a maximal R order where R is a Noetherian integrally closed domain and p a minimal prime of R. Then $\Lambda_p = \Lambda_P$ for a unique minimal prime P of Λ. Conversely, if Λ_P is the Goldie localization at an invertible prime P of Λ, then $\Lambda_p = \Lambda_P$, where $p = P \cap R$.*

Proof. Given any minimal prime p of R there is a unique minimal P of Λ lying above p (e.g., see [17]). First, we show that $R - p = C(P) \cap R$. Since R/p is canonically embedded in Λ/P it follows that $R - p \supseteq C(P) \cap R$. Now for $x \in R - p$, let $I = \{\lambda \in \Lambda \mid \lambda x \in P\}$. Then R/I is embedded in R/P as left and right R-module by multiplication by x. Since P is reflexive as Λ is an RI-order it follows by Lemma 1.3 that I is reflexive. Now as $P \subseteq I$ and P is a maximal reflexive $P = I$. Thus, we have $x \in C(P) \cap R$.

Now since $R - p \subseteq C(P) \cap R$, it follows that $\Lambda_p \subseteq \Lambda_P$ and by the maximality of Λ_p, $\Lambda_p = \Lambda_P$.

Conversely, if P is a minimal prime of Λ then $p = P \cap R$ is a minimal prime of R and P lies above p. As above $\Lambda_p = \Lambda_P$. ∎

4. ASANO'S LOCALIZATION

Let R be any order and P a prime ideal of R. By *Asano's Localization* of R at P, $A(P)$, we mean the set $\{q \in Q \mid qB \subseteq R$ for some invertible $B \nsubseteq P\}$. It is easily verified that $A(P)$ is a subring of Q containing R.

The following proposition is a "local" version of Theorem 3.1.

PROPOSITION 4.1. *Let R be a Noetherian RI-order and P an invertible prime ideal of R. Then,*

(1) *$A(P)$ is a left and right R-flat epimorphic ring extension of R.*

(2) *For I a right ideal of $A(P)$, $(I \cap R)\, A(P) = I$.*

(3) *$A(P)$ is a Noetherian RI-order with unique invertible ideal $\bar{P} = PA(P) = A(P)P$.*

Proof. Statements (1) and (2) can be verified in a fashion very similar to that of Statements (1) and (2) of Theorem 3.1.

Proof of (3). If $x \in PA(P)$ then $xB \subseteq P$ for some invertible ideal $B \nsubseteq P$ so $x \in PB^{-1}$. Since $PB^{-1} = B^{-1}P$, $x \in B^{-1}P \subseteq A(P)P$ so $\bar{P}$ is an ideal of $A(P)$.

Now if I is an ideal of $A(P)$ and $qI \subseteq I$, then $q(I \cap R)$ is a finitely generated right R-submodule of $A(P)$ so there is an invertible ideal B of R, $B \nsubseteq P$ such that $Bq(I \cap R) \subseteq R$. Clearly, $Bq(I \cap R) \subseteq (I \cap R)$ and since R is a maximal order $Bq \subseteq R$ so $q \in B^{-1} \subseteq A(P)$ and, therefore, $A(P)$ is a maximal order by Lemma 1.2. If I is a reflexive ideal of $A(P)$ and $q \in I^*$, then $q(I \cap R)$ is a finitely generated right R-submodule of $A(P)$, so there exists an invertible ideal B of R,

$B \not\subseteq P$, such that $Bq(I \cap R) \subseteq R$, hence $Bq \subseteq (I \cap R)^{-1}$ and $q \in B^{-1}(I \cap R)^{-1} \subseteq A(P)(I \cap R)^{-1}$. Since $(I \cap R)^{-1}$ is an invertible R-ideal, it follows that $II^* = A(P)$ by the maximality of $A(P)$ Similarly, $I^*I = A(P)$

Clearly, $\bar{P}$ is invertible whenever P is. If I is an invertible ideal of $A(P)$, $I^{-1} = A(P)(I \cap R)^{-1}$. However, $(I \cap R)^{-1} \subseteq A(P)$ unless $I \cap R \subseteq P$. In the former case, $I^{-1} \subseteq A(P)$, a contradiction. In the latter case, $I = (I \cap R)\, A(P) \subseteq \bar{P}$ which clearly implies that $\bar{P}$ is the unique maximal invertible ideal of $A(P)$. ∎

PROPOSITION 4.2. *Let R be a Noetherian RI-order and P an invertible prime ideal of R. Then,*

(1) $A(P) \subseteq R_P$.

(2) $PA(P) \cap R = P$.

(3) $A(P) = R_P$ *if and only if* $\forall\, c \in C(P)$, $cR \supseteq$ *ideal* ($Rc \supseteq$ *ideal*). *In particular, if R is bounded*, $A(P) = R_P$.

Proof. (1) Given $x \in A(P)$, $\exists$ an invertible $B \not\subseteq P$ with $xB \subseteq R$. Since P is prime and $B \not\subseteq P$, $B \cap C(P) \neq \varnothing \Rightarrow xc = r \in R$ for some $c \in C(P)$, $r \in R$ and hence, $x = rc^{-1}$. Thus, $x \in R_P$.

(2) Clear.

(3) If $R_P = A(P)$ and $c \in C(P)$, $\exists$ an invertible $B \not\subseteq P$ with $c^{-1}B \subseteq R$ or $B \subseteq cR$. Conversely, if $\exists$ an ideal $B \subseteq cR$, we can assume that B is the largest such B and hence invertible If $B \subseteq P \Rightarrow B \subseteq cP \Rightarrow BP^{-1} \subseteq cR \Rightarrow BP^{-1} \subseteq B$ which by maximality of $R \Rightarrow P^{-1} \subseteq R$ a contradiction. Thus, $B \not\subseteq P \Rightarrow c^{-1} \in A(P)$ and hence $R_P \subseteq A(P)$ The proof of the final assertion is clear. ∎

When R is a Noetherian RI-order, the importance of $A(P)$ stems from the fact that it always exists, is a localization in the sense of Silver, hence, a Noetherian RI-order, and that it has a unique maximal invertible ideal Of course, when R is bounded $A(P) = R_P$ and $A(P)$ is thus a classical localization as well.

LEMMA 4 3. *Let R be a Noetherian RI-order, P an invertible prime ideal and P' a prime ideal of R containing P but no other invertible prime ideal then $P'B = BP'$ for every invertible ideal $B \not\subseteq P$*

Proof. If B is an invertible ideal not contained in P then $B \not\subseteq P'$, since B is a product of prime invertibles none of which is P. Now $(BP'B^{-1})\, B \subseteq P'$ and since $B \not\subseteq P'$, $BP'B^{-1} \subseteq P'$ so $BP' \subseteq P'B$ and by symmetry $BP' = P'B$. ∎

COROLLARY 4 4. *If R is a Noetherian RI-order and P' is a prime ideal containing a unique invertible prime ideal P, then $P'A(P) = A(P)\, P'$.*

Proof. Since $A(P) = \bigcup B^{-1}$, where each B is an invertible ideal not contained in P, $P'B^{-1} = B^{-1}P'$ by the lemma so it follows that $P'A(P) = A(P)\, P'$. ∎

PROPOSITION 4.5. *Let R be a Noetherian RI-order and P an invertible prime ideal of R.*

(1) *If P' is a prime ideal of R containing P and no other invertible prime ideal, then $P'A(P) = A(P)P'$ is a prime ideal of $A(P)$.*

(2) *If T is a prime ideal of $A(P)$, then $T = T'A(P)$ for a unique prime $T' \supseteq P$. Moreover, $T' = T \cap R$ and T' contains no other invertible ideal of R.*

Proof. (1) Since $0 \to R/P' \to A(P) \otimes_R R/P'$ is exact by arguments similar to those in Corollary 3 14, the result now follows from Lemma 4.4 above and [20, Corollary 1.10]. In fact, this is the "local" version of Theorem 3.17

(2) "Localize" Theorem 3.17(2) ∎

THEOREM 4.6. *Let P' be a nonzero prime ideal of R. Then*

$$A(P') = \bigcap_{\substack{P \text{ invertible} \\ P \subseteq P'}} A(P),$$

Moreover,

$$R = \bigcap_{P \text{ invertible}} A(P).$$

Proof. Trivially, $A(P') = S$ iff P' contains no invertible primes In this case, the intersection on the right extends over the empty set yielding S as well. Thus, we can assume that P' contains an invertible prime and that

$$X = \Big(\bigcap_{P \subseteq P'} A(P) - A(P')\Big) \neq \varnothing .$$

Choose an invertible $B \supset P$ maximal with respect to the property that $\exists\ x \in X$ with $xB \subseteq R$ We claim that B is prime. Suppose this is not the case Then $B = B_1B_2$ for certain invertible B_i with $B \subsetneqq B_1$ and $B \subsetneqq B_2$. Clearly, $B \subseteq P'$ since $x \notin A(P')$. Moreover, $xB_1 \subseteq A(P')$. Otherwise, there would exist an $x' \in xB_1 \cap X$ with $x'B_2 \subseteq R$, contradicting the choice of B. Similarly, $xB_2 \subseteq A(P')$.

By definition of $A(P')$ there exists an invertible $C \not\subseteq P'$ with $xB_1C \subseteq R$. As above, $xC \not\subseteq A(P')$ is impossible but since $C^{-1} \subseteq A(P')$, $xC \subseteq A(P')$ is likewise impossible. Consequently, B is prime and since $B \subseteq P'$ necessarily one of the P's. However, since $xB = xP \subseteq R \Rightarrow x \in P^{-1} \cap A(P) = R$ by Proposition 4 2(2), a contradiction. Thus $X = \varnothing$ proving the first assertion.

To prove the second, suppose $x \in \bigcap A(P)$ Then for each P, $\exists\ B_P \not\subseteq P$ B_P invertible with $xB_P \subseteq R$ Set $B = \sum_P B_P$. Clearly, $xB \subseteq R$ and since B is not contained in any P, $B^* = R \Rightarrow x \in R$. ∎

5. Semi-Local Orders

An order R is called *semi-local* (*quasi-local*) if R is semisimple (simple) Artinian modulo its Jacobson radical A ring R is called (right) *p-connected* (after Bass) if projective (right) R-modules are generators (e.g., any quasi-local order). The goal of this section is to extend the results of Michler and Robson for semi-local hereditary Asano orders to global dimension 2, and at the same time, to generalize certain important results of Ramras [16] for classical two-dimensional orders over regular local rings. To avoid repetition and to simplify the statement of theorems, we shall assume throughout this section that an order R is a two-sided Noetherian semilocal, p-connected order with glb $R \leqslant 2$ (this is not an exercise in name calling!)

Theorem 5.1 (Fuller and Shutters [74]). *If R is a p-connected semi-local ring then there exists a primitive idempotent $e \in R$ such that every projective right (left) R-module is isomorphic to a direct sum of copies of $eR(Re)$.* ∎

Theorem 5 2 *Let R be an order Then for some $n > 0$, $R \approx M_n(k)$, where k is a right Ore domain. Moreover, projective right ideals of R are principal.*

Proof. Let U_R be a basic (uniform) right ideal of R Since $U = U^{**}$, U_R is necessarily projective and hence, a progenerator by p-connectivity. Thus, $k = \text{End } U_R$ is Morita equivalent to R and hence, semi-local and p-connected. Since k is a domain and Morita equivalent to R, projective k-modules are free by Theorem 5 1 Thus, ${}_kU \approx k^n$ for some $n > 0 \Rightarrow R \approx M_n(k)$ as claimed.

To show that projective right ideals of R are principal, it suffices to show that any projective k-submodule of k_k^n is generated by $\leqslant n$ elements. For, if P is a projective right ideal of R, $P \otimes_{k_n} k^n$ is a projective right k-submodule of k^n and $(P \otimes_{k_n} k^n) \otimes_k k^n \approx P$ in mod-k_n. If $P \otimes_{k_n} k^n$ is generated by $x_1, \ldots, x_m$, say where $m \leqslant n$, then viewing the x_i's as column vectors of length n, the $n \times n$ matrix having ith column x_i for $1 \leqslant i \leqslant m$ and remaining columns zero, say, generates $(P \otimes_{k_n} k^n) \otimes_k k^n$ as a right k_n-module. However, any projective submodule of k_k^n is free, say $\approx k^m$ and since k is an Ore domain, $m \leqslant n$. ∎

Corollary 5.3 (Robson). *Any semilocal hereditary Asano order is a principal right and left ideal ring*

Proof R is clearly p-connected. ∎

In order to continue our analysis of semi-local (maximal) orders, we shall examine the relationship between R and its equivalent orders.

Lemma 5.4. *Let R be an order and let $S \overset{\ell}{\sim} R$. Then $C = \{x \in R \mid Sx \subseteq R\}$ is projective and hence, a principal right ideal of R.*

Proof. Since glb $R \leqslant 2$, it suffices to show that C_R is reflexive Clearly, $S \subseteq C^*$ and hence $SC^{**} \subseteq C^*C^{**} \subseteq R \Rightarrow C^{**} \subseteq C \Rightarrow C$ is reflexive. ▮

THEOREM 5.5. *Let R be an order. If S is any maximal order equivalent to R, then S is Morita equivalent to R and hence R is maximal*

Proof Suppose $\alpha S\beta \subseteq R$ for units α, $\beta \in \underset{\sim}{Q}$. Then $S' = \alpha S\alpha^{-1}$ satisfies $S'\beta' \subseteq R$ for $\beta' = \alpha\beta$. Since S' is maximal and R is Morita equivalent to S if and only if it is Morita equivalent to S', we can assume that $S\alpha \subseteq R$ for α regular in R. Set $I = S\alpha R$. I is clearly an integral right R-ideal of R, and $S = O_\ell(I) \approx \text{End}\, I_R$ since $O_\ell(I) \sim R \sim S$, $O_\ell(I) \supseteq S$ and S is maximal. Similarly, $S = O_r(I^*) \approx \text{End}_R\, I^*$. Since R is p-connected and glb $R \leqslant 2$, ${}_RI^*$ is a progenerator $\Rightarrow S$ is Morita equivalent to R ▮

THEOREM 5 6. *Let R be a bounded order. Then if S is any maximal order equivalent to R, S is conjugate to R.*

Proof. By Theorem 5.5, S is Morita equivalent to $R \Rightarrow S$ is a bounded order as well. Thus, there exist regular elements $a \in R$ and $b \in S$ satisfying $Sa \subseteq R$ and $Rb \subseteq S$. Set $A = \{x \in R \mid Sx \subseteq R\}$ and $B = \{x \in S \mid Rx \subseteq S\}$. Then $A = sR$ and $B = tS$ for regular elements $s \in R$ and $t \in S$ by Lemma 5.4 Since $Ss \subseteq A = sR$ and $Rt \subseteq tS$, $R \subseteq tSt^{-1} \subseteq tsR\, s^{-1}t^{-1}$. Clearly, $R \sim ts\, R(ts)^{-1} \Rightarrow R = tSt^{-1}$ as claimed. ▮

Note added in proof It has been brought to the attention of the authors that there is some overlap between section 3 of this paper and that of Chamarie, M., Localisations dans les ordres maximaux, *Comm. in Algebra* **4** (1974), 279–293.

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