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A Nonabelian Normal Subgroup with a Core-Free Projective Image

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1. INTRODUCTION

In [5] Menegazzo proves the following beautiful theorem:

If π is a projectivity[1] *from a finite group G to a group G_1, $H \lhd G$ and*

(i) *G has odd order,*

(ii) *H^π is core-free in G_1,*

then H is abelian.

Hypothesis (ii) here is purely for notational convenience. For, denoting the preimage (under π) of the core of H^π in G_1 by N, then $N \lhd G$ (see Lemma 1.1). Thus π induces a projectivity from G/N to G_1/N^π and Menegazzo's theorem says that H/N is abelian provided G/N has odd order. On the other hand Menegazzo left open the question of whether hypothesis (i) is necessary. The purpose of this paper is to answer this question.

THEOREM A. *There are finite 2-groups G, G_1, a normal subgroup H of G and a projectivity $\pi: G \to G_1$ such that H^π is core-free in G_1 and H is not abelian.*

[1] That is, an isomorphism from the subgroup lattice of G to that of G_1.

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In the light of Menegazzo's theorem it is natural to ask if, in these situations, there is some bound (necessarily exceeding 1, by Theorem A) on the derived length of H. In fact this is the case and the derived length of H is always at most 3. This was proved by the first author [3] after the discovery of the counterexample of Theorem A. On the other hand we know of no example in which the derived length of H exceeds 2. Using recent work by Rips and Zacher [10], however, it is possible to show that the derived length of H is at most 3 *even when G is infinite* and this is the main theorem of [3].

The groups G and G_1, which we construct in order to prove Theorem A, have order 2^{13} and the normal subgroup H has order 2^7. Not surprisingly for groups of this order, it has not been easy to establish the existence of a projectivity π from G to G_1. Therefore it is natural to ask if there are smaller and less complicated examples, which would simplify the problem of finding π and proving that it *is* a projectivity. In fact we have been able to prove that there are no smaller examples. Theorems B and C are concerned with this fact.

Before stating Theorems B and C, we introduce some notation and recall a result about preimages (under projectivities) of cores. If H is a subgroup of a group G, write H_G for the core of H in G. If π is a projectivity from G to some group G_1, denote the subgroup $((H^\pi)_{G_1})^{\pi^{-1}}$ by $H_{\pi,G}$. The following Lemma was proved by Schmidt for finite groups [7] and generalised to infinite groups by Busetto [2].

LEMMA 1.1. *If $H \lhd G$, then $H_{\pi,G} \lhd G$.*

We can now state

THEOREM B. *Suppose that G and G_1 are groups, π: $G \to G_1$ is a projectivity and $H \lhd G$ with $H/H_{\pi,G}$ nonabelian. Then there is a subgroup X of G containing H such that X/H is cyclic and*

(i) *$X/H_{\pi,X}$ is a finite 2-group of order $\geqslant 2^{13}$,*

(ii) *$H/H_{\pi,X}$ is nonabelian of order $\geqslant 2^7$.*

Thus π induces a projectivity $X/H_{\pi,X} \to X^\pi/(H^\pi)_{X^\pi}$ and the nonabelian normal subgroup $H/H_{\pi,X}$ has core-free image.

The proof of this theorem quickly reduces to a consideration of finite 2-groups and will then follow from

THEOREM C. *Suppose that X and X_1 are finite 2-groups, π: $X \to X_1$ is a projectivity, $H \lhd X$ and X/H is cyclic. If H^π is core-free in X_1 and H is nonabelian, then* (i) $|X| \geqslant 2^{13}$, *and* (ii) $|H| \geqslant 2^7$.

To give a detailed proof of Theorem C would double the length of our paper. Therefore we content ourselves with a very brief summary of the main steps of the argument. (Details can be found in [4].) For any p-group G and any integer $r \geqslant 0$, we define as usual

$$\Omega_r(G) = \langle g \mid g \in G, g^{p^r} = 1 \rangle \quad \text{and} \quad \mho_r(G) = \langle g^{p^r} \mid g \in G \rangle.$$

Then the method of proof of Theorem C is as follows. One shows that $\Omega_1(H)$ is an indecomposable X-module. Also there is an element h in H such that $\Omega_1\langle h \rangle \lhd X$ and there is a subgroup Q of H such that

$$H = Q\langle h \rangle, \qquad Q \cap \langle h \rangle = 1.$$

Consider a minimal counterexample to the theorem. When H has exponent $\geqslant 2^4$, Q becomes a normal subgroup of H. Thus H is residually cyclic, i.e., abelian, contradicting the hypothesis. When H has exponent $\leqslant 2^3$, one obtains precise information about X and X_1, e.g., $Q \leqslant \Omega_2(X)$ and $|H'| = 2$. Also $\Omega_2(X)$ normalises $\langle h \rangle$ and $\Omega_2(X_1)$ normalises $\langle h \rangle^\pi$. However, while the kernel of the $\Omega_2(X)$-action on $\langle h \rangle$ is X-invariant, the kernel of the $\Omega_2(X_1)$-action on $\langle h \rangle^\pi$ is not X_1-invariant. This leads to the existence of a modular subgroup of X_1 whose preimage under π is not modular, which is clearly impossible. (A *modular* group is one whose subgroup lattice is modular.)

Deduction of Theorem B *from Theorem* C. Let G, G_1, π and H satisfy the hypotheses of Theorem B. By [5, Lemma 1],

$$(H^\pi)_{G_1} = \bigcap_{x \in S} (H^\pi)_{\langle H, x \rangle^\pi},$$

where $S = \{x \in G \mid |\langle x \rangle / (\langle x \rangle \cap H)| \text{ is a prime power or infinite}\}$. However, by [10, Corollario 1], if $\langle x \rangle$ is infinite and $\langle x \rangle \cap H = 1$, then $\langle x \rangle^\pi$ normalises H^π. Thus since $H/H_{\pi,G}$ is nonabelian and

$$H_{\pi,\langle H,x \rangle} \lhd \langle H, x \rangle, \tag{1}$$

by Lemma 1.1, there is an element x in G such that $|\langle x \rangle / (\langle x \rangle \cap H)|$ is a prime power and $H/H_{\pi,\langle H,x \rangle}$ is nonabelian. Let $X = \langle H, x \rangle$. Then we see from (1) that π induces a projectivity

$$X/H_{\pi,X} \to X^\pi/(H^\pi)_{X^\pi}.$$

We will show that $X/H_{\pi,X}$ is a finite 2-group of order at least 2^{13} and $H/H_{\pi,X}$ has order $\geqslant 2^7$. (Then $X^\pi/(H^\pi)_{X^\pi}$ will have the same order as $X/H_{\pi,X}$, by [9, Chap. 1, Theorem 12].)

Factoring by $H_{\pi,X}$ and $(H^\pi)_{X^\pi}$ in X and X^π, respectively, we may assume that $H_{\pi,X}=1$ and $(H^\pi)_{X^\pi}=1$. Now X/H is cyclic of prime power order p^n say, and clearly $n\geqslant 1$. Therefore $|X^\pi : H^\pi|$ is finite by [10, Theorem A]. Since H^π is core-free in X^π, it follows that X^π and hence X are finite. If $n=1$, then H is a maximal subgroup of X and hence H^π is a maximal subgroup of X^π. As the image of a normal subgroup of X, H^π is a Dedekind subgroup of X^π. (Dedekind subgroups are defined, for example, in [7] where they are called modular.) It follows from [6, Lemma 1] that X^π is nonabelian of order qr, where q and r are primes. This implies that H^π and hence H have prime order, contradicting the fact that H is not abelian.

Therefore $n\geqslant 2$ and the lattice of subgroups between H^π and X^π is a chain of length $\geqslant 2$. Then, by [6, Satz 1],

$$X^\pi \qquad \text{is a } q\text{-group},$$

for some prime q. Since X^π is not abelian, X is also a q-group and so $q=p$. Thus X and X^π are finite p-groups of the same order. By Menegazzo's theorem [5] we see that $p=2$. Since X/H is cyclic Theorem C shows that $|X|\geqslant 2^{13}$ and $|H|\geqslant 2^7$, as required. ∎

Sections 2–5 are devoted to the proof of Theorem A and with a brief summary of this proof we conclude our Introduction. Theorem B tells us that there is an example proving Theorem A with $G=H\langle a\rangle$, a finite 2-group. Also it is not difficult to show that $H\cap\langle a\rangle=1$ and that H cannot be a generalised quaternion group. Therefore we choose H such that $\Omega_1(H)$ has rank 2 and then it is possible to show that H must be metacyclic and modular. Theorem B also tells us that $|H|\geqslant 2^7$ and results obtained in proving Theorem C suggested that we take for H the group of order 2^7 presented by (2) in Section 2, and the element a of order 2^6. We define an action of a on H with $G=H\langle a\rangle$, again using our experience from Theorem C. To find a second group G_1 and a projectivity $\pi: G\to G_1$ such that $H_1=H^\pi$ is core-free in G_1, we were able to show that H_1 cannot be abelian or isomorphic to H. Therefore we define $H_1=\langle h_1, q_1 \mid h_1^{16}=q_1^8=1, h_1^{q_1}=h_1^5\rangle$ and form a product $G_1=H_1\langle a_1\rangle$, where $|a_1|=2^6$ and H_1 is core-free in G_1, again consistent with information obtained when proving Theorem C. Every projectivity between finite groups of the same order is induced by an element map. In Section 2 we define a bijection $\sigma: G\to G_1$ and in Section 3 we show that the image of σ restricted to each subgroup of $E=\langle H, a^2\rangle$ is a subgroup of $E_1=\langle H_1, a_1^2\rangle$. However, while Section 4 establishes the analogous result for all subgroups of G other than the cyclic ones outside E, it is easier for us to abandon element maps in order to handle these latter subgroups, where π is defined directly. The short Section 5 shows that π is surjective and a projectivity.

Baer's work [1] on projectivities from abelian groups is the starting point for our construction of π. The only other result on projectivities that we have been able to use is the following, due to Schmidt [7, Lemma 2.5].

LEMMA 1.2. *Let G be a group, Z and H subgroups of G with $Z \leqslant H$, and suppose that for every subgroup U of G either $U \leqslant H$ or $Z \leqslant U$. Let $\bar{Z}$ and $\bar{H}$ be subgroups of the group $\bar{G}$ with the same properties. If τ is a projectivity from H to $\bar{H}$ and σ is an isomorphism from the lattice of subgroups of G containing Z to the lattice of subgroups of $\bar{G}$ containing $\bar{Z}$ such that $U^\sigma = U^\tau$ for all subgroups between Z and H, then the map ρ defined by $U^\rho = U^\tau$ for $U \leqslant H$ and $U^\rho = U^\sigma$ for $U \nleqslant H$ is a projectivity from G to $\bar{G}$.*

2. THE GROUPS AND THE PROJECTIVITY OF THEOREM A

2.1. *Construction of the Groups*

We will construct a group G with a normal nonabelian subgroup H, a second group G_1 and a projectivity

$$\pi\colon G \to G_1$$

such that H^π is core-free in G_1. The groups G and G_1 will be finite of order 2^{13}, H will be metacyclic of order 2^7 and G/H will be cyclic.

Thus let

$$H = \langle h, q \mid h^{16} = q^8 = 1, h^q = h^9 \rangle, \tag{2}$$

a split extension of a cyclic group $\langle h \rangle$ of order 16 by a cyclic group $\langle q \rangle$ of order 8. Then

$$H' = \langle h^8 \rangle$$

has order 2. Also H has an automorphism α of order 8 defined by

$$h^\alpha = h^{-1}q^4, \qquad q^\alpha = h^2 q^{-1}.$$

Therefore there is a split extension G of H by a cyclic group $\langle a \rangle$ of order 64, presented as follows:

$$G = \langle a, h, q \mid a^{64} = h^{16} = q^8 = 1, h^q = h^9, h^a = h^{-1}q^4, q^a = h^2 q^{-1} \rangle. \tag{3}$$

This group G has order 2^{13}. The subgroup $\langle a^2, h, q\rangle$ (of order 2^{12}) has class 2 and hence all relations in this subgroup are easy consequences of

$$[h, q] = h^8, \tag{4}$$

$$[a^2, q] = h^4, \tag{5}$$

$$[a^2, h] = h^8. \tag{6}$$

The construction of G_1 proceeds as follows. Let elements b_1 and h_1 generate cyclic groups of order 16 and form their direct product

$$X_1 = \langle b_1\rangle \times \langle h_1\rangle.$$

The relation (6) shows that

$$X = \langle a^4, h\rangle = \langle a^4\rangle \times \langle h\rangle \cong X_1. \tag{7}$$

The subgroup $\langle b_1\rangle$ will be the image under π of $\langle a^4\rangle$; and $\langle h_1\rangle$ will be the image of $\langle h\rangle$ and X_1 the image of X.

The group X_1 has an automorphism β of order 4 defined by

$$b_1^\beta = b_1^{-3}h_1^8, \qquad h_1^\beta = h_1^5.$$

Thus there exists a split extension Y_1 of X_1 by a cyclic group $\langle q_1\rangle$ of order 8, presented by

$$Y_1 = \langle b_1, h_1, q_1 \mid b_1^{16} = h_1^{16} = q_1^8 = 1,\ h_1^{b_1} = h_1,\ b_1^{q_1} = b_1^{-3}h_1^8,\ h_1^{q_1} = h_1^5\rangle. \tag{8}$$

This group Y_1 has order 2^{11}. The subgroup $\langle q_1\rangle$ will be the image of $\langle q\rangle$ under π.

We make one final extension of Y_1 by a cyclic group of order 4. First, we define a map γ on the generators of Y_1. Let

$$b_1^\gamma = b_1, \quad h_1^\gamma = b_1^{-1}h_1^7q_1^4, \quad q_1^\gamma = h_1^{-2}q_1^{-1}. \tag{9}$$

From the presentation of Y_1 and elementary commutator identities we see that $Y_1' = \langle h_1^4, b_1^4\rangle$ and Y_1 has class 2. Then it is easy to check that γ preserves the relations of Y_1 and extends to an automorphism; moreover

$$\gamma^4 \quad \text{is conjugation by } b_1. \tag{10}$$

By the cyclic extension theorem (see, e.g., [8, p. 250]), there is a group $G_1 = Y_1\langle a_1\rangle$, where $Y_1 \lhd G_1$, G_1/Y_1 is cyclic of order 4 and $a_1^4 = b_1$. This group is presented as follows:

$$G_1 = \langle a_1, h_1, q_1 \mid a_1^{64} = h_1^{16} = q_1^8 = 1, \quad h_1^{a_1^4} = h_1, \quad a_1^{4q_1} = a_1^{-12}h_1^8,$$
$$h_1^{q_1} = h_1^5,\ h_1^{a_1} = a_1^{-4}h_1^7q_1^4,\ q_1^{a_1} = h_1^{-2}q_1^{-1}\rangle. \tag{11}$$

(Here we have used (8), (9), and (10).) The order of G_1 is 2^{13}, i.e., the same as the order of G. The cyclic subgroup $\langle a_1 \rangle$ will be the image of $\langle a \rangle$ under π. We note that

$$a^8 \text{ and } h^8 \text{ lie in the center of } G \tag{12}$$

and a_1^{16} lies in the centre of G_1.

Let

$$H_1 = \langle h_1, q_1 \rangle = \langle h_1 \rangle] \langle q_1 \rangle.$$

Here $\langle h_1 \rangle$ has order 16 and $\langle q_1 \rangle$ has order 8. This subgroup H_1 will be the image of $H(\lhd G)$ under π and it is easy to see that

$$H_1 \qquad \text{is core-free in } G_1.$$

For,

$$\Omega_1(H_1) = \langle h_1^8, q_1^4 \rangle = W,$$

say, and $W \cap W^{a_1} \cap W^{a_1^2} = 1$.

2.2. *Definition of* π

First we define an element map

$$\sigma\colon G \to G_1. \tag{13}$$

Every element of G can be written uniquely in the form

$$a^k h^j q^i, \tag{14}$$

where

$$0 \leqslant k \leqslant 63, \qquad 0 \leqslant j \leqslant 15, \qquad 0 \leqslant i \leqslant 7.$$

Similarly every element of G_1 can be written in the form

$$a_1^k h_1^j q_1^i, \tag{15}$$

where k, j, i are integers uniquely determined modulo 64, 16, 8, respectively. Writing the elements of G in the form (14), the map (13) is defined by

$$(a^k h^j q^i)^\sigma = a_1^{k'} h_1^{j'} q_1^{i'}, \tag{16}$$

where

$$k' = k(1+4i), \tag{17}$$

$$j' = j(1+4i), \tag{18}$$

$$i' = \begin{cases} i+2 & \text{if } i \text{ is odd,} \\ i+4jk & \text{if } i \text{ is even.} \end{cases} \tag{19}$$

It is routine to check that σ is a bijection.

Remark 1. Replacing k, j, i by congruent integers modulo 64, 16, 8, respectively, does not change the element (14). Also the right-hand sides of (18) and (19) will be unchanged modulo 16, 8, respectively, and therefore they can be used as the exponents of h_1 and q_1 in (16). However, the right-hand side of (17) will be invariant only modulo 32 and so it can be used as the exponent of a_1 in (16) only when k is even.

Remark 2. The term $4jk$ in the definition of i' should be viewed as a small adjustment to what will shortly emerge as a natural map to consider in order to attempt to construct π.

We are now ready to define π. It is easy to see that the elements (14) with k even form a subgroup E of index 2 in G. Similarly the elements (15) with k even form a subgroup E_1 of index 2 in G_1.

Every cyclic subgroup $\langle a^{k'}h^{j'}q^{i'}\rangle$, with k' odd, is generated by an element of the form ah^jq^i. If K is a subgroup of E or a noncyclic subgroup of G, define

$$K^\pi = K^\sigma.$$

$$\text{Otherwise } K = \langle ah^jq^i\rangle \text{ and we define } K^\pi = \langle (ah^jq^i)^\sigma\rangle. \tag{20}$$

(We have not checked to see if we can define $K^\pi = K^\sigma$ for all K, because such a calculation would be too tedious.)

3. Consideration of $\pi|_E$

3.1. *Cyclic Subgroups*

Let $B = \langle a^8, h^2, q\rangle$, $B_1 = \langle a_1^8, h_1^2, q_1\rangle$ ($\leqslant Y_1$). It is clear from (17), (18), and (19) that σ restricts to a bijection from B to B_1. The subgroup B is abelian and homogeneous of exponent 8 with basis $\{a^8, h^2, q\}$. The subgroup B_1 is the split extension of $\langle a_1^8, h_1^2\rangle = \langle a_1^8\rangle \times \langle h_1^2\rangle$ (homogeneous of exponent 8) by $\langle q_1\rangle \cong C_8$, where q_1 conjugates the elements of $\langle a_1^8, h_1^2\rangle$ to their 5th powers, as we easily see from (11). In particular B_1 is a modular group and it is a well-known fact that B and B_1 have isomorphic subgroup lattices. In [1] Baer shows how to construct a bijection from B to B_1 inducing a projectivity. It is not difficult to check that our map σ is Baer's map. However, while σ has its origins in the work of Baer, it is not necessary to check our claim here, because we will prove that $\sigma|_E$ induces a projectivity from E to E_1, and therefore (by restriction) a projectivity from B to B_1.

We show first that

$$\sigma \text{ maps cyclic subgroups of } E \text{ to cyclic subgroups of } E_1. \tag{21}$$

Therefore we need formulas for powers of elements of E and E_1. As we have already pointed out (before (4)) E has class 2. Then for any elements u, v of E, $(uv)^n = u^n v^n [v, u]^{n(n-1)/2}$. So it is easy to check that

$$(a^{2k} h^j q^i)^l = a^{2k_1} h^{j_1} q^{i_1}, \tag{22}$$

where

$$\begin{aligned} k_1 &\equiv kl \bmod 32, \\ j_1 &\equiv \{j + 2[i(2j-k) + 2jk](l-1)\}\, l \bmod 16, \\ i_1 &\equiv il \bmod 8. \end{aligned} \tag{23}$$

The corresponding formula for powers of the element $x_1 = a_1^{2k} h_1^j q_1^i$, which is only a little harder to obtain, is given by

$$x_1^m = a_1^{2k_0} h_1^{j_0} q_1^{i_0}$$

where

$$\begin{aligned} 2k_0 &\equiv 2km[1 + 2i(m-1)] \begin{cases} \bmod 32 & \text{if } k \text{ is odd}, \\ \bmod 64 & \text{if } k \text{ is even}, \end{cases} \\ j_0 &\equiv m\{j - 2[i(k+j) + 2jk](m-1)\} \bmod 16, \\ i_0 &\equiv im \bmod 8. \end{aligned} \tag{24}$$

Let $x = a^{2k} h^j q^i$. Then (21) will follow from

$$\langle x \rangle^\sigma = \langle x^\sigma \rangle. \tag{25}$$

The proof of (25) is direct when k is even. When k is odd, it is easier to show first that

$$\langle x \rangle^\sigma \leqslant \langle x^\sigma \rangle \langle a_1^{32} \rangle. \tag{26}$$

However, in this case the exponent of a_1 in x^σ has the form $2k'$, where k' is odd (by (17)). Then using the fact that a_1^{32} lies in the centre of G_1, it is easy to see from (24) that $a_1^{32} = (x^\sigma)^{16} \in \langle x^\sigma \rangle$. Thus (26) will imply $\langle x \rangle^\sigma \leqslant \langle x^\sigma \rangle$. Since x and x^σ both have order 32, (25) will then follow.

Let l be an integer. To prove (25) we show that there is an integer m such that

$$(x^l)^\sigma = (x^\sigma)^m \text{ (modulo } \langle a_1^{32} \rangle \text{ if } k \text{ is odd).}$$

By (22), $x^l = a^{2k_1} h^{j_1} q^{i_1}$, where k_1, j_1, i_1 satisfy (23). Recalling Remark 1 (after (19)), the form (15) for $(x^l)^\sigma$ has

$$a_1 \text{ exponent} = 2k_1(1+4i_1), \tag{27}$$

$$h_1 \text{ exponent} = j_1(1+4i_1) \tag{28}$$

and

$$q_1 \text{ exponent} = \begin{cases} i_1 + 2 & \text{if } i_1 \text{ is odd,} \\ i_1 & \text{if } i_1 \text{ is even,} \end{cases} \tag{29}$$

(from (17), (18), and (19)). Now write $x^\sigma = a_1^{2k_2} h_1^{j_2} q_1^{i_2}$. Then by (24) the form (15) for $(x^\sigma)^m$ (for any integer m) has

$$a_1 \text{ exponent} \equiv 2k_2 m[1 + 2i_2(m-1)] \begin{cases} \text{mod } 32 & \text{if } k_2 \text{ is odd,} \\ \text{mod } 64 & \text{if } k_2 \text{ is even,} \end{cases} \tag{30}$$

$$h_1 \text{ exponent} = m\{j_2 - 2[i_2(k_2 + j_2) + 2j_2 k_2](m-1)\}, \tag{31}$$

and

$$q_1 \text{ exponent} = i_2 m. \tag{32}$$

By (17), (18), (19) we have

$$k_2 = k(1+4i),$$

$$j_2 = j(1+4i),$$

$$i_2 = \begin{cases} i+2 & \text{if } i \text{ is odd,} \\ i & \text{if } i \text{ is even.} \end{cases}$$

Then it is easy to check that the three equations obtained by equating (27), (28), (29), respectively, with (30), (31), (32) have the following solution for m:–

when i and l are odd, $m = 3l - 2 + 2(l-1)(i+1)$;

when i is odd and l is even, $m = 5l + 2l(l+i)$;

when i is even, $m = l + 2il(l-1)$.

This establishes (25) and hence (21).

3.2. *Arbitrary Subgroups*

We show now that σ maps every subgroup of E to a *subgroup* of E_1. The following two resuls will achieve this. Write $N = \langle a^2, h \rangle$.

LEMMA 3.2.1. *If U is a subgroup of N and V is a subgroup of E, then $(UV)^\sigma = U^\sigma V^\sigma$.*

Proof. Let $u \in U$, $v \in V$. Then $u = a^{2k}h^j$ (by (6)) and $v = a^{2k_1}h^{j_1}q^{i_1}$. Again using (6) we have

$$uv = a^{2k+2k_1}h^{j+8jk_1+j_1}q^{i_1}$$

and hence

$$(uv)^\sigma = a_1^{(2k+2k_1)(1+4i_1)}h_1^{(j+8jk_1+j_1)(1+4i_1)}q_1^m,$$

where $m = i_1 + 2$ if i_1 is odd and $m = i_1$ if i_1 is even. From (11)

$$h_1^{a_1^2} = a_1^{32}h_1^9. \tag{33}$$

The fact that $\langle a_1^2, h_1 \rangle$ has class 2 then gives

$$[h_1^{(j+8jk_1)(1+4i_1)}, a_1^{2k_1(1+4i_1)}] = (a_1^{32}h_1^8)^{jk_1}$$

and so

$$(uv)^\sigma = (a_1^{2k(1+4i_1)}h_1^{j(1+4i_1)})(a_1^{2k_1(1+4i_1)+32jk_1}h_1^{j_1(1+4i_1)}q_1^m).$$

Thus if j, k_1 are not both odd, then $(uv)^\sigma = (u^{1+4i})^\sigma v^\sigma$. On the other hand if k_1 is odd, then $v^{16} = a^{32}$ by (22) and (23). If also j is odd, then $a_1^{32jk_1} = a_1^{32}$. Moreover for any element of G,

$$(a^{32}g)^\sigma = a_1^{32}g^\sigma \tag{34}$$

(by definition of σ). Hence in this case $(uv)^\sigma = (u^{1+4i})^\sigma(v^{17})^\sigma$. Therefore we obtain $(UV)^\sigma = U^\sigma V^\sigma$. ∎

Now let $N_1 = \langle a_1^2, h_1 \rangle$. Then we have

LEMMA 3.2.2. *σ induces a projectivity from N to N_1.*

Proof. From the definition of σ, it is clear that σ restricts to a bijection from N to N_1. We apply Lemma 1.2 to N and N_1 (with $\langle a^{32} \rangle$, $X = \langle a^4, h \rangle$ for Z, H, respectively, and $\langle a_1^{32} \rangle$, $X_1 = \langle a_1^4, h_1 \rangle$ for $\bar{Z}$, $\bar{H}$, respectively). By (7), $X \cong X_1$ and $\sigma\colon a^{4k}h^j \mapsto a_1^{4k}h_1^j$ defines an isomorphism $X \to X_1$. Thus, in particular, σ induces a projectivity from X to X_1. Similarly $N/\langle a^{32}\rangle \cong N_1/\langle a_1^{32}\rangle$ (by (6) and (33)) and

$$\sigma\colon \langle a^{32}\rangle\, a^{2k}h^j \mapsto \langle a_1^{32}\rangle\, a_1^{2k}h_1^j$$

defines such an isomorphism (by (34)). Suppose that $U \leqslant N$ and $U \nleqslant X$. Then (22) and (23) show that $\langle a^{32}\rangle \leqslant U$; and similarly if $U_1 \leqslant N_1$ and $U_1 \nleqslant X_1$, then from (24), $\langle a_1^{32}\rangle \leqslant U_1$. Thus Lemma 1.2 shows that σ induces a projectivity $N \to N_1$. ∎

Now let K be a subgroup of E. By (4) and (5), $N \lhd E$ and $E = N\langle q\rangle$. So $K = UV$ where $U = K \cap N$ and V is cyclic. By Lemma 3.2.1 $K^\sigma = U^\sigma V^\sigma$, and by Lemma 3.2.2 U^σ is a subgroup of E_1. Also V^σ is a subgroup of E_1, by (21). Again by (21), $(K^\sigma)^{-1} = K^\sigma$. Therefore

$$U^\sigma V^\sigma = K^\sigma = (K^\sigma)^{-1} = (V^\sigma)^{-1}(U^\sigma)^{-1} = V^\sigma U^\sigma$$

and it follows that K^σ is a subgroup of E_1. We have now shown that

σ and hence π, by (20), map each subgroup of E to a subgroup of E_1.

4. Consideration of π Applied to Subgroups Outside E

Let $x = a^k h^j q^i$ *where k is odd.* Then $x \notin E$, but $|G : E| = 2$ and so $x^2 \in E$. From Section 3 we know that $\langle x^2\rangle^\sigma$ is a subgroup of G_1. We will prove next that

$$\langle x^2\rangle^\sigma = \langle (x^\sigma)^2\rangle. \tag{35}$$

For this purpose it suffices to show that

(i) $|x| = |x^\sigma|$ and

(ii) $(x^\sigma)^2 \in \langle x^2\rangle^\sigma$.

To prove (i) we find from the presentation (3) that

$$x^{16} = a^{16k} \tag{36}$$

and hence $|x| = |a| = 64$, since k is odd. Also x^σ has a_1-exponent (in (16)) odd. Therefore consider an element of G_1 of the form $x_1 = a_1^\gamma h_1^\beta q_1^\alpha$, where γ is odd. From the presentation (11) of G_1 we obtain

$$x_1^{16} = a_1^{16\gamma + 32\beta}$$

and therefore $|x_1| = |a_1| = 64$. This establishes (i).

To prove (ii) we can work modulo $\langle a_1^{16} \rangle$, since $a_1^{16} \in \langle x^2 \rangle^\sigma$, by (36). It is necessary therefore to find an integer solution for λ of the congruence

$$((x^2)^\sigma)^\lambda \equiv (x^\sigma)^2 \bmod \langle a_1^{16} \rangle.$$

When i is odd we find that $\lambda = -1 - 2k(j+1)$ is a solution; and when i is even, $\lambda = 1 - 2kj$ is a solution. Thus (ii) and hence (35) follow.

Suppose that K is a *noncyclic* subgroup of G with $K \nleq E$. We will show that

$$K^\sigma \quad \text{is a subgroup of } G_1. \tag{37}$$

Clearly K contains an element of the form $x = a^k h^j q^i$, where k is odd. We claim that

$$F = \langle h^8, a^8 \rangle \leqslant K. \tag{38}$$

For, since G/H is cyclic and K is noncyclic, $K \cap H \neq 1$. Thus if $h^8 \notin K$, then $K \cap H$ contains an element of the form $h^{8j_1} q^4$ in $\Omega_1(H)$. But then K contains

$$[h^{8j_1} q^4, x] = [h^{8j_1} q^4, a^k h^j q^i] = [h^{8j_1} q^4, a^k] = [q^4, a^k] = h^8,$$

giving a contradiction. Therefore $h^8 \in K$. Also K contains $x^8 = a^{8k} h^{8i}$, and so $a^8 \in K$. Then (38) follows.

Now let $F_1 = \langle a_1^8, h_1^8 \rangle$. So $F_1 = F^\sigma$. Also, for all $x \in G$,

$$(Fx)^\sigma = F_1 x^\sigma. \tag{39}$$

For, let $f \in F$. Then $(fx)^\sigma \equiv f^\sigma x^\sigma \bmod \langle a_1^{32} \rangle$ and so $(fx)^\sigma \in F_1 x^\sigma$. Thus, by order considerations, (39) follows. By (12), F lies in the centre of G, and from the presentation of G_1, we see that $F_1 \lhd G_1$. Recall that K is a noncyclic subgroup of G and that $K \nleq E$. In order to prove (37) we distinguish three cases.

Case 1. *K/F is cyclic.* Then $K = \langle F, x \rangle$, where $x = a^k h^j q^i$ and k is odd. It suffices to show that

$$(Fx^{2r+1})^\sigma = F_1 (x^{2r})^\sigma x^\sigma, \tag{40}$$

for any integer r. For, recalling (35), $\langle x^2 \rangle^\sigma = \langle (x^\sigma)^2 \rangle$. Also any generator of $\langle x \rangle$ can be written as x^{2r+1}. Hence if (40) holds, then

$$(x^{2r+1})^\sigma \, F_1 (x^{2r})^\sigma \, x^\sigma \subseteq F_1 \langle x^\sigma \rangle.$$

Thus

$$\langle x \rangle^\sigma \subseteq F_1 \langle x^\sigma \rangle. \tag{41}$$

Therefore

$$K^\sigma = (F\langle x \rangle)^\sigma \subseteq F_1\langle x^\sigma \rangle$$

by (39) and (41). But by (35) and (39)

$$(F\langle x^2 \rangle)^\sigma = F_1\langle (x^\sigma)^2 \rangle.$$

Since $F\langle x^2 \rangle$ has index 2 in $F\langle x \rangle$ and $F_1\langle (x^\sigma)^2 \rangle$ has index 2 in $F_1\langle x^\sigma \rangle$, order considerations show that

$$K^\sigma = (F\langle x \rangle)^\sigma = F_1\langle x^\sigma \rangle.$$

Thus K^σ is a subgroup of G_1.

To prove (40), we find

$$x^2 \equiv a^{2k} h^{2ki} q^{4j} \bmod F.$$

Since the factors on the right-hand side of this congruence commute (as is easily seen from the presentation (3) of G), it follows that

$$x^{2r} \equiv a^{2kr} h^{2kir} q^{4jr} \bmod F.$$

Then (again by (3))

$$x^{2r+1} \equiv a^{2kr+k} h^{-2kir+j} q^{4jr+i} \bmod F.$$

Therefore

$$(x^{2r+1})^\sigma \equiv a_1^{k(2r+1)(1+4i)} h_1^{(-2kir+j)(1+4i)} q_1^{i_1} \bmod F_1,$$

where $i_1 = 4jr + i + 2$ if i is odd and $i_1 = 4jr + i + 4j$ if i is even. It follows that

$$\begin{aligned}(x^{2r+1})^\sigma &\equiv a_1^{2kr+k(1+4i)} h_1^{-2kir+j(1+4i)} q_1^{i_1} \bmod F_1 \\ &\equiv a_1^{2kr} h_1^{2kir} q_1^{4jr} a_1^{k(1+4i)} h_1^{j(1+4i)} q_1^{i_1-4jr} \bmod F_1 \\ &\equiv (x^{2r})^\sigma x^\sigma \bmod F_1.\end{aligned}$$

We have now proved (40) and hence Case 1 is complete.

Case 2. $K \cap H \leqslant \langle h^2, q^2 \rangle$. Let $v = a^{k_1} h^{j_1} q^{i_1}$, $w = h^{2j_2} q^{2i_2}$ be elements of G. Since h_1^2 and q_1 commute modulo F_1, we see that

$$(vw)^\sigma \equiv v^\sigma w^\sigma \bmod F_1. \tag{42}$$

Now $K/(K \cap H) \cong KH/H$ and therefore $K/(K \cap H)$ is cyclic and

$$K = V(K \cap H),$$

where V is cyclic. Thus from (42) it follows that

$$K^\sigma \equiv V^\sigma(K \cap H)^\sigma \bmod F_1;$$

i.e., $F_1 K^\sigma = F_1 V^\sigma(K \cap H)^\sigma$ and so, by (39),

$$K^\sigma = (F_1 V^\sigma)(K \cap H)^\sigma. \tag{43}$$

Applying case 1 to FV, we see that $(FV)^\sigma$ is a subgroup of G_1. Also (39) shows that $(FV)^\sigma = F_1 V^\sigma$; and from Section 3.2 we know that $(K \cap H)^\sigma$ is a subgroup of G_1. Now, by Section 3.1 and case 1, K^σ contains all powers, in particular the inverse, of each of its elements. Therefore from (43)

$$K^\sigma = (K^\sigma)^{-1} = (K \cap H)^\sigma(F_1 V^\sigma)$$

and hence K^σ is a subgroup of G_1.

Case 3. $K \cap H \nleqslant \langle h^2, q^2 \rangle$. We claim that

$$\langle a^4, h^4, q^4 \rangle \leqslant K. \tag{44}$$

For, since $K \cap H \nleqslant \langle h^2, q^2 \rangle$, K contains an element

$$u = h^{j_1} q^{i_1},$$

where at least one of j_1, i_1 is odd. Also, since $K \nleqslant E$, K contains an element

$$x = ah^j q^i.$$

Then from (3)

$$x^2 \equiv a^2 h^{2i} q^{4j} \bmod F. \tag{45}$$

Suppose that i_1 is odd. Without loss of generality we may assume that $i_1 = 1$. Thus K contains $[u, x^2]$; and modulo F

$$\begin{aligned}[u, x^2] \equiv [h^{j_1} q, a^2] &\equiv [h^{j_1}, a^2][q, a^2] \\ &\equiv [q, a^2] \qquad \text{(by (6))} \\ &\equiv h^4 \qquad \text{(by (5))}.\end{aligned}$$

Since $F \leqslant K$, it follows that $h^4 \in K$. Therefore

$$q^4 \in \langle u^4, h^4 \rangle \leqslant K.$$

Now suppose that i_1 is even. Then j_1 is odd and we may even assume that $j_1 = 1$. Hence $h^4 = u^4 \in K$. Also

$$[u, x] = [hq^{i_1}, ah^j q^i] = [h, ah^j q^i]^{q^{i_1}} [q^{i_1}, ah^j q^i].$$

Thus modulo F

$$[u, x] \equiv [h, a][q, a]^{i_1} \equiv h^{-2}q^4(h^2q^{-2})^{i_1} \quad \text{(from (3))}$$
$$\equiv h^{-2+2i_1}q^{4-2i_1}.$$

Therefore $h^{-2}q^{4-2i_1} \in K$. Then K contains

$$u^2h^{-2}q^{4-2i_1} = q^4.$$

It follows that, for all i_1, $\langle h^4, q^4 \rangle \leqslant K$. Now K contains x^4 and, by (45), $x^4 = a^4h^{4i}$. Thus $a^4 \in K$ and (44) follows.

Let $J = \langle a^4, h^4, q^4 \rangle$. Then $J \lhd G$. For, from (3) we see that $\langle h^4, q^4 \rangle \lhd G$. Also, from (5) and (6), a^4 is central in G modulo $\langle h^4, q^4 \rangle$. Similarly $J_1 = J^\sigma \lhd G_1$. For, from (8) it follows that

$$\mho_2(Y_1) = \langle a_1^{16}, h_1^4, q_1^4 \rangle \lhd G_1;$$

and, modulo $\mho_2(Y_1)$, a_1^4 is central in G_1.

Let $g \in G$. Then

$$(Jg)^\sigma = J_1 g^\sigma. \tag{46}$$

To see this, let $y \in J$. Using (39) we obtain $(yg)^\sigma \equiv g^\sigma \bmod J_1$. Therefore

$$(Jg)^\sigma = \bigcup_{y \in J} (yg)^\sigma \subseteq J_1 g^\sigma$$

and (46) follows.

The groups G/J and G_1/J_1 are isomorphic via the map induced by $a \mapsto a_1$, $h \mapsto h_1$, $q \mapsto q_1$ and σ induces this isomorphism. Therefore if $g_1, g_2 \in K$, then

$$g_1^\sigma g_2^\sigma \in J_1 K^\sigma = (JK)^\sigma \qquad \text{(by (46))}$$
$$= K^\sigma$$

by (44). Thus K^σ is a subgroup of G_1.

We have finally proved (37), i.e., for every noncyclic subgroup K of G with $K \nleqslant E$, K^σ is a subgroup of G_1.

5. Surjectivity of π

We now know that π, defined by (20), maps each subgroup of G to a subgroup of G_1 of the same order. Let U and V be subgroups of G with $U < V$. Then

$$U^\pi < V^\pi. \tag{47}$$

For, by (23) and (36), E has exponent 32 and G has exponent 64. Thus suppose that U is cyclic of order 64, generated by $u = ah^j q^i$. Then V is non-cyclic and so $V^\pi = V^\sigma$. But $u^\sigma \in V^\sigma$ and so $\langle u^\sigma \rangle \leqslant V^\sigma$, i.e., $U^\pi < V^\pi$.

Now suppose that V is cyclic of order 64, generated by $v = ah^j q^i$. Then $U \leqslant E \cap \langle v^2 \rangle$ and so

$$U^\pi = U^\sigma \leqslant \langle v^2 \rangle^\sigma = \langle (v^\sigma)^2 \rangle \qquad \text{(by (35))},$$
$$< \langle v^\sigma \rangle = V^\pi.$$

Finally suppose that neither U nor V is cyclic of order 64. Then $U^\pi = U^\sigma < V^\sigma = V^\pi$. We have now proved (47).

To prove that π is a projectivity from G to G_1, it is sufficient now to show that each subgroup of G_1 occurs in the image of π. This we achieve via the following result.

LEMMA 5.1. *Let G, G_1 be finite 2-groups. Suppose that π is a map from the subgroup lattice of G into the subgroup lattice of G_1 such that $U \leqslant V$ if and only if $U^\pi \leqslant V^\pi$ and*

(i) $|U| = |U^\pi|$, *all* $U \leqslant G$,

(ii) U^π *is cyclic whenever* U *is cyclic,*

(iii) $G^\pi = G_1$.

Then π is a projectivity from G to G_1.

Proof. Suppose that the lemma is false. Choose $K_1 \leqslant G_1$ with $|K_1|$ minimal subject to

(a) K_1 has no preimage under π and

(b) there is a subgroup $N \leqslant G$ with $N^\pi > K_1$ and $|N^\pi : K_1| = 2$.

This choice is possible by (iii). Also N is not cyclic, by (i) and (ii). Therefore there exist maximal subgroups $M_1 \neq M_2$ of N. Let $M = M_1 \cap M_2$. Then $|N : M| = 4$ and so $|N^\pi : M^\pi| = 4$, by (i). Since $M \lhd N$ and $M^\pi \lhd N^\pi$ and N/M, N^π/M^π are elementary of order 4, it follows that $K_1 \ngeqslant M^\pi$. Let $L_1 = M^\pi \cap K_1$. Then $L_1 < M^\pi$ and $L_1 \lhd N^\pi$ with N^π/L_1 elementary of order 8. Now $|M^\pi : L_1| = 2$ and therefore, by choice of K_1, there is a subgroup $L \leqslant G$ such that $L^\pi = L_1$.

We claim that

$$\text{there is an element } t \in N \text{ such that } t^2 \notin L. \qquad (48)$$

For, if not, $\mho_1(N) \leqslant L$ and then $L \lhd N$. Since $|N : L| = 8$, N/L is then elementary of order 8. Thus K_1 would have a preimage under π. Then (48) follows.

Let $T = \langle L, t \rangle$. If $T = N$, then $N = \langle M, t \rangle$ and N/M is cyclic, which is not the case. Therefore $T < N$ and $|T : L| = 4$, by (48). Thus $|T^\pi : L_1| = 4$, by (i). Now we see that

there is a unique subgroup strictly between T and L.

For, if there were two such subgroups, they would be normal in T and L would be their intersection, showing that T/L is elementary of order 4. But T/L is cyclic by definition.

Now $T^\pi/L_1 \leqslant N^\pi/L_1$ and so T^π/L_1 is elementary of order 4. Therefore there are three subgroups strictly between T^π and L_1 (all of index 4 in N^π) and there is only one subgroup strictly between T and L, contradicting our choice of K_1. ∎

Returning to the conclusion of the proof of Theorem A, we see that all the hypotheses of Lemma 5.1 are satisfied by our groups G and G_1 and the map π, defined in (3), (11), and (20). Therefore we have finally shown that $\pi: G \to G_1$ is a projectivity, $H \lhd G$, H is not abelian, and H^π is core-free in G_1. This completes the proof of Theorem A. ∎

Remark. Lemma 5.1 does not hold for finite p-groups when p is odd. For, let G be the nonabelian group of order p^3 and exponent p and let G_1 be the elementary abelian p-group of rank 3. It is not difficult to define a map π, from the lattice of subgroups of G to the lattice of subgroups of G_1, which is not a projectivity but which satisfies the hypotheses of Lemma 5.1.

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