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Symmetry of Positive Solutions of Semilinear Elliptic Equations in Infinite Strip Domains

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In this article, we apply the improved "moving plane" method to prove the symmetry of the solutions of the Dirichlet problem $-\Delta u + u = g(u)$ in infinite strip domains with zero boundary condition.

1. INTRODUCTION

Let $N = m + n$, $m \geqslant 2$, $n \geqslant 1$, $\boldsymbol{\omega}$ be a smooth bounded domain in $\mathbb{R}^m$, $\mathbf{A} = \boldsymbol{\omega} \times \mathbb{R}^n$ an infinite strip domain in $\mathbb{R}^N$, and the function g be under some suitable assumptions. In this article, we consider the following semilinear elliptic equation

$$\begin{cases} -\Delta u + u = g(u) & \text{in } \mathbf{A}, \\ u > 0 & \text{in } \mathbf{A}, \\ u = 0 & \text{on } \partial\mathbf{A}, \\ \lim_{|y| \to \infty} u(x, y) = 0 & \text{uniformly in } \quad x \in \boldsymbol{\omega}. \end{cases} \tag{1}$$

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It is well known that there is a solution of Eq. (1) in the whole space $\mathbb{R}^N$. Moreover, in an elegant paper [1], Gidas *et al.* proved that any solution of the same equation in $\mathbb{R}^N$ is radially symmetric with respect to a certain point in $\mathbb{R}^N$. Later, Kwong [4] proved that the solution of Eq. (1) in $\mathbb{R}^N$ is unique. The uniqueness of the solution of Eq. (1) in $\mathbb{R}^N$ is useful: it implies the existence of the solution of the same equation in $\mathbb{R}^N \backslash D$, where D is a bounded domain in $\mathbb{R}^N$.

Similarly, Lien *et al.* [7, Theorem 4.8] proved that there is a solution of Eq. (1) in the infinite strip domain **A**. However, the symmetric and the unique properties of a solution of the same equation in **A** are unknown. If it is unique, then we can prove the existence of the solution of the same equation in $\mathbf{A}\backslash E$, where E is a bounded domain in **A**. See Hsu and Wang [3] for a related result.

In this article, we apply the improved "moving plane" method given by Li [5] and Li and Ni [6] to prove that any solution of Eq. (1) in **S** is symmetric as follows. Let

$$\mathbf{S} = \{(x, t) \in \mathbf{B}^{N-1}(R) \times \mathbb{R} \mid x = (x_1, ..., x_{N-1}) \in \mathbf{B}^{N-1}(R), t \in \mathbb{R}\},$$

where $\omega = \mathbf{B}^{N-1}(R)$ is a ball with center at the origin and of radius R in $\mathbb{R}^{N-1}$. Then u is radially symmetric in x and axially symmetric in t. We also establish, in Section 2, the asymptotic behavior of positive solutions of Eq. (1) in domain **A**. Related results were also studied by Lopes [8].

2. ASYMPTOTIC BEHAVIOR IN A

Let λ_1 be the first eigenvalue and ϕ_1 the corresponding first positive eigenfunction of the Dirichlet problem $-\Delta\phi_1 = \lambda_1\phi_1$ in $\boldsymbol{\omega}$, $\phi_1 = 0$ on $\partial\boldsymbol{\omega}$.

(P1) $g(u) > 0$ as $u > 0$,

(P2) $g(u) = O(u^p)$ as $u \to 0$ for some $p > 1$.

PROPOSITION 1. *Suppose g satisfies* (P1) *and* (P2). *Let u be a solution of Eq.* (1). *Then for any* $0 < \delta < 1 + \lambda_1$ *there exist* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\alpha\phi_1(x)\, e^{-\sqrt{1+\lambda_1+\delta}\,|y|} \leqslant u(z) \leqslant \beta\phi_1(x)\, e^{-\sqrt{1+\lambda_1-\delta}\,|y|}, \qquad \textit{for} \quad z = (x, y) \in \mathbf{A}.$$

Proof. (1) Let $z_0 \in \partial\mathbf{A}$ and B be a small ball in **A** such that $z_0 \in \partial B$. Let

$$w_\delta(z) = \phi_1(x)\, e^{-\sqrt{1+\lambda_1+\delta}\,|y|} \qquad \text{for} \quad z = (x, y) \in \mathbf{A}.$$

Since $w_\delta(z) > 0$, $u(z) > 0$ for $z \in B$, $w_\delta(z_0) = 0$, $u(z_0) = 0$, by the Hopf boundary point lemma (see Gilbarg and Trudinger [2]), $(\partial w_\delta/\partial v)(z_0) < 0$, $(\partial u/\partial v)(z_0) < 0$, where v is the outward unit normal vector at z_0. Thus

$$\lim_{\substack{z \in \mathbf{A} \\ z \to z_0 \\ \text{normally}}} \frac{u(z)}{w_\delta(z)} = \frac{(\partial u/\partial v)(z_0)}{(\partial w_\delta/\partial v)(z_0)} > 0.$$

Note that

$$\frac{u(z)}{w_\delta(z)} > 0 \qquad \text{for} \quad z = (x, y) \in \mathbf{A}.$$

Thus

$$\frac{u(z)}{w_\delta(z)} > 0 \qquad \text{for} \quad z = (x, y) \in \bar{\mathbf{A}}.$$

For $0 < \delta < 1 + \lambda_1$, take $R > 0$ such that $\delta - (\sqrt{1 + \lambda_1 + \delta}\,(n-1)/|y|) \geqslant 0$ for $|y| \geqslant R$. Since $w_\delta(z)$ and $u(z)$ are in $C^1(\bar{\mathbf{A}})$, if we set

$$\alpha = \inf_{\substack{z \in \bar{\mathbf{A}} \\ |y| \leqslant R}} \frac{u(z)}{w_\delta(z)},$$

and $w(z) = \alpha w_\delta(z)$ for $z \in \bar{\mathbf{A}}$, then $\alpha > 0$ and

$$w(z) \leqslant u(z) \qquad \text{for} \quad z \in \bar{\mathbf{A}}, \quad |y| \leqslant R.$$

For $z \in \bar{\mathbf{A}}$, $|y| \geqslant R$, we have

$$\begin{aligned} \Delta(w-u)(z) - (w-u)(z) &= (\Delta w(z) - w(z)) + (-\Delta u(z) + u(z)) \\ &= w(z)\left(\delta - \frac{\sqrt{1+\lambda_1+\delta}\,(n-1)}{|y|}\right) + g(u) \geqslant 0. \end{aligned}$$

The maximum principle implies that $w - u \leqslant 0$ in $z \in \mathbf{A}$, $|y| \geqslant R$, and therefore

$$w(z) \leqslant u(z) \qquad \text{for} \quad z \in \mathbf{A}.$$

(2) For $0 < \delta < 1 + \lambda_1$, take $R' > 0$ such that $g(u) \leqslant (\delta/2) u$ for $|y| \geqslant R'$. Let

$$w_{-\delta}(z) = \phi_1(x)\, e^{-\sqrt{1+\lambda_1-\delta}\,|y|} \qquad \text{for} \quad z = (x, y) \in \mathbf{A},$$

$$\frac{1}{\beta} = \inf_{\substack{z \in \bar{\mathbf{A}} \\ |y| \leqslant R'}} \frac{w_{-\delta}(z)}{u(z)},$$

$$v(z) = \beta w_{-\delta}(z) \qquad \text{for} \quad z \in \bar{\mathbf{A}}.$$

For $z \in \mathbf{A}$, $|y| \geqslant R'$ we have

$$\begin{aligned} -\Delta(u-v)(z) + (u-v)(z) &= (-\Delta u(z) + u(z)) + (\Delta v(z) - v(z)) \\ &= g(u(z)) + \left(-\delta - \frac{\sqrt{1+\lambda_1-\delta}\,(n-1)}{|y|} \right) v(z) \\ &\leqslant \frac{\delta}{2}\,(u-v)(z), \end{aligned}$$

therefore

$$-\Delta(u-v)(z) + \left(1 - \frac{\delta}{2}\right)(u-v)(z) \leqslant 0.$$

As in part (1), we obtain that

$$u(z) \leqslant v(z) \qquad \text{for} \quad z \in \mathbf{A}. \quad \blacksquare$$

3. SYMMETRY OF THE SOLUTIONS

Let $\mathbf{S} = \{(x, t) \in \mathbf{B}^{N-1}(R) \times \mathbb{R} \mid x = (x_1, ..., x_{N-1}) \in \mathbf{B}^{N-1}(R),\ t \in \mathbb{R}\}$. Now we consider the following equation:

$$\begin{cases} -\Delta u + u = g(u) & \text{in } \mathbf{S}, \\ u > 0 & \text{in } \mathbf{S}, \\ u = 0 & \text{on } \partial \mathbf{S}, \\ \lim_{|t| \to \infty} u(x, t) = 0 & \text{uniformly in} \quad x \in \mathbf{B}^{-1}(R). \end{cases} \tag{2}$$

We apply the "moving plane" method to prove the symmetry of the solutions of Eq. (2).

THEOREM 2. *Assume that $g \in C^1$ satisfies* (P1) *and* (P2). *Let $u(x, t)$ be a C^2 solution of Eq.* (2). *Then u is radially symmetric in x and axially symmetric in t; that is to say, $u(x, t-\sigma) = u(|x|, |t-\sigma|)$ for some σ.*

Part I. u is axially symmetric with respect to some hyperplane $t=\sigma$.

Notations.

$S_\theta=\{(x,t)\in\mathbf{S} \mid x\in B^{N-1}(R),\ t=\theta\}$;
$\Gamma_\theta=\{(x,t)\in\mathbf{S} \mid x\in B^{N-1}(R),\ t<\theta\}$;

For any $(x,t)\in\mathbf{S}$, set $(x,t^\theta)=(x,2\theta-t)$; that is to say, (x,t^θ) is the reflection of (x,t) with respect to S_θ;

Let Θ be the collection of all $\theta\in\mathbb{R}$ such that the following statements hold:

$$\begin{cases} u(x,t)<u(x,t^\theta) & \text{for all } \quad (x,t)\in\Gamma_\theta,\\ u_t(x,t)>0 & \text{on } \mathbf{S}\cap S_\theta. \end{cases}$$

LEMMA 3. *There exists* $\theta_0>0$, *such that either* $(-\infty,-\theta_0]\subset\Theta$ *or* $u(x,t)\equiv u(x,t^{-\theta_0})$ *in* $\Gamma_{-\theta_0}$.

Proof. Given $\theta\in\mathbb{R}$, set $w^\theta(x,t)=u(x,t)-u(x,t^\theta)$ for $(x,t)\in\Gamma_\theta$, and $w^\theta(x,t)$ satisfies

$$\Delta w^\theta(x,t)+c_\theta(x,t)\,w^\theta(x,t)=0, \tag{3}$$

where $c_\theta(x,t)=(g(u(x,t))-g(u(x,t^\theta)))/(u(x,t)-u(x,t^\theta))-1=g'(\xi_\theta)-1$ where ξ_θ is in between $u(x,t)$ and $u(x,t^\theta)$.

Claim that there exists $\theta_0>0$ such that if $\theta\leqslant-\theta_0$, then $w^\theta(x,t)\leqslant 0$ in Γ_θ.

Otherwise, suppose $w^\theta(x,t)>0$ for some $(x,t)\in\Gamma_\theta$. Since $\lim_{t\to-\infty} w^\theta(x,t)=0$ uniformly in x, $w^\theta(x,t)$ achieves its maximum at $(x_\theta,t_\theta)\in\Gamma_\theta$. Then

$$\nabla w^\theta(x_\theta,t_\theta)=0, \qquad \{w^\theta_{ij}(x_\theta,t_\theta)\}\leqslant 0.$$

Note that by (P2), $\lim_{t\to 0^+} g'(t)=0$. Take $t_0>0$ such that if $0<t\leqslant t_0$, then $g'(t)<1$. Choose $\theta_0>0$ such that if $t\leqslant-\theta_0$, $u(x,t)\leqslant t_0$ uniformly in x. For $\theta\leqslant-\theta_0$, $(x_\theta,t_\theta)\in\Gamma_\theta$, then

$$\Delta w^\theta(x_\theta,t_\theta)\leqslant 0, \qquad (g'(\xi_\theta)-1)\,w^\theta(x_\theta,t_\theta)=c_\theta(x,t)\,w^\theta(x_\theta,t_\theta)<0,$$

contradicting Eq. (3). As a consequence of the maximum principle and the Hopf boundary point lemma, either $w^{-\theta_0}(x,t)\equiv 0$ in $\Gamma_{-\theta_0}$ or for $\theta\leqslant-\theta_0$, $w^\theta(x,t)<0$ in Γ_θ and $w^\theta_t(x,t)>0$ for $(x,t)\in\mathbf{S}\cap S_\theta$, or $u_t(x,t)>0$ for $(x,t)\in\mathbf{S}\cap S_\theta$.

LEMMA 4. *If* $(-\infty,\theta]\subset\Theta$, *then there exists* $\varepsilon>0$ *such that* $[\theta,\theta+\varepsilon)\subset\Theta$.

Proof. Suppose not. There exists a decreasing sequence $\theta_k \to \theta$ and a sequence $\{(x_k, t_k)\}$ of points in Γ_{θ_k} such that $w^{\theta_k}(x_k, t_k) = u(x_k, t_k) - u(x_k, t_k^{\theta_k}) > 0$. There is a subsequence $\{(x_k, t_k)\}$ such that $x_k \to \bar{x}$ as $k \to \infty$. There may arise two possibilities as shown in Cases 1 and 2:

Case 1. $t_k \to -\infty$. As shown in Lemma 3, we assume

$$\begin{aligned} w^{\theta_k}(x_k, t_k) &= \max_{(x,t) \in \bar{\Gamma}_{\theta_k}} w^{\theta_k}(x, t), \\ \nabla w^{\theta_k}(x_k, t_k) = 0, &\qquad \{w^{\theta_k}_{ij}(x_k, t_k)\} \leqslant 0. \end{aligned} \tag{4}$$

From $\lim_{t_k \to -\infty} u(x_k, t_k) = 0$, as in Lemma 3, we obtain a contradiction.

Case 2. $t_k \to \bar{t}$. We have $(x_k, t_k) \to (\bar{x}, \bar{t}) \in \overline{\Gamma_\theta}$, thus $w^\theta(\bar{x}, \bar{t}) \geqslant 0$. Clearly $(\bar{x}, \bar{t}) \notin \Gamma_\theta$ since $w^\theta(x, t) < 0$ in Γ_θ. If $(\bar{x}, \bar{t}) \in S_\theta$, then $u_t(\bar{x}, \bar{t}) < 0$, which contradicts $\theta \in \Theta$. Moreover, $(\bar{x}, \bar{t}) \notin \partial \mathbf{S} \cap \overline{\Gamma_\theta}$. Note that $w^\theta(x, t)$ satisfies Eq. (3), and by the Hopf boundary point lemma, we obtain $(\partial/\partial v)\, w^\theta(\bar{x}, \bar{t}) < 0$. On the other hand, taking the limit in (4), we obtain $\nabla w^\theta(\bar{x}, \bar{t}) = 0$, a contradiction. We conclude that Case 2 is impossible. ∎

Proof of Part I. Let $\sigma = \sup\{\theta \in \mathbb{R} \mid (-\infty, \theta) \subset \Theta\}$. Then $\sigma \notin \Theta$. If not, by Lemma 4 we would have $[\sigma, \sigma + \varepsilon) \subset \Theta$, which contradicts the definition of σ. By continuity we have $u(x, t) \leqslant u(x, t^\sigma)$ for all $(x, t) \in \Gamma_\sigma$. Then by the maximum principle we have $u(x, t) \equiv u(x, t^\sigma)$ for all $(x, t) \in \Gamma_\sigma$. This proves $u(x, t)$ is symmetric with respect to the hyperplane $t = \sigma$ for all $(x, t) \in \mathbf{S}$. ∎

Part II. u is radially symmetric in $\mathbf{B}^{N-1}(R)$.

Notations.

$T_\lambda = \{(x, t) = (x_1, x_2, ..., x_{N-1}, t) \in \mathbf{S} \mid x_1 = \lambda\}$;

$\Sigma_\lambda = \mathbf{S} \cap \{(x, t) \mid x_1 < \lambda\}$;

For any $(x, t) = (x_1, x_2, ..., x_{N-1}, t) \in \mathbf{S}$, set $(x^\lambda, t) = (2\lambda - x_1, ..., x_{N-1}, t)$; that is to say, (x^λ, t) is the reflection of (x, t) with respect to T_λ;

Let Λ be the collection of all $\lambda \in (-R, 0)$ such that the following statements hold:

$$\begin{cases} u(x, t) < u(x^\lambda, t) & \text{for all} \quad (x, t) \in \Sigma_\lambda, \\ u_{x_1}(x, t) > 0 & \text{on } \mathbf{S} \cap T_\lambda. \end{cases}$$

LEMMA 5. *For some* $0 < \delta < R$, $(-R, -R + \delta) \subset \Lambda$.

Proof. Given $\lambda \in (-R, 0)$, set $v^\lambda(x, t) = u(x, t) - u(x^\lambda, t)$ for $(x, t) \in \Sigma_\lambda$, then $v^\lambda(x, t) = 0$ for $(x, t) \in \mathbf{S} \cap T_\lambda$, and $v^\lambda(x, t)$ satisfies

$$\Delta v^\lambda(x, t) + c_\lambda(x, t)\, v^\lambda(x, t) = 0, \tag{5}$$

where $c_\lambda(x, t) = (g(u(x, t)) - g(u(x^\lambda, t)))/(u(x, t) - u(x^\lambda, t)) - 1 = g'(\zeta_\lambda) - 1$ where ζ_λ is in between $u(x, t)$ and $u(x^\lambda, t)$.

Note that by (P2), $\lim_{t \to 0^+} g'(t) = 0$. Take $t_0 > 0$ such that if $0 < t \leqslant t_0$, then $g'(t) < 1$. Since $\lim_{|x| \to R} u(x, t) = 0$, we can choose δ, $R > \delta > 0$ such that if $R - \delta < |x| < R$, $u(x, t) \leqslant t_0$ uniformly in t.

Claim that if $-R < \lambda < -R + \delta$, then $v^\lambda(x, t) \leqslant 0$ in Σ_λ.

Otherwise, suppose there exists λ such that $-R < \lambda < -R + \delta$, $v^\lambda(x, t) > 0$ for some $(x, t) \in \Sigma_\lambda$. Since $\lim_{|t| \to \infty} v^\lambda(x, t) = 0$ uniformly in x, $v^\lambda(x, t)$ achieves its maximum at $(x_\lambda, t_\lambda) \in \Sigma_\lambda$. Then

$$\nabla v^\lambda(x_\lambda, t_\lambda) = 0, \qquad \{v^\lambda_{ij}(x_\lambda, t_\lambda)\} \leqslant 0.$$

But

$$\Delta v^\lambda(x_\lambda, t_\lambda) \leqslant 0, \qquad (g'(\zeta_\lambda) - 1)\, v^\lambda(x_\lambda, t_\lambda) = c_\lambda(x, t)\, v^\lambda(x_\lambda, t_\lambda) < 0,$$

which contradicts Eq. (5). So for $-R < \lambda < -R + \delta$, $v^\lambda(x, t) \leqslant 0$ in Σ_λ. Applying the maximum principle and the Hopf bounbary point lemma, for $-R < \lambda < -R + \delta$, we get $v^\lambda(x, t) < 0$ in Σ_λ and $v^\lambda_{x_1}(x, t) > 0$ for $(x, t) \in \mathbf{S} \cap T_\lambda$. Hence $u_{x_1}(x, t) > 0$ for $(x, t) \in \mathbf{S} \cap T_\lambda$. Then $(-R, -R + \delta) \subset \Lambda$.

LEMMA 6. *If* $(-R, \lambda] \subset \Lambda$, *then there exists* $\tau > 0$ *such that* $[\lambda, \lambda + \tau) \subset \Lambda$.

Proof. Suppose not. There exists a decreasing sequence $\lambda_k \to \lambda$ and a sequence $\{(x_k, t_k)\}$ of points in Σ_{λ_k} such that $v^{\lambda_k}(x_k, t_k) = u(x_k, t_k) - u(x_k^{\lambda_k}, t_k) > 0$. There is a subsequence $\{(x_k, t_k)\}$ such that $x_k \to \bar{x} \in \overline{\mathbf{B}^{N-1}(R)}$. There may arise two possibilities as shown in Cases 1 and 2:

Case 1. $|t_k| \to \infty$. As shown in Lemma 5, we assume

$$v^{\lambda_k}(x_k, t_k) = \max_{(x, t) \in \overline{\Sigma_{\lambda k}}} v^{\lambda_k}(x, t),$$

$$\nabla v^{\lambda_k}(x_k, t_k) = 0, \qquad \{v^{\lambda_k}_{ij}(x_k, t_k)\} \leqslant 0.$$

From $\lim_{|t_k| \to \infty} u(x_k, t_k) = 0$, as in Lemma 5, we obtain a contradiction.

Case 2. $t_k \to \bar{t}$. We have $(x_k, t_k) \to (\bar{x}, \bar{t}) \in \overline{\Sigma_\lambda}$. Thus $v^\lambda(\bar{x}, \bar{t}) \geqslant 0$. Clearly $(\bar{x}, \bar{t}) \notin \Sigma_\lambda$ since $v^\lambda(x, t) < 0$ in Σ_λ. If $(\bar{x}, \bar{t}) \in T_\lambda$ then $u_{x_1}(\bar{x}, \bar{t}) < 0$, which contradicts $\lambda \in \Lambda$. Moreover, $(\bar{x}, \bar{t}) \notin \partial\mathbf{S} \cap \overline{\Sigma_\lambda}$ since if $(\bar{x}, \bar{t}) \in \partial\mathbf{S} \cap \overline{\Sigma_\lambda}$ then $0 = u(\bar{x}, \bar{t}) \geqslant u(\bar{x}^\lambda, \bar{t}) > 0$, a contraction. We conclude that Case 2 is impossible. ∎

Proof of Part II. Let $\mu = \sup\{\lambda \in (-R, 0) \mid (-R, \lambda) \subset \Lambda\}$. Then $\mu \notin \Lambda$. If not, by Lemma 6 we would have $[\mu, \mu + \varepsilon) \subset \Lambda$, which contradicts the definition of μ. We claim that $\mu = 0$. Suppose not; $\mu \in (-R, 0)$. By continuity we have $u(x, t) \leqslant u(x^{\mu}, t)$ for all $(x, t) \in \Sigma_{\mu}$. Then by the maximum principle we have $u(x, t) \equiv u(x^{\mu}, t)$ for all $(x, t) \in \Sigma_{\mu}$, which is impossible. Thus $\mu = 0$. By reversing the x_1 axis, we conclude that $u(x, t)$ is symmetric with respect to the hyperplane T_0 and $u_{x_1}(x, t) < 0$ for $x_1 > 0$. Since the x_1 direction can be chosen arbitrarily, we conclude that $u(x, t)$ is radially symmetric in $\mathbf{B}^{N-1}(R)$. ∎

Remark 1. In the ordinary differential equation case, Theorem 2 will admit more important properties: Let u be a C^2 solution of Eq. (2) in $\mathbb{R}$. Then it is obvious that u is not only symmetric with respect to a certain point in $\mathbb{R}$ but also unique up to translations.

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