A simple state-space design of an interactor for a non-square system via system matrix pencil approach

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Received 14 February 2001; accepted 18 December 2001

Submitted by P. Van Dooren

Abstract

The construction of an interactor cancelling the infinite zeros of a non-square proper transfer matrix is discussed in this paper along the line of approach of B.R. Copeland, M.G. Safonov [Int. J. Robust Nonlinear Control 2 (1992) 139]. In this paper, some new properties of the infinite eigenstructure of the system matrix pencil of the transfer matrix are presented. A simple state-space design of the interactor is proposed by using these properties. Then, some new features about state-space relations among the original system, the interactor, and the compensated system are presented. Also, several conditions about the invariance of the stabilizability or controllability between the state-space realizations of the original system and the compensated system are shown. Furthermore, the proposed method of designing the interactor in this paper can be applied to a tall or fat transfer matrix with full normal rank. © 2002 Elsevier Science Inc. All rights reserved.

Keywords: Interactor; System matrix pencil; Infinite eigenstructure; Non-square system; State space design

1. Introduction

An interactor and zeros at infinity of multivariable systems are the generalized concepts of a relative degree of a scalar transfer function. The interactor which
cancels the infinite zeros of a proper transfer matrix plays an important role in adaptive control [8], the factorization of transfer matrices with infinite zeros [2,17], singular LQ regulation [10], singular $H_\infty$ control [3,18], and sliding mode control [9,14], to name a few.

It is known that a real proper transfer matrix $G(s)$ has no infinite zeros if and only if $G(\infty)$ has full rank, see [2]. Therefore, the interactor studied here is defined as follows.

**Definition 1.** Square polynomial matrix $Z(s)$ is said to be an interactor of a real proper transfer matrix $G(s)$, if

$$\hat{G}(s) := G(s)Z(s)$$  \hspace{1cm} (1)

is proper and $\hat{G}(\infty)$ has full rank (either column or row).

Such definition has been used in [1,7,10,17]. Note that the above definition is different from [15] where the additional requirement of the lower left triangular structure of $Z(s)$ is needed. The reason for omitting the lower triangular structure is that it does not necessarily need to impose such structure on the interactor for solving many problems, see e.g. the aforementioned papers.

In what follows, we shall give a brief review of some methods for constructing the interactor. For a linear multivariable system, [11] presents a recursive algorithm for the calculation of a nilpotent interactor whose zeros are all zero. The proposed algorithm operates on the coefficients of the numerator of the right matrix fraction (RMF) description of the system. Such nilpotent interactor has been used to solve singular LQ regulation problem in [10] and singular filtering problem in [1] both for discrete-time systems. However, rather than the nilpotent interactor, for continuous-time system we usually need stable interactor whose zeros are in the open left half plane. For square systems, by using Moore–Penrose pseudoinverse, [6] proposes a very simple way to compute the spectral interactor matrix which has all its zeros at the origin and has the all-pass property in discrete-time.

It is known that the interactor can be constructed by using Silverman inversion algorithm [12]. The difficulty, as pointed out by Copeland and Safonov [2], is that it requires recursive operations on the Markov parameters of $G(s)$, which does not generally lend itself to numerically stable computations. Therefore, for a tall transfer matrix $G(s)$ with full normal column rank, [2] utilizes the relationship between the infinite zero structure of $G(s)$ in [13] and the infinite eigenstructure of the associated Rosenbrock system matrix to develop a direct, numerically reliable method for the computation of the infinite zero compensator (IZC), which has the same meaning of the interactor defined in this paper. Also, [2] shows that the IZC can be designed by calculating the infinite eigenstructure of the system matrix via a numerically stable algorithm in [4]. By showing that all the infinite zeros of $G(s)$ are unobservable in the cascade realization of $\hat{G}(s)$ in (1) with $Z(s)$ having no infinite zeros, [2] proves that $\hat{G}(\infty)$ is injective.
Along the line of the approach of [2], this paper studies the state-space design of the interactor for a non-square system. Though the algorithm for constructing the interactor is similar to the Algorithm 1 in [2] for constructing IZC, we achieve the following in comparison with [2]. First, by exploring the results of [2,4] expressed in Lemma 1 (termed as Van Dooren canonical form) and Lemma 2 in this paper, respectively, we obtain some new properties of the infinite eigenstructure of the system matrix of $G(s)$. The structures of the Van Dooren canonical form of the system matrix pencil and the transformation matrices associated with the form are depicted further, see Theorems 1 and 2 in this paper. Second, we show directly that a matrix which is block part of the transformation matrix associated with the Van Dooren canonical form is injective. Furthermore, such matrix is shown to be $\hat{G}(\infty)$ which is obtained by using the proposed interactor. In this way, we show alternatively that $\hat{G}(\infty)$ has full rank. Third, this paper presents several new features about state-space relations among the original system $G(s)$, the interactor $Z(s)$, and the compensated system $\hat{G}(s)$. Especially, we provide some conditions about the invariance of the stabilizability or controllability between the state-space realizations of $G(s)$ and $\hat{G}(s)$, see Theorems 4 and 5. Fourth, the design of the interactor for a fat transfer matrix with full normal row rank, which has not been treated in [2], is provided in this paper. Therefore, the designed interactor in this paper can be used to treat singular control problems without dimension constraint.

On the other hand, it has been shown that the interactor just serves as a tool for derivation in filling the gaps between the singular and nonsingular of both $(J, J')$-lossless factorization problems [16] and inner–outer factorization problems [17]. In fact, from final formulae of these two factorizations, we only need the calculation of $B_0, \hat{D}$ and $V$ in Theorem 1, and we do not need to construct the interactor actually. Therefore, the simpler the formulae of the interactor is, the simpler the derivation related to the application is. In this aspect, the interactor in a simple form in this paper is easier for use.

Hence, the result related to the construction of the interactor matrix proposed in this paper may be considered to be somewhat extension and complementary to [2]. And, the interactor and the properties of the infinite eigenstructure presented in this paper are useful for analyzing and numerically computing singular control problems with infinite zeros.

The paper is organized as follows: Kronecker theory of a singular pencil and the result of [4] are introduced in Section 2. The state-space designs of the interactor for a tall transfer matrix and a fat one are presented in Sections 3 and 4, respectively. Two examples are given in Section 5. Conclusion is made in Section 6.

In this paper, $\mathbb{C}$ denotes open complex plane. The set of all $m \times r$ constant real matrices is denoted by $\mathbb{R}^{m \times r}$. $I_r$ denotes the identity matrix of size $r \times r$. $0_{m \times p}$ denotes the zero matrix of size $m \times p$. The subscripts can be dropped, if these dimensions are clear from the context. $\lambda(A)$ and $\sigma(A)$ denote an eigenvalue and the set of all eigenvalues of a matrix $A$, respectively. $|A|$ denotes the determinant of $A$. 
Im $A$ and Ker $A$ denote the image space and null space of $A$, respectively. $A^T$ means the transpose of $A$. Real matrix $A$ is called unitary if $A^T A = AA^T = I$. We denote $C(sE - A)^{-1} B + D := \begin{bmatrix} -sE + A & B \\ C & D \end{bmatrix}, \quad E \neq I.

If $E = I$, the term $-sE + A$ in the above notation is replaced by $A$ for simplicity.

2. Preliminaries

In this section, we shall briefly review the result of Van Dooren [4].

From a numerical point of view, the computation of Kronecker canonical form is not recommended because of the possible bad conditioning of the transformation matrices [4]. The following lemma named as Van Dooren canonical form, which is obtained from Proposition 4.7 in [4], is given.

Lemma 1 [4]. For pencil $-sM + W$, there exist unitary transformation matrices $S_m$ and $T_w$ such that

$$S_m^T(-sM + W)T_w = \begin{bmatrix} -sM_\eta + W_\eta & 0 & 0 & 0 \\ * & -sM_f + W_f & 0 & 0 \\ * & * & -sM_\infty + W_\infty & 0 \\ * & * & * & -sM_\epsilon + W_\epsilon \end{bmatrix}, \quad (2)$$

where (i) the singular pencils $-sM_\eta + W_\eta$ and $-sE_\epsilon + A_\epsilon$ contain the Kronecker row and column structures of $-sM + W$, respectively; (ii) the regular pencils $-sM_f + W_f$ and $-sM_\infty + W_\infty$ contain all the finite and infinite elementary divisors of $-sM + W$, respectively.

3. Design of the interactor for a tall matrix

We shall study the design of the interactor of $G(s)$ with its state-space realization

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad A \in \mathbb{R}^{n \times n}, \quad D \in \mathbb{R}^{m \times p}. \quad (3)$$

Here, we assume that $G(s)$ satisfies that

A1 $G(s)$ has full normal column rank $p$.

Next in Section 4, we consider the case when $G(s)$ has full normal row rank $m$. Note that in both cases, according to Definition 1, $G(s)$ is post-multiplied by interactor $Z(s)$, and the full normal rank assumption is necessary for $G(s)$ to have an interactor.
Let the system matrix pencil of $G(s)$ in (3) be $-sPE + PA$, where
\[
P_E := \begin{bmatrix} I_n & 0 \\ 0 & 0_{m \times p} \end{bmatrix}, \quad P_A := \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]

Owing to assumption A1, $-sPE + PA$ has full normal column rank. Therefore, we can apply Lemma 1 to $-sPE + PA$.

Now we recall the result of in [2, Lemma 5].

**Lemma 2** [2]. There exist unitary transformations $S, T$ such that
\[
S^T (-sPE + PA) T = \begin{bmatrix} -sE_{\eta f} + A_{\eta f} & 0 \\ * & -sE_{\infty} + A_{\infty} \end{bmatrix},
\]
where the regular pencil $-sE_{\infty} + A_{\infty}$ contains all the infinite elementary divisors of $-sPE + PA$.

It is indicated in [2] that the proof of the above lemma is constructive and follows directly from [4, Algorithm 4.1]. Also, from [2, Lemma 6], $T$ in (5) has the following structure:
\[
T = \begin{bmatrix} T_{11\infty} & T_{12\infty1} & 0_{n \times p} \\ T_{21\infty} & T_{22\infty1} & T_{22\infty2} \end{bmatrix},
\]
where $T_{22\infty2} \in \mathbb{R}^{p \times p}$ is nonsingular, and the rest matrices in (6) are with compatible sizes.

By applying Lemma 1 and using the structure of $PE$ in (4), we obtain the following theorem which extends the above mentioned results of [2]. Its proof is omitted for brevity.

**Theorem 1.** The following properties hold for the quantities defined as in Lemma 2. $E_{\eta f}$ has full column rank. Also, $T$ and $E_{\infty}$ have the following structures:
\[
T = \begin{bmatrix} T_r & 0 \\ 0 & I_p \end{bmatrix}, \quad T_r \in \mathbb{R}^{n \times n},
\]
\[
E_{\infty} = \begin{bmatrix} E_1 & 0_{n_{\infty} \times p} \end{bmatrix}, \quad E_1 \in \mathbb{R}^{n_{\infty} \times (n_{\infty} - p)},
\]
respectively, $E_1$ has full column rank where $n_{\infty}$ is the size of square matrix $E_{\infty}$.

Comparing (6) and (7), we know that
\[
T_{22\infty2} = I_p, \quad T_{21\infty} = 0, \quad T_{22\infty1} = 0.
\]

With the structures of $T$ in (7) and $E_{\infty}$ in (8) and the fact that $E_{\eta f}$ has full column rank, we can obtain one of the main results of this paper, which provides necessary material for constructing the interactor in a simple form. To express such result, we need the following quantities.

Since $E_1$ in (8) has full column rank, the singular value decomposition of $E_1$ can be given as
\[
U_1^T E_1 W_1 = \begin{bmatrix}
\Sigma \\
0_{p \times (n-\infty-p)}
\end{bmatrix},
\]
(10)
where \( U_1 \in \mathbb{R}^{n \times n} \) and \( W_1 \in \mathbb{R}^{(n-\infty-p) \times (n-\infty-p)} \) are unitary matrices, and \( \Sigma \in \mathbb{R}^{(n-\infty-p) \times (n-\infty-p)} \) is a diagonal positive definite matrix. Constructing \( U_\infty \in \mathbb{R}^{n \times n} \) as
\[
U_\infty := \begin{bmatrix}
W_1 \Sigma^{-1} & 0 \\
0 & I_p
\end{bmatrix} U_1^T
\]
(11)
yields
\[
U_\infty E_\infty = \begin{bmatrix}
I_{(n-\infty-p)} & 0 \\
0 & 0_{p \times p}
\end{bmatrix}.
\]
(12)
In accordance with (5), decompose \( S \) and \( T \) defined as in Theorem 1 as follows:
\[
S =: \begin{bmatrix}
S_{\eta f} & S_\infty
\end{bmatrix},
T =: \begin{bmatrix}
T_{\eta f} & T_\infty
\end{bmatrix}.
\]
(13)
Moreover, decompose \( S_\infty U_\infty^{-1} \) and \( U_\infty A_\infty \) in accordance with (12) as follows:
\[
S_\infty U_\infty^{-1} := \begin{bmatrix}
V \\
V_1
\end{bmatrix} \begin{bmatrix}
B_0 \\
\hat{D}
\end{bmatrix},
V \in \mathbb{R}^{n \times (n-\infty-p)},
\hat{D} \in \mathbb{R}^{m \times p},
\]
(14)
\[
U_\infty A_\infty =: \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
A_{11} \in \mathbb{R}^{(n-\infty-p) \times (n-\infty-p)},
A_{22} \in \mathbb{R}^{p \times p}.
\]
(15)
Here, since \( U_1 \) and \( W_1 \) in (10) are unitary, it follows from (11) that
\[
U_\infty^{-1} = U_1 \begin{bmatrix}
\Sigma W_1^T & 0 \\
0 & I_p
\end{bmatrix}.
\]
(16)
Now we have:

**Theorem 2.** Suppose that \( G(s) \) in (3) satisfies the assumption \( A1 \). With the quantities as defined in (10)–(16), then the following relations about the infinite eigenstructure of \(-sP_E + P_A\) defined as in (4) hold: \( V_1 \) in (14) is a zero matrix, and
\[
\begin{bmatrix}
-sI + A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
V & 0 \\
0 & I_p
\end{bmatrix} = \begin{bmatrix}
V & B_0 \\
0 & \hat{D}
\end{bmatrix} \begin{bmatrix}
-sI + A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]
(17)
\[
\begin{bmatrix}
-sI + A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} \neq 0 \quad \forall s \in \mathbb{C}.
\]
(18)
Furthermore, \( \hat{D} \in \mathbb{R}^{m \times p} \) has full column rank.

**Proof.** See Appendix A. \( \square \)

After having obtained \( V, \hat{B}, \hat{D} \) and \( A_{ij} \) \((i, j = 1, 2)\) satisfying Theorem 2, we can construct the interactor as follows according to [17].

**Theorem 3.** Consider \( G(s) \) in (3) satisfying the assumption \( A1 \). With quantities as defined in Theorem 2, then
\[ Z(s) = \begin{bmatrix} -sI + A_{11} & A_{12} & -H \\ A_{21} & A_{22} & -I_p \\ 0 & I_p & 0 \end{bmatrix} \] (19)

is an interactor of \( G(s) \), where \( H \in \mathbb{R}^{(n - \infty) \times p} \) is free. In this case, the state-space realization of the compensated matrix is given as

\[ \hat{G}(s) = G(s)Z(s) = \begin{bmatrix} A & \hat{B} \\ C & D \end{bmatrix}, \] (20)

where

\[ \hat{B} := B_0 + VH. \] (21)

The following theorem illustrates some properties of the interactor in Theorem 3.

**Theorem 4.** The following properties hold for \( Z(s) \) described in Theorem 3.

(i) The inverse matrix of \( Z(s) \) in (19) is

\[ Z^{-1}(s) = \begin{bmatrix} -sI + \hat{A}_{11} & \hat{A}_{12} \\ A_{21} & A_{22} \end{bmatrix}, \] (22)

which implies that

\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & \hat{B} \\ C & D \end{bmatrix} \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ A_{21} & A_{22} \end{bmatrix}. \] (23)

where

\[ \hat{A}_{11} := A_{11} - HA_{21}, \quad \hat{A}_{12} := A_{12} - HA_{22}. \] (24)

And the state-space realization of \( Z^{-1}(s) \) in (22) is minimal and has no finite invariant zeros.

(ii) The zeros of \( Z(s) \) (the roots of \(|Z(s)| = 0\)) are the eigenvalue of \( \hat{A}_{11} \) and can be arbitrarily placed in \( \mathbb{C} \) by choosing \( H \).

(iii) Choose \( H \) such that all zeros of \( Z(s) \) are stable. Then the stabilizability of \((A, B)\) guarantees that of \((A, \hat{B})\).

(iv) Choose \( H \) such that all its zeros of \( Z(s) \) are different from the eigenvalues of \( A \), i.e.,

\[ \sigma(\hat{A}_{11}) \cap \sigma(A) = \emptyset. \] (25)

Then the controllability of \((A, B)\) guarantees that of \((A, \hat{B})\).

**Proof.** See Appendix B. \( \square \)
4. Interactor for a fat matrix

We shall design the interactor for a fat matrix. Such interactor can be used to treat singular $H_{\infty}$ control problems where the transfer matrix from the control input to the controlled output is a fat matrix with infinite zeros. Note that such singular $H_{\infty}$ control problems are not considered in [3,18].

We assume that

$$A_2 \quad G(s)$$ has full normal row rank $m$.

Since the case $m = p$ can be treated using the results in Section 3, we only need to consider the case $m < p$.

Without loss of generality, we assume that the square transfer matrix constituted by the first $m$ columns of $G(s)$ is full normal rank, i.e., $G(s)$ can be decomposed as

$$G(s) = \begin{bmatrix} G_1(s) & G_2(s) \end{bmatrix}$$

$$= \begin{bmatrix} A & B_1 & B_2 \\ C & D_1 & D_2 \end{bmatrix}, \quad D_1 \in \mathbb{R}^{m \times m}, \quad D_2 \in \mathbb{R}^{m \times (m-p)},$$

(26)

where square $G_1(s)$ has full normal rank $m$. In fact, we provide a numerically stable method to compute the decomposition (26). See Appendix C for the details.

Now we can apply the procedure in Section 3 to find an interactor $Z_1(s)$ for $G_1(s)$ such that

$$\hat{G}_1(s) := G_1(s)Z_1(s) = \begin{bmatrix} A & \hat{B}_1 \\ C & \hat{D}_1 \end{bmatrix}$$

holds with $\hat{D}_1$ being invertible. Thus,

$$Z(s) = \begin{bmatrix} Z_1(s) & 0 \\ 0 & I_{p-m} \end{bmatrix}$$

(28)

satisfies

$$G(s)Z(s) = \begin{bmatrix} A & \hat{B}_1 & B_2 \\ C & \hat{D}_1 & D_2 \end{bmatrix}.$$  

(29)

This follows that $Z(s)$ in (28) is an interactor for $G(s)$ in (26).

Similar to Theorem 4, we can obtain the following properties of the above designed interactor.

**Theorem 5.** Consider $G(s)$ in (26) with $G_1(s)$ having full normal rank $m$. Let $Z_1(s)$ in (27) be an interactor of $G_1(s)$. Then the following properties hold for $Z(s)$ defined in (28):

(i) Suppose that all zeros of $Z_1(s)$ in (27) are stable. Then the stabilizability of $(A, [B_1 \ B_2])$ guarantees that of $(A, [\hat{B}_1 \ B_2])$. 

(ii) Suppose that all zeros of $Z_1(s)$ in (27) are different from the eigenvalues of $A$. Then the controllability of $(A, [B_1 \ B_2])$ guarantees that of $(A, [\hat{B}_1 \ B_2])$.

**Proof.** The proof of Theorem 5 follows by mimicking the arguments used in the proof of statements (iii) and (iv) of Theorem 4. It is omitted for brevity. □

**Remark.** The discussion in this paper has been devoted to the right zero interactor for $G(s)$ according to Definition 1, where $G(s)$ is post-multiplied by the interactor. Dually, by considering $G^T(s)$, we can design the left interactor for $G(s)$, where $G(s)$ is pre-multiplied by the interactor.

### 5. Numerical example

Consider the following fat transfer matrix:

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+3} & \frac{1}{s+5} \\ \frac{1}{s+2} & \frac{1}{s+4} & \frac{1}{s+6} \end{bmatrix}.$$  

Here we use the result in Section 4 to design a right interactor $Z(s)$ for $G(s)$ with all stable zeros.

First, decompose $G(s)$ as

$$G(s) = \begin{bmatrix} G_1(s) & G_2(s) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C & D_1 & D_2 \end{bmatrix}.$$  

Next, we design an interactor for

$$G_1(s) = \begin{bmatrix} A & B_1 \\ C & D_1 \end{bmatrix}.$$  

According to (17), we obtain

$$\begin{bmatrix} -sI + A & B_1 \\ C & D_1 \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} V & B_{10} \\ 0 & \hat{D}_1 \end{bmatrix} \begin{bmatrix} -sI + A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$
with
\[
V = \begin{bmatrix}
0.5000 & 0.5471 & 0.5796 & -0.3388 \\
-0.5000 & 0.7986 & -0.1463 & 0.3014 \\
0.5000 & 0.0006 & 0.0725 & 0.8630 \\
-0.5000 & -0.2509 & 0.7984 & -0.2228 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0 
\end{bmatrix},
\]

\[
B_{10} = \begin{bmatrix}
0.0000 & 0.0000 \\
0.0000 & 0.0000 \\
0.0000 & 0.0000 \\
0.0000 & 0.0000 \\
0.0000 & 0.0000 
\end{bmatrix}, \quad \hat{D}_1 = \begin{bmatrix}
-0.5533 & -0.8329 \\
-0.8329 & 0.5533
\end{bmatrix}, \quad (30)
\]

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix}
-2.5000 & 0.0223 & 1.0519 & -0.3718 & 0.0000 & 0.0000 \\
0.0223 & -1.8266 & 0.7177 & -0.0741 & 1.3457 & -0.2503 \\
1.0519 & 0.7177 & -2.9441 & -0.6147 & 0.4333 & 0.8709 \\
-0.3718 & -0.0741 & -0.6147 & -2.7293 & -0.0374 & 1.0858 \\
0.2796 & -0.7593 & -0.9040 & -0.7267 & 0.0000 & 0.0000 \\
-1.3863 & -0.1531 & -0.1823 & -0.1466 & 0.0000 & 0.0000
\end{bmatrix}
\]

(31)

To construct an interactor $Z_1(s)$ with all stable zeros, for example, at $-0.5$, $-1.0$, $-1.5$ and $-2.5$, we choose

\[
H = \begin{bmatrix}
-0.1779 & 0.2300 \\
0.4208 & -0.0550 \\
0.8353 & -0.1102 \\
0.7615 & 0.2172
\end{bmatrix}
\]
in (19). Then, calculating (19), we obtain

\[
Z_1(s) = \begin{bmatrix}
0.1398s^3 + 1.0774s^2 + 2.0886s + 1.5185 \\
-0.1398s^3 - 1.3570s^2 - 4.5171s - 4.4841 \\
-0.6931s^3 - 4.2056s^2 - 7.7399s - 4.2801 \\
0.6931s^3 + 5.5919s^2 + 13.9318s + 10.4781
\end{bmatrix}
\]

Then the interactor $Z(s)$ can be obtained from (28). And the compensated $\hat{G}(s)$ is given by (29) with $\hat{D}_1$ in (30) and
\[ \hat{B}_1 = \begin{bmatrix} 1.3675 & -0.0526 \\ 1.5324 & -0.0773 \\ 0.6291 & 1.2945 \\ 0.8199 & 0.8592 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix} \]  

(32)

Since all zeros of \( Z_1(s) \) are different from the eigenvalues of \( A \), owing to Theorem 5, the controllability of \( (A, [B_1 B_2]) \) guarantees that of \( (A, [\hat{B}_1 B_2]) \). Such fact can be verified easily.

6. Conclusion

The following results have been obtained in this paper. First, some new properties of the infinite eigenstructure of the system matrix pencil of the transfer matrix have been shown. Second, we have shown directly that a matrix which is block part of the transformation matrix associated with the Van Dooren canonical form is injective. And, such matrix has been shown to be \( \hat{G}(\infty) \) which is obtained by using the interactor proposed in this paper. Third, we have presented some new features about state-space relations among the original system \( G(s) \), the interactor \( Z(s) \), and the compensated system \( \hat{G}(s) \). We have shown some conditions about the invariance of the stabilizability or controllability between the state-space realizations of \( G(s) \) and \( \hat{G}(s) \). Fourth, the method of designing the interactor can be applied to a tall or fat transfer matrix with full normal rank.

The properties of the infinite eigenstructure illustrated in Theorem 2 and the interactor in Theorems 3–5 are believed to be useful for solving singular control problems.

Appendix A. Proof of Theorem 2

Putting (13) into (5) and using \( SS^T = I \), we obtain

\[ (-sP_E + P_A)T_\infty = S_\infty (-sE_\infty + A_\infty). \]  

(A.1)

From (7), \( T_\infty \) can be expressed as

\[ T_\infty = \begin{bmatrix} V_0 & 0 \\ 0 & I_p \end{bmatrix}, \]  

(A.2)

where \( V_0 \) will be determined later. Now, putting (12), (14), (15) and (A.2) into (A.1), we obtain

\[ \begin{bmatrix} -sI + A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_0 & 0 \\ 0 & I_p \end{bmatrix} = \begin{bmatrix} V & B_0 \\ V_1 & \hat{D} \end{bmatrix} \begin{bmatrix} -sI + A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \]  

(A.3)
Hence, by comparing the coefficient matrix related to \( s \) in the both sides of (A.3), we have
\[
V_0 = V, \quad V_1 = 0.
\] (A.4)

Thus, (17) holds. Furthermore, (18) holds owing to the facts that \( U_\infty \) is invertible and \(-s E_\infty + A_\infty \) contains all the infinite elementary divisors of \(-s P_E + P_A\).

In what follows, we shall show that \( \hat{D} \) has full column rank. To this end, we first show that (5) can be diagonalized in the following way: There exist \( S_d \) and \( T_d \) such that
\[
\hat{S} := \begin{bmatrix} S_d & S_\infty \end{bmatrix}, \quad \hat{T} := \begin{bmatrix} T_d & T_\infty \end{bmatrix}
\] (A.5)
are invertible, and
\[
\hat{S}^{-1} \begin{bmatrix} -s P_E + P_A \end{bmatrix} \hat{T} = \begin{bmatrix} -s E_{\eta f} + A_{\eta f} & 0 \\ 0 & -s E_\infty + A_\infty \end{bmatrix}.
\] (A.6)

To begin with, denote block (2, 1) of (5) as \(-s F + L\). To show (A.5), it suffices to show that there exist matrices \( K \) and \( Y \) such that
\[
\begin{bmatrix} I & 0 \\ K & I \end{bmatrix} \begin{bmatrix} -s E_{\eta f} + A_{\eta f} \\ -s F + L \end{bmatrix} \begin{bmatrix} I \\ Y \end{bmatrix} = \begin{bmatrix} -s E_{\eta f} + A_{\eta f} \\ 0 \\ -s E_\infty + A_\infty \end{bmatrix},
\] (A.7)
i.e.,
\[
KE_{\eta f} + E_\infty Y + F = 0,
\] (A.8)
\[
KA_{\eta f} + A_\infty Y + L = 0
\] (A.9)
hold. The existence of above \( K \) and \( F \) can be shown directly by using the facts that \(|-s E_\infty + A_\infty| \neq 0 \) holds for \( \forall s \in \mathbb{C} \) and \( E_{\eta f} \) has full column rank.

Now, on the contrary, suppose that \( \hat{D} \) is not full column rank. Then, there exists \( \alpha \neq 0 \) such that \( \hat{D} \alpha = 0 \) holds. Thus, from (14), we have
\[
S_\infty U_\infty^{-1} \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} B_0 \alpha \\ 0 \end{bmatrix} \in \text{Im} \ P_E = \text{Im} \ (P_E \hat{T}).
\] (A.10)

On the other hand, from (A.6) and (12), we obtain
\[
P_E \hat{T} = \begin{bmatrix} S_d E_{\eta f} & S_\infty U_\infty^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \end{bmatrix}.
\]
Together with (A.10), we know that there exist \( \beta_1 \) and \( \beta_2 \) such that
\[
S_\infty U_\infty^{-1} \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = S_d E_{\eta f} \beta_1 + S_\infty U_\infty^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \beta_2,
\] (A.11)
which follows that
\[
\begin{bmatrix}
S_d & S_\infty
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & U_\infty^{-1}
\end{bmatrix}
\begin{bmatrix}
E_{\eta f} \beta_1 \\
\beta_2 \\
-\alpha
\end{bmatrix} = 0.
\]

Thus, \( \alpha = 0 \) holds which yields a contradiction. Therefore, \( \hat{D} \) has full column rank.

\( \square \)

Appendix B. Proof of Theorem 4

(i) By the following equivalent transformation on \( Z(s) \) in (19):

Row 1 \( = \) Row 1 \(- H \times \) Row 2,

we have

\[
Z(s) =
\begin{bmatrix}
-sI + \hat{A}_{11} & \hat{A}_{12} & 0 \\
A_{21} & A_{22} & -I_p \\
0 & I_p & 0
\end{bmatrix}.
\] (B.1)

which follows from the inverse formula of descriptor system that (22) holds.

It follows from (24) and (18) that

\[
\left|\begin{array}{cc}
-sI + \hat{A}_{11} & \hat{A}_{12} \\
A_{21} & A_{22}
\end{array}\right| = \left|\begin{array}{cc}
-sI + A_{11} & A_{12} \\
A_{21} & A_{22}
\end{array}\right| \neq 0 \quad \forall s \in \mathbb{C}
\] (B.2)

holds. This implies that the state-space realization of \( Z^{-1}(s) \) in (22) is minimal and has no finite invariant zeros.

(ii) Note that the zeros of \( Z(s) \) are the poles of \( Z^{-1}(s) \). Since the state-space realization of \( Z^{-1}(s) \) in (22) is minimal, the poles of \( Z^{-1}(s) \) are the eigenvalue of

\( \hat{A}_{11} = A_{11} - HA_{21} \). Since \( (A_{11}, A_{21}) \) is observable owing to (18), the zeros of \( Z(s) \) can be arbitrarily placed in \( \mathbb{C} \) by choosing \( H \).

(iii) On the contrary, suppose that \((A, \hat{B})\) is not stabilizable. Then there exist \( \xi_0 \neq 0 \) and \( \text{Re}[\lambda_0] \geq 0 \) such that

\[
\lambda_0 \xi_0^T \hat{A} = \xi_0^T \hat{A}, \quad \xi_0^T \hat{B} = 0.
\] (B.3)

It follows from (17) that

\[
AV = VA_{11} + B_0 A_{21}, \quad B = VA_{12} + B_0 A_{22}.
\] (B.4)

Using (24) and (B.4), we obtain

\[
AV = V \hat{A}_{11} + \hat{B} A_{21}, \quad B = V \hat{A}_{12} + \hat{B} A_{22}.
\] (B.5)

Thus, owing to (B.3) and (B.5), \( \lambda_0 \xi_0^T T = \xi_0^T A T = \xi_0^T (T \hat{A}_{11} + \hat{B} A_{21}) = \xi_0^T T \hat{A}_{11} \).

Since \( \hat{A}_{11} \) is stable, \( \xi_0^T T = 0 \). From (B.5), \( \xi_0^T B = \xi_0^T (T \hat{A}_{12} + \hat{B} A_{22}) = 0 \). This contradicts the fact that \((A, B)\) is stabilizable. Therefore, \((A, \hat{B})\) is stabilizable.
(iv) The following proof is similar to that of the condition (iii). On the contrary, suppose that \((A, \hat{B})\) is not controllable. Then there exist \(\xi_1 \neq 0\) and \(\lambda_1\) such that
\[
\lambda_1 \xi_1^T = \xi_1^T A, \quad \xi_1^T \hat{B} = 0, \quad (B.6)
\]
which implies together with (B.5) that \(\lambda_1 \xi_1^T T = \xi_1^T (T \hat{A}_{11} + \hat{B} A_{21}) = \xi_1^T T \hat{A}_{11}\). Since \(\sigma(\hat{A}_{11}) \cap \sigma(A) = \emptyset\), \(\lambda_1\) is not an eigenvalue of \(\hat{A}_{11}\). Therefore, \(\xi_1^T T = 0\) holds. From (B.5), \(\xi_1^T B = \xi_1^T (T \hat{A}_{12} + \hat{B} A_{22}) = 0\). This contradicts the fact that \((A, B)\) is controllable. Therefore, \((A, \hat{B})\) is controllable. \(\square\)

**Appendix C. On the decomposition of (26)**

Based on \(A_2\), there exists a full normal rank transfer matrix of size \(m \times m\) whose \(m\) columns are those of \(G(s)\). Decompose \(G(s)\) as \(G(s) = [g_1(s) \ g_2(s) \cdots \ g_p(s)]\) with \(g_j(s) (j = 1, \ldots, p)\) being \(j\)th column of \(G(s)\). In what follows, we develop an algorithm to find \(m\) columns from \(p\) columns of \(G(s)\) for constructing the square transfer matrix with full normal rank.

To this end, we need the numerically stable method to check whether a given transfer matrix is of full normal rank. Note that [5] provides a numerically stable algorithm for computations of zeros of arbitrary transfer matrices (both non-square and/or degenerate). Also, the normal rank of a transfer matrix can be computed as a byproduct of computations of zeros by using such algorithm of [5]. Here, we shall use such result of [5] in the following proposed algorithm.

**Step 0. Initialization:** \(F_0(s) := [ ]\) (empty matrix), \(j := 1\) and \(Te = 0_{p \times 1}\).

**Step 1.** Check whether \([F_{j-1}(s) \ g_j(s)]\) is full column rank or not by using the algorithm of [5]. If yes, then \(F_j(s) := [F_{j-1}(s) \ g_j(s)]\) and \(Te(j) = 1\). Otherwise, \(F_j(s) := F_{j-1}(s)\).

**Step 2.** If \(F_j(s)\) is square, then go to Step 5.

**Step 3.** \(j := j + 1\). If \(j > p\), go to Step 4. Otherwise, go to Step 1.

**Step 4.** Print that the assumption \(A_2\) does not hold. Stop.

**Step 5.** Stop.

It follows from the above algorithm that if \(Te(j) = 1\) then the \(j\)th column of \(G(s)\) is chosen for constructing the square transfer matrix with full normal rank. With such information contained in \(Te\), we can obtain the decomposition (26) by performing some column permutations of \(G(s)\). Therefore, we can assume that (26) holds without loss of generality.

**Acknowledgements**

The first author acknowledges funding of the Mazda Foundation’s Research Grant.
References


