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On simplicial maps and chainable continua

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Abstract

An operation d on simplicial maps between graphs is introduced and used to characterize simplicial maps which can be factored through an arc. The characterization yields a new technique of showing that some continua are not chainable and allows to prove that span zero is equivalent to chainability for inverse limits of trees with simplicial bonding maps.

Key words: Simplicial maps; Graphs; Factorization through an arc; Continua; Chainability; Span

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1. Introduction

By a *graph* we understand a one-dimensional, finite simplicial complex. If G is a graph then $\mathscr{V}(G)$ will denote the set of vertices and $\mathscr{E}(G)$ will denote the set of edges. By the order of a vertex v we understand the number of edges containing v. A vertex of order 1 is called an endpoint. Two points belonging to an edge are called adjacent. A *simplicial* map of a graph G_1 into a graph G_0 is a function from $\mathscr{V}(G_1)$ into $\mathscr{V}(G_0)$ taking every two adjacent vertices either onto a pair of adjacent vertices or onto a single vertex. A simplicial map is *light* if the image of each edge is nondegenerate.

In this paper the same notation is kept for a graph and for its geometric realization. We will assume that every graph is a subset of the three-dimensional Euclidean space and every edge is a straight linear closed segment between its vertices. In this convention a simplicial map is understood as an actual continuous mapping (linearly extended to the edges). But it is important to note that a graph, either abstract or geometric, has a fixed collection of vertices and any change in this collection changes the graph.

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A graph with a geometric realization homeomorphic to an arc is simply called an arc. Observe that two arcs are isomorphic if and only if they have the same number of vertices. A connected graph without a simple closed curve is called a tree. A tree consisting of three edges having a common vertex is called a simple triod. A graph with three vertices and three edges is called a simple triangle. If u and v are two adjacent vertices of a graph, by $\langle u, v\rangle$ we will denote the edge between u and v. Additionally, if u and v are two vertices of a tree, by $\langle u, v\rangle$ we will denote the arc between u and v.

A *continuum* is considered here to be a connected and compact metric space. A continuum is *chainable* if it is the inverse limit of a sequence of arcs (the bonding maps are continuous and do not have to be simplicial). A continuum is *tree-like* if it is the inverse limit of a sequence of trees. If X is a continuum denote by π_1 and π_2 the projections of $X\times X$ onto the first and the second components. Let ρ be the distance function in X. The *surjective span* of X, $\sigma^*(X)$, is the least upper bound of all real numbers ε for which there is a continuum Z contained in $X\times X$ such that $\pi_1(Z)=X=\pi_2(Z)$ and $\rho(x, y)\geqslant\varepsilon$ for each $(x, y)\in Z$. The span of X, $\sigma(X)$, is defined by the formula $\sigma(X)=\operatorname{Sup}\{\sigma^*(A)\mid A\subset X,\ A\neq\emptyset \text{ connected}\}$. See [6].

In 1964, Lelek proved that a chainable continuum has span zero [5]. It is unknown whether (surjective) span zero implies chainability [1, Problem #8]. Several powerful results concerning this and related problems were obtained by Oversteegen in [10, 11], and jointly by Oversteegen and Tymchatyn in [12–16]. Among other things, they proved that a positive answer to the problem would complete the classification of homogeneous plane continua [12].

In order to prove that a continuum is chainable one needs to arrange elements of a (sufficiently fine) open covering into a (coarser) chain. To this end some combinatorial type of tools seems to be required. Mohler and Oversteegen in [8] and Oversteegen in [9] considered tree-words (trees with vertices labeled by letters) and gave some conditions sufficient for reducibility of tree-words to chain-words. The question of reducibility to chain-words is equivalent to the question when a simplicial map between graphs can be factored through an arc. In this paper we introduce an operation d assigning to each simplicial map φ between graphs, a simplicial map $d[\varphi]$ between another pair of graphs. Using this operation we obtain a characterization of simplicial maps between graphs that can be factored through an arc. The characterization is then used to prove that surjective span zero is equivalent to chainability for inverse limits of trees with simplicial bonding maps. (A similar result, with surjective span replaced by span, was announced by Oversteegen at the Prague Topological Symposium, Czechoslovakia, 1986. See [10, 11].) The characterization is also used to develop a technique of showing that some continua are not chainable. As an illustration of the technique we give a new proof that classic atriodic continua by Ingram [3, 4] and Davis and Ingram [2], are not chainable. An extension of this technique will be used in [7] to give an example of an atriodic continuum which is 4-od-like but not triod-like.

2. Simplicial maps which can be factored through an arc

Definition 2.1. For a graph G, let $D(G)$ denote the graph such that

(i) the set of vertices of $D(G)$ consists of edges of G and

(ii) two vertices of $D(G)$ are adjacent if and only if they intersect (as edges of G).

In particular, in the trivial case, when G contains no edges, $D(G)$ is empty. Even though $\mathscr{V}(D(G)) = \mathscr{E}(G)$, it will be convenient to have a notation avoiding confusion between the same object being either a vertex or an edge. Therefore if $v \in \mathscr{V}(D(G))$ then by v^* we will understand the edge v of the graph G.

Example 2.2. Fig. 1 gives a few examples of the operation D. If the solid black graph is G, then the dashed line graph is $D(G)$. Vertices of $D(G)$ are located close to the centers of the corresponding edges of G.

Proposition 2.3. *If G is an arc (that is G is a graph and its geometric realization is homeomorphic to an arc) with $n > 2$ vertices, then $D(G)$ is an arc with $n - 1$ vertices.*

Definition 2.4. Let $\varphi : G_1 \to G_0$ be a simplicial map between graphs. For every (closed) edge $e \in \mathscr{E}(G_0)$, let $\mathscr{K}(e)$ denote the set of components of $\varphi^{-1}(e)$ which are mapped by φ onto e. Denote by $\mathscr{K}(\varphi)$ the union of all $\mathscr{K}(e)$. Let $D(\varphi, G_1)$ be the graph such that

(i) the vertices of $D(\varphi, G_1)$ are elements of $\mathscr{K}(\varphi)$, and

(ii) two vertices of $D(\varphi, G_1)$ are adjacent if and only if they intersect (as subgraphs of G_1).

Let $d[\varphi] : D(\varphi, G_1) \to D(G_0)$ be the map defined by the formula $d[\varphi](v) = \varphi(v)$ for every vertex v of $D(\varphi, G_1)$.

Every vertex $v \in \mathscr{V}(D(\varphi, G_1))$ is also a subgraph of G_1. To avoid confusion we will denote this subgraph by v^*.

Observe that $d[\varphi]$ may be empty. This will occur for example when G_1 is a point.

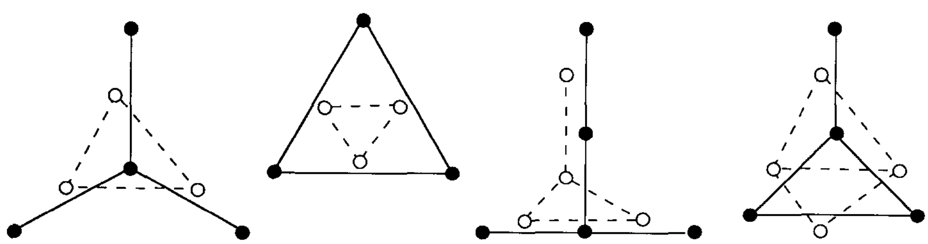
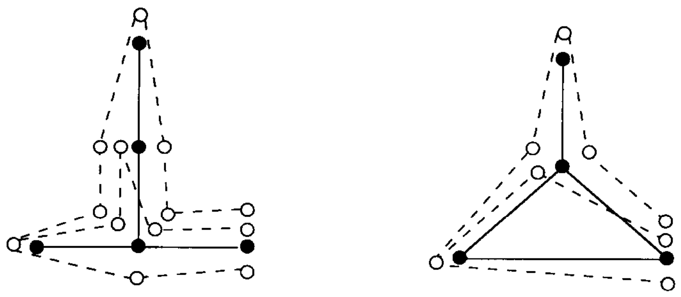


Fig. 1.



$\varphi: G_1 \to G_0$ $\qquad$ $d[\varphi] : D(\varphi, G_1) \to D(G_0)$

Fig. 2.

Example 2.5. Fig. 2 indicates how the operation D can be applied to the Ingram map [3]. The dashed line graph is the domain of the map while the solid black is the range and each vertex of the domain is mapped onto the nearest vertex of the range.

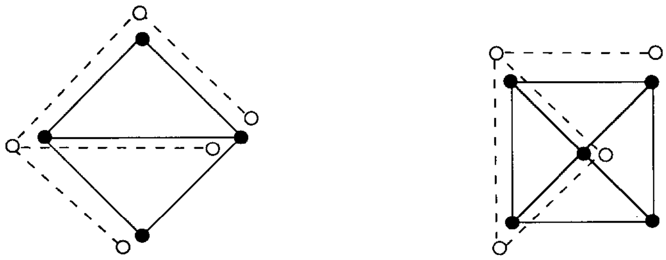
Proposition 2.6. *If $\varphi : G_1 \to G_0$ is a simplicial map, then $d[\varphi]$ is a light simplicial map.*

Proof. Let v_1 and v_2 be two adjacent vertices of $D(\varphi, G_1)$. Note that v_1^* and v_1^* intersect. Let e_1 and e_2 be the edges of G_0 such that v_1^* and v_2^* are components of $\varphi^{-1}(e_1)$ and $\varphi^{-1}(e_2)$, respectively. Since $v_1 \neq v_2$ and $v_1^* \cap v_2^* \neq \emptyset$, we have the result that $e_1 \neq e_2$ and $e_1 \cap e_2 \neq \emptyset$. Since $d[\varphi](v_i)$ is the vertex of $D(G_0)$ representing e_i, the vertices $d[\varphi](v_1)$ and $d[\varphi](v_2)$ are different and adjacent. □

Proposition 2.7. *Let φ be a simplicial map of an arc A with n vertices into a graph G. Then $D(\varphi, A)$ is either the empty set, or a point, or an arc with no more than $n - 1$ vertices.*

Proof. Let $a_1, a_2, \dots, a_n$ denote the sequence of consecutive vertices of A. For an arbitrary vertex $v \in D(\varphi, A)$, let $j(v)$ be an index such that $\langle a_{j(v)}, a_{j(v)+1}\rangle \subset v^*$ and $\varphi(\langle a_{j(v)}, a_{j(v)+1}\rangle)$ is an edge. The proposition follows from the following observation. If v and w are two different vertices of $D(\varphi, A)$ then either $w^* \subset \langle a_1, a_{j(v)}\rangle$ if $j(w) < j(v)$, or $w^* \subset \langle a_{j(v)+1}, a_n\rangle$ if $j(w) > j(v)$. □

Definition 2.8. Let $\varphi : G_1 \to G_0$ be a simplicial map between graphs. Then $d[d[\varphi]]$ will be denoted by $d^2[\varphi]$, and recursively $d[d^{n-1}[\varphi]]$ will be denoted by $d^n[\varphi]$. The domain of $d^n[\varphi]$ will be denoted by $D^n(\varphi, G_1)$ and the range by $D^n(G_0)$.



$d^2[\varphi] : D^2(\varphi, G_1) \to D^2(G_0)$ $d^3[\varphi] : D^3(\varphi, G_1) \to D^3(G_0)$

Fig. 3.

Example 2.9. Fig. 3 indicates further iterations of the operation d applied to the Ingram map. Like in the previous example the dashed line graph is the domain of the map while the solid black is the range and each vertex of the domain is mapped onto the nearest vertex of the range.

Definition 2.10. Let $\varphi : G_1 \to G_0$ and $\psi : G_2 \to G_1$ be simplicial maps between graphs. Let $d[\varphi, \psi] : D(\varphi \circ \psi, G_2) \to D(\varphi, G_1)$ be the map such that for every vertex v of $D(\varphi \circ \psi, G_2)$, $d[\varphi, \psi](v)$ is the vertex of $D(\varphi, G_1)$ containing $\psi(v^*)$. Let $d^n[\varphi, \psi] : D^n(\varphi \circ \psi, G_2) \to D^n(\varphi, G_1)$ denote the map defined by the formula $d^n[\varphi, \psi] = d[d^{n-1}[\varphi], d^{n-1}[\varphi, \psi]]$.

Proposition 2.11. *Let $\varphi : G_1 \to G_0$ and $\psi : G_2 \to G_1$ be simplicial maps between graphs. Then $d^n[\varphi, \psi]$ is a simplicial map and $d^n[\varphi \circ \psi] = d^n[\varphi] \circ d^n[\varphi, \psi]$.*

Proof. Let v be a vertex of $D(\varphi \circ \psi, G_2)$. Observe that $\varphi \circ \psi(v)$ is an edge of G_0. Denote this edge by e. Let C denote $\psi(v^*)$. Since C is a connected subgraph of G_1 and $\varphi(C) = e$, we have the result that $d[\varphi, \psi](v)$ is the only vertex of $D(\varphi, G_1)$ containing C. Observe that each of $d[\varphi \circ \psi](v)$ and $d[\varphi] \circ d[\varphi, \psi](v)$ is the element of $D(G_0)$ representing e. Now, if v_1 is a vertex of $D(\varphi \circ \psi, G_2)$ adjacent to v, then v^* and v_1^* intersect, and consequently $\psi(v^*)$ and $\psi(v_1^*)$ intersect. It follows that $d[\varphi, \psi](v)$ and $d[\varphi, \psi](v_1)$ are adjacent. So $d[\varphi, \psi]$ is a simplicial map. The proof for an arbitrary integer follows by induction. □

Theorem 2.12. *Let $\varphi : G_1 \to G_0$ be a simplicial map between graphs. Then φ can be factored through an arc if and only if $d[\varphi]$ can be factored through an arc.*

Proof. If φ can be factored through an arc, then it follows from Propositions 2.7 and 2.11 that $d[\varphi]$ can be factored through an arc.

Note that in order to prove the theorem in the other direction it is sufficient to prove it in the case when G_1 is connected. Observe also that the proof is trivial in the cases when $d[\varphi]$ is empty or $d[\varphi]$ maps $D(\varphi, G_1)$ into one point.

Suppose that that there is a nondegenerate arc I, and there are two simplicial maps $\tilde{\alpha}: D(\varphi, G_1) \to I$ and $\tilde{\beta}: I \to D(G_0)$ such that $d[\varphi] = \tilde{\beta} \circ \tilde{\alpha}$. We may assume that $\tilde{\alpha}$ maps $D(\varphi, G_1)$ onto I. Let $v_1, v_2, \ldots, v_n$ be the vertices of I ordered by one of the two natural orders on the arc I. Observe also that if for some i, $\tilde{\beta}(v_i) = \tilde{\beta}(v_{i+1})$, then the vertices v_i and v_{i+1} could be identified. So we may assume that $\tilde{\beta}(v_i) \neq \tilde{\beta}(v_{i+1})$ for $i = 1, \ldots, n-1$. For each $i = 1, \ldots, n$, let e_i denote the edge $(\tilde{\beta}(v_i))^*$ of G_0 which corresponds to $\tilde{\beta}(v_i)$. By our assumption e_i and e_{i+1} are two different edges. Since $\tilde{\beta}$ is simplicial e_i and e_{i+1} intersect at a vertex. Denote this vertex by w_i. Let w_0 be the vertex of e_1 different from w_1, let w_n be the vertex of e_n different from w_{n-1}. Let A denote the set $\{i = 1, \ldots, n \mid w_{i-1} \neq w_i\}$, and let B be the complement of A in $\{1, \ldots, n\}$. For each $i \in B$, let w_i' be the vertex of e_i different from w_i. Let J be an arc which is the union of subarcs $J_1, J_2, \ldots, J_n$ such that J_i is a single edge with vertices s_{i-1} and s_i for each $i \in A$, and J_i is the union of two edges with vertices $s_{i-1} - s_i' - s_i$ for each $i \in B$. Let $\beta: J \to G_0$ be the simplicial map defined by $\beta(s_i) = w_i$ for $i = 0, \ldots, n$, and $\beta(s_i') = w_i'$ for $i \in B$. In order to complete the proof we need to define a simplicial map $\alpha: G_1 \to J$ such that $\varphi = \beta \circ \alpha$.

Let V_i be the set of the vertices $v \in \mathscr{V}(G_1)$ which are contained in the union of vertices of $\tilde{\alpha}^{-1}(v_i)$. (Recall that $\tilde{\alpha}^{-1}(v_i)$ is a subset of $D(\varphi, G_1)$, and each vertex of $D(\varphi, G_1)$ is a subgraph of G_1.) Observe that $\mathscr{V}(G_1) = V_1 \cup V_2 \cup \cdots \cup V_n$ and $V_i \cap V_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. Note also that $\varphi(V_i \cap V_{i+1}) = w_i$. Let Y_i denote the set $\varphi^{-1}(w_i) \cap V_i$. For each $i \in B$, let U_i denote the set $V_i \cap V_{i+1}$, and let T_i denote the set $Y_i \setminus U_i$

Let u be an arbitrary vertex of U_i and let t be an arbitrary vertex of T_i. We will show that u and t are not adjacent. Suppose, to the contrary, that u and t are adjacent. Let x be the vertex of $D(\varphi, G_1)$ such that $u \in x^*$ and $\tilde{\alpha}(x) = v_{i+1}$.Since x^* is a component of $\varphi^{-1}(e_{i+1})$, w_i is a vertex of e_{i+1} and the edge between u and t is mapped by φ onto w_i, we have the result that $t \in x^*$ and consequently $t \in U_i$, a contradiction.

Define $\alpha: G_1 \to J$ in the following way: $\alpha(v) = s_i$ for $i \in A$ and $v \in Y_i$, $\alpha(v) = s_{i-1}$ for $i \in A$ and $v \in V_i \setminus Y_i$, $\alpha(v) = s_i$ for $i \in B$ and $v \in U_i$, $\alpha(v) = s_{i-1}$ for $i \in B$ and $v \in T_i$ and $\alpha(v) = s_i'$ for $i \in B$ and $v \notin U_i \cup T_i$. It can be readily verified that α is a simplicial map such that $\varphi = \beta \circ \alpha$. □

Theorem 2.13. *Let $\varphi: G_1 \to G_0$ be a simplicial map between graphs. Then φ can be factored through an arc if and only if there is an integer n such that $d^n[\varphi]$ is empty.*

Proof. Suppose that there is an arc I and there are simplicial maps $\alpha: G_1 \to I$ and $\beta: I \to G_0$ such that $\varphi = \beta \circ \alpha$. Let n be the number of vertices of I. By Proposition 2.7, the map $d^n[\beta]$ is empty. It follows from Proposition 2.11 that $d^n[\varphi]$ is also empty.

If $d^n[\varphi]$ is empty, then it can be factored through an arc, and the proof follows from Theorem 2.12. □

Proposition 2.14. *Let $\varphi: G_1 \to G_0$ be a simplicial map between graphs. Then every simple triangle contained in $D(\varphi, G_1)$ is mapped by $d[\varphi]$ onto a simple triangle in $D(G_0)$.*

Proof. Let $a, b, c \in \mathscr{V}(D(\varphi, G_1))$ form a simple triangle. Consider the subgraphs a^*, b^* and c^* of G_1 represented by a, b and c, respectively. If, for instance, $\varphi(a^*) = \varphi(b^*)$, then since a^* and b^* are components of $\varphi^{-1}(\varphi(a^*))$ and they intersect, we have that $a^* = b^*$ and consequently $a = b$. So $\varphi(a^*)$, $\varphi(b^*)$ and $\varphi(c^*)$ are three different edges of G_0. Since each two of them intersect, $\varphi(a^*)$, $\varphi(b^*)$ and $\varphi(c^*)$ form a simple triangle in $D(G_0)$. □

The following proposition follows readily from Proposition 2.14.

Proposition 2.15. *Let $\varphi: G_1 \to G_0$ be a simplicial map between graphs such that $D^n(\varphi, G_1)$ contains a simple triangle for some n. Then $D^m(\varphi, G_1)$ contains a simple triangle for every $m \geqslant n$.*

Proposition 2.16. *Let φ be a simplicial map of a tree G_1 into a graph G_0. Suppose that there is no simple triangle in $D(\varphi, G_1)$. Then $D(\varphi, G_1)$ is a tree. Moreover, if every arc contained in G_1 has at most n vertices then every arc contained in $D(\varphi, G_1)$ has at most $n-1$ vertices.*

Proof. Let $v_1, v_2, \ldots v_k$ be a sequence of vertices of $D(\varphi, G_1)$ such that v_i and v_{i+1} are two ends of an edge from $\mathscr{E}(D(\varphi, G_1))$ for $i = 1, \ldots, k-1$, and $v_{i-1} \neq v_{i+1}$ for $i = 2, \ldots, k-1$. To prove the proposition it is enough to show that $v_1, v_2, \ldots, v_k$ are distinct and that k is less than n.

The set $(v_i)^*$ is a subtree of G_1. Observe that $(v_{i-1})^* \cap (v_{i+1})^* = \emptyset$ for $i = 2, \ldots, k-1$, because otherwise v_{i-1}, v_i and v_{i+1} would form a simple triangle. There is a vertex $p_0 \in (v_1)^* \setminus (v_2)^*$. For each $i = 1, \ldots, k-1$, let p_i be a point of $(v_i)^* \cap (v_{i+1})^*$ such that the arc A_i between p_{i-1} and p_i meets $(v_{i+1})^*$ at p_i. There is a vertex $p_k \in (v_k)^* \setminus (v_{k-1})^*$. Observe that $p_i \neq p_{i+1}$ for $i = 1, \ldots, k-1$. Let A_k be the arc between p_{k-1} and p_k. Since A_{i+1} is contained in $(v_{i+1})^*$, we have that $A_i \cap A_{i+1} = \{p_i\}$ for $i = 1, \ldots, k-1$. Since G_1 is a tree, the union of $A_1, \ldots, A_k$ is an arc. Denote this arc by A. Since A has at least $k+1$ vertices, k is less than n. Observe that $(v_i)^* \cap A_{i+2} = \emptyset$, because $A_{i+2} \subset (v_{i+2})^*$ and $(v_i)^* \cap (v_{i+2})^* = \emptyset$. Since the intersection $v_i^* \cap A$ is connected and $v_i \neq v_{i+1}$, we have the result that $v_i \neq v_j$ for $i \neq j$. □

Proposition 2.17. *Let φ be a simplicial map of a tree G_1 into a graph G_0 and let ψ be a map of a tree G_2 into G_1. If $D(\varphi, G_1)$ is a tree then $D(\varphi \circ \psi, G_2)$ is a tree.*

Proof. Suppose that $D(\varphi \circ \psi, G_2)$ is not a tree. Then by Proposition 2.16, it contains a simple triangle T. By Proposition 2.14, $d[\varphi \circ \psi]$ maps T onto a simple triangle. Since $d[\varphi \circ \psi] = d[\varphi] \circ d[\varphi, \psi]$, $d[\varphi, \psi](T)$ is a simple triangle and $D(\varphi, G_1)$ is not a tree. □

Theorem 2.18. *Let G_1 be a tree such that every arc contained in G_1 has at most* $n+1$ *vertices. Let φ be a simplicial map of G_1 into a graph G_0. Then φ cannot be factored through an arc if and only if $D^n(\varphi, G_1)$ contains a simple triangle.*

Proof. If $D^n(\varphi, G_1)$ contains a simple triangle then, by Proposition 2.15, $D^m(\varphi, G_1)$ contains a simple triangle for every $m \geqslant n$. It follows from Theorem 2.13 that φ cannot be factored through an arc.

If $D^n(\varphi, G_1)$ does not contain a simple triangle then it follows from Proposition 2.15 that $D^i(\varphi, G_1)$ does not contain a simple triangle for $i = 1, \ldots, n$. Using n times Proposition 2.16 we get that $D^n(\varphi, G_1)$ is a tree such that every arc contained in $D^n(\varphi, G_1)$ has at most one $(n+1-n)$ vertex. Of course, this can only happen if $D^n(\varphi, G_1)$ is either empty or a point. Since $d^{n+1}[\varphi]$ is empty, Theorem 2.13 implies that φ can be factored through an arc. □

3. Inverse limits of trees with simplicial bonding maps.

In this section we use the operation d to prove that surjective span zero is equivalent to chainability for inverse limits of trees with simplicial bonding maps. It should be noted here that a similar result, with surjective span replaced by span, was announced by Oversteegen at the Prague Topological Symposium, Czechoslovakia, 1986. See [10, 11].

Lemma 3.1. *Let $\varphi : G_1 \to G_0$ be a simplicial map between connected graphs. Suppose that there are two simplicial maps $\tilde{\alpha}$ and $\tilde{\beta}$ from an arc I onto $D(\varphi, G_1)$ such that $d[\varphi](\tilde{\alpha}(v)) \neq d[\varphi](\tilde{\beta}(v))$ for every vertex v of $\mathscr{V}(I)$ and $d[\varphi](\tilde{\alpha}(e)) \neq d[\varphi](\tilde{\beta}(e))$ for every edge e of $\mathscr{E}(I)$. Then there are two simplicial maps α and β from an arc J onto G_1 such that $\varphi(\alpha(v)) \neq \varphi(\beta(v))$ for every vertex v from $\mathscr{V}(J)$ and $\varphi(\alpha(e)) \neq \varphi(\beta(e))$ for every edge e from $\mathscr{E}(J)$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of I ordered by one of the two natural orders on the arc I. Let A_i denote the subgraph $(\tilde{\alpha}(v_i))^*$ of G_1 represented by $\tilde{\alpha}(v_i)$ and B_i be the subgraph $(\tilde{\beta}(v_i))^*$ of G_1 represented by $\tilde{\beta}(v_i)$.

Claim 1. *Let $a_i', a_{i+1}' \in \mathscr{V}(A_i)$ and $b_i', b_{i+1}' \in \mathscr{V}(B_i)$ be such that $\varphi(a_i') \neq \varphi(b_i')$ and $\varphi(a_{i+1}') \neq \varphi(b_{i+1}')$. Then there is an arc J_i' with the endvertices c_i' and c_{i+1}', and there are simplicial maps α_i' of J_i' onto A_i and β_i' of J_i' onto B_i such that $\alpha_i'(c_i') = a_i'$, $\alpha_i'(c_{i+1}') = a_{i+1}'$, $\beta_i'(c_i') = b_i'$, $\beta_i'(c_{i+1}') = b_{i+1}'$, $\varphi(\alpha_i'(v)) \neq \varphi(\beta_i'(v))$ for every vertex v from $\mathscr{V}(J_i')$ and $\varphi(\alpha_i'(e)) \neq \varphi(\beta_i'(e))$ for every edge e from $\mathscr{E}(J_i')$.*

Observe that $\varphi(A_i)$ and $\varphi(B_i)$ are edges from $\mathscr{E}(G_0)$. Since $d[\varphi](\tilde{\alpha}(v_i)) \neq d[\varphi](\tilde{\beta}(v_i))$, we have that $\varphi(A_i) \neq \varphi(B_i)$. In the case when $\varphi(A_i)$ and $\varphi(B_i)$ are disjoint the claim is trivial. So we may assume that $\varphi(A_i)$ and $\varphi(B_i)$ have a common vertex p. Let a be the other vertex of $\varphi(A_i)$ and let b be the other vertex of $\varphi(B_i)$. Since $\varphi(a'_{i+1}) \neq \varphi(b'_{i+1})$, without loss of generality we may assume that $\varphi(a'_{i+1}) \neq p$. Since B_i is connected, there is an arc J' (possibly degenerate) with endpoints c'_i and d', and there is a simplicial map β' of J' into B_i such that $\beta'(c'_i) = b'_i$, $\varphi(\beta'(d')) = b$ and $\varphi(\beta'(v)) = \varphi(b'_i)$ for every vertex $v \in \mathscr{V}(J')$ different from d'. Let α' be the constant map of J' onto a'_i. Since A_i is connected, there is an arc J'' with endpoints d' and d'', and there is a simplicial map α'' of J'' onto A_i such that $\alpha''(d') = a'_i$ and $\alpha''(d'')) = a'_{i+1}$. Let β'' be the constant map of J'' onto $\beta'(d')$. There is an arc J''' with endpoints d'' and c'_{i+1}, and there is a simplicial map β''' of J''' onto B_i such that $\beta'''(d'') = \beta''(d'')$ and $\beta'''(c'_{i+1}) = b'_{i+1}$. Let α''' be the constant map of J''' onto a'_{i+1}. Define J'_i as the union of J', J'' and J'''. Define α'_i as the union of α', α'' and α'''. Finally, let β'_i be the union of β', β'' and β'''. It is easy to see that so defined J'_i, α'_i and β'_i satisfy the claim.

Claim 2. *Let $a_k \in \mathscr{V}(A_k)$ and $b_k \in \mathscr{V}(B_k)$ be such that $\varphi(a_k) \neq \varphi(b_k)$. Then there is an arc J_k with the end vertices c_k and c_{k+1}, and there are simplicial maps α_k of J_k onto $A_k \cup A_{k+1}$ and β_k of J_k onto $B_k \cup B_{k+1}$ such that $\alpha_k(c_k) = a_k$, $\beta_k(c_k) = b_k$, $\alpha_k(c_{k+1}) \in \mathscr{V}(A_{k+1})$, $\beta_k(c_{k+1}) \in \mathscr{V}(B_{k+1})$, $\varphi(\alpha_k(v)) \neq \varphi(\beta_k(v))$ for every vertex v from $\mathscr{V}(J_k)$ and $\varphi(\alpha_k(e)) \neq \varphi(\beta_k(e))$ for every edge e from $\mathscr{E}(J_k)$.*

Let a be a point of $A_k \cap A_{k+1}$ and let b be a point of $B_k \cap B_{k+1}$. We will consider the following two cases: $\varphi(a) \neq \varphi(b)$ and $\varphi(a) = \varphi(b)$.

Case 1: $\varphi(a) \neq \varphi(b)$. Use Claim 1 with $i = k$, $a'_i = a_k$, $b'_i = b_k$, $a'_{i+1} = a$ and $b'_{i+1} = b$. Then use Claim 1 again with $i = k+1$, $a'_i = a$, $b'_i = b$, $a'_{i+1} = a$ and $b'_{i+1} = b$. Define J_k as the union of J'_k and J'_{k+1}. Set $c_k = c'_k$ and $c_{k+1} = c'_{k+2}$. Define α_k as the union of α'_k and α'_{k+1}. Finally, let β_k be the union of β'_k and β'_{k+1}. It is easy to see that so defined J_k, α_k and β_k satisfy the claim.

Case 2: $\varphi(a) = \varphi(b) = p$. Observe that p is a common vertex of the edges $\varphi(A_k)$, $\varphi(A_{k+1})$, $\varphi(B_k)$ and $\varphi(B_{k+1})$. Let a', a'', b' and b'' denote the other vertices of the edges $\varphi(A_k)$, $\varphi(A_{k+1})$, $\varphi(B_k)$ and $\varphi(B_{k+1})$, respectively. Since $d[\varphi](\tilde{\alpha}(v_k)) \neq d[\varphi](\tilde{\beta}(v_k))$ and $d[\varphi](\tilde{\alpha}(v_{k+1})) \neq d[\varphi](\tilde{\beta}(v_{k+1}))$, we have that $a' \neq b'$ and $a'' \neq b''$. Since $d[\varphi](\tilde{\alpha}([v_k, v_{k+1}])) \neq d[\varphi](\tilde{\beta}([v_k, v_{k+1}]))$, we have that either $a' \neq b''$ or $b' \neq a''$. Without loss of generality we may assume that $b' \neq a''$. Since $\varphi(a_k) \neq \varphi(b_k)$, either $\varphi(a_k) = p$ and $\varphi(b_k) = b'$ or $\varphi(a_k) = a'$ and $\varphi(b_k) = p$. Since B_k is connected, there is an arc J' (degenerate if $\varphi(b_k) = b'$) with endpoints c_k and d', and there is a simplicial map β' of J' into B_k such that $\beta'(c_k) = \mathrm{b}_k$, $\varphi(\beta'(d')) = b'$ and $\varphi(\beta'(v)) = \varphi(b_k)$ for every vertex $v \in \mathscr{V}(J')$ different from d'. Let α' be the constant map of J' onto a_k. Since $A_k \cup A_{k+1}$ is connected, there is an arc J'' with endpoints d' and d'', and there is a simplicial map α'' of J'' onto $A_k \cup A_{k+1}$ such that $\alpha''(d') = a_k$, $\alpha''(d'') \in \mathscr{V}(A_{k+1})$ and $\varphi(\alpha''(d'')) = a''$. Let β'' be the constant map of J'' onto $\beta'(d')$. There is an arc J''' with endpoints d'' and c_{i+1}, and there is a simplicial map β''' of J''' onto $B_k \cup B_{k+1}$ such that $\beta'''(d'') = \beta''(d'')$ and $\beta'''(c_{k+1}) \in \mathscr{V}(B_{k+1})$. Let α''' be the constant map of J''' onto $\alpha''(d'')$.

Define J_k as the union of J', J'' and J'''. Define α_k as the union of α', α'' and α'''. Finally, let β_k be the union of β', β'' and β'''. It is easy to see that so defined J_k, α_k and β_k satisfy the claim.

There are points $a_1 \in \mathscr{V}(A_1)$ and $b_1 \in \mathscr{V}(B_1)$ such that $\varphi(a_1) \neq \varphi(b_1)$. Use Claim 2 for $k = 1$ to get J_1, α_1 and β_1. Set $a_2 = \alpha_1(c_2)$ and $b_2 = \beta_1(c_2)$. Use Claim 2 for $k = 2$ to get J_2, α_2 and β_2. Continue the procedure to get $J_3, \ldots, J_{n-1}$, $\alpha_3, \ldots, \alpha_{n-1}$ and $\beta_3, \ldots, \beta_{n-1}$. Define J as the union of $J_1, J_2, \ldots, J_{n-1}$. Define α as the union of $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. Finally, let β be the union of $\beta_1, \beta_2, \ldots, \beta_{n-1}$. It is easy to see that so defined J, α and β satisfy the lemma. □

Theorem 3.2. *Let (T_n, φ_n^m) be an inverse system of trees with simplicial bonding maps ($T_n \overset{\varphi_n^m}{\leftarrow} T_m$ for $n < m$). Let X denote the inverse limit $\varprojlim(T_n, \varphi_n^m)$. Suppose that there is a positive integer n such that for each integer $m > n$, the map φ_n^m cannot be factored through an arc. Then the surjective span of X is positive ($\sigma^*(X) > 0$).*

Proof. Without loss of generality we may assume that $\varphi_i^j(T_j) = T_i$ for every $i < j$. Let α_m and β_m be two simplicial maps from an arc J_m onto T_m. We will say that the triple (α_m, β_m, J_m) belongs to the class $\mathscr{K}_m$ if $\varphi_n^m(a_m(v)) \neq \varphi_n^m(\beta_m(v))$ for every vertex v from $\mathscr{V}(J_m)$ and $\varphi_n^m(\alpha_m(e)) \neq \varphi_n^m(\beta_m(e))$ for every edge e from $\mathscr{E}(J_m)$.

Claim 1. *$\mathscr{K}_m \neq \emptyset$ for $m > n$.*

By Theorem 2.18, there is an integer k such that $D^k(\varphi_n^m, T_m)$ contains a simple triangle with vertices a, b, c. By Proposition 2.14, $d^k[\varphi_n^m](a)$, $d^k[\varphi_n^m](b)$ and $d^k[\varphi_n^m](c)$ form a simple triangle in $D^k(T_n)$. Let $\tilde{\alpha}_1$ be a simplicial map of an arc I_1 with an endpoint p onto $D^k(\varphi_n^m, T_m)$ such that $\tilde{\alpha}_1(p) = a$. There is a simplicial map $\tilde{\beta}_1$ of I_1 into the triangle a, b, c such that $d^k[\varphi_n^m](\tilde{\alpha}_1(v)) \neq d^k[\varphi_n^m](\tilde{\beta}_1(v))$ for every vertex $v \in \mathscr{V}(I_1)$ and $d^k[\varphi_n^m](\tilde{\alpha}_1(e)) \neq d^k[\varphi_n^m](\tilde{\beta}_1(e))$ for every edge $e \in \mathscr{E}(I_1)$. Let $\tilde{\beta}_2$ be a simplicial map of an arc I_2 meeting I_1 at the common endpoint p onto $D^k(\varphi_n^m, T_m)$ such that $\tilde{\beta}_2(p) = \tilde{\beta}_1(p)$. There is a simplicial map $\tilde{\alpha}_2$ of I_1 into the triangle a, b, c such that $d^k[\varphi_n^m](\tilde{\alpha}_2(v)) \neq d^k[\varphi_n^m](\tilde{\beta}_2(v))$ for every vertex $v \in \mathscr{V}(I_2)$ and $d^k[\varphi_n^m](\tilde{\alpha}_2(e)) \neq d^k[\varphi_n^m](\tilde{\beta}_2(e))$ for every edge $e \in \mathscr{E}(I_2)$. Let $I = I_1 \cup I_2$, $\tilde{\alpha} = \tilde{\alpha}_1 \cup \tilde{\alpha}_2$ and $\tilde{\beta} = \tilde{\beta}_1 \cup \tilde{\beta}_2$. Observe that I is an arc mapped by $\tilde{\alpha}$ and $\tilde{\beta}$ onto $D^k(\varphi_n^m, T_m)$ such that $d^k[\varphi_n^m](\tilde{\alpha}(v)) \neq d^k[\varphi_n^m](\tilde{\beta}(v))$ for every vertex $v \in \mathscr{V}(I)$ and $d^k[\varphi_n^m](\tilde{\alpha}(e)) \neq d^k[\varphi_n^m]\tilde{\beta}(e))$ for every edge $e \in \mathscr{E}(I)$. Now, the claim follows from Lemma 3.1 used k times.

For $(\alpha_m, \beta_m, J_m) \in \mathscr{K}_m$, consider the set $Z_m = (\alpha_m \times \beta_m)(J_m) \subset T_m \times T_m$. Let C_m denote the collection of all such sets Z_m. Observe that $(\varphi_m^j \times \varphi_m^j)(Z_j) \in C_m$ for each $j > m$ and each $Z_j \in C_j$. Since C_m is finite for each $m > n$, there is a sequence $Z^{n+1}, Z^{n+2}, Z^{n+3}, \ldots$ such that $Z^m \in C_m$ for each $m > n$, and $(\varphi_m^j \times \varphi_m^j)(Z^j) = Z^m$ for each $j > m$. Let Z denote the inverse limit $\varprojlim(Z^m, \varphi_m^j \times \varphi_m^j)$. Observe that Z is a continuum contained in $X \times X$ such that $\pi_1(Z) = X = \pi_2(Z)$, where π_1 and π_2 are the projections of $X \times X$ onto the first and the second components. Denote by φ_m the projection of X onto T_n and let ρ denote the distance function on X. For each point $(x, y) \in Z$, we have that $\varphi_m(x) \neq \varphi_m(v)$. Since Z is compact there is a positive number ε such that $\rho(x, y) \geqslant \varepsilon$ for each $(x, y) \in Z$. Thus $\sigma^*(X) \geqslant \varepsilon > 0$. □

Theorem 3.3. *Let (T_n, φ_n^m) be an inverse system of trees with simplicial bonding maps ($T_n \overset{\varphi_n^m}{\leftarrow} T_m$ for $n < m$). Let X denote the inverse limit $\varprojlim(T_n, \varphi_n^m)$. Then the following conditions are equivalent.*

(i) *X is chainable.*

(ii) $\sigma^*(X) = 0$.

(iii) *For every positive integer n there is an integer $m > n$ such that φ_n^m can be factored through an arc.*

Proof. The implication (i) ⇒ (ii) was proven by Lelek in [5]. The implication (ii) ⇒ (iii) follows from Theorem 3.2. The implication (iii) ⇒ (i) is obvious. □

4. Lifting of light simplicial maps

In this section we introduce a notion of ultra light simplicial maps and prove that a factorization through a tree can be lifted through an ultra light map.

Definition 4.1. Let $\varphi: G_1 \to G_0$ be a simplicial map between graphs. We say that φ is *ultra light* if it is light and v^* is an edge of G_1 for each $v \in \mathscr{V}(D(\varphi, G_1))$.

Observe that φ is ultra light if and only if it is light and, for each $e \in \mathscr{E}(G_0)$, each component of $\varphi^{-1}(e)$ is either a vertex or an edge of G_1. Therefore $D(\varphi, G_1)$ can be naturally identified with $D(G_1)$.

Proposition 4.2. *Suppose $\varphi: G_1 \to G_0$ is a simplicial ultra light map between graphs. Then $d\,[\varphi]: D(\varphi, G_1) \to D(G_0)$ is also ultra light.*

Proof. By Proposition 2.6, $d[\varphi]$ is light. Let b be an edge of $D(G_0)$ and let C be a nondegenerate component of $(d[\varphi])^{-1}(b)$. Since C is nondegenerate and connected, it contains two adjacent vertices c' and c''. We will show that C contains no other vertices. Note that $(c')^*$ and $(c'')^*$ are two different edges of G_1 intersecting at a common vertex, which will be denoted by v. Denote by v' and v'' the remaining vertices of $(c')^*$ and $(c'')^*$, respectively. Since φ is ultra light $\varphi(v')$, $\varphi(v)$ and $\varphi(v'')$ are three different vertices of G_0. Let b' and b'' denote the vertices of $D(G_0)$ representing $\langle \varphi(v'), \varphi(v) \rangle$ and $\langle \varphi(v), \varphi(v'') \rangle$, respectively. Observe that b' and b'' are the vertices of b, $d[\varphi](c') = b'$ and $d[\varphi](c'') = b''$. Suppose that C contains a vertex other than c' and c''. In this case, without loss of generality, we may assume that there is a vertex c of C such that $c' \neq c \neq c''$ and c is adjacent to c'. It means that c^* and $(c')^*$ are two intersecting edges of G_1.

Since $d[\varphi]$ is light, $d[\varphi](c)=b''$ and consequently $\varphi(c^*)=\langle\varphi(v),\ \varphi(v'')\rangle$. It follows that v' is not a vertex of c^*, and thus v is the common vertex of c^* and $(c')^*$. But, then $c^*\cup(c'')^*$ is connected and mapped by φ onto the edge $\langle\varphi(v),\ \varphi(v'')\rangle$, which is impossible, because φ is ultra light. □

Theorem 4.3. *Let G_0, G_1 and G_2 be connected graphs and let T be a tree. Suppose $\varphi: G_1\to G_0$, $\psi: G_2\to G_1$, $\lambda: G_2\to T$ and $\sigma: T\to G_0$ are simplicial light maps such that φ is ultra light, $\lambda(G_2)=T$ and $\varphi\circ\psi=\sigma\circ\lambda$. Then there is a simplicial map $\sigma': T\to G_1$ such that $\psi=\sigma'\circ\lambda$.*

Proof. First we will prove the following claim.

Claim. *Suppose v and v' are vertices of G_2 such that $\lambda(v)=\lambda(v')$. Then $\psi(v)=\psi(v')$.*

Since G_2 is connected, G_2 contains an arc A with endpoints v and v'. Let n denote the number of vertices of A. We will prove the claim by induction with respect to n. Suppose that for each pair of vertices w and w' of G_2 such that $\lambda(w)=\lambda(w')$ and G_2 contains an arc B with endpoints w and w' and with less than n vertices, we have the result that $\psi(w)=\psi(w')$. If $n=1$, then $v=v'$ and the claim is obvious. If $n=2$ and $\psi(v)\neq\psi(v')$, then $\psi(v)$ and $\psi(v')$ are adjacent vertices of G_1, which is impossible, because $\varphi(\psi(v))=\sigma(\lambda(v))=\sigma(\lambda(v'))=\varphi(\psi(v'))$ and φ is light. So we may assume that $n>2$. Suppose that there is a vertex s of A such that $v\neq s\neq v'$ and $\lambda(s)=\lambda(v)$. In this case we have by induction the result that $\psi(v)=\psi(s)$ and $\psi(s)=\psi(v')$. So we may assume that $\lambda(s)\neq\lambda(v)$ for each vertex s of A different from v and v'. Let u be the vertex of A adjacent (in A) to v and let u' be the vertex of A adjacent (in A) to v'. Since $n>2$, $u\neq v'$ and $u'\neq v$. Let B denote the subarc of A joining u and u'. Consider the points $\lambda(u)$ and $\lambda(u')$. Note that $\lambda(u)\neq\lambda(v)\neq\lambda(u')$. Since T is a tree and each of the points $\lambda(u)$ and $\lambda(u')$ is adjacent to $\lambda(v)$, we have the result that either $\lambda(u)=\lambda(u')$ or $\lambda(v)$ separates T between $\lambda(u)$ and $\lambda(u')$. In the last case there exist a vertex s of B such that $\lambda(s)=\lambda(v)$, which contradicts our assumption. So $\lambda(u)=\lambda(u')$, and by the inductive hypothesis we have the result that $\psi(u)=\psi(u')$. Now, suppose that $\psi(v)\neq\psi(v')$. Then $\langle\psi(v),\ \psi(u)\rangle$ and $\langle\psi(u'),\ \psi(v')\rangle$ are two distinct intersecting edges of G_1 that are mapped by φ onto one edge $\langle\varphi(\psi(v)),\ \varphi(\psi(u))\rangle=\langle\sigma(\lambda(v)),\ \sigma(\lambda(u))\rangle$, a contradiction because φ is ultra light. Hence the claim is true.

Since $\lambda(G_2)=T$, for each vertex t of T there is a vertex $v\in\mathscr{V}(G_2)$ such that $\lambda(v)=t$. Define $\sigma'(t)=\psi(v)$. To complete the proof it is enough to show that σ' is a simplicial map. Let u and u' be a pair of adjacent vertices of T. Since T is a tree and $\lambda(G_2)=T$, there are two adjacent vertices s and s' of G_2 such that $\lambda(s)=u$ and $\lambda(s')=u'$. Using the claim we infer that $\sigma'(u)=\psi(s)$ and $\sigma'(u')=\psi(s')$, so $\sigma'(u)$ and $\sigma'(u')$ either coincide or are adjacent and consequently σ' is a simplicial map. □

5. Factorization through an arc and compositions of map

In this section we will show how to use the operation d to prove that some inverse limits with simplicial bonding maps are not chainable. In view of Theorem 3.3, it suffices to show that an composition of the bonding maps cannot be factored through an arc. We do that by applying some iteration of d to the inverse system and observing that the system we get is essentially the same as before but one map shorter. We illustrate the technique on examples of classic atriodic continua by Ingram [3, 4], and Davis and Ingram [2]. A similar proof will be used in [7] to get an example of an atriodic continuum which is simple 4-od-like but not simple triod-like.

Definition 5.1. We will say that a graph G' *subdivides* a graph G if G' is a graph obtained from G by adding vertices on some of its edges. More precisely, G' is a graph such that $\mathscr{V}(G) \subset \mathscr{V}(G')$ and for every edge $e \in \mathscr{E}(G)$ there is an arc (e, G') contained in G' such that

(i) (e, G') has the same endpoints as e,

(ii) $(d, G') \cap (e, G') = d \cap e$ for $d, e \in \mathscr{E}(G)$ and $d \neq e$, and

(iii) every vertex from $\mathscr{V}(G')$ belongs to some (e, G') and every edge from $\mathscr{E}(G')$ is an edge of some (e, G').

If v is a vertex of G and e is an edge of G containing v, then by (v, e, G') we denote the edge of (e, G') containing v.

Proposition 5.2. *If G' is a graph subdividing a graph G and G'' is a graph subdividing G', then G'' subdivides G.*

Definition 5.3. Let $\varphi : G_1 \to G_0$ be a simplicial map between graphs. Let G_0' be a graph subdividing G_0 and let φ' be a simplicial map of a graph G_1' subdividing G_1 onto G_0'. We will say that φ' is a *subdivision* of φ *matching* G_0' provided that $\varphi'(v) = \varphi(v)$ for each vertex $v \in \mathscr{V}(G_1)$, and for each edge $e \in \mathscr{E}(G_1)$ we have that

– if $\varphi(e)$ is degenerate then $(e, G_1') = e$, and

– if $\varphi(e)$ is an edge of G_0 then φ' is an isomorphism of (e, G_1') onto $(\varphi(e), G_0')$.

Proposition 5.4. *Let $\varphi : G_1 \to G_0$ be a simplicial map between graphs. Let G_0' be a graph subdividing G_0. Then there is a subdivision φ' of φ matching G_0'. Moreover, φ' is unique up to an isomorphism.*

Definition 5.5. Suppose G is a graph and S is a function from $\mathscr{V}(G)$ into the set of nonempty subsets of $\mathscr{E}(G)$. We say that S is an *edge selection on* G if v is a vertex of e for each $v \in \mathscr{V}(G)$ and each $e \in S(v)$.

Suppose G_0 and G_1 are graphs, S is an edge selection on G_1 and φ is a simplicial map from a subdivision G_1' of G_1 into G_0. We say that φ is consistent on S provided that there is a simplicial isomorphism λ from a subdivision H_1 of G_1 onto $D(\varphi, G_1')$ such that

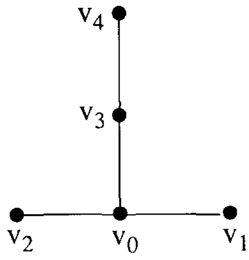


Fig. 4.

(i) $(v, e, G_1') \subset [\lambda(v)]^*$ for each $v \in \mathscr{V}(G_1)$ and each $e \in S(v)$, and

(ii) $[\lambda(v)]^* \subset (e, G_1')$ for each $e \in \mathscr{E}(G_1)$ and $v \in \mathscr{V}((e, H_1)) \setminus \mathscr{V}(G_1)$.

λ will be called a consistency isomorphism.

Example 5.6. We will consider again (see Example 2.5) the Ingram map from [3]. This time it will be important to us that the map takes the extended triod into itself, or rather, the domain is a subdivision of the range. Let T indicate the extended triod with its vertices named as in Fig. 4.

Fig. 5 indicates the Ingram map from a tree T' subdividing T onto T. We will denote this map by I. The dashed line graph is the domain of the map while the solid black is the range and each vertex of the domain is mapped onto the nearest vertex of the range. Note that $I(v_0) = v_2$, $I(v_1) = I(v_2) = I(v_4) = v_1$ and $I(v_3) = v_4$.

Let $\sigma : T \to T$ denote the symmetry of T about the axis $v_0 - v_3 - v_4$, that is $\sigma(v_0) = v_0$, $\sigma(v_1) = v_2$, $\sigma(v_2) = v_1$, $\sigma(v_3) = v_3$ and $\sigma(v_4) = v_4$. Let $\tilde{I}$ denote the composition $\sigma \circ I$.

Let S be an edge selection on T defined in the following way: $S(v_0) = \{\langle v_0, v_2 \rangle, \langle v_0, v_3 \rangle\}$ and $S(v_i)$ consists of all edges of T containing v_i for $=$ 1, 2, 3, 4. Observe that both I and $\tilde{I}$ are consistent on S. Let λ and $\tilde{\lambda}$ denote the consistency isomorphisms for I and $\tilde{I}$, respectively. Denote the map $d[\mathrm{I}] \circ \lambda$ by I_1, and $d[\tilde{\mathrm{I}}] \circ \tilde{\lambda}$ by $\tilde{I}_1$. Fig. 6 indicates I_1. As usual, the dashed line graph is the domain of the map while the solid black is the range and each vertex of the

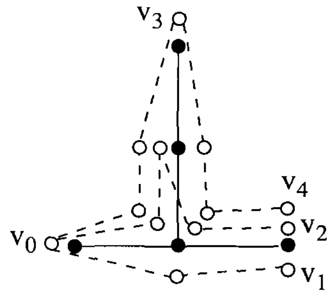


Fig. 5.

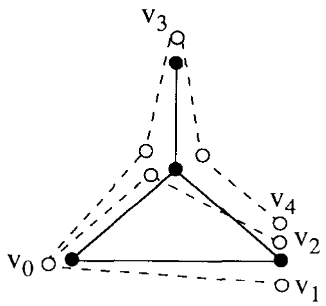


Fig. 6.

domain is mapped onto the nearest vertex of the range. A figure for $\tilde{I}_1$ would be like Fig. 6 reflected about a vertical line.

Again, observe that both I_1 and $\tilde{I}_1$ are consistent on S. (Note that $\langle v_0, v_1\rangle \notin S(v_0)$.) Let λ' and $\tilde{\lambda}'$ denote the consistency isomorphisms for I_1 and $\tilde{I}_1$, respectively. Denote the map $d[I_1]\circ\lambda'$ by I_2, and $d[\tilde{I}_1]\circ\tilde{\lambda}'$ by $\tilde{I}_2$. Fig. 7 indicates I_2. A figure for $\tilde{I}_2$ would be like Fig. 7 reflected about a vertical line. Observe that both I_2 and $\tilde{I}_2$ are ultra light.

Definition 5.7. Suppose that G_1 and G_2 are graphs. Let S_1 and S_2 be edge selections on G_1 and G_2, respectively. Let G_2' be a subdivision of G_2 and let $\psi: G_2' \to G_1$ be a simplicial map. We say that ψ *preserves* (S_1, S_2) provided that
(i) $\psi((v, e, G_2')) \in S_1(\psi(v))$ for each $v \in \mathscr{V}(G_2)$ and each $e \in S_2(v)$ and
(ii) for each two different edges $e, e' \in \mathscr{E}(G_2')$ intersecting at a common vertex v we have that either $\psi(e) \in S_1(\psi(v))$ or $\psi(e') \in S_1(\psi(v))$.

Example 5.8. Let $I: T' \to T$ and $\tilde{I}: T' \to T$ denote the Ingram maps defined in Example 5.6. Let S be the edge selection defined in the same example. Observe that both I and $\tilde{I}$ preserve (S, S).

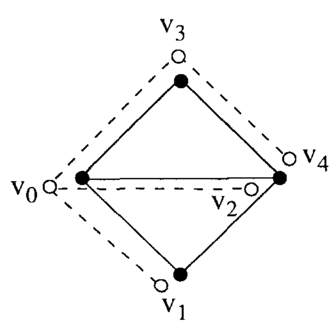


Fig. 7.

Lemma 5.9. *Suppose G_0 is a graph and G_1 and G_2 are trees. Let S_1 and S_2 be edge selections on G_1 and G_2, respectively. Let G_1' be a tree subdividing G_1 and let G_2' be a tree subdividing G_2. Suppose $\varphi: G_1' \to G_0$ and $\psi: G_2' \to G_1$ are light simplicial maps such that φ is consistent on S_1 and ψ preserves (S_1, S_2). Let $\lambda_1: H_1 \to D(\varphi, G_1')$ be a consistency isomorphism for φ, where H_1 is a subdivision of G_1. Let $\psi': G_2'' \to G_1'$ be a simplicial subdivision of ψ matching G_1' and let $\psi'': H_2 \to H_1$ be a subdivision of ψ matching H_1. Then $\varphi \circ \psi'$ is consistent on S_2 with a consistency isomorphism $\lambda_2: H_2 \to D(\varphi \circ \psi', G_2'')$ such that $\lambda_1 \circ \psi'' = d[\varphi, \psi'] \circ \lambda_2$.*

Proof. Let v_2 be a vertex of H_2. Denote by v_1 the point $\psi''(v_2) \in \mathscr{V}(H_1)$. Let C_1 denote $[\lambda_1(v_1)]^*$ and e_0 denote $\varphi(C_1)$. Observe that e_0 is an edge of G_0 and C_1 is a component of $\varphi^{-1}(e_0)$. We will define $\lambda_2(v_2)$ by considering the cases where $v_2 \in \mathscr{V}(G_2')$ and $v_2 \in \mathscr{V}(H_2) \setminus \mathscr{V}(G_2')$.

Case 1: $v_2 \in \mathscr{V}(G_2')$. In this case $v_1 = \psi(v_2) \in \mathscr{V}(G_1)$. We will prove that

(i) there is an edge $e_2' \in \mathscr{E}(G_2')$ containing v_2 such that $\psi(e_2') \in S_1(v_1)$. In case where $v_2 \in \mathscr{V}(G_2)$, let e_2 be an edge from $S_2(v_2)$ and let $e_2' = (v_2, e_2, G_2')$. Since ψ preserves (S_1, S_2), we have the result that $\psi(e_2') \in S_1(v_1)$.
In case where $v_2 \in \mathscr{V}(G_2') \setminus \mathscr{V}(G_2)$, let e_2 be the edge of G_2 such that v_2 is a vertex of (e_2, G_2'). Let e_2' and e_2'' be the two edges of (e_2, G_2') containing v_2. Since ψ preserves (S_1, S_2), we have the result at least one of these two edges, say e_2', has the property that $\psi(e_2') \in S_1(v_1)$. Thus (i) holds.

Denote $\psi(e_2')$ by e_1. Observe that $(v_1, e_1, G_1') \subset [\lambda_1(v_1)]^*$. Since φ is light, $\varphi((v_1, e_1, G_1')) = e_0$. Let C_2 be the component of $(\varphi \circ \psi')^{-1}(e_0)$ containing v_2. Since $\psi'((v_2, e_2', G_2'')) = (v_1, e_1, G_1')$, $(v_2, e_2', G_2'') \subset C_2$ and therefore $e_0 = \varphi(\psi'(C_2))$. Let $\lambda_2(v_2)$ be the element of $D(\varphi \circ \psi', G_2'')$ representing C_2.

We will prove additionally that if $v_2 \in \mathscr{V}(G_2') \setminus \mathscr{V}(G_2)$, then $C_2 \subset (e_2', G_2'') \cup (e_2'', G_2'')$. Suppose this is not true. Then there are two edges a and b of G_2' meeting at a common vertex v such that $\psi(v) \neq v_1$ and $(v, a, G_2'') \cup (v, b, G_2'') \subset C_2$. Since ψ preserves (S_1, S_2), without loss of generality, we may assume that $\psi(a) \in S_1(\psi(v))$. Since λ_1 is a consistency isomorphism $(\psi(v), \psi(a), G_1') \subset [\lambda_1(\psi(v))]^*$. Observe that $\psi'((v, a, G_2'')) = (\psi(v), \psi(a), G_1')$. So $\psi'((v, a, G_2''))$ is an edge contained in both $[\lambda_1(\psi(v))]^*$ and $[\lambda_1(v_1)]^*$. It follows that $\lambda_1(\psi(v)) = \lambda_1(v_1)$, a contradiction because λ_1 is an isomorphism and $\psi(v) \neq v_1$.

Case 2: $v_2 \in \mathscr{V}(H_2) \setminus \mathscr{V}(G_2')$. Let $e_2 \in \mathscr{E}(G_2')$ be the edge such that $v_2 \in (e_2, H_2)$. Observe that $\psi(e_2)$ is an edge of G_1. Denote this edge by e_1. Since ψ' is a subdivision of ψ matching G_1', ψ' maps (e_2, G_2'') isomorphically onto (e_1, G_1'). Since $v_1 \in \mathscr{V}((e_1, H_1)) \setminus \mathscr{V}(G_1)$ and consequently $C_1 = [\lambda_1(v_1)]^* \subset (e_1, G_1')$ there is exactly one component C_2 of $(\psi')^{-1}(C_1) \cap (e_2, G_2'')$ such that $\psi'(C_2) = C_1$. We will show that C_2 is a component of $(\varphi \circ \psi')^{-1}(e_0)$. Clearly, $C_2 \subset (\varphi \circ \psi')^{-1}(e_0)$. Suppose C_2 is not a component of $(\varphi \circ \psi')^{-1}(e_0)$. Then there is an edge $a \in \mathscr{E}(G_2')$ meeting e_2 at a common vertex v such that $a \neq e_2$ and $\psi'((v, a, G_2'')) \subset C_1$. Since $v \in \mathscr{V}(G_2')$ and $v_2 \in \mathscr{V}(H_2) \setminus \mathscr{V}(G_2')$, $\psi(v) \neq \psi'((v_2) = v_1$. Since G_1 is a tree and C_1 is connected, $\psi'((v, e_2, G_2'')) \subset C_1$. Since ψ preserves (S_1, S_2), either $\psi(a) \in$

$S_1(\psi(v))$ or $\psi(e_2) \in S_1(\psi(v))$. In either case we have the result that $C_1 = [\lambda_1(\psi(v))]^*$ and $\lambda_1(\psi(v)) = \lambda_1(v_1)$, which is impossible because λ_1 is an isomorphism. Thus C_2 is a component of $(\varphi \circ \psi')^{-1}(e_0)$. Let $\lambda_2(v_2)$ be the element of $D(\varphi \circ \psi', G_2')$ representing C_2.

Clearly, λ_2 is a simplicial map satisfying (i) and (ii) of Definition 5.5 and such that $\lambda_1 \circ \psi'' = d[\varphi, \psi'] \circ \lambda_2$. Observe also that $v_2 \in [\lambda_2(v_2)]^*$ for each $v_2 \in G_2'$. We will prove that λ_2 is an isomorphism.

Let w be an arbitrary vertex of $D(\varphi \circ \psi', G_2'')$ and let e'' be an edge of G_2'' contained in w^*. There is edge $e' \in \mathscr{E}(G_2')$ such that $e'' \subset (e', G_2'')$. Let U be the union of $[\lambda_2(v)]^*$ where $v \in \mathscr{V}((e', H_2))$. Since U is connected and it contains the endpoints of e', there is $v \in (e', H_2)$ such that $e'' \subset [\lambda_2(v)]^*$. Observe that $\lambda_2(v) = w$ and thus λ_2 is surjective.

To conclude the proof it remains to show that λ_2 is a bijection. Clearly, it will be enough to prove that λ_2 restricted to $\mathscr{V}(G_2')$ is a bijection. Let c be a vertex of $D(\varphi \circ \psi', G_2'')$ and let C denote the set c^*. Suppose that v_2 and v_2' are two different vertices of G_2' such that $\lambda_2(v_2) = c = \lambda_2(v_2')$. Observe that $v_2 \in C$ and $v_2' \in C$. Since φ is light and λ_1 is an isomorphism, either $\psi(v_2) = \psi(v_2')$ or $[\lambda_1(\psi(v_2))]^* \cap [\lambda_1(\psi(v_2'))]^*$ does not contain an edge. Since $\psi'(C) \subset [\lambda_1(\psi(v_2))]^* \cap [\lambda_1(\psi(v_2'))]^*$ and ψ' is light we have the result that $\psi(v_2) = \psi(v_2')$. Observe that v_2 and v_2' are not adjacent in G_2', because ψ is light. Since C is connected and G_2'' is a tree, $\langle v_2, v_2' \rangle \subset C$. Let a and b be the two edges of G_2' contained in $\langle v_2, v_2' \rangle$ intersecting at some vertex v. Since $\lambda_2(v)$ was defined in such a way that either $(v, a, G_2'') \subset [\lambda_2(v)]^*$ or $(v, b, G_2'') \subset [\lambda_2(v)]^*$, we have the result that $\lambda_2(v) = \lambda_2(v_2)$ and consequently $\lambda_1(\psi(v)) = \lambda_1(v_1)$ for each $v \in \langle v_2, v_2' \rangle \cap \mathscr{V}(G_2')$. Since λ_1 is an isomorphism $\psi(v) = \psi(v_2)$ for each $v \in \langle v_2, v_2' \rangle \cap \mathscr{V}(G_2')$. This is impossible, because ψ is light. □

Definition 5.10. Let n be a positive integer and let N denote either the set $\{0, 1, \dots, n\}$ or the set of all nonnegative integers. Denote by N_1 the set $N \setminus \{0\}$. Let $G_0, G_1, G_2, \dots$ be a sequence of graphs with N as the set of indices. Let Σ be a sequence of simplicial maps $\varphi_1, \varphi_2, \dots$ such that for each $j \in N_1$, φ_j maps a graph G_j' subdividing G_j into G_{j-1}. Using inductively Proposition 5.4, we can define a sequence of simplicial maps $\psi_1, \psi_2, \dots$ such that $\psi_1 = \varphi_1$ and for each $j \in N_1 \setminus \{1\}$, ψ_j subdivides ψ_j matching the domain of ψ_{j-1}. For each $j \in N_1$, denote by Σ_j the domain of ψ_j. Set $\Sigma_0 = G_0$. For every two integers i and j from N such that $i > j$, let Σ_j^i denote the composition $\psi_{j+1} \circ \cdots \circ \psi_j$ mapping Σ_i into Σ_j. We will say that the inverse system $\{\Sigma_j, \Sigma_j^i\}$ is generated by the sequence Σ.

Let S_j be an edge selection on G_j for $j \in N_1$. We will say that Σ preserves the sequence $S_1, S_2, \dots$ if φ_j preserves (S_{j-1}, S_j) for each $j \in N_1 \setminus \{1\}$.

We say that two inverse (possibly finite) systems $\{K_j, \kappa_j^i\}$ and $\{H_j, \eta_j^i\}$ are isomorphic if there is a sequence of isomorphisms $\lambda_0, \lambda_1, \dots$, where $\lambda_j : K_j \to H_j$ such that $\lambda_j \circ \kappa_j^i = \eta_j^i \circ \lambda_i$ for $i > j \geqslant 0$.

Theorem 5.11. *Let n be a positive integer and let N denote either the set* $\{0, 1, \dots, n\}$ *or the set of all nonnegative integers. Let* N_1 *denote the set* $N \setminus \{0\}$. *Let* G_0 *be a graph and let* $G_1, G_2, \dots$ *be a sequence of trees with* N_1 *as the set of indices. Let* S_j *be an edge selection on* G_j *for* $j \in N_1$. *Let* Σ *be a sequence of simplicial maps* $\varphi_1, \varphi_2, \dots$ *such that for each* $j \in N_1$, φ_j *maps a graph* G_j' *subdividing* G_j *into* G_{j-1}. *Suppose* φ_1 *is consistent on* S_1 *and* Σ *preserves the sequence* $S_1, S_2, \dots$. *Let* $\lambda_1 : H_1 \to D(\varphi_1, G_1')$ *be a consistency isomorphism for* φ_1, *where* H_1 *is a subdivision of* G_1. *Then the system* $\{D(\Sigma_0^j, \Sigma_j), d[\Sigma_0^j, \Sigma_j^i]\}$ *is isomorphic to the system generated by the sequence* $d[\varphi_1] \circ \lambda_1, \varphi_2, \varphi_3, \dots$.

Proof. For each $j \in N_1 \setminus \{1\}$, let $\psi_j : H_j \to H_{j-1}$ be a simplicial subdivision φ_j of matching H_{j-1}. Let H_0 denote $D(G_0)$ and let $\psi_1 = d[\varphi_1] \circ \lambda_1$. Note that the system $\{H_j, \psi_j\}$ is generated by the sequence $d[\varphi_1] \circ \lambda_1, \varphi_2, \varphi_3, \dots$.

Applying Lemma 5.9 repeatedly, we infer that, for each $j \in N_1 \setminus \{1\}$, there is a consistency isomorphism λ_j of H_j onto $D(\Sigma_0^j, \Sigma_j)$ such that $\lambda_{j-1} \circ \psi_j = d[\Sigma_0^{j-1}, \Sigma_{j-1}^j] \circ \lambda_j$.

Let λ_0 be the identity on $D(G_0)$. Observe that the sequence $\lambda_0, \lambda_1, \lambda_2, \dots$ defines an isomorphism between $\{H_j, \psi_j\}$ and $\{D(\Sigma_0^j, \Sigma_j), d[\Sigma_0^j, \Sigma_j^i]\}$. □

Example 5.12. Let $I : T' \to T$ and $\tilde{I} : T' \to T$ denote the Ingram maps defined here in Example 5.6. Let Σ be an infinite sequence of simplicial maps $\varphi_1, \varphi_2, \dots$ each of which is either I or $\tilde{I}$. By $\{\Sigma_j, \Sigma_j^i\}$ we denote the system generated by Σ. Ingram proved that the inverse limit of Σ has positive span and therefore is not chainable (see [3, 4]). We will give here an alternate proof of this statement.

First we will prove that for each choice of $\varphi_1, \varphi_2, \dots$ we have

Claim. Σ_0^n *cannot be factored through an arc.*

Clearly, the claim is true if $n = 1$. Now, suppose that the claim is true for each sequence of $n - 1$ maps each of which is either I or $\tilde{I}$. In particular, we assume that the claim is true for the sequence $\varphi_2, \dots, \varphi_n$.

Let $I_1, \tilde{I}_1, I_2, \tilde{I}_2, \lambda, \tilde{\lambda}, \lambda'$ and $\tilde{\lambda}'$ be as in Example 5.6. If $\varphi_1 = I$ then set $\lambda_1 = \lambda$, $\psi_1 = I_1$ and $\lambda_1' = \lambda'$. Otherwise, if $\varphi_1 = \tilde{I}$ then set $\lambda_1 = \tilde{\lambda}$, $\psi_1 = \tilde{I}_1$ and $\lambda_1' = \tilde{\lambda}'$. Use Theorem 5.11 to get the result that the system $\{D(\Sigma_0^j, \Sigma_j), d[\Sigma_0^j, \Sigma_j^i]\}_{j=0}^n$ is isomorphic to the system generated by the sequence $d[\varphi_1] \circ \lambda_1, \varphi_2, \varphi_3, \dots, \varphi_n$. Use Theorem 5.11 again to infer that the system $\{D^2(\Sigma_0^j, \Sigma_j), d^2[\Sigma_0^j, \Sigma_j^i]\}_{j=0}^n$ is isomorphic to the system generated by the sequence $d[\psi_1] \circ \lambda_1', \varphi_2, \varphi_3, \dots, \varphi_n$. Let Γ denote the sequence $d[\psi_1] \circ \lambda_1', \varphi_2, \varphi_3, \dots, \varphi_n$ and let $\{\Gamma_j, \Gamma_j^i\}_{j=0}^n$ denote the system generated by Γ.

Suppose Σ_0^n can be factored through an arc. Then, by Theorem 2.12, $d^2[\Sigma_0^n]$ and consequently Γ_0^n can be factored through an arc. Since the map $\Gamma_0^1 = d[\psi_1] \circ \lambda_1'$ is either I_2 or $\tilde{I}_2$, it is ultra light (see Example 5.6). By Theorem 4.3, Γ_1^n can be factored through an arc. Since the domain of Γ_0^1 is T, the system $\{\Gamma_j, \Gamma_j^i\}_{j=1}^n$ is generated by $\varphi_2, \dots, \varphi_n$ and according to our assumption Γ_1^n cannot be factored through an arc. This contradiction proves the claim.

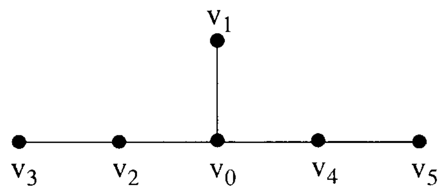


Fig. 8.

It follows from Theorem 3.3 that the inverse limit of the system $\{\Sigma_j, \Sigma_j^i\}$ is not chainable and has positive span.

Proposition 5.13. *Suppose $\varphi : G_1 \to G_0$ is a simplicial map between graphs. Let G_0' be a graph subdividing G_0 and let $\varphi' : G_1' \to G_0'$ be a subdivision of φ matching G_0'. Then φ can be factored through an arc if and only if φ' can be factored through an arc.*

Proof. Observe that clearly, if φ can be factored through an arc, then φ' also can be factored through an arc. Suppose that there is an arc A' and there are simplicial maps $\alpha' : G_1' \to A'$ and $\beta' : A' \to G_0'$ such that $\beta' \circ \alpha' = \varphi'$. Let $V = \{v \in \mathscr{V}(A') \mid \beta'(v) \in \mathscr{V}(G_0)\}$. Let A denote the graph with V as its set of vertices such that two vertices $v_1, v_2 \in V$ are adjacent if the subarc of A' between v_1 and v_2 does not contain other points of V. Clearly, A is an arc. Let $\beta : A \to G_0$ be such that $\beta(v) = \beta'(v)$ for each $v \in V$. Note that β is a simplicial map. Observe that $\alpha'(v) \in V$ for each $v \in \mathscr{V}(G_1)$. Let $\alpha : G_1 \to A$ be such that $\alpha(v) = \alpha'(v)$ for each $v \in \mathscr{V}(G_1)$. One can verify that α is a simplicial map and $\beta \circ \alpha = \varphi$. □

Example 5.14. We will consider here the continuum defined in [2] by Davis and Ingram. Davis and Ingram showed that the continuum has positive span and therefore is not chainable. We will give here an alternate proof of this statement.

Let T indicate the extended triod with its vertices named as in Fig. 8.

Fig. 9 indicates the Davis–Ingram map from a tree T' subdividing T onto T. The map will be denoted here by δ. As usual, the dashed line graph is the domain of the map while the solid black is the range and each vertex of the domain is mapped onto the nearest vertex of the range. Note that $\delta(v_0) = v_2$, $\delta(v_1) = v_3$, $\delta(v_2) = \delta(v_4) = v_4$ and $\delta(v_3) = \delta(v_5) = v_5$.

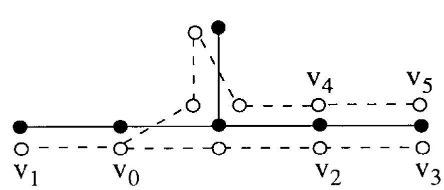


Fig. 9.

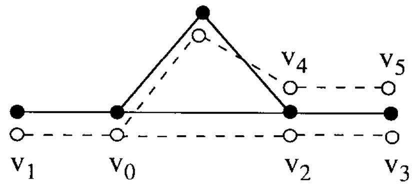


Fig. 10.

Let $\sigma : T \to T$ denote the symmetry of T about the axis v_0-v_1, that is $\sigma(v_0) = v_0$, $\sigma(v_1) = v_1$, $\sigma(v_2) = v_4$, $\sigma(v_3) = v_5$, $\sigma(v_4) = v_2$ and $\sigma(v_5) = v_3$. Let $\tilde{\delta}$ denote the composition $\sigma \circ \delta$.

Let S be an edge selection on T defined in the following way: $S(v_0) = \{\langle v_0, v_2\rangle, \langle v_0, v_4\rangle\}$, $S(v_1) = \{\langle v_0, v_1\rangle\}$, $S(v_2) = \{\langle v_0, v_2\rangle\}$, $S(v_3) = \{\langle v_2, v_3\rangle\}$, $S(v_4) = \{\langle v_0, v_4\rangle\}$ and $S(v_5) = \{\langle v_4, v_5\rangle\}$. Observe that both δ and $\tilde{\delta}$ preserve (S, S). Observe also that both δ and $\tilde{\delta}$ are consistent on S. Let λ and $\tilde{\lambda}$ denote the consistency isomorphisms for δ and $\tilde{\delta}$, respectively. Denote the map $d[\delta] \circ \lambda$ by δ_1, and $d[\tilde{\delta}] \circ \tilde{\lambda}$ by $\tilde{\delta}_1$. Figs. 10 and 11 indicate (in the usual convention) δ_1 and $\tilde{\delta}_1$, respectively. Note that both δ_1 and $\tilde{\delta}_1$, are ultra light.

Let Σ be an infinite sequence of simplicial maps $\varphi_1, \varphi_2, \ldots$ each of which is either δ or $\tilde{\delta}$. By $\{\Sigma_j, \Sigma_j^i\}$ we denote the system generated by Σ. (If $\varphi_i = \delta$ for each $i = 1, 2, \ldots,$ the system $\{\Sigma_j, \Sigma_j^i\}$ is identical with the one described in [2].) We will prove for each choice of $\varphi_1, \varphi_2, \ldots$ we have that

Claim. *Σ_0^n cannot be factored through an arc.*

Clearly, the claim is true if $n = 1$. Now, suppose that the claim is true for each sequence of $n - 1$ maps each of which is either δ or $\tilde{\delta}$. In particular, we assume that the claim is true for the sequence $\varphi_2, \ldots, \varphi_n$.

If $\varphi_1 = \delta$ then set $\lambda_1 = \lambda$, otherwise, if $\varphi_1 = \tilde{\delta}$ then set $\lambda_1 = \tilde{\lambda}$. Let Γ denote the sequence $d[\varphi_1] \circ \lambda_1, \varphi_2, \varphi_3, \ldots, \varphi_n$ and let $\{\Gamma_j, \Gamma_j^i\}_{j=0}^n$ denote the system generated by Γ. Use Theorem 5.11 to get the result that the system $\{D(\Sigma_0^i, \Sigma_j), d[\Sigma_0^j, \Sigma_j^i]\}_{j=0}^n$ is isomorphic to $\{\Gamma_j, \Gamma_j^i\}_{j=0}^n$.

Suppose Σ_0^n can be factored through an arc. Then, by Theorem 2.12, $d[\Sigma_0^n]$ and consequently Γ_0^n can be factored through an arc. Since the map $\Gamma_0^1 = d[\varphi_1] \circ \lambda_1$ is

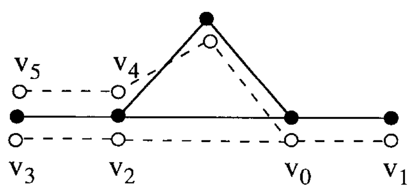


Fig. 11.

either δ_1 or $\tilde{\delta}_1$, it is ultra light. By Theorem 4.3, Γ_1^n can be factored through an arc. Since the domain of Γ_0^1 is a graph subdividing T, the system $\{\Gamma_j, \Gamma_j^i\}_{j=1}^n$ is generated by subdivisions of $\varphi_2, \ldots, \varphi_n$ and, according to our assumption and Proposition 5.13, Γ_1^n cannot be factored through an arc. This contradiction proves the claim.

It follows from Theorem 3.3 that the inverse limit of the system $\{\Sigma_j, \Sigma_j^i\}$ is not chainable and has positive span.

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