

Contents lists available at ScienceDirect

Information and Computation

journal homepage: www.elsevier.com/locate/ic

Soliton automata with constant external edges

Miklós Krész

Department of Computer Science, Juhász Gyula Faculty of Education, University of Szeged, Szeged, Hungary

ARTICLE INFO

Article history:

Received 6 July 2007

Revised 21 February 2008

Available online 6 June 2008

Keywords:

Molecular electronics

Soliton automata

Matchings in graphs

Automata product

ABSTRACT

Soliton automata are the mathematical models of certain possible molecular switching devices. Both from theoretical and practical point of view, it is a central question to describe soliton automata with constant external edges. Extending a result of Dassow and Jürgensen, we characterize soliton automata in this special case.

© 2008 Elsevier Inc. All rights reserved.

1. Introduction

Molecular computing is an emerging field in current theoretical and application-oriented research [25] as well. The idea of molecular memories goes back to Feynman's pioneer paper (cf. [14]), in which he proposes building small machines and then using those machines to build still smaller machines and so on, down to the molecular level. During the past decades several promising concepts have been worked out for unconventional computing; among these nonlinear media that exhibit self-localized mobile patterns in their evolution are potential candidates for the role of universal dynamical computers. The computational model called soliton cellular automata (see e.g. [23,24,26]) uses soliton interactions in the design of collision-based logic gates. The word soliton ("solitary wave") is applied to certain types of waves traveling relatively large distance with little energy loss. For a survey of unconventional architectures on the principles of molecular computing, see [1].

Other alternatives of molecular computers are based on the design of conventional digital circuits on the molecular level ([10]). The idea of this approach is: if we build up the electronic elements chemically from the molecular level, it would be possible to make circuits thousands of times smaller. These molecular circuits would use chemical molecules as electronic switches and be interconnected by some sort of ultra-fine conducting wires. One interesting possibility of these conductors was proposed by Carter ([9]) and is about using single strands of the electrically conductive plastic polyacetylene. In this case, soliton waves are induced by electrons travelling along polyacetylene in little packets. Hence, molecular scale electronic devices constructed from molecular switches and polyacetylene chains are called *soliton circuits*.

The practical research in soliton circuits (see e.g. [15,16]) has evoked the need to develop an applied mathematical arsenal in order to obtain a detailed understanding of the behavior of these circuits. The mathematical model of soliton circuits called soliton automata was introduced in [11], but it was not until [4] that matching theory ([22]) was recognized as the fundamental theoretical background for the study of this model. The underlying object of a soliton automaton is the topological model of the corresponding molecule chain called soliton graph. In this model, a soliton graph comes with a perfect internal matching, i.e. a matching that covers all the vertices with degree at least two. These vertices—called internal—model carbon atoms, whereas vertices with degree one—called external—represent a suitable chemical interface with the outside world.

E-mail address: kresz@jgypk.u-szeged.hu

Soliton graphs and automata have been systematically studied on the grounds of matching theory (see e.g. [3,4,6,8]). Perhaps the most significant contribution along this line is [6], where soliton graphs are decomposed into elementary components, and these components are grouped into pairwise disjoint families based on how they can be reached by alternating paths starting from external vertices. This decomposition is carried over to soliton automata in [5], using quasi-direct and α_0 -products of their component automata. The above structural results have significant algorithmic consequences for the verification and simulation of soliton circuits, as it was outlined in [19]. Moreover, the basic building elements of soliton circuits were characterized by the description of complete systems of soliton automata (cf. [18]).

It seems to be a fundamental question to determine the computational power of soliton automata. Several special cases have been described with respect to their transition monoids. (see e.g. [11,12,13]). In this paper we generalize the results of [12] where deterministic soliton automata with a single external vertex were characterized. First, we extend this result by considering nondeterministic automata. It will turn out that the transition monoids of soliton automata with a single external vertex are special null monoids. Then, as a generalization, we describe the class of soliton automata with constant external edges by products of automata characterized in the first part of the paper.

Though studying soliton automata with constant external edges is interesting in itself, the main reason for considering this special case is that these automata play a central role in the decomposition of soliton automata (cf. [5]). With the help of this decomposition, the descriptive complexity of soliton automata can be significantly reduced ([20]).

2. Perfect internal matchings in graphs

As it was mentioned in the introduction, the fundamental technique for the analysis of soliton automata is based on the concept of perfect internal matchings. Therefore, in this section we review some of the basic notions concerning graphs and matchings. Our notation and terminology will be compatible with that of [22], except that “point” and “line” will be replaced by the more conventional terminology “vertex” and “edge”, respectively.

By a graph we shall mean a finite undirected graph in the most general sense, i.e. with multiple edges and loops allowed. For a graph G , $V(G)$ and $E(G)$ will denote the set of vertices and the set of edges of G , respectively. The concept of walk, cycle and path can be defined in a usual way. If all edges in a walk are distinct, the walk is called a *trail*. The subtrail of a trail α between vertices v_i and v_j is denoted by $\alpha[v_i, v_j]$, while the notation α^{-1} will be used to represent the reverse of α .

If the vertex set of a graph G can be partitioned into two disjoint non-empty sets, $V(G) = A \cup B$, such that all edges of G connect a vertex of A to a vertex of B , we call G *bipartite* and refer to $A \cup B$ as the *bipartition* of G .

A vertex $v \in V(G)$ is called *external* if its degree $d(v)$ is one, *internal* if $d(v) \geq 2$, and *isolated* otherwise. The sets of external and internal vertices of G will be denoted by $Ext(G)$ and $Int(G)$, respectively. External edges are those of $E(G)$ that are incident with at least one external vertex, and internal edges are those connecting two internal vertices. Graph G is called *open* if it has at least one external vertex, otherwise G is called *closed*. For the demonstration of the above terminology see Fig. 1, where the external vertices are u and v , while the external edges are e and f .

A *matching* M of graph G is a subset of $E(G)$ such that no vertex of G occurs more than once as an endpoint of some edge in M . Again, it is understood by this definition that loops are not allowed to participate in M . The endpoints of the edges contained in M are said to be *covered* by M . A *perfect internal matching* is a matching that covers all of the internal vertices.

An edge $e \in E(G)$ is *allowed* (*mandatory*) if e is contained in some (respectively, all) perfect internal matching(s) of G . *Forbidden edges* are those that are not allowed. We shall also use the term *constant edge* to identify an edge that is either forbidden or mandatory.

Now consider the graph G in Fig. 1 again. It is easy to see that the set $\{e, h_1, h_2\}$ determines a perfect internal matching in G and g being its unique forbidden edge.

By the usual definition, a *subgraph* G' of G is just a collection of vertices and edges of G . Since in our treatment we are particular about external vertices, we do not want to allow that new external vertices (i.e. ones that are not present in G) emerge in G' . Therefore, whenever this happens, so that vertex $v \in Int(G)$ becomes external in G' , we shall augment G' with a loop edge around v . This augmentation will be understood automatically in all subgraphs of G . Finally, for a subgraph G' and matching M of G , $M_{(G')}$ will denote the restriction of M to G' .

Let G be an open graph and M be a perfect internal matching of G , fixed for the rest of this section. An edge $e \in E(G)$ is said to be *M -positive* (*M -negative*) if $e \in M$ (respectively, $e \notin M$). An *M -alternating path* (*cycle*) in G is a path (respectively, even-length cycle) stepping on M -positive and M -negative edges in an alternating fashion. An *M -alternating loop* is an odd-length cycle having the same alternating pattern of edges, except that exactly one vertex has two negative edges incident with it. Let us agree that, if the matching M is understood or irrelevant in a particular context, then it will not be explicitly indicated in these terms. An *external alternating path* is one that has an external endpoint. If both endpoints of the path are external, then it is called a *crossing*. An alternating path is *positive* if it is such at its internal endpoints, meaning that the edges incident with those endpoints are positive. In Fig. 1, $\gamma = u, e, w, f, v$ is an alternating crossing and $\beta = z_1, l_1, z_4, h_2, z_3, l_2, z_2, h_1, z_1$ is an alternating cycle with respect to the perfect internal matching $M = \{f, l_1, l_2\}$.

An internal vertex v of G is called *accessible* from external vertex w in M (or simply v is *M -accessible* from w), if there exists a positive external M -alternating path connecting w and v . Furthermore, an alternating cycle is said to be *M -accessible* from w if some of its vertices is accessible from w in M . Generally it is not true that if a vertex is accessible from an external vertex w in a perfect internal matching, then it is accessible from w in all perfect internal matchings. Nevertheless, as it was

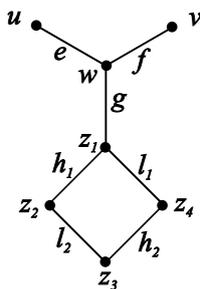


Fig. 1. Example graph with perfect internal matchings.

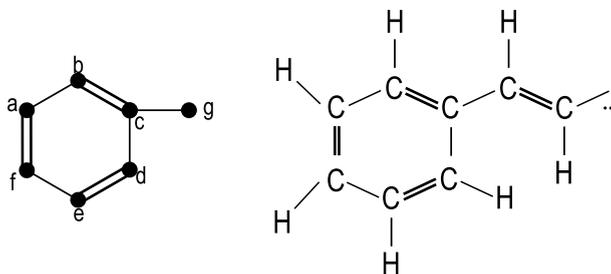


Fig. 2. A soliton graph with one of its interpretations.

proved in [6], the accessibility without specifying the external vertex is invariant with respect to perfect internal matchings. It is therefore meaningful to say that vertex v is accessible in G without specifying the perfect internal matching M and the external vertex w .

An M -alternating unit is either a crossing or an alternating cycle with respect to M . Switching on an alternating unit amounts to changing the sign of each edge along the unit. It is easy to see that the operation of switching on an M -alternating unit α creates a new perfect internal matching $S(M, \alpha)$ for G . Moreover, as it was proved in [4], every perfect internal matching M of G can be transformed into any other perfect internal matching M' by switching on a group of pairwise disjoint alternating units. A set of pairwise disjoint M -alternating units will be called an M -alternating network, and the alternating network determined by the symmetric difference of perfect internal matchings M and M' will be denoted by $N(M, M')$. We will also refer to $N(M, M')$ as the mediator alternating network between M and M' . The following important observation on alternating units was proved in [4].

Proposition 1 ([5]). *An edge e of a graph G having a perfect internal matching is not constant if and only if there exists an alternating unit passing through e in every perfect internal matching of G .*

3. Soliton graphs and automata

In this section, following [11], we introduce the concept of soliton automata as the mathematical model of switching at the molecular level by so-called “soliton valves”. Towards this goal, we first define the topological model of the underlying structure, which is a graph representing a molecule chain in which solitons travel along. In this simple model, vertices correspond to the atoms or certain groups of atoms, whereas the edges represent chemical bonds or chains of bonds. It is assumed that the molecules consist of carbon and hydrogen atoms only, and that among the neighbors of each carbon atom there exists a unique one to which the atom is connected by a double bond. The above property is captured by perfect internal matchings, where the edges contained in the given matching corresponding to the double bonds. Therefore, a *soliton graph* is defined as an open graph having a perfect internal matching. Since perfect internal matchings represent the states of the corresponding molecule chain, it is justified to refer them as *states* of the given soliton graph.

A soliton graph G models the underlying molecular structure as follows: Each internal vertex v represents a C atom or a C–H group depending on whether $d(v)$ is 3 or 2, respectively. A single (double) edge (v, w) in a given state represents a (CH)-chain with alternating double and single bonds which connects the C atoms of v and w and which begins and ends with a single (respectively, double) bond. As the length of such chains does not affect the logic of the model we draw them as length 1 chains; physico-chemical reasons may require different lengths for actual realizations. Finally, external vertices represent the connection to surrounding structures. Fig. 2 shows an example of a soliton graph and a possible chemical interpretation.

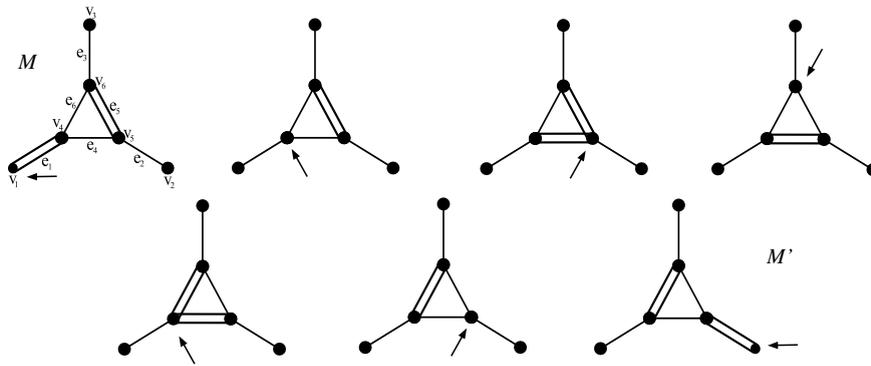


Fig. 3. A soliton walk.

It is clear that in the above model the degree of each internal vertex is at most 3. This restriction was applied in the original definition ([11], but omitting the above condition makes it possible to use more general techniques and constructions for soliton automata. Nevertheless, it can be proved (cf. [17]) that soliton graphs and automata in this generalized sense (soliton graphs without restrictions on the degree of the internal vertices) are equivalent to the original concept. Therefore, in the rest of the paper, by soliton graph, we mean the ones without degree restrictions.

For the study of the logical aspects of soliton switching we need to give a graph theoretic formalization of the state transitions induced by soliton waves. Ignoring the physico-chemical details, the effect of a soliton wave propagating along a polycetylene chain is to exchange all single and double bonds. This logical aspect is captured by the concept of soliton walk. Intuitively, a soliton walk is a backtrack-free walk which starts and ends at an external vertex, and alternates on matching covered and uncovered edges. However, the status of the traversed edges are exchanged dynamically step by step while making the walk.

Before the mathematical definition we give an informal description of soliton walks through the example in Fig. 3, which illustrates the effects in the graph model of a given molecule chain.

The initial state of the system is represented by perfect internal matching M . The walk $\alpha = v_1, e_1, v_4, e_4, v_5, e_5, v_6, e_6, v_4, e_4, v_2$ corresponding to the given soliton wave results in the sequence in the figure. In each of them the “position of the soliton” is indicated by an arrow. We note that, though during the walk, a step does not necessarily results in a perfect internal matching, but by the time the walk is finished, a new state M' of G is reached.

The walk starts at vertex v_1 and after traversing edge e_1 , the double bond is exchanged for single one; thus the status of the corresponding edge is exchanged. Then, the status of the traversed edge is exchanged dynamically. During the walk, if the soliton is about to continue its way on an uncovered edge—like in the second step at vertex v_4 —, then it might have several alternatives for the next step among the adjacent edges (e.g. both e_4 and e_6 could be chosen). However, if a situation of two adjacent covered edges occurs—like in the third step at vertex v_5 —, then the walk must continue on the appropriate covered edge. (Remember, that a soliton walk is backtrack-free.)

The concept of soliton walks can be formalized in soliton graphs in the following way. The collection of external alternating walks in G with respect to some state M , and the concept of switching on such walks are defined recursively as follows.

(i) The walk $\alpha = v_0 e v_1$, where $e = (v_0, v_1)$ with v_0 being external, is an external M -alternating walk, and switching on α results in the set $S(M, \alpha) = M \Delta \{e\}$. (The operation Δ is symmetric difference of sets.)

(ii) If $\alpha = v_0 e_1 \dots e_n v_n$ is an external M -alternating walk ending at an internal vertex v_n , and $e_{n+1} = (v_n, v_{n+1})$ is such that $e_{n+1} \in S(M, \alpha)$ iff $e_n \in S(M, \alpha)$, then $\alpha' = \alpha e_{n+1} v_{n+1}$ is an external M -alternating walk and

$$S(M, \alpha') = S(M, \alpha) \Delta \{e_{n+1}\}.$$

It is required, however, that $e_{n+1} \neq e_n$, unless $e_n \in S(M, \alpha)$ is a loop.

It is clear by the above definition that $S(M, \alpha)$ is a perfect internal matching iff the endpoint v_n of α is external, too. In this case, we say that α is a soliton walk.

Example. Consider the graph G of Fig. 1 again, and let $M = \{e, h_1, h_2\}$. Then $\gamma = uewgz_1 h_1 z_2 l_2 z_3 h_2 z_4 l_1 z_1 gwfv$ is a possible soliton walk from u to v with respect to M . Switching on γ then results in $S(M, \gamma) = \{f, l_1, l_2\}$.

Graph G gives rise to a soliton automaton $\mathcal{A}_G = (S_G, X \times X, \delta)$, the set S_G of states of which consists of the perfect internal matchings of G . The input alphabet $X \times X$ for \mathcal{A}_G is the set of all (ordered) pairs of external vertices in G —i.e. $X = Ext(G)$ —, and the transition function δ is defined by $\delta(M, (v, w)) = \{S(M, \alpha) \mid \alpha \text{ is an } M\text{-alternating soliton walk from } v \text{ to } w\}$. Nevertheless, if no soliton walk exists from v to w in M , then $\delta(M, (v, w)) = \{M\}$. Finally, as usual, we extend the transition function for any word $y \in (X \times X)^*$ —including the empty word ε —by $\delta(M, \varepsilon) = \{M\}$ and $\delta(M, ya) = \delta(\delta(M, y), a)$ with $a \in X \times X$.

Example. Consider the graph G in Fig. 1 again. This graph is a soliton graph having states: $s_h^e = \{e, h_1, h_2\}$, $s_l^e = \{e, l_1, l_2\}$, $s_h^f = \{f, h_1, h_2\}$, and $s_l^f = \{f, l_1, l_2\}$. The transitions of \mathcal{A}_G are the following:

$$\begin{aligned} \delta(s_h^e, (u, v)) &= \delta(s_l^e, (u, v)) = \{s_h^f, s_l^f\}, \\ \delta(s_h^e, (v, u)) &= \delta(s_l^e, (v, u)) = \{s_h^e, s_l^e\}, \\ \delta(s_h^e, (v, u)) &= \{s_h^f\}, \quad \delta(s_l^e, (v, u)) = \{s_l^f\}, \\ \delta(s_h^f, (u, v)) &= \{s_h^e\}, \quad \delta(s_l^f, (u, v)) = \{s_l^e\}, \\ \delta(s_h^f, (u, u)) &= \{s_h^e\}, \quad \delta(s_l^f, (u, u)) = \{s_l^e\}, \\ \delta(s_h^f, (v, v)) &= \{s_h^e\}, \quad \delta(s_l^f, (v, v)) = \{s_l^e\}. \end{aligned}$$

As an example, the transition $s_h^e \rightarrow s_l^f$ on input (u, v) is induced by the soliton walk:

$$uewgz_1h_1z_2l_2z_3h_2z_4l_1z_1gwfv.$$

One of the central goals of this paper is to describe *constant soliton automata*, i.e. soliton automata associated with a graph having constant external edges only (*constant soliton graphs*), in terms of products of smaller automata. Each of these automata will be either a soliton automaton itself or constructed from a soliton automaton in a certain way. Therefore, the input alphabet of each automaton is a Cartesian power X^2 over a set X . For automata with alphabet of this type we strengthen the concept of isomorphism.

Definition 2. Let X be an alphabet and for $i = 1, 2$, let $\mathcal{A}_i = (S_i, X \times X, \delta_i)$ be an automaton. We say that \mathcal{A}_1 and \mathcal{A}_2 are *strongly isomorphic* if there exists a bijection $\psi : S_1 \rightarrow S_2$ which satisfies the equation

$$\{\psi(s') \mid s' \in \delta_1(s, (x, x'))\} = \delta_2(\psi(s), (x, x'))$$

for every $s \in S_1$ and every $x, x' \in X$.

Recall from [11] that an edge e of G is *impervious* if there is no external alternating walk passing through e in any state of G . It is clear that impervious edges have no effect on the operations of soliton automata. Thus, without loss of generality, we can restrict our investigation to soliton graphs without impervious edges. Therefore, throughout the paper, G will denote a soliton graph without impervious edges.

Moreover, note that, without loss of generality, we can assume that all constant external edges of a soliton graph G are mandatory. Indeed, attaching an extra mandatory edge to each forbidden external edge of G results in a graph G' for which \mathcal{A}_G and $\mathcal{A}_{G'}$ are strongly isomorphic. We shall use this assumption throughout the paper without any further reference.

In [5] the transitions of soliton automata have been characterized in terms of alternating trails and networks. First, we quote the result describing the transitions between distinct states.

Definition 3. Let M be a state of G and $v, w \in \text{Ext}(G)$. An M -transition network Γ from v to w is a nonempty M -alternating network such that all elements of Γ , except one crossing from v to w if $v \neq w$, are alternating cycles accessible from v in M .

Theorem 4 ([5]). Let M, M' be distinct states of soliton automaton $\mathcal{A}_G = ((S_G, (X \times X), \delta)$. Then for any pair of external vertices $(v, w) \in X \times X$, $M' \in \delta(M, (v, w))$ holds iff $N(M, M')$ is an M -transition network from v to w .

For the characterization of *self-transitions*, transitions from a state to itself, we need the following concepts.

A *soliton trail* α is an external alternating walk, stepping on positive and negative edges in such a way that α is either a path, or it returns to itself only in the last step, traversing a negative edge. The trail α is a *c-trail* (*l-trail*) if it does return to itself, closing up an even-length (respectively, odd-length) cycle. That is, $\alpha = \alpha_1 + \alpha_2$, where α_1 is a path and α_2 is a cycle. These two components of α are called the *handle* and *cycle*, in notation, α_h and α_c . See Fig. 4 for illustrating the above concepts. In this figure, as well as in the further ones throughout the paper, double lines indicate edges that belong to the given matching.

An M -alternating double soliton *c-trail* α from external vertex v is a pair of M -alternating soliton *c-trails* $\alpha = (\alpha^1, \alpha^2)$ from v such that $E(\alpha_h^1) \cap E(\alpha_c^2) = \emptyset$, $E(\alpha_h^2) \cap E(\alpha_c^1) = \emptyset$, and either $\alpha_c^1 = \alpha_c^2$ or $V(\alpha_c^1) \cap V(\alpha_c^2) = \emptyset$. The maximal common external subpath—denoted by α_h —of α_h^1 and of α_h^2 is called the *handle* of α . The internal endpoint of α_h is called the *branching vertex* of α . Fig. 5 presents simple examples for the above definition.

Theorem 5 ([5]). For any state M of soliton automaton $\mathcal{A}_G = ((S_G, (X \times X), \delta)$ and for any external vertex $v \in X$ of G , $M \in \delta(M, (v, v))$ iff one of the following conditions holds:

- (i) G does not contain an M -alternating soliton *c-trail* from v .

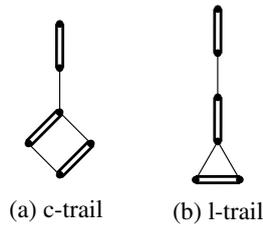


Fig. 4. Soliton trails.

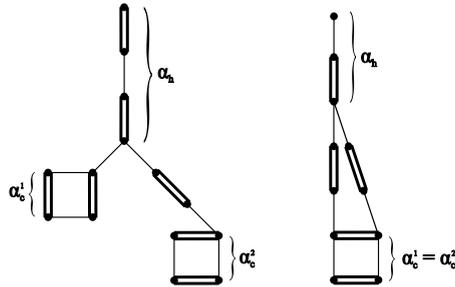


Fig. 5. Example for double soliton c-trails.

- (ii) G contains an M -alternating soliton l -trail from v .
- (iii) G contains an M -alternating double soliton c -trail from v .

The following important observation on soliton l -trails has been proved in [19].

Proposition 6 ([19]). *Let M be a state of soliton graph G and $v \in \text{Ext}(G)$ such that each edge of G is traversed by an external M -alternating trail starting from v . Then G contains a soliton l -trail from v iff G is non-bipartite.*

Making use of the above result, we can refine Theorem 5. To this end we will use the following notation: For any soliton graph G , state M of G and $v \in \text{Ext}(G)$, let $G[M, v]$ denote the graph determined by the edges traversed by an M -alternating trail starting from v . Since for any maximal external M -alternating trail α , $M_{(\alpha)}$ is clearly a perfect internal matching in the graph determined by α , $G[M, v]$ is also a soliton graph with $\text{Ext}(G[M, v]) = \text{Ext}(G) \cap V(G[M, v])$ and $M_{(G[M, v])} \in S(G[M, v])$.

Theorem 7. *For any state M of soliton automaton $\mathcal{A}_G = ((S_G, X \times X), \delta)$ and for any external vertex $v \in X$ of G , $M \in \delta(M, (v, v))$ iff one of the following conditions holds:*

- (a) $G[M, v]$ is a non-bipartite graph.
- (b) $G[M, v]$ is a bipartite graph containing an $M_{(G[M, v])}$ -alternating double soliton c -trail from v .
- (c) $G[M, v]$ is a bipartite graph not containing an $M_{(G[M, v])}$ -alternating cycle.

Proof. Immediate by Theorem 5 and Proposition 6. \square

4. Nondeterministic soliton automata with a single external vertex

As a first step towards the characterization of nondeterministic soliton automata with a single external vertex, the transition between distinct states is described below as a straightforward consequence of Theorem 4.

Theorem 8. *If G is a soliton graph with a single external vertex v , then $M_2 \in \delta(M_1, (v, v))$ holds for any distinct states M_1, M_2 of $\mathcal{A}_G = (S_G, X \times X, \delta)$.*

Proof. Immediate by Theorem 4 and by the observation that the mediator alternating network between M_1 and M_2 consists of alternating cycles accessible from v in M_1 . \square

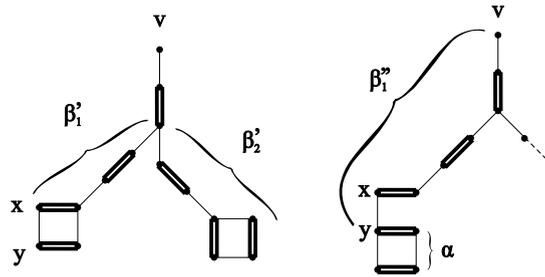


Fig. 6. The case $V(\alpha) \cap V(\beta_h) = \emptyset$ in the proof of Proposition 13.

For the analysis of self-transitions, according to Theorem 5, we need to investigate soliton trails. In reaching the above goal the following concept will play a central role.

Definition 9. States M_1 and M_2 of soliton graph G are called *compatible*, if M_1 and M_2 cover the same external vertices.

Proposition 10. Let M and M' be compatible states of soliton graph G and let α be an M -alternating crossing between external vertices v and w . Then there exists an M' -alternating crossing α' connecting v and w .

Proof. Let β_1, \dots, β_k ($k \geq 0$) be the alternating cycles constituting the mediator alternating network between M and M' ; and construct the graph $G' = \alpha + \beta_1 + \dots + \beta_k$. Then it is clear that $Ext(G') = \{v, w\}$, $M'_{(G')} \in S(G')$, and the external edges incident with v and w are non-constant in G' . Therefore, making use of Proposition 1, we easily obtain that v and w are connected by an $M'_{(G')}$ -alternating crossing, as required. \square

Proposition 11. Let M and M' be compatible states of a soliton graph G . Then for any external vertex $v \in Ext(G)$, $G[M, v] = G[M', v]$ holds.

Proof. By symmetry, it is enough to prove the claim in one direction. To this end let e be an edge of G traversed by an M -alternating trail α starting from v . If e is external, then the statement follows directly from Proposition 10. Suppose now that e is internal. In that case one endpoint of e , let it be denoted by w , is M -accessible from v either by α or by the appropriate prefix of α . Now let extend G by a new external edge (w, u) such that $u \notin V(G)$. By the above observation there is an M -alternating crossing between v and u in $G + (w, u)$. Now applying Proposition 10, we obtain that v and u is connected by an M' -alternating crossing β . Since M and M' are compatible, we conclude that $\beta[v, w]$ is a positive M' -alternating alternating path in G . Therefore either β or $\beta + e$ will provide an M' -alternating trail starting from v and traversing e ; as required. \square

Corollary 12. Let M_1 and M_2 be compatible states of soliton graph G and $v \in Ext(G)$. Then G contains an M_1 -alternating soliton l -trail from v iff it contains an M_2 -alternating soliton l -trail from v .

Proof. Immediate by Propositions 6 and 11. \square

Proposition 13. Let G be a bipartite soliton graph, M be a state of G and $v \in Ext(G)$. Then G contains an M -alternating double soliton c -trail from v iff there exists an M' -alternating double soliton c -trail from v for all states M' compatible with M .

Proof. Let $\beta = (\beta^1, \beta^2)$ be an M -alternating double soliton c -trail from v with branching vertex w , α be an M -alternating cycle, and $M' = S(M, \alpha)$. Since any mediator alternating network Γ between compatible states consists of alternating cycles only, if we prove that an M' -alternating double soliton c -trail from v also exists, then we are ready by a straightforward induction argument on $|\Gamma|$. For this goal, consider first the case of $V(\alpha) \cap V(\beta_h) = \emptyset$.

If $V(\alpha) \cap V(\beta) = \emptyset$, then our statement is trivial. Otherwise, for $k = 1, 2$, let β'_k denote the suffix of β^k from w to its internal endpoint, and if $V(\beta^k) \cap V(\alpha) \neq \emptyset$, then let β''_k denote the prefix of β^k from v to the first vertex common with α . (See an example in Fig. 6.) Then an M' -alternating double soliton c -trail $\gamma = (\gamma^1, \gamma^2)$ can be constructed in the following way: if $V(\alpha) \cap V(\beta'_k) \neq \emptyset$ for $k = 1, 2$, then let $\gamma^1_h = \beta''_1$, $\gamma^2_h = \beta''_2$, and $\gamma^1_c = \gamma^2_c = \alpha$. Otherwise, i.e. $V(\alpha) \cap V(\beta'_k) \neq \emptyset$ and $V(\alpha) \cap V(\beta'_{3-k}) = \emptyset$ for some $k \in \{1, 2\}$, let $\gamma^k_h = \beta''_k$, $\gamma^k_c = \alpha$, and $\gamma^{3-k} = \beta^{3-k}$.

By the preceding paragraph, we can assume for the rest of the proof that $V(\alpha) \cap V(\beta_h) \neq \emptyset$. In this case, starting from v , let u denote the first vertex at which β_h overlaps with α , and let u' be the vertex of $V(\alpha) \cap V(\beta)$ such that the positive M -alternating path $\beta' = \alpha[u, u']$ is maximal as a subpath in β . (Observe that $\beta_h[v, u]$ is negative at the u end.) Moreover, let α' be the negative M -alternating subpath of α connecting u and u' , i.e. $E(\alpha') = E(\alpha) \setminus E(\beta')$. See Fig. 7 for an example.

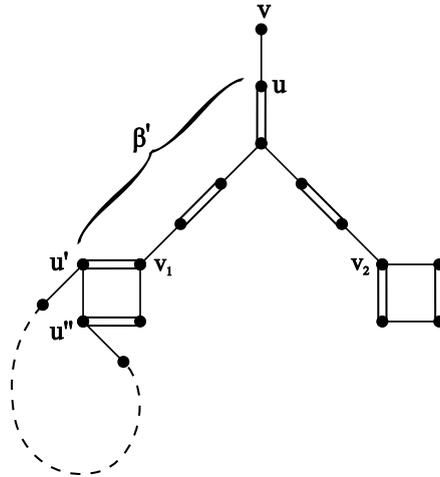


Fig. 7. The case $V(\alpha) \cap V(\beta_h) \neq \emptyset$ in the proof of Proposition 13.

From now on, assume that β is an M -alternating double soliton c -trail from v , such that the subpath β' constructed above is maximal. Then the following holds.

Claim A. α' is edge-disjoint from β .

In order to prove the above claim, let us assume by contradiction, that starting from u' , the next vertex u'' of α' having the property that $u'' \in V(\beta)$ and u'' is different from u . (See Fig. 7 again.) It is clear that $\alpha'[u', u'']$ is a negative alternating path, so that u' and u'' belong to distinct bipartition class of G . Furthermore, we may suppose without loss of generality that $u' \in V(\beta^1)$. We know by the choice of β' that $\beta^1[v, u']$ is positive at its u' end, but it is also easy to observe that if $u'' \in V(\beta^k)$ ($k \in \{1, 2\}$), then $\beta^k[v, u'']$ is negative at its u'' end. Indeed, if $\beta^k[v, u'']$ terminated in a positive edge at u'' , then both u' and u'' would be accessible from v in the bipartite graph $G + (u', u'')$. However, u' and u'' belong to distinct bipartition class, which is a contradiction.

For $k = 1, 2$, now let v_k denote the internal endpoint of β_h^k , and let v'' be the vertex adjacent to u'' by positive edge in M . (See Fig. 7 again.) We will show that an M -alternating double soliton c -trail $\gamma = (\gamma^1, \gamma^2)$ starting from v can be constructed such that the positive M -alternating path $\alpha[u, v'']$ is a subpath of γ^1 . For this, we distinguish four cases.

Case A/1: $u'' \in V(\beta_h^k)$ ($k \in \{1, 2\}$), and either $u' \in V(\beta_h^1)$ or $\beta_c^1 \neq \beta_c^2$.

Note that, in this case, if $\beta_c^1 \neq \beta_c^2$, then $k = 2$. Now it is easy to check that the trails defined below constitute an M -alternating double soliton c -trail γ such that $\alpha[u, v'']$ is a positive M -alternating subpath of γ^1 , as required.

$$\gamma_h^1 = \beta^1[v, u'] + \alpha'[u', u''] + \beta_h^k[u'', v_k]$$

$$\gamma_h^2 = \beta_h^k$$

$$\gamma_c^1 = \gamma_c^2 = \beta_c^k.$$

Case A/2: $u'' \in V(\beta_c^k)$ ($k \in \{1, 2\}$) such that either $u' \in V(\beta_h^1)$, or $\beta_c^1 \neq \beta_c^2$ with $k = 2$.

Then let

$$\gamma_h^1 = \beta^1[v, u'] + \alpha'[u', u''],$$

$$\gamma_h^2 = \beta_h^k,$$

$$\text{and } \gamma_c^1 = \gamma_c^2 = \beta_c^k.$$

Again, considering all possible alternatives of this case, we obtain that $\gamma = (\gamma^1, \gamma^2)$ is a double soliton c -trail with the required properties.

Case A/3: $u', u'' \in V(\beta_c^1)$.

Now a suitable $\gamma = (\gamma^1, \gamma^2)$ is defined as follows.

$$\gamma_h^1 = \beta_h^1,$$

$$\gamma_h^2 = \beta_h^2,$$

$$\gamma_c^1 = \beta^1[v_1, u'] + \alpha'[u', u''] + \beta^1[u'', v_1].$$

and

$$\gamma_c^2 = \gamma_c^1, \text{ if } \beta_c^1 = \beta_c^2$$

$$\gamma_c^2 = \beta_c^2, \text{ if } \beta_c^1 \neq \beta_c^2.$$

Case A/4: $u' \in V(\beta_c^1)$, $u'' \in V(\beta_h^2)$, and $\beta_c^1 = \beta_c^2$.

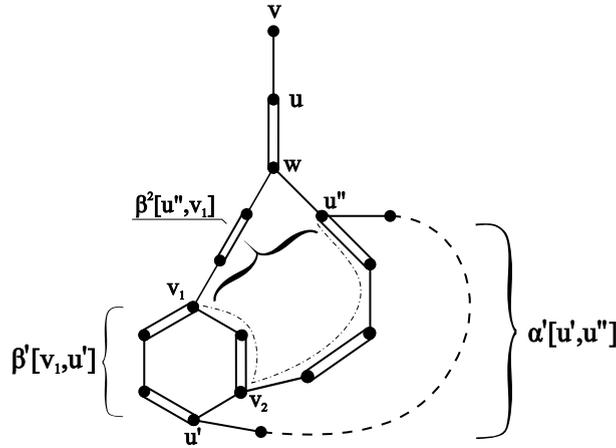


Fig. 8. Case A/4 in the proof of Proposition 13.

Now the construction of $\gamma = (\gamma^1, \gamma^2)$ below is represented in Fig. 8.

$$\begin{aligned} \gamma_h^1 &= \beta_h^1, \\ \gamma_h^2 &= \beta_h^2[v, u''], \\ \gamma_c^1 &= \gamma_c^2 = \beta'[v_1, u'] + \alpha'[u', u''] + \beta^2[u'', v_1]. \end{aligned}$$

Now considering all the possible combinations of the locations of u' and u'' , below we summarize that Cases A/1 – A/4 are indeed sufficient to cover all alternatives. (Remember that $u' \in V(\beta^1)$.)

- (i) If $u'' \in V(\beta_h^k)$ ($k \in \{1, 2\}$) and $u' \in V(\beta_h^1)$, then consider Case A/1.
- (ii) If $u'' \in V(\beta_h^k)$ ($k \in \{1, 2\}$) and $u' \in V(\beta_c^1)$, then consider Case A/4 ($\beta_c^1 = \beta_c^2$) and Case A/1 ($\beta_c^1 \neq \beta_c^2$).
- (iii) If $u'' \in V(\beta_c^k)$ ($k \in \{1, 2\}$) and $u' \in V(\beta_h^1)$, then consider Case A/2.
- (iv) If $u'' \in V(\beta_c^k)$ ($k \in \{1, 2\}$) and $u' \in V(\beta_c^1)$, then consider Case A/3 ($k = 1$) and Case A/2 ($k = 2$).

Therefore, we obtained in all possible cases that $\alpha[u, v'']$ is a positive M -alternating subpath of γ^1 . However, the length of $\alpha[u, v'']$ is greater than that of β' , since $\alpha[u, v'']$ is constructed as $\beta' + \alpha'[u', u''] + (u'', v'')$. The above fact contradicts the choice of β , thus Claim A is proved.

By the above claim, we can suppose for the rest of the proof that α' is edge-disjoint from β' . As earlier, assume that $u' \in V(\beta^1)$ and for $i \in \{1, 2\}$, let v_i denote the internal endpoint of β_h^i . We will construct an M' -alternating double soliton c -trail $\delta = (\delta^1, \delta^2)$ in the subgraph determined by β and α' . For this, we must deal with three cases and several subcases.

Case 1: $v_1 = v_2$.

In this case $\beta_c^1 = \beta_c^2$, for which we use the notation β_c . Now starting from v , let y denote the last vertex of β_h^1 such that y is incident with an edge in $E(\beta_h^2) - E(\beta_h^1)$, and let x denote the last vertex of β_h^1 preceding y with $x \in V(\beta_h^2)$. (See Fig. 9.) Below we give the construction of $\delta = (\delta^1, \delta^2)$, for which, based on the location of u' , we distinguish four subcases.

Subcase 1a: $u' \in V(\beta_h^1[v, x])$.

Then

$$\begin{aligned} \delta_h^1 &= \beta_h[v, u] + \alpha' + \beta_h^1[u', v_1], \\ \delta_h^2 &= \beta_h[v, u] + \alpha' + \beta_h^1[u', x] + \beta_h^2[x, v_2], \\ \delta_c^1 &= \delta_c^2 = \beta_c. \end{aligned}$$

Subcase 1b: $u' \in V(\beta_h^1[x, y])$.

Then

$$\begin{aligned} \delta_h^1 &= \beta_h[v, u] + \alpha' + \beta_h^1[u', v_1], \\ \delta_h^2 &= \beta_h[v, u] + \alpha' + (\beta_h^1)^{-1}[u', x] + \beta_h^2[x, v_2], \\ \delta_c^1 &= \delta_c^2 = \beta_c. \end{aligned}$$

Subcase 1c: $u' \in V(\beta_h^1[y, v_1])$.

Then

$$\delta_h^1 = \beta_h[v, u] + \alpha' + \beta_h^1[u', v_1],$$

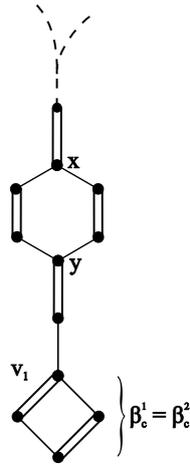


Fig. 9. Case 1 in the proof of Proposition 13.

$$\begin{aligned} \delta_h^2 &= \beta_h[v,u] + \alpha' + (\beta_h^1)^{-1}[u',y] \\ \delta_c^1 &= \beta_c, \\ \delta_c^2 &= \beta_h^1[x,y] + (\beta_h^2)^{-1}[y,x]. \end{aligned}$$

Subcase 1d: $u' \in V(\beta_c^1)$.

In this case, let β'' denote the negative M -alternating subpath from u' to v_1 in β_c determined by the edges not contained in β' .

Then

$$\begin{aligned} \delta_h^1 &= \beta_h[v,u] + \alpha' + (\beta')^{-1}[u',y], \\ \delta_h^2 &= \beta_h[v,u] + \alpha' + \beta'' + (\beta_h^1)^{-1}[v_1,y] \\ \delta_c^1 &= \delta_c^2 = \beta_h^1[x,y] + (\beta_h^2)^{-1}[y,x]. \end{aligned}$$

Now it is easy to check that $\delta = (\delta^1, \delta^2)$ is indeed a suitable M' -alternating double soliton c -trail in all of the above cases.

Case 2: $v_1 \neq v_2$ and $\beta_c^1 = \beta_c^2$. In this case, consider the bipartite graph G' determined by α' and β , and apply the method of [7] contracting the redexes of G' . Recall from [7] that a redex r consists of two adjacent edges $e = (u,z)$ and $f = (z,w)$ such that $u \neq w$ are both internal and $d(z) = 2$. Contracting r in G' means creating a new graph G_r from G' by deleting z and merging u and w into one vertex. Now applying the above method for all redexes of G' in an iterative way, we obtain a graph G_1 having no redexes. Then let $M_1 (M'_1)$ denote the restriction of M (respectively, M') to the edges of G_1 . It is easy to see that β is contracted to an M_1 -alternating double soliton c -trail $\gamma = (\gamma^1, \gamma^2)$ such that γ_h^1 and γ_h^2 have the same internal endpoint. Therefore, according to Case 1, an M'_1 -alternating double soliton c -trail also exists, which obviously becomes an M' -alternating double soliton c -trail after unfolding all the redexes.

Case 3: $\beta_c^1 \neq \beta_c^2$.

To handle this case, we need to further break it down into two subcases. (Remember that $u' \in V(\beta^1)$.)

Subcase 3a: $u' \in V(\beta_h^1)$.

Consider the subgraph G_h determined by β_h^1 and β_h^2 . Then v_1 and v_2 are obviously accessible in G_h , consequently there exists an $M'_{(G_h)}$ -alternating path $\alpha_1 (\alpha_2)$ from v to v_1 (respectively, v_2). Therefore, the external alternating trails $\delta^1 = \alpha_1 + \beta_c^1$ and $\delta^2 = \alpha_2 + \beta_c^2$ form an M' -alternating double soliton c -trail from v .

Subcase 3b: $u' \in V(\beta_c^1)$.

Now starting from v , let x denote the last vertex of β_h^1 which is also on β_h^2 , and let β'' denote the negative M -alternating subpath from u' to v_1 in β_c^1 determined by the edges not contained in β' . Then a suitable M' -alternating double soliton c -trail $\delta = (\delta^1, \delta^2)$ is defined as follows.

$$\begin{aligned} \delta_h^1 &= \beta_h[v,u] + \alpha' + (\beta')^{-1}[u',x] + \beta_h^2[x,v_2], \\ \delta_h^2 &= \beta_h[v,u] + \alpha' + \beta'' + (\beta_h^1)^{-1}[v_1,x] + \beta_h^2[x,v_2] \\ \delta_c^1 &= \delta_c^2 = \beta_c^2. \end{aligned}$$

In summary, Cases 1 – 3 cover all the possible alternatives and we obtained in all cases that an M' -alternating double soliton c -trail δ can be constructed, as required. Therefore the proof is complete. \square

By the above result it is meaningful to say for a bipartite soliton graph G with a single external vertex v that "G contains a double soliton c -trail " without specifying any state.

Now we are ready to prove our main result concerning self-transitions.

Theorem 14. *Let M, M' be compatible states of soliton automaton $\mathcal{A}_G = (S_G, X \times X, \delta)$ and $v \in X$ be an external vertex of G . Then $M \in \delta(M, (v, v))$ iff $M' \in \delta(M', (v, v))$.*

Proof. Let G_v denote the subgraph $G[M, v] = G[M', v]$ (see Proposition 11) and apply Theorem 7 for G_v . It is clear that if condition (c) holds with respect to M or M' , then $M = M'$, and there is nothing to prove. The statement is also obvious if G_v is non-bipartite, while in other case the theorem follows from Proposition 13. \square

Making use of the above results, we obtain a characterization of soliton automata with a single external vertex.

Definition 15. Let $\mathcal{A} = (S, X, \delta)$ be an automaton such that its alphabet is a singleton, i.e. $X = \{x\}$. We say that \mathcal{A} is a *full (semi-full) automaton* if for each $s \in S$, $\delta(s, x) = S$ (respectively, $\delta(s, x) = S - \{s\}$ with $|S| > 1$). Moreover, a full automaton with a single state is called *trivial*.

Theorem 16. *Let G be a soliton graph with a single external vertex v . Then \mathcal{A}_G is either a full or a semi-full automaton. Moreover, \mathcal{A}_G is semi-full iff G is a bipartite graph without double soliton c -trails.*

Proof. Since any maximal alternating trail in G is necessarily either an l -trail or a c -trail, G is either non-bipartite or it contains alternating cycle with respect to any of its states. Therefore the argument is straightforward by using Theorems 5 and 8 with Propositions 6 and 13. \square

Having characterized the structure of soliton graphs and automata with a single external vertex, we can describe the transition monoids of these automata. Since it was proved in [12] that the transition monoid in deterministic case is isomorphic with the symmetric group of order 2 (or order 1, in the trivial case of an automaton having a single state), we will assume that the considered soliton automata are not deterministic.

Nevertheless, before stating the closing result of this section, we briefly review the necessary concepts from the theory of automata and semigroups. For any automaton $\mathcal{A} = (S, X, \delta)$ and $w \in X^*$, we can define the *relation δ_w induced by w on S* , i.e. $(s, s') \in \delta_w$ iff $s' \in \delta(s, w)$ for $s, s' \in S$. Then the set $\mathcal{T}(\mathcal{A})$ of relations δ_w on S which are induced by some $w \in X^*$, with the usual composition of relations is a monoid $\mathcal{T}(\mathcal{A})$, the *transition monoid* of \mathcal{A} .

A monoid (A, \circ) with $|A| \geq 2$ is called a *null monoid* if there exists an element $a \in A$ such that $x \circ y = a$ for all $x, y \in A - \{e\}$, where e denotes the identity element. In that case a is called the *zero element*. It is clear that a null monoid with two elements— A consists of the identity element and the zero element—is uniquely determined up to isomorphism, which is called the *trivial null monoid*.

Now we are ready to state the final result of this section.

Theorem 17. *Let G be a soliton graph with a single external vertex v such that \mathcal{A}_G is not deterministic. Then $\mathcal{T}(\mathcal{A}_G)$ is a null monoid with at most 3 elements. Moreover, $\mathcal{T}(\mathcal{A}_G)$ is nontrivial iff G is a bipartite graph without double soliton c -trails.*

Proof. The identity element of $\mathcal{T}(\mathcal{A}_G)$ is the relation induced by the empty word. If G is nonbipartite or bipartite with a double soliton c -trail, then based on Theorem 16, the relation induced by (v, v) is S_G^2 , which clearly serves as a zero element. Based on the above fact, $\mathcal{T}(\mathcal{A}_G)$ has no additional elements, consequently $\mathcal{T}(\mathcal{A}_G)$ is trivial.

Consider now the case of G being bipartite without double soliton c -trails. Then applying Theorem 16 again, we obtain that the relation induced by (v, v) is $\rho_v = \{(q, q') \mid q \neq q' \in S_G\}$. However, since \mathcal{A}_G is not deterministic, it is easy to see that $S_G \geq 3$, and for $n \geq 2$, $(v, v)^n$ induces S_G^2 . Therefore, we obtained that $\mathcal{T}(\mathcal{A}_G)$ is indeed a null-monoid with 3 elements. \square

5. Elementary decomposition of soliton graphs

Having characterized soliton automata with a single external vertex, our goal is to describe the class of constant soliton automata in terms of products of full automata. For this end, we will make use of the structure theory of soliton graphs and automata worked out in [5,6]. In this section we provide a brief summary of the most significant results obtained in [6]. Again, let us fix a soliton graph G for the forthcoming discussion.

Recall from [4,22] that a graph is *elementary* if its allowed edges form a connected subgraph containing all the external vertices. In general, the allowed edges of G determine a number of connected components as subgraphs in G . The full subgraphs of G induced by these components are called the *elementary components* of G . An elementary component is called *external* if it contains external vertices, and *internal* if this is not the case. A *mandatory elementary component* is a single mandatory edge $e \in E(G)$, which might have a loop around one or both of its endpoints. It is easy to see that each external elementary component of a constant soliton graph is mandatory.

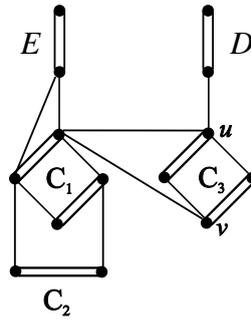


Fig. 10. Elementary components in a soliton graph.

The concept of *canonical equivalence* was originally introduced for elementary graphs only (cf. [4,22]). In [6], it was proved that the following extension of the concept results in an equivalence on $Int(G)$ for any soliton graph G .

Let $v, w \in Int(G)$ be internal vertices. Then $u \sim v$ if u and v belong to the same elementary component and an extra edge e connecting u and v becomes forbidden in $G + e$. The classes determined by \sim are called *canonical classes* and the corresponding canonical partition of $Int(G)$ is denoted by $\mathcal{P}(G)$.

The structure of elementary components in a soliton graph G has been analysed in [6]. To summarize the main results of this analysis, we first need to review some of the key concepts introduced in that paper. An internal elementary component C is *one-way* if all external alternating paths (with respect to any perfect internal matching M) enter C in vertices belonging to the same canonical class of C . This unique class, as well as the vertices belonging to this class, are called *principal*. Furthermore, every external elementary component is considered a priori one-way (with no principal canonical class, of course). An elementary component is *two-way* if it is not one-way.

Example. The graph of Fig. 10. has five elementary components, among which D and E are mandatory external, while C_1 , C_2 and C_3 are internal. Component C_3 is one-way with the canonical class $\{u, v\}$ being principal, while C_1 and C_2 are two-way.

Let C be an elementary component of G , and M be a state. An M -alternating C -ear is a negative M -alternating path or loop having its two endpoints, but no other vertices, in C . The endpoints of the ear will necessarily be in the same canonical class of C .

We say that elementary component C' is *two-way accessible* from component C with respect to any perfect internal matching M , in notation $C \rho C'$, if C' is covered by a C -ear. As it was shown in [6], the two-way accessible relationship is matching invariant. A *family* of elementary components in G is a block of the partition induced by the smallest equivalence relation containing ρ . A family \mathcal{F} is called *external* if it contains an external elementary component, otherwise \mathcal{F} is *internal*. It was proved in [6] that any family contains a unique one-way elementary component called the *root* of the family.

Example. Our example graph in Fig.10. has three families: $\mathcal{F}_1 = \{E, C_1, C_2\}$, $\mathcal{F}_2 = \{D\}$, $\mathcal{F}_3 = \{C_3\}$. Families \mathcal{F}_1 and \mathcal{F}_2 are external, whereas \mathcal{F}_3 is internal. The roots of the families are E , D and C_3 .

For two distinct families \mathcal{F}_1 and \mathcal{F}_2 , \mathcal{F}_2 is said to *follow* \mathcal{F}_1 , in notation $\mathcal{F}_1 \mapsto \mathcal{F}_2$, if there exists an edge in G connecting any non-principal vertex in \mathcal{F}_1 with a principal vertex belonging to the root of \mathcal{F}_2 . The reflexive and transitive closure of \mapsto is denoted by \mapsto^* . As a main result of [6] it was shown that the relation \mapsto^* is a partial order among the families, by which the external families are maximal elements.

6. Constant soliton automata

In this section, synthesizing the results on the characterization of soliton automata with a single external vertex and the structure theory of soliton graphs, we will describe the class of constant soliton automata in terms of products of full automata. As a first step, for constant soliton automata we establish the connection between soliton transitions and the structure theory presented in Section 5. Let us fix a constant soliton graph G for the forthcoming discussion.

Definition 18. For any external vertex v of G , let G_v denote the full subgraph of G induced by the vertices belonging to elementary components of families \mathcal{F} for which $\mathcal{F}_v \mapsto^* \mathcal{F}$, where \mathcal{F}_v denotes the external family containing v .

Proposition 19. For any external vertex v and state M of G , graph G_v is identical with $G[M, v]$.

Proof. Adapting Lemma 3.8 in [5], we obtain that an edge e is viable by an external alternating trail starting from v iff e is incident with a non-principal vertex of an elementary component belonging to G_v . \square

Definition 20. Let v be an external vertex of G and let $\{C_1, \dots, C_n\}$ ($n > 0$) be the set of the elementary components of G_v with C_1 being its mandatory external elementary component. Graph G_v is a *component-chain graph* if it can be decomposed in the *chain-form* $G_v = C_1 + (w_1, v_2) + C_2 + (w_2, v_3) + \dots + (w_{n-1}, v_n) + C_n$ such that for each $i \in [n - 1]$, $(w_i, v_{i+1}) \in E(G)$ with $w_i \in V(C_i)$ and $v_{i+1} \in V(C_{i+1})$. Furthermore, G_v is called *rigid* if it does not contain a double soliton c -trail.

Now we are ready to characterize the self-transitions with respect to G_v .

Proposition 21. Let v be an external vertex of G , M be a state of G and let δ denote the transition function of \mathcal{A}_G . Then $M \notin \delta(M, (v, v))$ iff G_v is a bipartite rigid component-chain graph.

Proof. The ‘If’ part is straightforward by Theorem 7 and Proposition 19. In order to prove the ‘Only if’ part, observe first that G_v must be rigid bipartite by Theorem 7 and Proposition 19 again. In that case each family in G_v consists of a single elementary component, in other case the given family would span a nonbipartite subgraph (see [5]). Therefore if G_v was not a component chain-graph, then either two families are connected by two edges or there exists a family \mathcal{F} such that two distinct families \mathcal{F}_1 and \mathcal{F}_2 are originated from \mathcal{F} by \mapsto . However, since any external alternating path reaching the families in the order determined by \mapsto ([6]), in both cases two distinct external alternating paths can be constructed from v to the referred vertices of the “branching” family. Because of the same reason both paths can be extended to appropriate soliton c -trails having cycles below the “branching” family. Therefore we would obtain a double soliton c -trail in both cases, which is a contradiction. \square

With the help of the above concepts, we can define the appropriate automata product by which constant soliton automata will be characterized.

Definition 22. Consider the automata $\mathcal{A}_t = (S_t, X_t, \delta_t)$ ($t \in [m], m \in \mathbb{N}$) and let X be an alphabet. Suppose that there exists a partial order \leq among X and the automata $\mathcal{A}_1, \dots, \mathcal{A}_m$ such that the set of maximal elements is equal to X . Furthermore, let Y be a subset of X such that for each $y \in Y$, the elements \mathcal{A}_j with $\mathcal{A}_j \leq y$ constitute a chain. Then the X^2 -chain product of $\mathcal{A}_1, \dots, \mathcal{A}_m$ with respect to Y and feedback function ϕ_{\leq} is the automaton $\mathcal{A} = (S, X \times X, \delta)$ defined in the following way.

- (a) $S = S_1 \times \dots \times S_m$.
- (b) The function $\phi_{\leq} = (\phi_1, \dots, \phi_m)$ is given in such a way that for each $i \in [m]$, $\phi_i : X \times X \rightarrow X_i \cup \{\varepsilon\}$ is a mapping subject to the following conditions:
 - (b/1) If x and y are distinct elements of X , then $\phi_i(x, y) = \varepsilon$.
 - (b/2) If $\phi_i(x, x) \neq \varepsilon$ for an $x \in X$, then $\mathcal{A}_i \leq x$.
- (c) For every $x, y \in X$ and $s = (s_1, \dots, s_m) \in S$,

$$\delta(s, (x, y)) = \begin{cases} \delta_1(s_1, \phi_1(x, y)) \times \dots \times \delta_m(s_m, \phi_m(x, y)) - \{s\}, & \text{if } x = y \in Y \\ \delta_1(s_1, \phi_1(x, y)) \times \dots \times \delta_m(s_m, \phi_m(x, y)), & \text{otherwise} \end{cases}$$

We will show that any constant soliton automaton can be decomposed into full automata by an X^2 -chain product. In order to reach the above goal, we need the following simple observation on *essentially elementary soliton automata*, by which we mean automata associated with a graph consisting of an elementary component D and that of an external edge attached to D .

Proposition 23. Any essentially elementary soliton automaton \mathcal{A}_G is either a full or a semi-full automaton.

Proof. Since any essentially elementary soliton automaton has a single external vertex, the claim is immediate by Theorem 16. \square

For the product construction, we will need the following technical notion.

Definition 24. Let $\mathcal{A} = (S, X \times X, \delta)$ be an automaton. The *self-transition extension* of \mathcal{A} is the automaton $\mathcal{A}^e = (S, X \times X, \delta^e)$, where for any state $s \in S$ of \mathcal{A} and for any input $(x, x') \in X \times X$,

$$\delta^e(s, (x, x')) = \begin{cases} \delta(s, (x, x')), & \text{if } x \neq x' \\ \delta(s, (x, x')) \cup \{s\}, & \text{otherwise} \end{cases}$$

Now we can give the necessary decomposition.

Proposition 25. Let C_1, \dots, C_k be the internal elementary components of G and for $j \in [k]$, let \mathcal{A}_j denote the full automaton with $|S_{C_j}|$ number of states. Then \mathcal{A}_G is strongly isomorphic with an X^2 -chain product \mathcal{A} of $\mathcal{A}_1, \dots, \mathcal{A}_k$, where $X = \text{Ext}(G)$.

Proof. During the proof we will use the notations of Definition 22 with the same parameters. Furthermore, it is clear by part (c) of Definition 22 that we only need to prove that \mathcal{A}_G^e and \mathcal{A}^e are strongly isomorphic.

Let Y denote the set of external vertices v for which G_v is a rigid bipartite component-chain graph. Moreover, let ϕ_{\leq} be determined by relation \mapsto^* . More precisely, for $i, j \in [k]$, if \mathcal{F}_i and \mathcal{F}_j denote the families containing C_i and C_j , respectively, then $\mathcal{A}_j \leq \mathcal{A}_i$ iff $\mathcal{F}_j \mapsto^* \mathcal{F}_i$ and $\mathcal{A}_j \leq v$ iff $\mathcal{F}_v \mapsto^* \mathcal{F}_j$ for $v \in X$ and v belonging to \mathcal{F}_v .

Now for each $1 \leq j \leq k$, let C'_j be a soliton graph obtained from C_j by attaching a new external edge to some vertex of C_j . Consider the self-transition extension $\mathcal{A}_{C'_j}^e = (S_{C'_j}, \{x, x\}, \delta_j^e)$ of automaton $\mathcal{A}_{C_j} = (S_{C_j}, \{x, x\}, \delta_j)$ defined by $\delta_j^e(M, (x, x)) = \delta_j(M, (x, x)) \cup \{M\}$ for any state $M \in S_{C'_j}$.

By Proposition 23, $\mathcal{A}^e(C'_j)$ is strongly isomorphic with the full-automaton \mathcal{A}_j . Capitalizing on the above fact, because of technical reasons, when we refer to the transition function (or the states) of \mathcal{A}_j , we will mean the ones of $\mathcal{A}^e(C'_j)$. This assumption will be used throughout the proof without any further reference.

Let δ_G and δ denote the transition function of \mathcal{A}_G and that of the product automaton \mathcal{A} . Moreover, let $y_1, y_2 \in \text{Ext}(G)$ and $M \in S_G$ be arbitrary. Since the mapping $\psi(M) = (M_{(C_1)}, \dots, M_{(C_k)})$ is clearly a bijection between S_G and $S_{C_1} \times \dots \times S_{C_k}$, we only have to prove that

$$\{\psi(M^*) \mid M^* \in \delta_G^e(M, (y_1, y_2))\} = \delta^e(\psi(M), (y_1, y_2))$$

Then based on part (c) of Definition 22, it is clear that the following holds for the right side of the above basic equality.

$$\delta^e(\psi(M), (y_1, y_2)) = \delta_1^e(M_{(C_1)}, z_1) \times \dots \times \delta_k^e(M_{(C_k)}, z_k) \tag{1}$$

Now, in order to study the transition functions and the left side of the basic equality, we distinguish between two cases.

Case 1: No soliton walk exists from y_1 to y_2 with respect to M .

It is easy to see that the above situation is equivalent to the condition of $y_1 \neq y_2$. By Definition 22, we obtain that for each $i \in [k]$, $z_i = \varepsilon$. Consequently, $\delta_i^e(M_{(C_i)}, z_i) = \{M_{(C_i)}\}$ ($i \in [k]$) holds, which results in the following.

$$\{\psi(M^*) \mid M^* \in \delta_G^e(M, (y_1, y_2))\} = \{\psi(M)\} = \delta_1^e(M_{(C_1)}, z_1) \times \dots \times \delta_k^e(M_{(C_k)}, z_k) \tag{2}$$

Now the basic equation is obtained by combining (1) and (2).

Case 2: There is a soliton walk from y_1 to y_2 with respect to M . It is clearly equivalent with the condition of $y_1 = y_2$.

Now, for any M -alternating network Γ , let $S(M, \Gamma)$ denote the state obtained from M by switching on the alternating units constituting Γ , and let $E(\Gamma)$ denote the set of edges contained in some alternating unit of Γ . Furthermore, let $\mathcal{T}(M, y_1, y_2)$ denote the set of M -transition networks from y_1 to y_2 , and let $\mathcal{T}^*(M, y_1, y_2) = \mathcal{T}(M, y_1, y_2) \cup \emptyset$.

Finally, for any elementary component C , let $\mathcal{T}_C^*(M, y_1, y_2)$ denote the set of M -alternating networks Γ for which $\Gamma \in \mathcal{T}^*(M, y_1, y_2)$ and $E(\Gamma) \subseteq E(C)$.

Then, making use of Theorem 4, we obtain the following for the left side of (1).

$$\{\psi(M^*) \mid M^* = S(M, \Gamma), \Gamma \in \mathcal{T}^*(M, y_1, y_2)\} = \{S(M_{(C_1)}, \Gamma_1) \mid \Gamma_1 \in \mathcal{T}_{C_1}^*(M, y_1, y_2)\} \times \dots \times \{S(M_{(C_k)}, \Gamma_k) \mid \Gamma_k \in \mathcal{T}_{C_k}^*(M, y_1, y_2)\} \tag{3}$$

Now comparing (2) and (3), we conclude that the proof becomes complete, if we show that for any $i \in [k]$,

$$\{S(M_{(C_i)}, \Gamma_i) \mid \Gamma_i \in \mathcal{T}_{C_i}^*(M, y_1, y_2)\} = \delta_i^e(M_{(C_i)}, z_i) \tag{4}$$

For the above goal, let $i \in [k]$ be arbitrary. Then applying Definition 22, we obtain that $z_i = (x_i, x_i)$ iff C_i belongs to G_{y_1} . Now Eq. (4) easily follows, and the proof is complete. \square

Analyzing the construction of the product in the above proposition, it is clear that the mandatory internal elementary components have role only in the self-transitions. More exactly, if a loop is present in a mandatory internal elementary component C , then for each vertex v with G_v containing C , there exists a self-transition from v in all states. Therefore, if for each external vertex v identified by the above way, we add a loop around the internal vertex of the mandatory external elementary component containing v , we obtain a graph associated with a strongly isomorphic automaton. Repeating the above process for all loops belonging to a mandatory internal elementary component, we obtain a graph G' with \mathcal{A}_G and $\mathcal{A}_{G'}$ being strongly isomorphic.

Now applying the decomposition described in the proof of Proposition 25 for $\mathcal{A}_{G'}$ and removing the trivial component automata (having a single state) from the product, the resulted product automaton \mathcal{A}' will be evidently strongly isomorphic with the original product automaton, if \mathcal{A}_G is *nontrivial* (having at least two states). We will refer to this automaton as the *reduced product automaton* of \mathcal{A}_G . Based on the above argument, we obtained the following result.

Proposition 26. *Let \mathcal{A}_G is nontrivial, and let \mathcal{A}_G^r denote its reduced product automaton. Then \mathcal{A}_G and \mathcal{A}_G^r are strongly isomorphic.*

It is easy to see that the above claim does not hold for trivial soliton automata. Indeed, in that case all the internal elementary components are mandatory, thus the reduction procedure results in an empty automaton. Since trivial constant soliton automata are well-characterized, we restrict our analysis to nontrivial automata in the rest of the paper.

Corollary 27. Any nontrivial constant soliton automaton is strongly isomorphic with an X^2 -chain product of nontrivial full automata, where $X = \text{Ext}(G)$.

Proof. Immediate by Propositions 25 and 26. \square

In order to obtain a complete characterization of constant automata, now we show that the reverse direction is also true.

Proposition 28. Let X be an arbitrary alphabet, and let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be nontrivial full automata. Then any X^2 -chain product of $\mathcal{A}_1, \dots, \mathcal{A}_k$ is strongly isomorphic with a nontrivial constant soliton automata.

Proof. For each $1 \leq i \leq k$, let G_i be a closed elementary graph consisting of as many parallel edges as the number of states of \mathcal{A}_i . It is clear that each edge uniquely determines a state, Now construct a soliton graph G in such a way that one vertex of G_i is distinguished as principal vertex and each elementary graph will form an internal family. The external families are determined by mandatory edges corresponding to the elements of X . Now the graph is built in such a way that the edges connecting distinct elementary components must correspond to the relation \leq in such a way that the relation \mapsto^* in the resulted graph reflects this order. The elementary components below the vertices of Y are supposed to be connected by a single edge in such a way that a chain is built. Finally, a loop is added to each external mandatory component corresponding to an element of $X \setminus Y$.

It is easy to see that the family structure of the resulted constant soliton graph G will reflect the partial order \leq and applying Proposition 25 for \mathcal{A}_G , we obtain a strongly isomorphic product automaton. The proof is now complete. \square

Combining Corollary 27 and Proposition 28, as a final result, we obtain the characterization of nontrivial constant soliton automata.

Theorem 29. The class of nontrivial constant soliton automata and the class of automata obtained by X^2 -chain products of nontrivial full automata coincide up to strong isomorphism.

7. Conclusion

We have provided a complete characterization of soliton automata with constant external edges. We have described the underlying graph structure of the special case of single external vertex in terms of soliton trails. With the help of these results we proved that any nondeterministic soliton automaton in this special case is either a full or a semi-full automaton, the latter holds only if the underlying graph is bipartite and it does not contain a double soliton c -trail. Then, we concluded that the transition monoid of such a (not deterministic) automaton is a null monoid with 2 or 3 elements, depending on whether the given automaton is full or semi-full. Finally, generalizing the concept, we introduced constant soliton automata and characterized their class by products of full automata.

Acknowledgments

The author thank the anonymous referees for their constructive comments on a preliminary version of the paper ([21]).

References

- [1] A. Adamatzky (Ed.), Collision-Based Computing, Springer-Verlag, 2002.
- [2] M. Bartha, The Gallai-Edmonds algebra of graphs, *Congressus Numerantium* 123 (1997) 205–219.
- [3] M. Bartha, E. Gombás, A structure theorem for maximum internal matchings in graphs, *Information Processing Letters* 40 (1991) 289–294.
- [4] M. Bartha, E. Gombás, On graphs with perfect internal matchings, *Acta Cybernetica* 12 (1995) 111–124.
- [5] M. Bartha, M. Krész, Elementary decomposition of soliton automata, *Acta Cybernetica* 14 (2000) 631–652.
- [6] M. Bartha, M. Krész, Structuring the elementary components of graphs having a perfect internal matching, *Theoretical Computer Science* 299 (2003) 179–210.
- [7] M. Bartha, M. Krész, Flexible matchings, *Lecture Notes in Computer Science* 4271 (2006) 313–324.
- [8] M. Bartha, M. Krész, Tutte type theorems in graphs having a perfect internal matching, *Information Processing Letters* 91 (2004) 277–284.
- [9] F.L. Carter, The molecular device computer: point of departure for large scale cellular automata, *Physica D* 10 (1984) 175–194.
- [10] F.L. Carter, et al., Soliton switching and its implications for molecular electronics, *Molecular Electronic Devices II*, Marcel Dekker Inc., 1987, pp. 149–182.
- [11] J. Dassow, H. Jürgensen, Soliton automata, *Journal of Computer and System Sciences* 40 (1990) 154–181.
- [12] J. Dassow, H. Jürgensen, Soliton automata with a single exterior node, *Theoretical Computer Science* 84 (1991) 281–292.
- [13] J. Dassow, H. Jürgensen, Soliton automata with at most one cycle, *Journal of Computer and System Sciences* 46 (1993) 155–197.
- [14] R.P. Feynman, There's plenty of room at the bottom, in: H.D. Gilbert (Ed.), *Miniaturization*, Reinhold, New York, 1961, p. 282.
- [15] M.P. Groves, C.F. Carvalho, R.H. Prager, Switching the polyacetylene soliton, *Materials Science and Engineering C* 3 (1995) 181–185.
- [16] M.P. Groves, Soliton circuit design using molecular gate arrays, in: *Proceedings of the 20th Australasian Computer Science Conference*, 1997, pp. 245–252.
- [17] M. Krész, Soliton automata: a computational model on the principle of graph matchings, Ph.D. thesis, University of Szeged, Szeged, Hungary, 2004.

- [18] M. Krész, Isomorphically complete systems of soliton automata, in: *Proceedings of the 11th International Conference on Automata and Formal Languages*, 2005, pp. 150–163.
- [19] M. Krész, Simulation of soliton circuits, *Lecture Notes in Computer Science* 3845 (2005) 347–348.
- [20] M. Krész, Graph decomposition and descriptonal complexity of soliton automata, *Journal of Automata, Languages and Combinatorics* 12 (2007) 237–263.
- [21] M. Krész, Nondeterministic soliton automata with a single external vertex, in: *Proceedings of the 1st International Conference on Languages, Automata Theory and Applications*, 2007, pp. 319–330.
- [22] L. Lovász, M.D. Plummer, *Matching Theory*, North Holland, Amsterdam, 1986.
- [23] T.S. Papatheodorou, M.J. Ablowitz, Y.G. Saridakis, A rule for fast computation and analysis of soliton automata, *Studies in Applied Mathematics* 79 (1988) 173–184.
- [24] J.K. Park, K. Steiglitz, W.P. Thurston, Soliton-like behaviour in automata, *Physica D* 19 (1986) 423–432.
- [25] T. Sienko, A. Adamatzky, N. Rambidi, M. Conrad (Eds.), *Molecular Computing*, The MIT Press, London, 2003.
- [26] K. Steiglitz, I. Kamal, A. Watson, Embedded computation in one-dimensional automata by phase coding solitons, *IEEE Transactions on Computers* 37 (1988) 138–145.