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Note

Hamiltonian index is NP-complete

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ABSTRACT

In this paper we show that the problem to decide whether the hamiltonian index of a given graph is less than or equal to a given constant is NP-complete (although this was conjectured to be polynomial). Consequently, the corresponding problem to determine the hamiltonian index of a given graph is NP-hard. Finally, we show that some known upper and lower bounds on the hamiltonian index can be computed in polynomial time.

1. Introduction

By a *graph* we mean a simple loopless finite undirected graph $G = (V(G), E(G))$. If we admit G to have multiple edges, we say that G is a *multigraph*. A graph is *trivial* if it has only one vertex. For a graph G and a nonnegative integer k, we denote $V_k(G) = \{x \in V(G) \mid d_G(x) = k\}$, where $d_G(x)$ is the degree of x in G. A graph G is *d-regular* if $V(G) = V_d(G)$, and in the special case $d = 3$ we say that G is *cubic*. The *maximum degree* (*minimum degree*) of a graph G is denoted by $\Delta(G)$ ($\delta(G)$), and G is said to be *even* if every vertex of G has even degree.

A spanning subgraph of a graph G is also referred to as a *factor* of G. A subgraph of G is called *eulerian* if it is connected and even, and G is *supereulerian* if G has an eulerian factor. A 2-*factor* is a factor with all vertices of degree 2. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. When we simply say F is an *induced subgraph* of G, it means that F is induced by its set of vertices.

If $F \subset G$ is a connected subgraph of G, we say that a graph H is obtained from G by *contracting* the subgraph F, if $V(H) = (V(G) \setminus V(F)) \cup \{v_F\}$ (where $v_F \notin V(G)$) and $E(H) = (E(G) \setminus E(F)) \cup \{v_F u : u \in V(G) \setminus V(F)$ and $xu \in E(G)$ for some $x \in V(F)\}$.

The *line graph* $L(G)$ of a graph $G = (V(G), E(G))$ has $E(G)$ as its vertex set, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges have a common vertex in G. The *m-iterated line graph* $L^m(G)$ is defined recursively by $L^0(G) = G$, $L^1(G) = L(G)$ and $L^m(G) = L(L^{m-1}(G))$. Chartrand [2] showed that if a connected graph G is not a path, then $L^m(G)$ is hamiltonian (and hence also has a 2-factor, an even factor and an eulerian factor) for some integer m. The *hamiltonian index*

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(2-*factor index*, *even factor index*, *supereulerian index*) of a graph G, denoted by $h(G)$ ($f(G)$, $ef(G)$, $s(G)$), is the smallest integer m such that $L^m(G)$ contains a hamiltonian cycle (2-factor, even factor, spanning eulerian subgraph), respectively.

For further graph-theoretical notations and terminology not defined here we refer the reader to [7], and for complexity concepts we refer to [3].

In [11], it was conjectured that there is an algorithm to determine the hamiltonian index of a graph G with $h(G) \geq 2$ in polynomial time. In Section 2, we will prove that the problem to determine the hamiltonian index $h(G)$ of a graph G is NP-hard even for graph G with large $h(G)$, and we state two NP-complete formulations of the problem (which disproves the conjecture if P $\neq$ NP). In Section 3 we show that the upper and lower bounds for $h(G)$, $f(G)$, $ef(G)$ and $s(G)$, given in [8,10,9,12], can be determined in polynomial time.

2. NP-completeness of the hamiltonian index

Let G be a graph. For any two subgraphs H_1 and H_2 of G, define the *distance* $d_G(H_1, H_2)$ between H_1 and H_2 as the minimum of the distances $d_G(v_1, v_2)$ over all pairs with $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$. If $d_G(e, H) = 0$ for an edge e of G, we say that H *dominates* e. A subgraph H of G is called *dominating* if it dominates all edges of G. There is a characterization of graphs G with $h(G) \leq 1$ which involves the existence of a dominating eulerian subgraph in G.

Theorem 1 (*Harary and Nash-Williams, [5]*)**.** *Let G be a graph with at least three edges. Then $h(G) \leq 1$ if and only if G has a dominating eulerian subgraph.*

A *branch* in G is a nontrivial path in G with endvertices in $V(G) \setminus V_2(G)$ and with interior vertices (if any) in $V_2(G)$. We use $\mathcal{B}(G)$ to denote the set of branches of G and $\mathcal{B}_1(G)$ to denote the subset of $\mathcal{B}(G)$ in which at least one endvertex has degree one. For a subgraph H of G, $\mathcal{B}_H(G)$ denotes the set of branches of G whose edges are all in H.

The following theorem can be considered as an analogue of Theorem 1 for the k-iterated line graph $L^k(G)$ of a graph G.

Theorem 2 (*Xiong and Liu, [11]*)**.** *Let G be a connected graph that is not a path and let $k \geq 2$ be an integer. Then $h(G) \leq k$ if and only if $EU_k(G) \neq \emptyset$ where $EU_k(G)$ denotes the set of those subgraphs H of G which satisfy the following five conditions:*

(I) *H is an even graph;*
(II) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;
(III) $d_G(H_1, H - H_1) \leq k - 1$ *for every induced subgraph H_1 of H with $\emptyset \neq V(H_1) \subsetneq V(H)$;*
(IV) $|E(B)| \leq k + 1$ *for every branch $B \in \mathcal{B}(G) \setminus \mathcal{B}_H(G)$;*
(V) $|E(B)| \leq k$ *for every branch B in $\mathcal{B}_1(G)$.*

For a graph G and a positive integer $m \geq 2$, let $\Gamma_1^{(m)}(G), \Gamma_2^{(m)}(G), \ldots, \Gamma_s^{(m)}(G)$ denote the components of the graph obtained from G by removing edges and interior vertices of all branches of length at least m (some $\Gamma_i^{(m)}(G)$ can be trivial), and $H^{(m)}(G)$ the graph obtained from G by contracting the subgraphs $\Gamma_1^{(m)}(G), \Gamma_2^{(m)}(G), \ldots, \Gamma_s^{(m)}(G)$ to distinct vertices. The vertex of $H^{(m)}(G)$ obtained by contracting a subgraph $\Gamma_i^{(m)}(G)$ of G will be denoted by Γ_i. (Note that $\Gamma_i^{(m)}(G)$ are induced subgraphs of G, and $H^{(m)}(G)$ is a graph, i.e., has no parallel edges).

Now we construct a (multi)graph $\tilde{H}^{(m)}(G)$ from $H^{(m)}(G)$ by the following construction:

(1) Delete all cycles with all vertices except one of degree 2 (i.e., all cycles that are endblocks).
(2) If $\Gamma_i, \Gamma_j \in V(H^{(m)}(G))$ are connected by n_1 branches of length m or $m + 1$ and m_1 branches of length at least $m + 2$ with $m_1 + n_1 \geq 3$, then delete some of them in such a way that there remain n_2 branches of length m or $m + 1$ and m_2 branches of length at least $m + 2$, where

$$(m_2, n_2) = \begin{cases} (2, 0) & m_1 \text{ even},\ n_1 = 0; \\ (1, 0) & m_1 \text{ odd},\ n_1 = 0; \\ (1, 1) & n_1 = 1; \\ (0, 2) & n_1 \geq 2. \end{cases}$$

(3) Delete all end-branches of length m.
(4) Replace each non-end branch of length m or $m + 1$ by a single edge.

Theorem 3 (*Hong et al. [6]*)**.** *If G is a graph with $\Delta(G) \geq 3$ and $h(G) \geq 2$, then*

$$h(G) = \min\{m \geq 2 : \tilde{H}^{(m)}(G) \text{ has a spanning eulerian subgraph}\}.$$

The Hamiltonian Problem (HP), i.e., the problem to decide whether a given graph is hamiltonian, is one of the classical NP-complete problems. The following two results show that the HP remains NP-complete even if restricted to cubic graphs (the Cubic Hamiltonian Problem, CHP), or to line graphs (the Line Graph Hamiltonian Problem, LHP).

Theorem 4 (*Garey et al. [4]*)**.** *It is NP-complete to decide whether a given cubic graph is hamiltonian.*

Theorem 5 (*Bertossi, [1]*). *It is NP-complete to decide whether a given line graph is hamiltonian.*

It is easy to see that the LHP remains NP-complete even if restricted to graphs G with maximum degree $\Delta(G) \leq 3$ (this problem will be denoted 3-LHP).

Corollary 6. *It is NP-complete to decide whether the line graph of a given graph G with $\Delta(G) \leq 3$ is hamiltonian.*

Proof. It is obvious that this problem is in NP. To see the NP-completeness, we reduce the CHP to 3-LHP. Given a cubic graph G, let $G^{(1)}$ be the subdivision of G, i.e. the graph obtained by subdividing each edge of G by a vertex of degree 2. Then $L(G^{(1)})$ is isomorphic to the graph obtained from G by replacing each vertex by a triangle (sometimes also called the inflation of G), and it is straightforward to check that G is hamiltonian if and only if $L(G^{(1)})$ is hamiltonian. □

In this paper, we will consider the following decision problem.

HAMIND

Instance: *Graph G and a positive integer k.*

Question: *Does $L^k(G)$ have a hamiltonian cycle?*

It is easy to observe that the decision problem HAMIND is polynomially equivalent with the optimization problem of finding the value of $h(G)$, hence the NP-completeness of HAMIND implies that the problem of finding $h(G)$ is NP-hard.

We will show that HAMIND remains NP-complete also for fixed value of k, and when restricted to input graphs of small degrees.

(D, k)-**HAMIND**

Instance: *Graph G with $\Delta(G) \leq D$.*

Question: *Does $L^k(G)$ have a hamiltonian cycle?*

Theorem 7. *$(3, k)$-HAMIND is NP-complete for any fixed $k \geq 1$.*

Proof. For $k = 1$ the result follows from Corollary 6. Hence we suppose that $k \geq 2$.

1. It is easy to observe that $(3, k)$-HAMIND is in NP since, for fixed k, the construction of $L^k(G)$ is polynomial, hence the same argument as for the fact that HP $\in$ NP applies here.
2. We transform the CHP to $(3, k)$-HAMIND. Let G be a cubic graph and let $G(k)$ be the kth subdivision of G, i.e. the graph obtained from G by subdividing every edge of G by k vertices of degree 2. We show that $L^k(G(k))$ is hamiltonian if and only if G is hamiltonian.

By the construction of $G(k)$, all subgraphs $\Gamma_i^{(k)}(G(k))$ are trivial, and each of them corresponds to a vertex v_i of G. Hence $H^{(k)}(G(k)) = G(k)$. For any $\Gamma_i, \Gamma_j \in V(H^{(k)}(G(k)))$, either Γ_i, Γ_j are connected by exactly one branch of length $k+1$ (if $v_i, v_j \in V(G)$ are adjacent in G), or there is no such branch (if $v_iv_j \notin E(G)$). Hence $\tilde{H}^{(k)}(G(k)) = G$. It is straightforward to check that $L^{k-1}(G(k))$ contains 3-element cutsets consisting of vertices of degree 2, hence $L^{k-1}(G(k))$ cannot be hamiltonian. Thus, $L^k(G(k))$ is hamiltonian if and only if $h(G(k)) = k$. By Theorem 3, this is if and only if $\tilde{H}^{(k)}(G(k))$ has a spanning eulerian subgraph but $\tilde{H}^{(k-1)}(G(k))$ has no spanning eulerian subgraph. But $\tilde{H}^{(k)}(G(k)) = G$, G is cubic, and a spanning eulerian subgraph of a cubic graph is a hamiltonian cycle. □

Note that HAMIND is NP-complete also in the formulation when k is a part of input data. This is clear as concerns the transformation of the CHP to HAMIND, however, the fact that the problem is in NP is not so obvious. It is an easy calculation to observe that if G is d-regular, then $L^k(G)$ is a regular graph of degree $2^k(d-2)+2$. This simple example shows that the construction of $L^k(G)$ is, in general, not polynomial if k is not a constant. We therefore state the fact that HAMIND $\in$ NPC as a separate result. We will prove NP-completeness also in the special case under restriction to graphs with maximum degree 3.

D-**HAMIND**

Instance: *Graph G with $\Delta(G) \leq D$ and an integer $k \geq 2$.*

Question: *Does $L^k(G)$ have a hamiltonian cycle?*

Theorem 8. *3-HAMIND is NP-complete.*

Proof. 1. By Theorem 2, $L^k(G)$ is hamiltonian, i.e. $h(G) \leq k$, if and only if there is a subgraph $H \in EU_k(G)$ satisfying conditions (I)–(V) of Theorem 2. It is easy to see that conditions (I), (II), (IV) and (V) can be verified in polynomial time; it remains to show that (III) is also polynomially verifiable.

Let $H^{(1)}, \ldots, H^{(s)}$ be components of H. Clearly, if for some $H^{(j)}$, both $V(H_1) \cap V(H^{(j)})$ and $V(H_1) \cap V(H^{(j)} - H_1)$ are nonempty, then trivially $d_G(H_1, H - H_1) = 1 \leq k-1$. Hence it is sufficient to verify condition (III) for all induced subgraphs $H_1 \subset H$ such that every component of H_1 is a component of H. This can be easily done by the following construction. Let $\overline{H}$ be the graph with $V(\overline{H}) = \{H^{(1)}, \ldots, H^{(s)}\}$ and $E(\overline{H}) = \{H^{(i)}H^{(j)} \mid d_G(H^{(i)}, H^{(j)}) \leq k\}$. Then clearly H satisfies (III) if and only if $\overline{H}$ is connected. Hence 3-HAMIND is in NP.

2. A reduction of CHP to 3-HAMIND is obtained in the same way as in the proof of Theorem 7. □

3. Sharp upper and lower bounds for $h(G), f(G), ef(G)$ and $s(G)$

In [8,10,9,12], sharp upper and lower bounds for $h(G), f(G), ef(G)$ and $s(G)$ were given. In this section we show that these bounds can be determined in polynomial time. We will need the following notation. For any subset S of $\mathcal{B}(G)$, $G-S$ denotes the subgraph obtained from G by deleting all edges and internal vertices of branches of S. A subset S of $\mathcal{B}(G)$ is called a *branch cut* if $G-S$ has more components than G. A minimal branch cut is called a *branch-bond*. Obviously, for a connected graph G, a subset S of $\mathcal{B}(G)$ is a branch-bond if and only if $G-S$ has exactly two components. Let $\mathcal{BB}(G)$ denote the set of branch-bonds of G. A branch-bond is said to be *odd* if it consists of an odd number of branches. The *length* of a branch-bond $S \in \mathcal{BB}(G)$, denoted by $l(S)$, is the length of a shortest branch in S. Let $\mathcal{BB}_1(G) = \mathcal{B}_1(G)$, $\mathcal{BB}_2(G) = \{S \in \mathcal{BB}(G) \setminus \mathcal{BB}_1(G) : |S| = 1\}$, and let $\mathcal{BB}_3(G) = \{S \in \mathcal{BB}(G) : |S| \geq 3 \text{ and } S \text{ is odd}\}$. For $i \in \{1, 2, 3\}$, define

$$h_i(G) = \begin{cases} \max\{l(S) : S \in \mathcal{BB}_i(G)\} & \text{if } \mathcal{BB}_i(G) \neq \emptyset, \\ 0 & \text{if } \mathcal{BB}_i(G) = \emptyset. \end{cases}$$

The following results are all related to the parameter $h_i(G)$.

Theorem 9 (*Xiong et al. [9]*). *Let G be a connected graph that is not a path. Then*

(i) $h(G) \leq \max\{h_1(G), h_2(G) + 1, h_3(G) + 1\}$;
(ii) $h(G) \geq \max\{h_1(G), h_2(G) + 1, h_3(G) - 1\}$ *if* $h(G) \geq 1$.

These bounds are all sharp.

Theorem 10 (*Xiong and Li, [10]*). *Let G be a connected graph that is not a path such that* $\max\{h_1(G), h_2(G) - 1, h_3(G) - 1\} \geq 2$. *Then* $f(G) = \max\{h_1(G), h_2(G) - 1, h_3(G) - 1\}$.

Theorem 11 (*Xiong, [8]*). *Let G be a connected graph that is not a path such that* $\max\{h_1(G), h_2(G) - 1, h_3(G) - 1\} \geq 1$. *Then* $ef(G) = \max\{h_1(G), h_2(G) - 1, h_3(G) - 1\}$.

Theorem 12 (*Xiong and Yan, [12]*). *Let G be a connected graph that is not a path. Then*

(i) $s(G) \leq \max\{h_1(G), h_2(G) + 1, h_3(G)\}$;
(ii) $s(G) \geq \max\{h_1(G), h_2(G) + 1, h_3(G) - 1\}$ *if* $s(G) \geq 1$.

These bounds are all sharp.

Theorem 13 (*Xiong, [8]*). *Let G be a claw-free graph. Then G has an even factor if and only if* $h_1(G) = 0$ *and* $h_i(G) \leq 1$ *for* $i = 2, 3$.

It is easy to observe that there are polynomial time algorithms to determine $h_i(G)$ for $i \leq 2$. In the following, we prove that there is also a polynomial time algorithm to determine $h_3(G)$ for any graph G.

Lemma 14. *Given a graph G and an integer L, it can be decided in polynomial time whether* $h_3(G) \geq L$.

Proof. We first prove an auxiliary statement.

Claim 15. *Given a multigraph G, it can be decided in polynomial time whether G has a bond S of odd cardinality* $|S| \geq 3$.

Proof. Clearly, G has such a bond S if and only if some of its blocks has such a bond. Hence we can suppose that G is 2-connected. Specifically, $\delta(G) \geq 2$. If G has a vertex v of odd degree, then the edges vx with $x \in V(G) \setminus \{v\}$ form a bond S of odd cardinality $|S| \geq 3$, and if all vertices are of even degree, then G is eulerian and hence cannot have a bond of odd cardinality. □

Now we can complete the proof of Lemma 14. Remove from G all branches in $\mathcal{BB}_3(G)$ of length at least L, and consider the resulting components $C_1, \ldots, C_k$. Create an auxiliary multigraph G^* that contains a corresponding vertex for every component C_i, and a corresponding edge for every branch of length at least L in $\mathcal{BB}_3(G)$. Then $h_3(G) \geq L$ if and only if G^* has a bond S of odd cardinality $|S| \geq 3$. Lemma 14 then follows from Claim 15. □

From Lemma 14 we immediately obtain the following.

Theorem 16. *There is a polynomial time algorithm to determine* $h_3(G)$ *for every graph G.*

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