Paths and Edge-Connectivity in Graphs

HARUKO OKAMURA

Faculty of Engineering, Osaka City University, Osaka 558, Japan

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Mader proved that for every k-edge-connected graph $G$ ($k \geq 4$), there exists a path joining two given vertices such that the subgraph obtained from $G$ by deleting the edges of the path is $(k-2)$-edge-connected. A generalization of this and a sufficient condition for existence of 3, 4, or 5 terminus $k$ edge-disjoint paths in graphs are given. © 1984 Academic Press, Inc.

1. INTRODUCTION

We consider finite undirected graphs passibly with multiple edges but without loops. Let $G$ be a graph and let $V(G)$ and $E(G)$ be the sets of vertices and edges of $G$, respectively. For two distinct vertices $x$ and $y$, let $I_{\delta}(x, y)$ be the maximal number of edge-disjoint paths between $x$ and $y$, and let $I_{\delta}(x, x) = \infty$. For an integer $k \geq 1$, let $\Gamma(G, k)$ be

$$\{X \subseteq V(G) \mid \text{for each } x, y \in X, \lambda_{\delta}(x, y) \geq k\}.$$

Let $(s_1, t_1), ..., (s_k, t_k)$ be pairs of vertices of $G$. When is the following statement true?

(1.1) There exist edge-disjoint paths $P_1, ..., P_k$ such that $P_i$ has ends $s_i, t_i$ ($1 \leq i \leq k$).

Seymour [7] and Thomassen [8] characterised such graphs when $k = 2$, and Seymour [7] when $|\{s_1, ..., s_k, t_1, ..., t_k\}| = 3$.

For integers $k \geq 1$ and $n \geq 2$, set

$$g(k) = \min\{m \mid \text{if } G \text{ is } m\text{-edge-connected, then (1.1) holds}\},$$

$$\lambda'(k, n) = \min\{m \mid |\{s_1, ..., s_k, t_1, ..., t_k\}| \leq n \text{ and }$$

$$\{s_1, ..., s_k, t_1, ..., t_k\} \in \Gamma(G, m), \text{ then (1.1) holds}\}.$$
\[ \lambda(k, n) = \min \{ m \mid \text{if } |\{s_1, \ldots, s_k, t_1, \ldots, t_k\}| \leq n \text{ and } \lambda_G(s_i, t_i) \geq m \ (1 \leq i \leq k), \text{then (1.1) holds} \}, \]

and set

\[ \lambda'(k) = \lambda'(k, 2k) = \lambda'(k, m) \ (m > 2k) \quad \text{and} \quad \lambda(k) = \lambda(k, 2k). \]

Then for each \( k \geq 1 \),

\[ \lambda'(k, 3) = \lambda(k, 3) \quad \text{and} \quad \lambda(k) \geq \lambda'(k) \geq g(k) \geq k. \]

For \( n \geq 4 \) and even integer \( k \geq 2 \),

\[ g(k) > k \quad \text{and} \quad \lambda(k) \geq \lambda(k, n) \geq \lambda'(k, n) > k \]

(see Fig. 1 in which \( k/2 \) represents the number of parallel edges).

Thomassen [8] gave Conjecture 1, and we give Conjecture 2 slightly stronger than Conjecture 1.

**CONJECTURE 1.** For each integer \( k \geq 1 \),

\[ g(k) = \begin{cases} k & \text{if } k \text{ is odd}, \\ k + 1 & \text{if } k \text{ is even}. \end{cases} \]

**CONJECTURE 2.** For each integer \( k \geq 1 \),

\[ \lambda(k) = \begin{cases} k & \text{if } k \text{ is odd}, \\ k + 1 & \text{if } k \text{ is even}. \end{cases} \]

Clearly \( \lambda(1) = 1 \). Cypher [1] proved \( \lambda(2) = 3 \) and \( \lambda(k) \leq k + 2 \) \( (k = 3, 4, 5) \), and \( \lambda(3) = 3 \) was announced in [5] and proved in [6] by the author. Enomoto and Saito [2] proved \( g(4) = 5 \), and independently Hirata, Kubota, and Saito [3] proved \( \lambda(4) = 5 \) and \( \lambda(k) \leq 2k - 3 \) \( (k \geq 6) \).

![Figure 1](image-url)
The following theorems are useful when we consider the edge-disjoint paths problem.

**Theorem 1.** Suppose that \( k \geq 4 \) is an integer, \( G \) is a graph, \( \{s, t\} \subseteq T \subseteq V(G) \) and \( T \in \Gamma(G, k) \). Then

1. For each nonseparating edge \( e \) incident to \( s \), there exists a path \( P \) between \( s \) and \( t \) passing through \( e \) such that
   \[
   T \in \Gamma(G - E(P), k - 2) \quad \text{and} \quad \{s, t\} \in \Gamma(G - E(P), k - 1).
   \]

2. For each vertex \( a \in T - \{s, t\} \) of degree less than \( 2k \) and for each edge \( f \) incident to \( a \), there exists a path \( P \) between \( s \) and \( t \) not passing through \( a \) such that
   \[
   T \in \Gamma(G - E(P), k - 2), \quad \{s, t, a\} \in \Gamma(G - E(P), k - 1), \quad \{s, a\} \text{ or } \{t, a\} \in \Gamma(G - E(P) - f, k - 1).
   \]

3. For each vertex \( a \) with \( \lambda_G(s, a) < k \), there exists a path \( P \) between \( s \) and \( t \) not passing through \( a \) such that
   \[
   T \in \Gamma(G - E(P), k - 2), \quad \{s, t\} \in \Gamma(G - E(P), k - 1), \quad \text{and for } x = s, t,
   \]
   \[
   \lambda_{G - E(P)}(x, a) - \lambda_G(x, a).
   \]

4. For each nonseparating edges \( e \) and \( e' \) incident to \( s \), there exists a cycle \( C \) passing through \( e \) and \( e' \) such that
   \[
   T \in \Gamma(G - E(C), k - 2).
   \]

(Here \( G - E(P) \) denotes the subgraph obtained from \( G \) by deleting the edges of \( P \).)

**Corollary 1.** For every \( k \)-edge-connected graph \( G \) \((k \geq 4) \) and for every vertices \( x, y \) of \( G \), there exists a path \( P \) between \( x \) and \( y \) such that \( G - E(P) \) is \((k - 2)\)-edge-connected.

Theorem 1 is a generalization of an unpublished result of Mader given in Corollary 1. Since \( \lambda(3) = 3 \), from Corollary 1 it follows that \( g(4) = 5 \).
**Theorem 2.** Suppose that $k \geq 4$ and $n \geq 2$ are integers, $G$ is a graph and $T = \{s_1, ..., s_n, t_1, ..., t_n\} \subseteq V(G)$. If $T \in I(G, k)$ and for each $1 \leq i \leq n$, 
\[ \lambda_G(s_i, t_i) \geq k, \]
then for some $1 \leq j < l \leq n$, there exist disjoint paths $P_1$ between $s_j$ and $t_j$ and $P_2$ between $s_l$ and $t_l$ such that 
\[ \{s_j, t_j\} \in \Gamma \left( G - \bigcup_{i=1}^{2} E(P_i), k - 2 \right) \quad (1 \leq i \leq n). \]

**Theorem 3.** Suppose that $n \geq 4$ is an integer and $k \geq 3$ is an odd integer. If for each odd integer $1 \leq m \leq k$,
\[ \lambda'(m, n) = m, \]
then 
\[ \lambda(k, n) = k \quad \text{and} \quad \lambda(k + 1, n) = k + 2. \]

From Theorem 3 it follows that $\lambda(4) = 5$.

**Theorem 4.** Suppose that $k \geq 2$ is an integer, $G$ is a graph, $\{a_1, a_2\} \subseteq T \subseteq V(G)$, $|T| \leq 3$ and $T \in I'(G, k)$. Then there exists a path $P$ between $a_1$ and $a_2$ such that $T \in I(G - E(P), k - 1)$.

**Theorem 5.** Suppose that $k \geq 3$ is an odd integer, $G$ is a graph, $\{a_1, a_2, a_3\} \subseteq T \subseteq V(G)$, $a_2 \neq a_3$ and $T \in I'(G, k)$. Then

1. If $|T| \leq 4$, then there exists a path $P$ between $a_1$ and $a_2$ such that $T \in I(G - E(P), k - 1)$.

2. For $m = 2, 3$ if $|T| \leq 4$ and for $m = 3$ if $|T| = 5$ and $k \geq 5$, there exist edge-disjoint paths $P_1$ between $a_1$ and $a_2$ and $P_2$ between $a_1$ and $a_m$ such that $T \in I(G - \bigcup_{i=1}^{2} E(P_i), k - 2)$.

**Theorem 6.** For each integer $k \geq 1$,
\[ \lambda(k, 3) = k \quad \text{and} \quad \lambda(k, 4) = \lambda(k, 5) = \begin{cases} k & \text{if } k \text{ is odd}, \\ k + 1 & \text{if } k \text{ is even}. \end{cases} \]

In Theorem 5(2) if $m = 2$ and $|T| = 5$, then the conclusion does not always hold. Figure 2 gives a counterexample with $k = 7$.

When $k$ is odd and $|\{s_1, ..., s_k, t_1, ..., t_k\}| \geq 4$, if for some $1 \leq i \leq k$, 
\[ \lambda_G(s_i, t_i) < k, \]
then (1.1) does not always hold. Figure 3 gives a counterexample.
Notations and Definitions

Let $X, Y \subseteq V(G)$, $F \subseteq E(G)$, $\{x, y\} \subseteq V(G)$, and $e \in E(G)$. We often denote $\{x\}$ by $x$ and $\{e\}$ by $e$. The subgraph of $G$ induced by $X$ is denoted by $\langle X \rangle_G$ and the subgraph obtained from $G$ by deleting $X$ ($F$) is denoted by $G - X$ ($G - F$). We denote by $\partial_G(X, Y)$ the set of edges with one end in $X$ and the other in $Y$, and $\partial_G(X)$ denotes $\partial_G(X, V(G) - X)$. We denote by $\lambda_G(X, Y)$ the maximal number of edge-disjoint paths with one end in $X$ and the other in $Y$. We call $\lambda_G(X)$ an $n$-cut if $\lambda_G(X) = n$ and $\langle X \rangle_G$ and $\langle V(G) - X \rangle_G$ are both connected. An $n$-cut $\lambda_G(X)$ is called nontrivial if $|X| \geq 2$ and $|V(G) - X| \geq 2$, trivial otherwise. We denote by $d_G(x)$ the degree of $x$ and $N_G(x)$ denotes the set of vertices adjacent to $x$. We regard a path and a cycle as subgraphs of $G$. A path $P = P[x, y]$ denotes a path between $x$ and $y$, and for $x', y' \in V(P)$, $P(x', y')$ denotes a subpath of $P$ between $x'$ and $y'$.

2. Proof of Theorem 1

For a vertex $w \in V(G)$ and $b, c \in N_G(w)$, we let $G^b,c_w$ be the graph $(V(G), (E(G) \cup e) - \{f, g\})$, where $e$ is a new edge with ends $b$ and $c$, $f \in \partial_G(w, b)$ and $g \in \partial_G(w, c)$. We require

Lemma 2.1 (Mader [4]). Suppose that $w$ is a nonseparating vertex of a
graph $G$ with $d_G(w) \geq 4$ and with $|N_G(w)| \geq 2$. Then there exist $b, c \in N_G(w)$ such that for each $x, y \in V(G) - w$,

$$\lambda_{G_{w,c}}(x, y) = \lambda_G(x, y).$$

We prove Theorem 1 by induction on $|E(G)|$. If $|T| = 1$, then $s = t$ and the results holds, and so we may assume that $|T| \geq 2$ and $s \neq t$. If $G$ is not 2-connected, then we can deduce the results by using induction on some blocks. Thus we may assume that $G$ is 2-connected.

Case 1. $G$ has a nontrivial $k$ cut $\partial_G(X) = \{e_1, \ldots, e_k\} (X \subseteq V(G))$ separating $T$.

Let $H (K)$ be the graph obtained from $G$ by contracting $V(G) - X (X)$ to a new vertex $u (v)$. Set $T_H = (X \cap T) \cup u$ and $T_K = (T - X) \cup v$. Let $s \in X$. Note that $\{e_1, \ldots, e_k\}$ is contained in $E(H)$ and also in $E(K)$.

(1) Let $t \in X$. By induction $H$ has a path $P[s, t]$ such that $e \in E(P)$, $T_H \in \Gamma(H - E(P), k - 2)$, and $\{s, t\} \in \Gamma(H - E(P), k - 1)$. If $u \in V(P)$, then $P$ is a required path of $G$. If $u \notin V(P)$, then we may let $\{e_1, e_2\} \subseteq E(P)$. By induction $K$ has a cycle $C$ such that $\{e_1, e_2\} \subseteq E(C)$ and $T_K \in \Gamma(K - E(C), k - 2)$. Now we can construct a required path of $G$. Let $t \in V(G) - X$. $H$ has a path $P_1[s, u]$ such that $e \in E(P_1)$, $T_H \in \Gamma(H - E(P_1), k - 2)$ and $\{s, u\} \in \Gamma(H - E(P_1), k - 1)$. We may let $e_1 \in E(P_1)$. $K$ has a path $P_2[v, t]$ such that $e_1 \in E(P_2)$, $T_K \in \Gamma(K - E(P_2), k - 2)$ and $\{v, t\} \in \Gamma(K - E(P_2), k - 1)$. Now we can construct a required path of $G$.

(2) and (3). If $\{a, t\} \subseteq X$, $a \in X$ and $t \in V(G) - X$, or $\{a, t\} \subseteq V(G) - X$, then we can deduce the results similarly to (1). Let $a \in V(G) - X$ and $t \in X$. By induction for each $1 \leq i \leq k$, $H$ has a path $P_i[s, t]$ such that $u \notin V(P_i)$, $T_H \in \Gamma(H - E(P_i), k - 2)$, $\{s, t, u\} \in \Gamma(H - E(P_i), k - 1)$ and for $x = s$ or $t$, $\{x, u\} \in \Gamma(H - E(P_i) - e_i, k - 1)$ (say $x = t$ for $i = 1$). Let $a \in T$. $K$ has a path $P[a, v]$ such that $f \in E(P)$, $T_K \in \Gamma(K - E(P), k - 2)$ and $\{a, v\} \in \Gamma(K - E(P), k - 1)$. We may let $e_1 \in E(P)$. Since

$$\lambda_{H - E(P)}(t, u) = k - 1 = \lambda_{K - e_1, f}(v, a),$$

we have

$$\lambda_{G - E(P)}(t, a) = k - 1,$$

and so $P_1$ is a required path of $G$.

Let $\lambda_G(s, a) < k$. For some $1 \leq i \leq k$ (say for $i = 1$),

$$\lambda_{K - e_i}(v, a) = \lambda_K(v, a) = \lambda_{G}(t, a) = \lambda_{G}(s, a).$$
Since
\[ \lambda_{H-E(P_1)-e_1}(t, u) = k - 1 \quad \text{and} \quad \lambda_{H-E(P_1)-e_1}(v, a) = \lambda_{G}(t, a), \]
we have
\[ \lambda_{G-E(P_1)}(t, a) = \lambda_{G}(t, a). \]

Then
\[ \lambda_{G-E(P_1)}(s, a) \geq \min\{\lambda_{G-E(P_1)}(s, t), \lambda_{G-E(P_1)}(t, a)\} = \lambda_{G}(s, a), \]
and so \( P_1 \) is a required path of \( G \).

(4) Similar to (1).

Case 2. In (3), \( G \) has a nontrivial \( \lambda_{G}(s, a) - \text{cut} \) \( \partial_{G}(X) \) (\( X \subseteq V(G) \)) separating \( s \) and \( a \).

Let \( s \in X \) and \( a \in V(G) - X \). Since \( \lambda_{G}(s, a) < k \), \( T \subseteq X \). Let \( H \) be the graph obtained from \( G \) by contracting \( V(G) - X \) to \( a \). Then by induction (3) holds in \( H \), and so in \( G \).

Case 3. Case 1 or 2 does not occur.

Let \( T_1 \) be \( T \) for (1), (2), and (4) and \( T \cup a \) for (3). If an edge \( g \) of \( G \) is not incident to any vertex of \( T_1 \), then we can apply induction on \( G - g \). Thus we may assume that each edge is incident to a vertex of \( T_1 \). Let \( x \in V(G) - T_1 \) if such an \( x \) exists. If \( d_{G}(x) \geq 4 \), then by Lemma 2.1 there exist \( b, c \in N_{G}(x) \) such that for each \( y, z \in V(G) - x \),
\[ \lambda_{G^{b,c}}(y, z) = \lambda_{G}(y, z). \]

By induction the results hold in \( G^{b,c} \), thus we may let \( d_{G}(x) = 3 \). If \( |N_{G}(x)| = 2 \), then for some \( y \in T, |\partial_{G}(x, y)| = 2 \) and for \( h \in \partial_{G}(x, y) \) with \( h \neq e \), we can apply induction on \( G - h \). Thus we may let \( |N_{G}(x)| = 3 \).

Assume first that \( |T'| = 2 \). Then \( V(G) = T \) for (1), (2), and (4), and so the results follows. For (3)
\[ d_{G}(a) = |\partial_{G}(a, s)| + |\partial_{G}(a, t)| + |V(G) - T_1| \]
and
\[ d_{G}(s) = |\partial_{G}(s, a)| + |\partial_{G}(s, t)| + |V(G) - T_1|. \]

Since \( d_{G}(a) < k \leq d_{G}(s) \), we have
\[ |\partial_{G}(s, t)| > |\partial_{G}(a, t)| \geq 0. \]
Thus the result easily follows.
Let $|T| \geq 3$.

(1) Let $w \in T - \{s, t\}$. By Lemma 2.1 there exist $b, c \in N_G(w)$ such that for each $x, y \in V(G)$,

$$\lambda_{G^b,c}(x, y) = \lambda_G(x, y).$$

Set $G' = G^b,c$ and $T' = T - w$. By induction $G'$ has a path $P'[s, t]$ such that $e \in E(P')$, $T' \in \Gamma(G' - E(P'), k - 2)$ and $\{s, t\} \in \Gamma(G' - E(P'), k - 1)$. Let $P_1$ be the corresponding path in $G$.

$T - w \in \Gamma(G - E(P_1), k - 2)$ and $\{s, t\} \in \Gamma(G - E(P_1), k - 1)$.

For a path $P$ of $G$, let $A(P)$ be

$$\{x \mid x \in V(P) \cap N_G(w), E(P) \cap \partial_G(w, x) \neq \emptyset \text{ or } x \notin T\}.$$

Let $|A(P_1)| \leq 2$. Then in $G - E(P_1)$ there exist $k - 2$ edges $g_1, \ldots, g_{k-2}$ incident to $w$ such that the other end of $g_i$ is in $T$ or adjacent to a vertex of $T - w$ ($1 \leq i \leq k - 2$). Thus

$$\lambda_{G - E(P_1)}(w, T - w) \geq k - 2.$$

Hence $T \in \Gamma(G - E(P_1), k - 2)$, and $P_1$ is a required path. If $|A(P_1)| \geq 3$, then starting at $s$ along $P_1$, let $x_1$ and $x_2$ be the first and the last vertices of $A(P_1)$, respectively. Let $P_2$ be the path obtained by combining $P_1(s, x_1), g_1, g_2$ and $P_1(x_2, t)$, where $g_i \in \partial_G(w, x_i)$ ($i = 1, 2$). Then for each $y, z \in V(G)$,

$$\lambda_{G - E(P_2)}(y, z) \geq \lambda_{G - E(P_1)}(y, z).$$

Moreover $|A(P_2)| = 2$. Thus $P_2$ is a required path.

(2) Let $|T| = 3$. We may let $T = \{s, t, a\}$. If for some $y \in V(G) - T$, $\partial_G(a, y) = \{f\}$, then the path $P[s, t]$ with $E(P) \subseteq \partial_G(y)$ is a required path. We may let $f \in \partial_G(a, x)$ for $x = s$ or $t$, say $x = s$. If $\partial_G(s, t) \neq \emptyset$, then a path $P[s, t]$ with $|E(P)| = 1$ is a required path. If $\partial_G(s, t) = \emptyset$, then $|V(G)| > |T|$, because $d_G(a) < 2k$ and $\lambda_G(s, t) \geq k$;

$$\lambda_G(a, t) = \lambda_{G - f}(a, t),$$

and so for some $y \in V(G) - T$, the path $P[s, t]$ with $E(P) \subseteq \partial_G(y)$ is a required path. If $|T| \geq 4$, then we choose $w \in T - \{s, t, a\}$ and we can deduce the result similarly as (1).

(3) For some $w \in T - \{s, t\}$, we define $G'$ and $T'$ similarly as in (1). Then $G'$ has a path $P'[s, t]$ such that $a \in V(P')$, $T' \in \Gamma(G - E(P'), k - 2)$, $\{s, t\} \in \Gamma(G - E(P'), k - 1)$ and for $x = s, t$, $\lambda_{G - E(P')}^{x, a}(x, a) = \lambda_G(x, a)$. Let $P$, be the path of $G$ corresponding to $P'$. We define $A(P_1)$ similarly as in (1).
Then we may assume $A(P_1) \leq 2$ (see the proof of (1)). If $\partial_G(w, a) = \emptyset$, then the result follows. Let $\partial_G(w, a) \neq \emptyset$. Since

$$|\partial_G(w) - \partial_G(w, a)| + \min(|\partial_G(a) - \partial_G(a, w)|, |\partial_G(a, w)|) \geq k$$

and

$$\lambda_G - E(P_1)(s, a) = \lambda_G(s, a) = d_G(a),$$

we have

$$\lambda_G - E(P_1)(w, T - w) \geq k - 2.$$ 

Now the result follows.

(4) Similar to (1).

3. PROOF OF THEOREM 2

**Lemma 3.1.** Suppose that $k \geq 4$ and $n \geq 1$ are integers, $G$ is a graph, $T = \{s_1, ..., s_n, t_1, ..., t_m\} \subseteq V(G)$, $\lambda_G(s_i, t_i) \geq k$ $(1 \leq i \leq n)$, $a \in V(G)$, and $d_G(a) < k$. If for each $X \subseteq V(G)$ such that $\partial_G(X)$ separates $T \cup a$, $|\partial_G(X)| \geq d_G(a)$, then for some $1 \leq j \leq n$, there exists a path $P[s_j, t_j]$ such that $\{s_i, t_i\} \in \Gamma(G - E(P), k - 2)$ $(1 \leq i \leq n)$ and $\lambda_G - E(P)(s_j, a) = d_G(a)$.

**Proof.** We proceed by induction on $|E(G)|$. If $T \in \Gamma(G, k)$, then from Theorem 1 the result follows, and so we may assume that for some $X \subseteq V(G)$, $\partial_G(X)$ separates $T$ and $|\partial_G(X)| < k$. Choose $X$ with this property such that $|\partial_G(X)|$ is minimum. We may assume that $a \in V(G) - X$ and $T \cap X = \{s_1, ..., s_n, t_1, ..., t_m\}$. Let $H$ be the graph obtained from $G$ by contracting $V(G) - X$ to a new vertex $u$. By induction for some $1 \leq j \leq m$, $H$ has a path $P[s_j, t_j]$ such that $\{s_i, t_i\} \in \Gamma(H - E(P), k - 2)$ $(1 \leq i \leq m)$ and $\lambda_H - E(P)(s_j, u) = d_H(u)$. It easily follows that $\{s_i, t_i\} \in \Gamma(G - E(P), k - 2)$ $(1 \leq i \leq n)$ and $\lambda_G - E(P)(s_j, a) = d_G(a)$, and so Lemma 3.1 is proved.

Now we prove Theorem 2. Since $T \notin \Gamma(G, k)$, for some $X \subseteq V(G)$, $\partial_G(X)$ separates $T$ and $|\partial_G(X)| < k$. Choose $X$ with this property such that $|\partial_G(X)|$ is minimum. We may assume that

$$T \cap X = \{s_1, ..., s_m, t_1, ..., t_m\} \quad \text{and} \quad T - X = \{s_{m+1}, ..., s_n, t_{m+1}, ..., t_n\}.$$ 

Let $H(K)$ be the graph obtained from $G$ by contracting $V(G) - X$ (X) to a new vertex $u (v)$. By Lemma 3.1 for some $1 \leq j \leq m$, $H$ has a path $P_1[s_j, t_j]$ such that $\{s_i, t_i\} \in \Gamma(H - E(P_1), k - 2)$ $(1 \leq i \leq m)$ and $\lambda_H - E(P)(s_j, u) = d_H(u)$, and for some $m + 1 \leq l \leq n$, $K$ has a path $P_2[s_l, t_l]$ such that $\{s_i, t_i\} \in \Gamma(K - E(P_2), k - 2)$ $(1 \leq i \leq n)$ and $\lambda_K - E(P)(s_l, a) = d_K(a)$.
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\( \Gamma(K - E(P_2), k - 2) \) \( (m + 1 \leq i \leq n) \) and \( \lambda_{K - E(P_2)}(s_i, v) = d_K(v) \). Now it easily follows that

\[
\{s_i, t_i\} \in \Gamma \left( G - \bigcup_{i=1}^{2} E(P_i), k - 2 \right) \quad (1 \leq i \leq n),
\]

and so Theorem 2 is proved.

\section*{4. Proof of Theorem 3}

For each odd integer \( 1 \leq m \leq k \), since \( \lambda'(m, n) = m \), by Theorem 1 it follows that \( \lambda'(m + 1, n) = m + 2 \). Let \( \alpha = \emptyset \) or 1 and \( \beta = 2\alpha \). We prove \( \lambda(k + \alpha, n) = k + \beta \) by induction on \( k \). We may assume \( k + \alpha \geq 4 \). Suppose that \( G \) is a graph, \( T = \{s_1, \ldots, s_{k + \alpha}, t_1, \ldots, t_{k + \alpha}\} \subseteq V(G) \), \( |T| \leq n \) and \( \lambda_o(s_i, t_i) \geq k + \beta \) \((1 \leq i \leq k + \alpha)\). We prove that for \( k + \alpha \) instead of \( k \), (1.1) holds in \( G \). Then Theorem 3 is proved. Since \( \lambda'(k + \alpha, n) = k + \beta \), we may assume that \( T \notin \Gamma(G, k) \). Then by Theorem 2 for some \( 1 \leq j < l \leq k + \alpha \), there exist disjoint paths \( P_1[s_j, t_j] \) and \( P_2[s_j, t_j] \) such that \( \{s_i, t_i\} \in \Gamma(G - \bigcup_{i=1}^{2} E(P_i), k + \beta - 2) \) \((1 \leq i \leq k + \alpha)\). By induction \( \lambda(k + \alpha - 2, n) = k + \beta - 2 \). Hence \( G - \bigcup_{i=1}^{2} E(P_i) \) has edge-disjoint paths \( P_3[s_3, t_3], \ldots, P_{k + \alpha}[s_{k + \alpha}, t_{k + \alpha}] \), and so the result follows.

\section*{5. Proof of Theorem 4}

We proceed by induction on \( |E(G)| \). We may let \( a_1 \neq a_2 \) and \( |T| = 3 \). If \( G \) has a nontrivial \( k \)-cut \( \partial G(X) \) \((X \subseteq V(G)) \) separating \( T \), then we define \( H, K, u, \) and \( v \) similarly as in the proof of Theorem 1. We may let \( |T \cap X| = 2 \). By induction for \( H \) and \( (T \cap X) \cup u \) instead of for \( G \) and \( T \), the result holds. Thus the result follows. Hence we may assume that each edge is incident to a vertex of \( T \). If there exists \( x \in V(G) - T \), then we may assume that \( d_G(x) = 3 \) and \( N_o(x) = T \) (see the proof of Theorem 1), and so the path \( P[a_1, a_2] \) with \( E(P) \subseteq \partial_G(x) \) is a required path. If \( V(G) = T \), then the result easily follows.

\section*{6. Proof of Theorem 5}

We call a graph \( G \) elemental for \( V_1 \subseteq V(G) \) if \( V(G) = V_1 \cup V_2 \), \( V_1 \cap V_2 = \emptyset \) and for each \( x \in V_2 \), \( d_G(x) = 3 \), \( |N_o(x)| = 3 \) and \( N_o(x) \subseteq V_1 \). We call a graph \( G \) elemental for \( V_1 \subseteq V(G) \) and an integer \( k \geq 1 \) if \( G \) is elemental for \( V_1 \) and for each \( x \in V_1 \), \( d_G(x) = k \). For integers \( p \geq 0 \) and \( q \geq 0 \), we say that a graph \( G \) is \( G(p, q) \) if \( G \) is elemental for some \( V_1 = \ldots \)
\( \{x_1, x_2, x_3\} \subseteq V(G), |V(G) - V_1| = q, \) and \( |\partial_G(x_i, x_j)| = p \) \((1 \leq i < j \leq 3)\). Let \( G \) be an elemental graph for \( V_1 \subseteq V(G) \). We call a subgraph \( S \) an elemental star if \( V(S) \subseteq V_1, |V(S)| = 2, \) and \( |E(S)| = 1, \) or if for some \( x \in V(G) - V_1, V(S) = N_G(x) \cup x, \) and \( E(S) = \partial_G(x). \)

We require the following lemmas.

**Lemma 6.1.** Suppose that \( k \geq 3 \) is an integer, \( G \) is an elemental graph for \( T \subseteq V(G) \) and \( k, T \in \Gamma(G, k), G \) has no nontrivial \( k \)-cut separating \( T, \) and that \( S_1, S_2, S_3 \) are elemental stars of \( G. \) If \( V(S_1) \cap V(S_2) \cap V(S_3) = \emptyset, \) then \( T \in \Gamma(G - \bigcup_{i=1}^3 E(S_i), k - 2). \)

**Proof.** Assume that \( X \subseteq V(G), |X| \leq |V(G) - X|, \) and \( X \) separates \( T. \) Set \( G' = G - \bigcup_{i=1}^3 E(S_i). \) If \( |X| = 1, \) then let \( X = \{x\}. \) Since \( d_G(x) > d_G(x) - 2 = k - 2, \) we have \( |\partial_G(X)| \geq k - 2. \) If \( |X| \geq 2, \) then \( |\partial_G(X)| \geq k + 1, \) and so \( |\partial_G(X)| \geq k - 2. \) Now Lemma 6.1 is proved.

**Lemma 6.2.** Suppose that \( k \geq 2 \) is an integer, \( G \) is an elemental graph for \( T = \{x_1, x_2, x_3, x_4\} \subseteq V(G) \) and \( k, |T| = 4 \) and \( T \in \Gamma(G, k). \) Then

1. One of the following holds:
   
   (i) \( \partial_G(x_1, x_2) \neq \emptyset, \partial_G(x_1, x_3) \neq \emptyset, \) or for some \( y \in V(G) - T, N_G(y) = \{x_1, x_2, x_3\}. \)
   
   (ii) \( k \) is even, \( |\partial_G(x_2, x_3)| = k/2, \) and
   
   \(|\{y \in V(G) - T \mid N_G(y) = \{x_1, x_1, x_4\}\}| = k/2 \quad (i = 2, 3). \)

2. One of the following holds:
   
   (i) For each \( 1 \leq i < j \leq k, G \) has an elemental star \( S \) containing \( x_i \) and \( x_j. \)
   
   (ii) \( k \) is even and \( G \) is the graph obtained from four cycle by replacing each edge by \( k/2 \) parallel edges.

3. If \( G \) has no nontrivial \( k \)-cut separating \( T, \) then
   
   (i) \( \partial_G(x_1, x_2) \neq \emptyset \) or \( G \) has two elemental stars containing \( x_1 \) and \( x_2. \)
   
   (ii) One of the following holds.
   
   (a) \( G \) has edge-disjoint paths \( P_1[x_1, x_2] \) and \( P_2[x_1, x_3] \) such that for \( i = 2 \) or \( 4, \)

   \[ \{x_1, x_3\} \in \Gamma(G - \bigcup_{j=1}^2 E(P_j), k - 1) \text{ and } T \in \Gamma(G - \bigcup_{j=1}^2 E(P_j), k - 2). \]

   (b) For each \( e \in \partial_G(x_3) - \partial_G(x_3, x_2), G \) has edge-disjoint paths \( P_1[x_1, x_2] \) and \( P_2[x_1, x_3] \) such that \( e \in E(P_2) \) and \( T \in \Gamma(G - \bigcup_{i=1}^2 E(P_i), k - 2). \)
Proof. For $1 \leq i, j \leq 4$, set

$$p_{i,j} = |\partial_G(x_i, x_j)|,$$

$$R_i = \{ y \in V(G) - T | N_G(y) = T - x_i \},$$

$$r_i = |R_i|.$$ 

(1) Assume $p_{1,2} = p_{1,3} = r_4 = 0$. Then

$$d_G(x_1) = k = p_{1,4} + r_2 + r_3,$$

$$d_G(x_4) - k = p_{1,4} + p_{2,4} + p_{3,4} + r_1 + r_2 + r_3.$$ 

Thus

$$p_{2,4} = p_{3,4} = r_1 = 0.$$ 

Since $T \in \Gamma(G, k)$, we have

$$|\partial_G(\{x_2, x_3\})| = r_2 + r_3 \geq k.$$ 

Thus

$$p_{1,4} = 0.$$ 

By comparing $d_G(x_i)$ with $d_G(x_j)$ for $1 \leq i < j \leq 3$, we have

$$r_2 = r_3 = p_{2,3}.$$ 

Now (ii) follows.

(2) Assume $p_{1,2} = r_3 = r_4 = 0$. Then by comparing $d_G(x_1) + d_G(x_2)$ with $d_G(x_3) + d_G(x_4)$, we have

$$r_1 = r_2 = p_{3,4} = 0.$$ 

Now by comparing $d_G(x_i) = k = p_{1,3} + p_{2,3}$ with $d_G(x_i)$ for $i = 1, 2$, we have

$$p_{1,4} = p_{2,3} \quad \text{and} \quad p_{2,4} = p_{1,3}.$$ 

Moreover,

$$|\partial_G(\{x_1, x_4\})| = p_{1,3} + p_{2,4} = 2p_{1,3} \geq k,$$

$$|\partial_G(\{x_1, x_3\})| = p_{1,4} + p_{2,3} = 2p_{1,4} \geq k.$$ 

Thus

$$p_{1,3} = p_{2,3} = p_{2,4} = p_{1,4},$$ 

and so (ii) follows.
(3)(i) We assume \( p_{1,2} = r_4 = 0 \), and then prove \( r_3 \geq 2 \). Since any cut separating \( \{x_1, x_3\} \) and \( \{x_2, x_4\} \) or separating \( \{x_1, x_4\} \) and \( \{x_2, x_3\} \) has more than \( k \) edges, we have

\[
(6.1) \quad p_{1,4} + p_{2,3} + p_{3,4} + r_1 + r_2 + r_3 \geq k + 1,
\]

and

\[
(6.2) \quad p_{1,3} + p_{2,4} + p_{3,4} + r_1 + r_2 + r_3 \geq k + 1.
\]

By comparing \( d_G(x_3) + d_G(x_4) \) with (6.1) and (6.2), we have

\[
r_3 \geq 2.
\]

(ii) If there exists an \( f \in \partial_G(x_1, x_3) \), then by Theorem 1, \( G \) has a path \( P[x_3, x_2] \) such that \( f \in E(P) \), \( \{x_3, x_2\} \in \Gamma(G - E(P), k - 1) \) and \( T \in \Gamma(G - E(P), k - 2) \), and so (a) follows. Thus we may let

\[
p_{1,3} = p_{1,2} = 0,
\]

then by (1)

\[
r_4 > 0.
\]

If \( r_3 > 0 \), then for \( y_1 \in R_4 \) and \( y_2 \in R_3 \),

\[
\{x_3, x_4\} \in \Gamma \left( G - \bigcup_{i=1}^{2} \partial_G(y_i), k - 1 \right) \quad \text{and} \quad T \in \Gamma \left( G - \bigcup_{i=1}^{2} \partial_G(y_i), k - 2 \right),
\]

and so (a) follows. Thus we may let

\[
r_3 = 0.
\]

Then by (1) and (3)

\[
p_{1,4} > 0 \quad \text{and} \quad r_4 \geq 2.
\]

Let \( y \) be another end of \( e \), then \( y = x_4 \) or \( y \in R_i \) (\( i = 1, 2 \) or 4). In each case (b) easily follows.

**Lemma 6.3.** Suppose that \( k \geq 3 \) is an odd integer, \( G \) is a graph, \( \{x_1, x_2, x_3\} \subseteq T \subseteq V(G), x_i \neq x_j \) (\( 1 \leq i < j \leq 3 \)), \( T \in \Gamma(G, k) \) and \( e \in E(G) \). If one of (i) or (ii) below holds, then for \( m = 2, 3 \), \( G \) has edge-disjoint paths \( P_1[x_1, x_2] \) and \( P_2[x_1, x_m] \) such that \( e \in E(P_1) \cup E(P_2) \) and \( T \in \Gamma(G - \bigcup_{i=1}^{2} E(P_i), k - 2) \).

(i) \( e \in \partial_G(x_1, x_2) \),

(ii) \( e \in \partial_G(x_1, y) \) for some \( y \in V(G) - T \) with \( d_G(y) = 3 \) and with \( N_G(y) = \{x_1, x_2, x_3\} \).
Assume that (i) holds. By Theorem 1 if \( m = 2 \), then \( G \) has a cycle \( C \) such that \( e \in E(C) \) and \( T \in \Gamma(G - E(C), k - 2) \), and if \( m = 3 \), then \( G \) has a path \( P[x_2, x_3] \) such that \( e \in E(P) \) and \( T \in \Gamma(G - E(P), k - 2) \).

Assume that (ii) holds. We may assume that \( G \) is 2-connected. If \( d_G(x_3) = d > k \), then we replace \( x_3 \) by \( d \) vertices of degree \( k \) (Fig. 4 gives an example with \( d = 8 \) and \( k = 5 \)), producing a new graph \( G' \). In \( G' \) we assign \( x_3 \) on \( N_{G'}(y) \setminus \{x_1, x_2\} \). If the result holds in \( G' \), then clearly the result holds in \( G \), and so we may assume that \( d_G(x_3) = k \). Let \( f \in \partial_G(x_3) - \partial_G(y, x_3) \). By Theorem 1 \( G \) has a path \( P[x_1, x_2] \) such that \( x_3 \notin V(P) \), \( T \in \Gamma(G - E(P), k - 2) \), \( \{x_1, x_2\} \in \Gamma(G - E(P), k - 1) \) and \( \{x_i, x_3\} \in \Gamma(G - E(P) - f, k - 1) \) (\( i = 1 \) or \( 2 \)). Then \( y \notin V(P) \), because \( d_G(x_3) = k \) and \( d_G(y) = 3 \). Moreover, \( T \in \Gamma(G - E(P) - y, k - 2) \). Thus the result follows.

Now we prove Theorem 5. We may assume that \( G \) is 2-connected, \( d_G(x) = k \) for each \( x \in T \) (see the proof of Lemma 6.3 and Fig. 4, in this case we can assign \( x \) on any vertex of new \( d_G(x) \) vertices of degree \( k \)) and that \( d_G(y) = 3 \) for each \( y \in V(G) \setminus T \) (see Case 3 in the proof of Theorem 1). We proceed by induction on \( |E(G)| \). If \( |T| < 3 \), then the results follow from Theorem 4. Thus let \( |T| \geq 4 \).

**Case 1.** \( G \) has a nontrivial \( k \)-cut \( \partial_G(X) = \{e_1, \ldots, e_k\} \) (\( X \subseteq V(G) \)) separating \( T \).

We define \( H, K, u, v, T_U, \) and \( T_K \) similarly as Case 1 in the proof of Theorem 1. If \( |X \cap T| = 1 \), then the results hold in \( K \), and so in \( G \). Thus let \( |X \cap T| \geq 2 \) and \( |T - X| \geq 2 \).

We require the following:

(6.3) **If** \( G \) **has a nontrivial** \( k \)-**cut** \( \partial_G(Y) = \{f_1, \ldots, f_k\} \) (\( Y \subseteq X \)) **separating** \( T \), **then we may assume that** \( (X - Y) \cap T \neq \emptyset \).

**Proof.** Assume \( (X - Y) \cap T = \emptyset \). Let \( b_i \) (\( c_i \)) be the end of \( e_i \) (\( f_i \)) in \( V(G) - X \) (\( Y \)) (\( 1 \leq i \leq k \)). We may assume that the graph obtained from \( \langle X - Y \rangle_G \) by adding \( b_1, \ldots, b_k, c_1, \ldots, c_k, e_1, \ldots, e_k, f_1, \ldots, f_k \) has edge-disjoint paths \( P_i[b_1, c_1], \ldots, P_k[b_k, c_k] \). Let \( G' \) be the graph obtained from \( G - (X - Y) \) by adding new edges \( g_1, \ldots, g_k \), where \( g_i \) has ends \( b_i \) and \( c_i \).

![Figure 4](image-url)
(1 < i < k). Then |E(G')| < |E(G)|, and the results of Theorem 5 hold in G', and so in G. Now (6.3) is proved.

(6.4) If |X ∩ T| = 2 (|T − X| = 2), then we may assume that H (K) is $G(p, q)$ ($G(p', q')$) for some integers $p$ and $q$ ($p'$ and $q'$).

Proof. Assume |X ∩ T| = 2. If H has a nontrivial $k$-cut $\partial H(Y)$ ($Y \subseteq V(H) − u$) separating $T_H$, then by (6.3) $(X − Y) \cap T \neq \emptyset$, and so |T ∩ Y| = 1. Then by taking $Y$ instead of $X$ the results of Theorem 5 hold. Thus we may assume that an end of each edge of $H$ is in $T_H$. Hence the result easily follows.

We return to the proof of Theorem 5. By Lemma 6.3 we may assume the following.

(6.5) $\partial_G(a_1, a_i) = \emptyset$ ($i = 2, m$) and for each $y \in V(G) − T$, $\{a_1, a_2, a_m\} \not\subseteq N_G(y)$.

Let $a_1 \in X$.

1. Now |X ∩ T| − |T − X| = 2. If $a_2 \in X$, then by (6.4) the result easily follows. Thus let $a_2 \in V(G) − X$. Since $p + q > (k + 1)/2$ and $p' + q' > (k + 1)/2$, for some $1 < i < k$, $H$ has an elemental star $S_1$ containing $a_1$ and $e_i$ and $K$ has an elemental star $S_2$ containing $a_2$ and $e_i$. Then $T \in \Gamma(G − \bigcup_{i=1}^2 E(S_i), k − 1)$.

2. Subcase 1.1. $\{a_2, a_m\} \subseteq X$.

$H$ has required paths. If one of them passes through $u$, then we can deduce the result by using Theorem 1(4) on $K$.

Subcase 1.2. $\{a_2, a_m\} \subseteq V(G) − X$ and |X ∩ T| = 2.

Set $X ∩ T = \{a_1, a_s\}$. If |T| = 4, then $a_4$ does not exist. By (6.4) $H$ is $G(p, q)$. Thus if one of (6.6) or (6.7) below holds, then the result follows.

(6.6) For some $e_i \in \partial_H(u, a_1)$, $K$ has edge-disjoint paths $P_1[v, a_2]$ and $P_2[v, a_m]$ such that $e_i \in E(P_1) \cup E(P_2)$ and $T_x \in \Gamma(K − \bigcup_{i=1}^2 E(P_i), k − 2)$.

(6.7) For some $e_i, e_j \in \partial_H(u) − \partial_H(u, a_1)$, $K$ has edge-disjoint paths $P_1[v, a_2]$ and $P_2[v, a_m]$ such that $\{e_i, e_j\} \subseteq E(P_1) \cup E(P_2)$ and $T_x \in \Gamma(K − \bigcup_{i=1}^2 E(P_i), k − 2)$.

If $p = 0$, then $\partial_H(u, a_1) = \emptyset$, and so (6.7) follows. Thus let $p > 0$. If |T − X| = 2, then by (6.4) $K$ is $G(p', q')$, and so (6.6) follows. Thus let |T − X| = 3 and $m = 3$. Set $T − X = \{a_2, a_3, a_4\}$.
Subcase 1.2.1. $K$ has a nontrivial $k$-cut $\partial_K(Y)$ $(Y \subseteq V(K) - v)$ separating $T_K$.

By (6.3) we may let $|Y \cap T_K| = |T_K - Y| = 2$. Let $K_1$ and $K_2$ be the graphs obtained from $K$ by contracting $Y$ and $V(K) - Y$ to a vertex respectively. Then similarly as (6.4) $K_i$ is $G(p_i, q_i)$ for some integers $p_i$ and $q_i$ $(i = 1, 2)$. Let $M$ be

$$\{x_1, x_2\} \subseteq V(K) - T_K | \partial_x(x_1, x_2) \neq \emptyset\},$$

and let $M'$ be

$$\{x | \text{for some } N \in M, x \in N\}.$$

For each $N \in M$, $N \cap V(K_i) \neq \emptyset$ $(i = 1, 2)$,

$$d_{K-N}(a_j) = d_{K-N}(v) = k - 1 \quad (j = 2, 3, 4) \quad \text{and} \quad T_K \in \Gamma(K - N, k - 1).$$

If $k = |M|$, then $p_1 = p_2 = 0$ and the result easily follows, and so let $k > |M|$. $K - M'$ is elemental for $T_K$ and $k - |M|$. Assume that $k - |M|$ is even and $K - M'$ is the graph obtained from four cycle by replacing each edge by $(k - |M|)/2$ parallel edges. For each cycle $C$ of $K - M'$ such that $|V(C)| = |E(C)| = 4$, we have $T_K \in \Gamma(G - E(C), k - 2)$. If $\partial(a_1, a_4) \neq \emptyset$, then (6.6) follows, and if not, then by (6.5) $a_1$ is adjacent to $p$ vertices of $M'$. If $|M| > 2$, then (6.6) follows. Thus assume $1 \geq |M| \geq p > 1$. Since $(k - |M|)/2 \geq (5 - 1)/2 = 2$, for some $1 \leq i < j \leq k$,

$$\{e_i, e_j\} \subseteq \partial_{K}(u) - \partial_{K}(u, a_s),$$

and $K$ has a four cycle $C$ such that $|V(C)| = |E(C)| = 4$ and $\{e_i, e_j\} \subseteq E(C)$. Hence (6.7) follows.

By Lemma 6.2(2) we may assume that for each two vertices of $T_K$, $K - M'$ has an elemental star containing them. Set $a_0 = v$, and for $i, j = 0, 2, 3, 4$, set

$$p_{i,j} = |\partial_{K}(a_i, a_j)|, \quad r_i = |x \in V(K) - T_K | N_K(x) = T_K - a_i|.$$

For $i, j = 0, 2, 3, 4$, since $|\partial_{K}(\{a_i, a_j\})| \geq k$,

$$p_{i,j} \leq (k - 1)/2.$$
If $a_4 \in Y$, then (6.6) easily follows, and thus let $T_H - Y = \{a_0, a_4\}$. Since $p_{0,4} \geq |\partial_G(a_1, a_4)| = p > 0$, by Lemma 6.2(1) we have

\[ p_{4,2} > 0, \quad p_{4,3} > 0, \quad \text{or} \quad r_0 > 0, \]

and

\[ p_{0,2} > 0, \quad p_{0,3} > 0, \quad \text{or} \quad r_4 > 0. \]

If $r_0 > 0$, $r_4 > 0$, $p_{0,2} \cdot p_{3,4} > 0$, or $p_{0,3} \cdot p_{2,4} > 0$, then (6.6) follows (note that $K_i$ is $G(p_1, q_i)$ for $i = 1, 2$). Thus we may assume that

\begin{equation}
\tag{6.8}
p_{0,2} > 0, \quad p_{2,4} > 0 \quad \text{and} \quad r_0 = r_4 = p_{0,3} = p_{3,4} = 0.
\end{equation}

Assume $|M| = 0$. Then

\[ d_G(a_3) = p_{2,3} + r_2 \quad \text{and} \quad p_{2,3} \leq (k - 1)/2, \]

and so

\begin{equation}
\tag{6.9}
r_2 \geq (k + 1)/2 = p + 1.
\end{equation}

By comparing $d_G(a_2)$ with $d_G(a_4)$ we have

\[ p_{0,2} + p_{2,3} = p_{0,4} + r_2. \]

Thus

\begin{equation}
\tag{6.10}
p_{0,2} > p_{0,4} \geq p.
\end{equation}

From (6.9) and (6.10), (6.7) follows.

Now we may let $|M| > 0$. Since $\{a_2, a_3\} \subseteq Y$, we have

\[ |\partial_k(Y)| = k = d_k(a_2) - d_k(a_3) - 2p_{2,3} - |M| = 2k - 2p_{2,3} - |M|, \]

and so

\[ 2p_{2,3} + |M| = k. \]

Since $d_G(a_3) - k = p_{2,3} + r_2 + |M|$, $r_2 = p_{2,3}$.

Since $d_G(a_3) = 2r_2 + |M|$, $d_G(a_4) = p_{0,4} + p_{2,4} + r_2 + r_3 + |M|$, and $p_{2,4} > 0$ (by (6.8)), we have

\begin{equation}
\tag{6.11}
r_2 \geq a_{0,4} + 1 \geq p + 1.
\end{equation}
By comparing \( d_G(a_2) \) with \( d_G(a_4) \), we have
\[
p_{0,2} = p_{0,4}.
\]
Thus
\[
(6.12) \quad p_{0,2} + |M| \geq p + 1.
\]
From (6.11) and (6.12), (6.7) follows.

Subcase 1.2.2. \( K \) has no nontrivial \( k \)-cut separating \( T_K \).

We may assume that an end of each edge of \( K \) in \( T_K \) and \( K \) is elemental for \( T_K \). The proof is similar as the case \( |M| = 0 \) in the proof of Subcase 1.2.1.

Subcase 1.3. \( \{a_2, a_m\} \subseteq V(G) - X \) and \( |X \cap T| = 3 \).

Now \( m = 3 \). By (6.4) \( K \) is \( G(p', q') \). Set \( X \cap T = \{a_1, a_4, a_5\} \). If \( H \) has a nontrivial \( k \)-cut \( \partial_H(Y) \) \( (Y \subseteq V(H) - u) \) separating \( T_H \), then we may let \( |Y \cap T_H| = 2 \). Then for \( Y \) or \( V(G) - Y \) instead of \( X \), Subcase 1.1 or 1.2 occurs. Thus we may assume that this is not the case and \( H \) is elemental for \( T_H \). If either (6.13) or (6.14) holds, then the result follows:

\[
(6.13) \quad \text{For some } e_i \in \partial_K(v) - \bigcup_{i=2}^{3} \partial_K(v, a_i), \text{ } H \text{ has edge-disjoint paths } P_1[a_1, u] \text{ and } P_2[a_1, u] \text{ such that } e_i \in E(P_1) \cup E(P_2) \text{ and } T_H \in \Gamma(H - \bigcup_{i=1}^{2} E(P_i), k - 2).\]

\[
(6.14) \quad \text{For } l = 2 \text{ or } 3 \text{ and for some } e_i \in \partial_K(v, x_i) \text{ and } e_j \in \partial_K(v) - \partial_K(v, x_i), \text{ } H \text{ has edge-disjoint paths } P_1[a_1, u] \text{ and } P_2[a_1, u] \text{ such that } \{e_i, e_j\} \subseteq E(P_1) \cup E(P_2) \text{ and } T_H \in \Gamma\left(H - \bigcup_{i=1}^{2} E(P_i), k - 2\right).\]

Set \( a_0 = u \) and for \( i, j = 0, 1, 4, 5, \) set
\[
p_{i,j} = |\partial_H(a_1, a_j)|,
\]
\[
R_i = |x \in V(H) - T_H | N_H(x) = T_H - a_i|,
\]
\[
r_i = |R_i|.
\]
By (6.5) \( p_{0,1} = 0 \).

Assume \( p_{1,4} = p_{1,5} = 0 \). If \( r_0 \leq (k - 1)/2 \), then
\[
r_4 + r_5 = d_G(a_4) - r_0 \geq (k + 1)/2 \geq p' + 1,
\]
and so (6.13) or (6.14) follows. Thus let \( r_0 \geq (k + 1)/2 \). Since \( d_G(a_0) = p_{0,4} + p_{0,5} + r_1 + r_4 + r_5 \) and \( d_G(a_5) = p_{0,5} + p_{4,5} + r_0 + r_1 + r_4 \), we have
\[
p_{0,4} + r_5 = p_{4,5} + r_0.
\]
Hence

\[ d_\alpha(a_4) = k \geq p_{0,4} + r_o + r_s \geq 2r_0 > k, \]

a contradiction.

Now we may let \( p_{1,i} > 0 \) for \( i = 4 \) or \( 5 \), say \( i = 4 \). Since \( p_{0,1} = 0 \) and by Lemma 6.2(3), we have

\[ r_4 + r_s \geq 2. \]

For each \( x \in R_4 \cup R_5 \), if \( x \) is adjacent to a vertex of \( V(K) - T_K \) in \( G \), then (6.13) follows, thus assume that \( \partial_\alpha(x, a_i) \neq \emptyset \) (\( i = 2 \) or \( 3 \)). For each \( x, y \in R_4 \cup R_5 \), if \( \partial_\alpha(x, a_2) \neq \emptyset \) and \( \partial_\alpha(y, a_3) \neq \emptyset \), then (6.14) follows, thus assume that for \( i = 2 \) or \( 3 \), \( \partial_\alpha(x, a_i) = \partial_\alpha(y, a_i) = \emptyset \), say \( i = 3 \), and that \( r_4 + r_s \leq p' \).

Assume \( r_4 > 0 \). For some \( e_i \in \partial_K(v) - \partial_K(v, a_2) \), \( e_i \) is incident to \( a_4 \) or a vertex of \( R_1 \) in \( G \), because

\[ p' + q' \geq (k + 1)/2 > p_{0,5}. \]

Thus (6.14) follows.

Now we may assume that \( r_4 = 0 \), \( r_s > 0 \), and \( p_{1,5} = 0 \). Thus \( p_{0,1} = p_{1,5} = r_4 = 0 \), contrary to Lemma 6.2(1).

Subcase 1.4. \( a_2 \in X \) and \( a_m \in V(G) - X \).

Now \( m = 3 \).

Subcase 1.4.1. \( |X \cap T| = 2 \).

By (6.4) \( H = G(p, q) \), and by (6.5) \( p = 0 \). Since \( |T_K| \leq 4 \), by induction \( K \) has a path \( P[v, a_2] \) such that \( T_K \in \Gamma(K - E(P), k - 1) \). Let \( e_i \in E(P) \). \( H \) has an elemental star \( S_1 \) containing \( a_1 \) and \( e_1 \). Let \( S_2 \) be another elemental star of \( H \). Then \( T_H \in \Gamma(H - \bigcup_{i=1}^k E(S_i), k - 2) \), and so the result follows.

Subcase 1.4.2. \( |X \cap T| = 3 \) and \( |T - X| = 2 \).

Assume that \( H \) has a nontrivial \( k \)-cut \( \partial_H(Y) = \{f_1, \ldots, f_k\} \) (\( Y \subseteq V(H) - u \)) separating \( T_H \). Then we may assume that \( |Y \cap T_H| = 2, a_2 \in Y \) and \( a_1 \in X - Y \). Let \( H_1 (H_2) \) be the graph obtained from \( H \) by contracting \( V(H) - Y (Y) \) to a new vertex \( u_i (u_2) \). Then similarly as (6.4) \( H_i \) is \( G(p_i, q_i) \) for some integers \( p_i \) and \( q_i \) (\( i = 1, 2 \)). If \( p_2 = 0 \), then the result easily follows. If \( p_2 > 0 \), then we may let \( \{f_1, e_1\} \subseteq \partial_\alpha(a_1) \) and we can easily deduce the result.

Now we may assume that \( H \) has no nontrivial \( k \)-cut separating \( T_H \) and \( H \) is elemental for \( T_H \). Set \( X \cap T = \{a_1, a_2, u, a_4\} \) and \( T - X = \{a_3, a_5\} \). For \( a_1, a_2, u, a_4 \) instead of \( x_1, x_2, x_3, x_4 \), (a) or (b) of Lemma 6.2(3) holds. If (a) holds, then the result easily follows, thus assume that (b) holds. Since

\[ |\partial_H(u) - \partial_H(u, a_2)| \geq (k + 1)/2 \quad \text{and} \quad p' + q' \geq (k + 1)/2, \]

for some \( 1 \leq i \leq k \),

\[ e_i \in \partial_H(u) - \partial_H(u, a_2) \quad \text{and} \quad e_i \in \partial_K(v) - \partial_K(v, a_3), \]

and so the result follows.
Case 2. $G$ has no nontrivial $k$-cut separating $T$.

We may assume that $G$ is elemental for $T$. If $|T| = 4$, then by Lemma 6.1 the result follows. Thus let $|T| = 5$ and $m = 3$. Set $T = \{a_1, a_2, a_3, a_4, a_5\}$ and for $1 \leq i, j, l \leq 5$, set

$$p_{i,j} = |\partial_G(a_i, a_j)|,$$

$$R(i,j,l) = \{x \in V(G) - T| N_G(x) = \{a_i, a_j, a_l\}\},$$

$$r(i,j,l) = |R(i,j,l)|.$$

We require

(6.15) For each distinct $1 \leq i, j, l \leq 5$, $G$ has an elemental star containing $\{a_i, a_j\}$ or $\{a_i, a_l\}$.

Proof. Assume that each elemental star of $G$ does not contain $\{a_1, a_2\}$ nor $\{a_2, a_3\}$. Then

$$d_G(a_1) = p_{1,4} + p_{1,5} + r(1,4,5).$$

Since $p_{i,j} \leq (k - 1)/2$ for each $i, j$, we have $r(1,4,5) > 0$. Let $F$ be the cut of $G$ separating $\{a_1, a_4, a_5\} \cup R(1, 4, 5)$ from the rest of the graph, then

$$|F| \leq d_G(a_4) + d_G(a_5) - (p_{1,4} + p_{1,5} + 2r(1, 4, 5)) < k,$$

a contradiction. Now (6.15) is proved.

We return to the proof of Theorem 5. By (6.5)

$$p_{1,2} = p_{1,3} = r(1, 2, 3) = 0.$$

If $r(1, 2, i) > 0$ and $r(1, 3, j) > 0$ ($i, j = 4$ or 5), then the result follows. Thus and by (6.15) we may assume that

$$r(1, 2, 4) > 0 \quad \text{and} \quad r(1, 3, i) = 0 \quad (i = 4, 5).$$

By (6.15)

$$p_{i,5} + r(i, 5, 2) + r(i, 5, 4) > 0 \quad (i = 1, 3).$$

If $p_{1,5} > 0$, $p_{3,5} > 0$, $r(1, 5, 2) \cdot r(3, 5, 4) > 0$, or $r(1, 5, 4) \cdot r(3, 5, 2) > 0$, then by Lemma 6.1 the result follows. Thus we may assume that for $(i,j) = (2,4)$ or $(4,2)$,

$$p_{1,5} = p_{3,5} = 0, \quad r(1, 5, i) = r(3, 5, i) = 0,$$
and
\[ r(1, 5, j) \cdot r(3, 5, j) > 0. \]

Assume \( r(1, 5, 2) = r(3, 5, 2) = 0 \). Then
\[ d_G(x_1) = p_{1,4} + r(1, 2, 4) + r(1, 4, 5), \]
and
\[ d_G(x_4) \geq p_{1,4} + r(1, 2, 4) + r(1, 4, 5) + r(3, 4, 5) > k, \]
a contradiction. Thus
\[ r(1, 5, 4) = r(3, 5, 4) = 0. \]
Since \( r(1, 2, 5) > 0 \), by the same argument we have
\[ \rho_{1,4} = \rho_{3,4} = 0. \]
Thus
\[ d_G(x_1) = r(1, 2, 4) + r(1, 2, 5) \]
and
\[ d_G(x_2) \geq r(1, 2, 4) + r(1, 2, 5) + r(2, 3, 5) > k, \]
a contradiction.

7. Proof of Theorem 6

Suppose that \( k \geq 1 \) is an integer, \( G \) is a graph, \( T = \{s_1, \ldots, s_k, t_1, \ldots, t_k\} \subseteq V(G) \) and \( T \in \Gamma(G, k) \). We prove that if \( |T| = 3 \), or if \( k \) is odd and \( |T| = 4 \) or \( 5 \), then (1.1) holds. We proceed by induction on \( k \).

Assume \( |T| = 3 \). By Theorem 4 \( G \) has a path \( p[s_k, s_k] \) such that \( T \in \Gamma(G - E(P), k - 1) \). By induction for \( k - 1 \), (1.1) holds in \( G - E(P) \), and so for \( k \), (1.1) holds in \( G \).

Assume that \( k \geq 5 \) is odd and \( |T| = 4 \) or \( 5 \). For some \( 1 \leq i < j \leq k \), if \( |T| = 4 \), then
\[ s_i = s_j \text{ or } t_j, \]
and if \( |T| = 5 \), then
\[ s_i = s_j \text{ or } t_j \text{ and } \{s_i, t_i\} \neq \{s_j, t_j\}. \]
say for \( i = k - 1 \) and \( j = k \). By Theorem 5 \( G \) has edge-disjoint paths 
\[ P_1[s_{k-1}, t_{k-1}] \text{ and } P_2[s_k, t_k] \] 
such that \( T \subseteq I(G - \bigcup_{i=1}^{2} E(P_i), k - 2) \). By 
induction for \( k - 2 \), (1.1) holds in \( G - \bigcup_{i=1}^{2} E(P_i) \), and so for \( k \), (1.1) holds 
in \( G \).

Thus for each integer \( k \geq 1 \),
\[ \lambda'(k, 3) = \lambda(k, 3) = k, \]
and for each odd integer \( k \geq 1 \),
\[ \lambda'(k, 4) = \lambda'(k, 5) = k. \]

By Theorem 3 for each odd integer \( k \geq 1 \),
\[ \lambda(k, 4) = \lambda(k, 5) = k \quad \text{and} \quad \lambda(k + 1, 4) = \lambda(k + 1, 5) - k + 2. \]

Now Theorem 6 is proved.

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