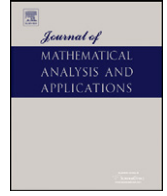




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Inverse problems for periodic generalized Jacobi matrices

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ABSTRACT

Some inverse problems for semi-infinite periodic generalized Jacobi matrices are considered. In particular, a generalization of the Abel criterion is presented. The approach is based on the fact that the solvability of the Pell–Abel equation is equivalent to the existence of a certainly normalized *J*-unitary 2×2 -matrix polynomial (the monodromy matrix).
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1. Introduction

In 1826, Abel proved that the square root \sqrt{R} of the polynomial *R* of degree $2n$ without multiple zeros can be expanded into a periodic continued fraction, that is

$$\sqrt{R} - U = \frac{1}{p_0 + \frac{1}{p_1 + \frac{1}{\ddots}}} = \frac{1}{p_0} + \frac{1}{p_1} + \cdots + \frac{1}{p_{s-1}} + \frac{1}{p_0} + \cdots, \tag{1.1}$$

where *U* and p_0, \dots, p_{s-1} are polynomials, if and only if the Pell–Abel equation

$$X^2 - RY^2 = 1 \tag{1.2}$$

has the polynomial solutions *X* and *Y* (the Abel criterion). Actually, the basic idea of Abel was to find out if there exists a polynomial ρ such that the integral

$$\int \frac{\rho(t)}{\sqrt{R(t)}} dt$$

can be expressed in terms of the elementary functions. It turned out that such a polynomial ρ exists if and only if the Pell–Abel equation (1.2) is solvable in polynomials. The proof of these results and much more modern information on this topic from the algebraic point of view can be found in [15].

Another appearance of the Pell–Abel equation is intimately related to extremal polynomials. For example, the Chebyshev polynomials T_j and U_j of the first and second kind satisfy the following relation

$$T_j^2(\lambda) - (\lambda^2 - 1)U_j^2(\lambda) = 1.$$

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This observation can be generalized to the case of the polynomials that least deviate from zero on several intervals [16]. Furthermore, by setting

$$R(\lambda) = (\lambda^2 - 1) \prod_{j=1}^{n-1} (\lambda - \alpha_j)(\lambda - \beta_j),$$

where $-1 < \alpha_1 < \beta_1 < \dots < \alpha_{n-1} < \beta_{n-1} < 1$, we have that Eq. (1.2) is solvable in polynomials if and only if the set

$$[-1, 1] \setminus \bigcup_{j=1}^{n-1} (\alpha_j, \beta_j)$$

coincides with the spectrum of a bi-infinite periodic classical Jacobi matrix [16]. In turn, such a spectrum gives rise to periodic solutions of the Toda lattice (for example, see [17, Chapter 12]).

On the other hand, we have seen semi-infinite classical Jacobi matrices at the beginning although it was implicitly. Indeed, in some special cases the continued fraction (1.1) can be a J -fraction generating a classical Jacobi matrix [1]. However, J -fractions do not cover all the possible cases of periodic fractions of the form (1.1). That is why it is more natural to consider semi-infinite generalized Jacobi matrices associated with continued fractions of the form (1.1). Such matrices were introduced in [8] and the theory was further developed in [7,9]. In particular, direct spectral problems for the periodic generalized Jacobi matrices were considered in [7]. The goal of the present paper is to solve some inverse problems for semi-infinite generalized Jacobi matrices associated with periodic continued fractions. It is done in the following way. First, a one-to-one correspondence between certainly normalized 2×2 -matrix polynomials (the *monodromy matrices*) and the periodic generalized Jacobi matrices in question is established. Then we show that the solvability of the Pell–Abel equation is equivalent to the existence of a normalized J -unitary 2×2 -matrix polynomial, that is the monodromy matrix of the underlying periodic generalized Jacobi matrix. The latter statement generalizes the Abel criterion. Namely, we give necessary and sufficient conditions for functions of the form

$$\frac{\sqrt{R(\lambda)} - U(\lambda)}{V(\lambda)}$$

to be the m -functions of semi-infinite periodic generalized Jacobi matrices or, equivalently, to admit the periodic continued fraction expansions.

Finally, it should be mentioned that some inverse spectral problems for finite generalized Jacobi matrices were studied in [6].

2. Preliminaries

2.1. P -fractions

Let φ be a nonrational function holomorphic at a neighborhood of infinity and having the property

$$\overline{\varphi(\bar{\lambda})} = \varphi(\lambda).$$

So, φ has the following representation in a neighborhood of infinity

$$\varphi(\lambda) = - \sum_{j=0}^{\infty} \frac{s_j}{\lambda^{j+1}}, \tag{2.1}$$

where the *moments* s_j are real. A number $n_j \in \mathbb{N}$ is called a *normal index* of the sequence $\mathbf{s} := \{s_j\}_{j=0}^{\infty}$ if $\det(s_{i+k})_{i,k=0}^{n_j-1} \neq 0$. Since φ is not rational, there exists an infinite number of normal indices of \mathbf{s} (see [10, Section 16.10.2]). Also, without loss of generality we will always assume that the sequence \mathbf{s} is *normalized*, i.e. the first nonvanishing moment s_{n_1-1} has modulus 1.

As is known [14], the series (2.1) leads to the following infinite continued fraction

$$-\frac{\varepsilon_0}{p_0(\lambda)} - \frac{\varepsilon_0 \varepsilon_1 b_0^2}{p_1(\lambda)} - \dots - \frac{\varepsilon_{j-1} \varepsilon_j b_{j-1}^2}{p_j(\lambda)} - \dots, \tag{2.2}$$

where $\varepsilon_j = \pm 1$, $b_j > 0$ and $p_j(\lambda) = \lambda^{k_j} + p_{k_j-1}^{(j)} \lambda^{k_j-1} + \dots + p_1^{(j)} \lambda + p_0^{(j)}$ are real monic polynomials of degree k_j (see also [3,5]). Note, that $n_j = k_0 + k_1 + \dots + k_{j-1}$.

The continued fraction (2.2) is called a P -fraction. Actually, the P -fraction can be considered as a sequence of the linear-fractional transformations [12, Section 5.2]

$$T_j(\omega) := \frac{-\varepsilon_j}{p_j(\lambda) + \varepsilon_j b_j^2 \omega}$$

having the following matrix representation

$$\mathcal{W}_j(\lambda) = \begin{pmatrix} 0 & -\frac{\varepsilon_j}{b_j} \\ \varepsilon_j b_j & \frac{p_j(\lambda)}{b_j} \end{pmatrix}, \quad j \in \mathbb{Z}_+. \tag{2.3}$$

The superposition $T_0 \circ T_1 \circ \dots \circ T_j$ of the linear-fractional transformations corresponds to the product of the matrices $\mathcal{W}_l(\lambda)$

$$\mathcal{W}_{[0,j]}(\lambda) = \left(w_{ik}^{(j)}(\lambda) \right)_{i,k=1}^2 := \mathcal{W}_0(\lambda)\mathcal{W}_1(\lambda) \dots \mathcal{W}_j(\lambda). \tag{2.4}$$

It is well known that the entries of $\mathcal{W}_{[0,j]}$ can be expressed in terms of denominators and numerators of the convergents to the P -fraction (2.2). To give these formulas explicitly define the polynomials $P_{j+1}(\lambda)$, $Q_{j+1}(\lambda)$ by the equalities

$$\begin{pmatrix} -Q_0 \\ P_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -Q_{j+1}(\lambda) \\ P_{j+1}(\lambda) \end{pmatrix} := \mathcal{W}_{[0,j]}(\lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad j \in \mathbb{Z}_+.$$

The relation $\mathcal{W}_{[0,j]}(\lambda) = \mathcal{W}_{[0,j-1]}(\lambda)\mathcal{W}_j(\lambda)$ yields

$$\mathcal{W}_{[0,j]}(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathcal{W}_{[0,j-1]}(\lambda) \begin{pmatrix} 0 \\ \varepsilon_j b_j \end{pmatrix} = \begin{pmatrix} -\varepsilon_j b_j Q_j(\lambda) \\ \varepsilon_j b_j P_j(\lambda) \end{pmatrix}, \quad j \in \mathbb{N}.$$

So, the matrix $\mathcal{W}_{[0,j]}(\lambda)$ has the form

$$\mathcal{W}_{[0,j]}(\lambda) = \begin{pmatrix} -\varepsilon_j b_j Q_j(\lambda) & -Q_{j+1}(\lambda) \\ \varepsilon_j b_j P_j(\lambda) & P_{j+1}(\lambda) \end{pmatrix}, \quad j \in \mathbb{Z}_+. \tag{2.5}$$

Further, the equality

$$\begin{pmatrix} -Q_{j+1}(\lambda) \\ P_{j+1}(\lambda) \end{pmatrix} = \mathcal{W}_{[0,j-1]}(\lambda)\mathcal{W}_j(\lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{b_j} \mathcal{W}_{[0,j-1]}(\lambda) \begin{pmatrix} -\varepsilon_j \\ p_j(\lambda) \end{pmatrix}, \quad j \in \mathbb{N},$$

shows that the polynomials $P_j(\lambda)$, $Q_j(\lambda)$ are solutions of the difference equation

$$\varepsilon_{j-1} \varepsilon_j b_{j-1} u_{j-1} - p_j(\lambda) u_j + b_j u_{j+1} = 0, \quad j \in \mathbb{N},$$

obeying the initial conditions

$$\begin{aligned} P_0(\lambda) &= 1, & P_1(\lambda) &= \frac{p_0(\lambda)}{b_0}, \\ Q_0(\lambda) &= 0, & Q_1(\lambda) &= \frac{\varepsilon_0}{b_0}. \end{aligned}$$

According to (2.5), the $(j + 1)$ -th convergent of the P -fraction (2.2) is equal to

$$f_j := T_0 \circ T_1 \circ \dots \circ T_j(0) = -Q_{j+1}(\lambda)/P_{j+1}(\lambda).$$

The relations (2.5), (2.4), and (2.3) imply the following relation [9]

$$\varepsilon_j b_j (Q_{j+1}(\lambda) P_j(\lambda) - Q_j(\lambda) P_{j+1}(\lambda)) = 1, \quad j \in \mathbb{Z}_+. \tag{2.6}$$

2.2. Generalized Jacobi matrices

Let $p(\lambda) = p_n \lambda^n + \dots + p_1 \lambda + p_0$ be a monic scalar real polynomial of degree n , i.e. $p_n = 1$. Let us associate with the polynomial p its symmetrizer E_p and let the companion matrix C_p be given by

$$E_p = \begin{pmatrix} p_1 & \dots & p_n \\ \vdots & \ddots & \\ p_n & & \mathbf{0} \end{pmatrix}, \quad C_p = \begin{pmatrix} 0 & \dots & 0 & -p_0 \\ 1 & & \mathbf{0} & -p_1 \\ & \ddots & & \vdots \\ \mathbf{0} & & 1 & -p_{n-1} \end{pmatrix}.$$

As is known, $\det(\lambda - C_p) = p(\lambda)$ and the matrices E_p and C_p are related by

$$C_p E_p = E_p C_p^\top. \tag{2.7}$$

Now we are in a position to associate a tridiagonal block matrix with (2.2).

Definition 2.1. (See [8,13].) Suppose we are given a P -fraction of the form (2.2). Let p_j be real scalar monic polynomials of degree k_j

$$p_j(\lambda) = \lambda^{k_j} + p_{k_j-1}^{(j)}\lambda^{k_j-1} + \dots + p_1^{(j)}\lambda + p_0^{(j)},$$

and let $\varepsilon_j = \pm 1$, $b_j > 0$, $j \in \mathbb{Z}_+$. The tridiagonal block matrix

$$H = \begin{pmatrix} A_0 & \tilde{B}_0 & & \mathbf{0} \\ B_0 & A_1 & \tilde{B}_1 & \\ & B_1 & A_2 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{pmatrix} \tag{2.8}$$

where $A_j = C_{p_j}$ and $k_{j+1} \times k_j$ matrices B_j and $k_j \times k_{j+1}$ matrices \tilde{B}_j are given by

$$B_j = \begin{pmatrix} 0 & \dots & b_j \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix}, \quad \tilde{B}_j = \begin{pmatrix} 0 & \dots & \tilde{b}_j \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix}, \quad \tilde{b}_j = \varepsilon_j \varepsilon_{j+1} b_j, \quad j \in \mathbb{Z}_+, \tag{2.9}$$

will be called a *generalized Jacobi matrix* associated with the P -fraction (2.2).

Remark 2.2. The papers [8,9,13] are only concerned with the case of generalized Jacobi matrices which are finite rank perturbations of classical Jacobi matrices.

Now, introducing the following shortened matrices

$$H_{[0,j]} = \begin{pmatrix} A_0 & \tilde{B}_0 & & \\ B_0 & A_1 & \ddots & \\ & \ddots & \ddots & \tilde{B}_{j-1} \\ & & B_{j-1} & A_j \end{pmatrix}, \quad H_{[1,j]} = \begin{pmatrix} A_1 & \tilde{B}_1 & & \\ B_1 & A_2 & \ddots & \\ & \ddots & \ddots & B_{j-1} \\ & & B_{j-1} & A_j \end{pmatrix}, \tag{2.10}$$

where $j \in \mathbb{Z}_+ \cup \{\infty\}$, one can obtain the connection between the polynomials P_j , Q_j and the shortened Jacobi matrices $H_{[0,j]}$, $H_{[1,j]}$, respectively [8]:

$$P_j(\lambda) = (b_0 \dots b_{j-1})^{-1} \det(\lambda - H_{[0,j-1]}),$$

$$Q_j(\lambda) = \varepsilon_0 (b_0 \dots b_{j-1})^{-1} \det(\lambda - H_{[1,j-1]}).$$

Furthermore, the following statement holds true.

Proposition 2.3. (See [8,9].) Let $j \in \mathbb{N}$. Then

- i) The polynomials P_j and P_{j+1} have no common zeros.
- ii) The polynomials Q_j and Q_{j+1} have no common zeros.
- iii) The polynomials P_j and Q_j have no common zeros.

In what follows we are only interested in periodic generalized Jacobi matrices.

Definition 2.4. Let $s \in \mathbb{N}$. A generalized Jacobi matrix satisfying the properties

$$A_{js+k} = A_k, \quad B_{js+k} = B_k, \quad \varepsilon_{js+k} = \varepsilon_k, \quad j \in \mathbb{Z}_+, \quad k \in \{0, \dots, s-1\}$$

will be called an s -periodic generalized Jacobi matrix.

Let $\ell^2_{[0,\infty)}$ denote the Hilbert space of complex square summable sequences (w_0, w_1, \dots) equipped with the usual inner product. Evidently, any s -periodic generalized Jacobi matrix generates a bounded linear operator in $\ell^2_{[0,+\infty)}$. To see some more properties of periodic generalized Jacobi matrices let us define the symmetric matrix G by the equality

$$G = \text{diag}(G_0, G_1, \dots), \quad G_j = \varepsilon_j E_{p_j}^{-1}, \quad j \in \mathbb{Z}_+. \tag{2.11}$$

Clearly, the operator G defined on $\ell^2_{[0,\infty)}$ is bounded and self-adjoint. Moreover, G^{-1} is a bounded linear operator in $\ell^2_{[0,\infty)}$.

Let $\mathfrak{H}_{[0,\infty)}$ be a space of elements of $\ell^2_{[0,\infty)}$ provided with the following indefinite inner product

$$[x, y] = (Gx, y)_{\ell^2_{[0,\infty)}}, \quad x, y \in \ell^2_{[0,\infty)}. \tag{2.12}$$

Let us recall [4] that a pair $(\mathfrak{H}, [\cdot, \cdot])$ consisting of a Hilbert space \mathfrak{H} and a sesquilinear form $[\cdot, \cdot]$ on $\mathfrak{H} \times \mathfrak{H}$ is called a space with indefinite inner product. A space with indefinite metric $(\mathfrak{H}, [\cdot, \cdot])$ is called a Krein space if the indefinite scalar product $[\cdot, \cdot]$ can be represented as follows

$$[x, y] = (Jx, y)_{\mathfrak{H}}, \quad x, y \in \mathfrak{H},$$

where the linear operator J satisfies the following conditions

$$J = J^{-1} = J^*.$$

The operator J is called the fundamental symmetry. So, one can see that the space $\mathfrak{H}_{[0,\infty)}$ is the Krein space with the fundamental symmetry $J = \text{sign } G$ (see [4] for details). Moreover, the property (2.7) implies the following.

Proposition 2.5. (See [7,8].) *The s -periodic generalized Jacobi matrix defines a bounded self-adjoint operator H in the Krein space $\mathfrak{H}_{[0,\infty)}$, that is,*

$$[Hx, y] = [x, Hy], \quad x, y \in \mathfrak{H}_{[0,\infty)}.$$

The main tool for the spectral analysis of periodic generalized Jacobi operators is the following matrix

$$T(\lambda) := \mathcal{W}_{[0,s-1]}(\lambda) = \begin{pmatrix} -\varepsilon_{s-1}b_{s-1}Q_{s-1}(\lambda) & -Q_s(\lambda) \\ \varepsilon_{s-1}b_{s-1}P_{s-1}(\lambda) & P_s(\lambda) \end{pmatrix}. \tag{2.13}$$

The matrix T is called the monodromy matrix. In fact, using the Floquet theory [7] we can get the description of the spectrum of the matrix H . Indeed, let $w_1 = w_1(\lambda)$ and $w_2 = w_2(\lambda)$ be the roots of the characteristic equation $\det(T(\lambda) - w) = 0$. Introducing the following notations

$$E := \{ \lambda \in \mathbb{C} : |w_1(\lambda)| = |w_2(\lambda)| \},$$

$$E_p := \{ \lambda \in \mathbb{C} : P_{s-1}(\lambda) = 0, |b_{s-1}Q_{s-1}(\lambda)| > |P_s(\lambda)| \}$$

we can give the description of spectra of periodic generalized Jacobi operators.

Theorem 2.6. (See [7].) *The spectrum $\sigma(H)$ of the s -periodic generalized Jacobi operator H has the form*

$$\sigma(H) = E \cup E_p, \quad \sigma_p(H) = E_p,$$

where $\sigma_p(H)$ denotes the point spectrum of H , i.e. eigenvalues.

It should be mentioned that for the classical Jacobi matrices this result was obtained in [11].

Remark 2.7. (See [7].) One can also easily get another description of E

$$E = \{ \lambda \in \mathbb{C} : (P_s(\lambda) - \varepsilon_{s-1}b_sQ_{s-1}(\lambda)) \in [-2, 2] \}.$$

2.3. m -Functions of periodic generalized Jacobi matrices

Recall that the m -functions are one of the central tools in studying linear difference operators. We start with the definition of the m -function.

Definition 2.8. Let H be a bounded generalized Jacobi matrix. The function m defined by

$$m(\lambda) = [(H - \lambda)^{-1}e, e], \quad e = (1, 0, 0, \dots)^\top \tag{2.14}$$

is called the m -function (or, the Weyl function) of the operator H .

Next, by using the Frobenius formula, one can see that the m -function m and the m -function

$$m_{[1,\infty)}(\lambda) = [(H_{[1,\infty)} - \lambda)^{-1}e, e] = (G_{[1,\infty)}(H_{[1,\infty)} - \lambda)^{-1}e, e)$$

of $H_{[1,\infty)}$ (for the definition of $H_{[1,\infty)}$ and $G_{[1,\infty)}$ see (2.10)) are related by the equality [8]

$$m(\lambda) = \frac{-\varepsilon_0}{p_0(\lambda) + \varepsilon_0 b_0^2 m_{[1,\infty)}(\lambda)}, \quad |\lambda| > \|H\| \geq \|H_{[1,\infty)}\|. \tag{2.15}$$

The standard technique of dealing with periodic continued fractions enables us to calculate the m -function explicitly.

Proposition 2.9. *Let H be an s -periodic generalized Jacobi matrix. Then its m -function has the following form*

$$m(\lambda) = \frac{-(P_s(\lambda) + \varepsilon_{s-1}b_{s-1}Q_{s-1}(\lambda)) + \sqrt{(P_s(\lambda) - \varepsilon_{s-1}b_{s-1}Q_{s-1}(\lambda))^2 - 4}}{2\varepsilon_{s-1}b_{s-1}P_{s-1}(\lambda)}, \tag{2.16}$$

where the cut is E and the branch is chosen in such a way that $m(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

Proof. Due to (2.15) and the s -periodicity of H , we have that

$$m(\lambda) = -\frac{\varepsilon_0}{p_0(\lambda)} - \frac{\varepsilon_0\varepsilon_1b_0^2}{p_1(\lambda)} - \dots - \frac{\varepsilon_{s-1}\varepsilon_{s-2}b_{s-2}^2}{p_{s-1}(\lambda) + \varepsilon_{s-1}b_{s-1}^2m(\lambda)}.$$

According to (2.4) and (2.13) the latter relation takes the form

$$m(\lambda) = -\frac{\varepsilon_{s-1}b_{s-1}Q_{s-1}(\lambda)m(\lambda) + Q_s(\lambda)}{\varepsilon_{s-1}b_{s-1}P_{s-1}(\lambda)m(\lambda) + P_s(\lambda)}. \tag{2.17}$$

Next, (2.17) can be rewritten as follows

$$\varepsilon_{s-1}b_{s-1}P_{s-1}(\lambda)m^2(\lambda) + (P_s(\lambda) + \varepsilon_{s-1}b_{s-1}Q_{s-1}(\lambda))m(\lambda) + Q_s(\lambda) = 0. \tag{2.18}$$

Now, (2.16) follows from (2.18) and (2.6). The choice of the cut and the branch is due to Definition 2.8 and Theorem 2.6. \square

Theorem 2.10. (Cf. [7].) *The m -function of the periodic generalized Jacobi matrix H determines H uniquely.*

Proof. In fact, consecutive applications of the relation (2.15) lead to the continued fraction (2.2), which determines H uniquely. \square

3. Inverse problems

3.1. The monodromy matrices

Note that knowing the monodromy matrix T gives the complete information about the spectrum of the corresponding s -periodic generalized Jacobi matrix H . Moreover, it is clear that one can reconstruct the s -periodic generalized Jacobi matrix H by its monodromy matrix $T = \mathcal{W}|_{[0, s-1]}$ since by the construction we have that

$$-\frac{Q_s(\lambda)}{P_s(\lambda)} = -\frac{\varepsilon_0}{p_0(\lambda)} - \frac{\varepsilon_0\varepsilon_1b_0^2}{p_1(\lambda)} - \dots - \frac{\varepsilon_{s-2}\varepsilon_{s-1}b_{s-2}^2}{p_{s-1}(\lambda)}$$

and b_{s-1} can be extracted from T directly. Now, one of the most natural questions is the following: which matrices can be the monodromy matrices? To answer this question let us note first that the monodromy matrix

$$T(\lambda) = \begin{pmatrix} t_{11}(\lambda) & t_{12}(\lambda) \\ t_{21}(\lambda) & t_{22}(\lambda) \end{pmatrix}$$

satisfies the following properties

(T1) $t_{11}, t_{12}, t_{21}, t_{22}$ are polynomials with real coefficients such that

$$t_{11}(\lambda)t_{22}(\lambda) - t_{12}(\lambda)t_{21}(\lambda) = 1, \quad \lambda \in \mathbb{R};$$

(T2) $\deg t_{12} < \deg t_{22}, \deg t_{21} < \deg t_{22}$;

(T3) the absolute values of the leading coefficients of t_{21} and t_{22} are equal and the leading coefficient of t_{22} is positive.

These properties give us a tip to introduce the following definition.

Definition 3.1. The 2×2 -matrix polynomial T is called admissible if it satisfies the conditions (T1)–(T3).

Remark 3.2. Consider the following 2×2 -matrix

$$J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

It is said that 2×2 matrix polynomial is J -unitary on the real line \mathbb{R} if the following equality holds true

$$T(\lambda)JT^*(\bar{\lambda}) = J, \quad \lambda \in \mathbb{R}.$$

Clearly, every admissible matrix polynomial is J -unitary. Moreover, every J -unitary 2×2 -matrix polynomial on the real line can be normalized by multiplications from left and right by constant J -unitary matrices to an admissible matrix (for more details see [9, Section 3.7]).

The answer to the above-stated question is given by the following statement.

Theorem 3.3. (Cf. [9].) *Every admissible matrix polynomial T is the monodromy matrix of a unique periodic generalized Jacobi matrix.*

Proof. Actually, this statement can be proved by re-examining [9, Section 3.7]. First, note that the admissible matrix T generates the shortened generalized Jacobi matrix $H_{[0,s-1]}$. Indeed, let us consider the P -fraction expansion

$$\frac{t_{12}(\lambda)}{t_{22}(\lambda)} = -\frac{\varepsilon_0}{p_0(\lambda)} - \frac{\varepsilon_0\varepsilon_1b_0^2}{p_1(\lambda)} - \cdots - \frac{\varepsilon_{s-2}\varepsilon_{s-1}b_{s-2}^2}{p_{s-1}(\lambda)},$$

which, according to the definition, generates the shortened generalized Jacobi matrix $H_{[0,s-1]}$. So, one can recover the sequences $P_0, P_1, \dots, P_{s-1}, b_{s-1}P_s$ and $Q_0, Q_1, \dots, Q_{s-1}, b_{s-1}Q_s$ such that $t_{12}/t_{22} = -Q_s/P_s$. Choosing $b_{s-1} > 0$ to have the equalities $t_{12} = -Q_s$ and $t_{22} = P_s$ we conclude that we know the matrix

$$\mathcal{W}_{[0,s-1]}(\lambda) = \begin{pmatrix} -\varepsilon_{s-1}b_{s-1}Q_{s-1}(\lambda) & -Q_s(\lambda) \\ \varepsilon_{s-1}b_{s-1}P_{s-1}(\lambda) & P_s(\lambda) \end{pmatrix}$$

and the number b_{s-1} . Also, the periodicity implies that $\varepsilon_0 = \varepsilon_s$ and, so, one can determine $\tilde{b}_{s-1} = \varepsilon_{s-1}\varepsilon_s b_{s-1}$. Thus, the admissible matrix T gives rise to the following s -periodic generalized Jacobi matrix

$$H = \begin{pmatrix} H_{[0,s-1]} & \tilde{B}_{s-1} & & \mathbf{0} \\ B_{s-1} & H_{[0,s-1]} & \tilde{B}_{s-1} & \\ & B_{s-1} & H_{[0,s-1]} & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{pmatrix}.$$

Finally, it follows from (T1)–(T3), Proposition 2.3, and (2.6) that the monodromy matrix $\mathcal{W}_{[0,s-1]}$ of H coincides with T (for more details see the proof of [9, Theorem 3.23]). \square

Remark 3.4. Due to Remark 3.2 and formula (2.4), Theorem 3.3 can be applied for getting factorization results for J -unitary matrix polynomials [2,9].

3.2. The Abel criterion

Formula (2.16) leads to the conclusion that the m -function of the periodic generalized Jacobi matrix has the following algebraic form

$$\varphi(\lambda) = \frac{\sqrt{R(\lambda)} - U(\lambda)}{V(\lambda)}, \tag{3.1}$$

where $R, U,$ and V are polynomials such that

$$\deg R = 2n, \quad \deg U = n, \quad \deg V < \deg U. \tag{3.2}$$

Besides, the m -function has the property that $\varphi(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

Clearly, the function φ can be also represented by means of some other choice of polynomials $R, U,$ and V . So, without loss of generality we always assume that the polynomials $R, U,$ and V are chosen in such a way that the polynomial V has the minimal degree between all such polynomials in the representation (3.1) of the given function φ . Now the natural question is to ask under what conditions on $R, U,$ and V a function of the form (3.1) can be the m -function of a periodic generalized Jacobi matrix? One can find the answer in the following statement.

Theorem 3.5. *Let $R, U,$ and V be polynomials subject to (3.2). Let us also suppose that one of the branches of the function $\varphi = (\sqrt{R} - U)/V$ possesses the property $\varphi(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. Then there is a cut such that the corresponding branch of φ is the m -function of a periodic generalized Jacobi matrix if and only if there exist real polynomials X, Y, Z satisfying the following relations*

$$X^2 - RY^2 = 1, \tag{3.3}$$

$$(U^2 - R)Y = VZ. \tag{3.4}$$

Proof. The “if” part. Let $\varphi = (\sqrt{R} - U)/V$ be the m -function of a periodic generalized Jacobi matrix. Then φ satisfies the following equation

$$\varepsilon_{s-1}b_{s-1}P_{s-1}(\lambda)\varphi^2(\lambda) + (P_s(\lambda) + \varepsilon_{s-1}b_{s-1}Q_{s-1}(\lambda))\varphi(\lambda) + Q_s(\lambda) = 0. \tag{3.5}$$

Substituting $\varphi = (\sqrt{R} - U)/V$ to (3.5) one gets

$$\begin{aligned} &\sqrt{R}(-2\varepsilon_{s-1}b_{s-1}P_{s-1}U + P_sV + \varepsilon_{s-1}b_{s-1}Q_{s-1}V) + \varepsilon_{s-1}b_{s-1}P_{s-1}(R^2 + U^2) \\ &- (P_s + \varepsilon_{s-1}b_{s-1}Q_{s-1})UV + Q_sV^2 = 0. \end{aligned} \tag{3.6}$$

Observe that the equality $\alpha\sqrt{R} + \beta = 0$, where α and β are some polynomials, implies $\alpha = 0$ and $\beta = 0$. So, it follows from (3.6) that

$$P_sV = 2\varepsilon_{s-1}b_{s-1}P_{s-1}U - \varepsilon_{s-1}b_{s-1}Q_{s-1}V, \tag{3.7}$$

$$Q_sV^2 = (P_s + \varepsilon_{s-1}b_{s-1}Q_{s-1})UV - \varepsilon_{s-1}b_{s-1}P_{s-1}(R^2 + U^2). \tag{3.8}$$

Combining (3.7) and (3.8) gives

$$Q_sV^2 = \varepsilon_{s-1}b_{s-1}P_{s-1}(U^2 - R). \tag{3.9}$$

It is easy to see from (2.16) as well as from (2.18) that $P_{s-1} = \tilde{P}_{s-1}V$. Taking this observation into account one can rewrite (3.7) and (3.9) as follows

$$P_s = 2\varepsilon_{s-1}b_{s-1}\tilde{P}_{s-1}U - \varepsilon_{s-1}b_{s-1}Q_{s-1}, \tag{3.10}$$

$$Q_sV = \varepsilon_{s-1}b_{s-1}\tilde{P}_{s-1}(U^2 - R). \tag{3.11}$$

Now, by setting $Z = Q_s$ and $Y = \varepsilon_{s-1}b_{s-1}\tilde{P}_{s-1}$ the relation (3.11) becomes (3.4).

In order to get (3.3), let us multiply formula (2.6) by V

$$\varepsilon_{s-1}b_{s-1}(VQ_sP_{s-1} - VQ_{s-1}P_s) = 1.$$

Substituting (3.10) and (3.11) to the latter equality we arrive at

$$\varepsilon_{s-1}b_{s-1}(\varepsilon_{s-1}b_{s-1}\tilde{P}_{s-1}^2V(U^2 - R) - \varepsilon_{s-1}b_{s-1}V(2U\tilde{P}_{s-1} - Q_{s-1})Q_{s-1}) = V.$$

Further calculations lead to the relation

$$b_{s-1}^2(\tilde{P}_{s-1}^2(U^2 - R) - 2U\tilde{P}_{s-1}Q_{s-1} + Q_{s-1}^2) = 1,$$

which can be rewritten as follows

$$b_{s-1}^2(\tilde{P}_{s-1}U - Q_{s-1})^2 - b_{s-1}^2R\tilde{P}_{s-1}^2 = 1. \tag{3.12}$$

Setting $X = \varepsilon_{s-1}b_{s-1}(\tilde{P}_{s-1}U - Q_{s-1})$ and recalling that $Y = \varepsilon_{s-1}b_{s-1}\tilde{P}_{s-1}$ one sees that (3.12) is exactly (3.3).

The “only if” part. Suppose that the system (3.3), (3.4) is satisfied for some polynomials X , Y , and Z . Then the “if” part gives us a hint to introduce the following polynomials

$$t_{11} = X - YU, \quad t_{12} = YV, \quad t_{21} = -Z, \quad t_{22} = X + YU,$$

where the signs of the leading coefficients of X , Y , and Z are taken to have the leading coefficient t_{22} positive and $\deg t_{11} < n$. Next it is easy to check

$$\det T = t_{11}t_{22} - t_{12}t_{21} = X^2 - Y^2U^2 + ZYV = X^2 - (RY^2 - ZYV) + ZYV = X^2 - RY^2 = 1.$$

It remains to note that

$$\deg t_{21} = \deg Z = \deg(U^2 - R)Y/V < \deg X = \deg t_{22}$$

in order to see that the matrix $T = (t_{ij})_{i,j=1}^2$ is admissible. The rest follows from Theorem 3.3. \square

Remark 3.6. If $V = 1$ then Theorem 3.5 reduces to the classical Abel criterion of the periodicity of continued fractions representing \sqrt{R} .

Remark 3.7. Every solvable system (3.3), (3.4) leads to an admissible matrix polynomial. The converse is also true. In turn, every admissible matrix polynomial can be constructed by (2.3), (2.4).

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