Grüss-Type Inequalities

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A connection between Grüss inequality and the error of best approximation is revealed. A Grüss-type inequality that unifies the continuous and discrete versions of the classical Grüss inequalities is established.

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G. Grüss [6] proved an interesting and useful inequality that gives an estimate of the difference between the integral of the product of two functions and the product of their integrals, as follows:

**Grüss inequality.** Suppose \( f, g : [a, b] \to \mathbb{R} \) are integrable, 
\[ m_f \leq f(x) \leq M_f \text{ and } m_g \leq g(x) \leq M_g, \text{ for all } x \in [a, b]. \]

Then 
\[
\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right| \\
\leq \frac{1}{4} (M_f - m_f)(M_g - m_g).
\]

Moreover, the equality holds when 
\[ f(x) = g(x) = \operatorname{sign} \left| x - \frac{a + b}{2} \right|, \]

with \( M_f = M_g = 1 \) and \( m_f = m_g = -1 \).

Many generalizations of this inequality can be seen in [8] and the references therein. Applications of the Grüss type inequalities have been found in statistics, coding theory, and numerical analysis (see, e.g., [3, 4]).
recent paper, Fink [5] proved more general inequalities of the Grüss type. He showed that a standard proof of the Grüss inequality is based on the observation that the left-hand side can be written in terms of a double integral of the product of the difference functions $f(x) - f(y)$ and $g(x) - g(y)$ followed by an application of the Cauchy–Schwarz inequality. Although this type of proof is very concise, it leaves no clue as to why such an inequality holds. In particular, it is not clear from the proof how the right-hand side can be interpreted. The proofs do not give any idea of the interpretation of the right-hand side of the Grüss inequality.

This paper was motivated by the discovery of a very elementary proof ([7]) of the Grüss inequality and the desire to understand the meaning of the quantity on the right-hand side of the inequality. As a result, we find several inequalities that either imply or generalize the Grüss inequality. More importantly, continuous version and discrete version of Grüss-type inequalities are unified into one general result (see Theorem 5). The unified version is useful for applications of these inequalities in statistics where we do not need to distinguish the continuous case from the discrete one.

Let $(X, \mu)$ be a measure space with $\mu(X) = 1$. Let $L_{p}(\mu)$ denote the space of real-valued functions $f$ defined on $X$ such that $\|f\|_p$ are $\mu$-integrable when $1 \leq p < \infty$ and $f$ is essentially bounded on $X$ when $p = \infty$. We write $\|\cdot\|_{L_p(\mu)}$ for the corresponding $L_p$ norms with measure $\mu$.

To motivate our formulation, we first prove a few simple results that will lead to our Grüss-type inequality.

**Lemma 1.** Let $f \in L_1(\mu)$. Then

$$\int_{X} \left| f - \int_{X} f \, d\mu \right| \, d\mu \leq \|f\|_{L_2(\mu)}. \quad (1)$$

**Proof.** Using $\mu(X) = 1$, we have $\|f\|_{L_1(\mu)} \leq \|f\|_{L_2(\mu)}$, and so

$$\int_{X} |f - \int_{X} f \, d\mu| \, d\mu \leq \left\{ \int_{X} \left( f - \int_{X} f \, d\mu \right)^2 \, d\mu \right\}^{1/2}. $$

Now the right-hand side is equal to

$$\left\{ \int_{X} f^2 \, d\mu - \left( \int_{X} f \, d\mu \right)^2 \right\}^{1/2},$$

which is less than $\|f\|_{L_2(\mu)}$. 

An immediate, useful consequence of Lemma 1 can be stated as follows.

**Corollary 2.** Suppose $f \in L_{\infty}(\mu)$ and $x \in X$ with

$$m_f \leq f(x) \leq M_f, \text{ a.e.}$$

If $\int f(x) \, d\mu = 0$, then

$$\int |f(x)| \, d\mu \leq \frac{1}{2} (M_f - m_f).$$
Proof. For any constant $c$, apply Lemma 1 to the translation of $f(x)$, $f(x) - c$. This gives
\[ \int_X |f - \int_X f \, d\mu| \, d\mu \leq \|f - c\|_{L^2(\mu)}, \tag{2} \]
since the left-hand side of (1) is translation invariant. The right-hand side of (2) is less than $\|f - c\|_{L^\infty(\mu)}$. So we obtain
\[ \int_X |f - \int_X f \, d\mu| \, d\mu \leq \frac{\|f - c\|_{L^\infty(\mu)}}{2}. \tag{3} \]
for any constant $c$. Now we choose $c = (M_f + m_f)/2$. Then it is easy to see that the right-hand side of (3) is less than $(M_f - m_f)/2$. Finally, the condition $\int_X f \, d\mu = 0$ will imply the desired result.

We next generalize Lemma 1 to two functions.

**Lemma 3.** Let $f, g \in L^2(\mu)$. Then
\[ \left| \int_X fg \, d\mu - \int_X f \, d\mu \int_X g \, d\mu \right| \leq \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)}. \tag{4} \]

Proof. We use the fact that $L^2(\mu)^\ast = L^2(\mu)$, where $L^2(\mu)^\ast$ is the dual of $L^2(\mu)$, to obtain
\[
\sup_{\|g\|_{L^2(\mu)} = 1} \left| \int_X \left( f - \int_X f \, d\mu \right) g \, d\mu \right| = \left\| f - \int_X f \, d\mu \right\|_{L^2(\mu)}
\]
\[
= \left\{ \int_X f^2 \, d\mu - \left( \int_X f \, d\mu \right)^2 \right\}^{1/2} \leq \|f\|_{L^2(\mu)}.
\]
Hence,
\[ \left| \int_X \left( f - \int_X f \, d\mu \right) g \, d\mu \right| \leq \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)}. \]

The following corollary gives Grüss inequality on general measure spaces (see [2, Proposition 3]).

**Corollary 4 (Grüss inequality geneeralized).** Suppose $f, g \in L^\infty(\mu)$ and $x \in X$ with
\[ m_f \leq f(x) \leq M_f \text{ and } m_g \leq g(x) \leq M_g \text{ a.e.} \]

Then
\[ \left| \int_X fg \, d\mu - \int_X f \, d\mu \int_X g \, d\mu \right| \leq \frac{1}{4} (M_f - m_f)(M_g - m_g). \]
Proof. As in the proof of Corollary 2, we exploit the translation invariance of the left-hand side of (4). For any constants $c$ and $d$, replace $f$ and $g$ by $f(x) - c$ and $g(x) - d$, respectively, in (4). We then obtain

$$\left| \int f g \, d\mu - \int f \, d\mu \int g \, d\mu \right| \leq \|f - c\|_{L^2(\mu)} \|g - d\|_{L^2(\mu)},$$

which is less than

$$\|f - c\|_{L^\infty(\mu)} \|g - d\|_{L^\infty(\mu)}.$$

Now let $c = (M_f + m_f)/2$ and $d = (M_g + m_g)/2$. As in the proof of Corollary 2, we see that the last expression is no greater than

$$\frac{1}{4} (M_f - m_f)(M_g - m_g).$$

This finishes the proof.

An important observation is that in our proofs of Corollaries 2 and 4, the translation invariance of the left-hand side of the inequalities in (1) and (4), respectively, allows us to replace the quantity on the right-hand side by the one that makes $\|f - c\|_{L^\infty(\mu)}$ and $\|g - d\|_{L^\infty(\mu)}$ as small as possible among all possible choices of constants $c$ and $d$. This reveals the connection of Grüss inequality to the best approximation to functions by constants. We now use this point of view to further generalize Grüss inequality. Let $\mathcal{P}_n$ denote the set of polynomials of degree at most $n$ and define

$$E_{n,p}(f) := \inf \{ \|f - P\|_{L_p(\mu)} : P \in \mathcal{P}_n \},$$

the error of the best approximation to $f$ by polynomials of degree at most $n$ in $L_p(\mu)$-norm. Then we can strengthen the Grüss inequality in Corollary 4 as follows.

**Theorem 5.** Let $f, g \in L_2(\mu)$. Then

$$\left| \int_X f g \, d\mu - \int_X f \, d\mu \int_X g \, d\mu \right| \leq E_{0, 2}(f)E_{0, 2}(g).$$

**Proof.** As in the proof of Corollary 4, we use $f - c$ and $g - d$ in place of $f$ and $g$ in (4), respectively. Then we obtain

$$\left| \int f g \, d\mu - \int f \, d\mu \int g \, d\mu \right| \leq \|f - c\|_{L^2(\mu)} \|g - d\|_{L^2(\mu)}.$$

Now, taking infimum among all possible constants $c$ and $d$ will give us the desired result.

Theorem 5 yields various types of Grüss inequalities. We first use it to derive a stronger version of Corollary 4.
Corollary 6 (Corollary 4 revised). Suppose \( f, g \in L_\infty(\mu) \). Then

\[
\left| \int_X fg d\mu - \int_X f d\mu \int_X g d\mu \right| \leq E_{0, \infty}(f)E_{0, \infty}(g). \tag{5}
\]

Furthermore, the equality holds if and only if

\[
f(x), g(x) = a[\pm \chi_A(x) \mp \chi_{X \setminus A}(x)] + b
\]

for some constants \( a \) and \( b \) and a measurable set \( A \subseteq X \) satisfying \( \mu(A) = 1/2 \).

Proof. By Theorem 5, the inequality (5) follows from the fact that \( E_{0, 2}(f) \leq E_{0, \infty}(f) \).

The case of equality can be checked by a straightforward calculation. \( \blacksquare \)

We next derive a discrete version of Grüss inequality from Theorem 5. There are other discrete versions of the inequality. The following version, due to Biernacki, Pidek, and Ryll-Nardzewski ([8, Chapter X]) is deduced from Theorem 5.

Corollary 7. Let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be real numbers such that \( a \leq a_i \leq A \) and \( b \leq b_i \leq B \) for \( i = 1, 2, \ldots, n \). Then there holds

\[
\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \frac{1}{n} \sum_{i=1}^{n} b_i \right| \leq \frac{1}{n} \left( 1 - \frac{1}{n} \right) \left( A - a \right) \left( B - b \right), \tag{6}
\]

where \( \lfloor x \rfloor \) denotes the integer part of \( x \).

Proof. Let \( X \) be \( \{1, 2, \ldots, n\} \) and let \( \mu \) be the normalized counting measure on \( X \); i.e., \( \mu(S) \) is equal to \( |S|/n \), where \( |S| \) denotes the number of elements in \( S \subseteq X \). Let \( f(i) = a_i \) and \( g(i) = b_i \) for \( i = 1, 2, \ldots, n \). Then Theorem 5 yields

\[
\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \frac{1}{n} \sum_{i=1}^{n} b_i \right| \leq \min_c \left( \frac{1}{n} \sum_{i=1}^{n} (a_i - c)^2 \right)^{1/2}
\times \min_d \left( \frac{1}{n} \sum_{i=1}^{n} (b_i - d)^2 \right)^{1/2}.
\]

Elementary calculation shows that the right-hand side is equal to the square root of

\[
\left[ \frac{1}{n} \sum_{i=1}^{n} a_i^2 - \frac{(\sum_{i=1}^{n} a_i)^2}{n^2} \right] \left[ \frac{1}{n} \sum_{i=1}^{n} b_i^2 - \frac{(\sum_{i=1}^{n} b_i)^2}{n^2} \right]. \tag{7}
\]

We now find the maximum value of the expression in (7) when \( a \leq a_i \leq A \) and \( b \leq b_i \leq B \), \( i = 1, 2, \ldots, n \). As a convex quadratic function in \( a_1, a_2, \ldots, a_n \), the first factor is maximized for \( a \leq a_i \leq A, \ i = 1, 2, \ldots, n \), if and only if \( a_i = a \) or \( A, i = 1, 2, \ldots, n \). Assume there are \( k \ a_i \)'s equal
to $a$ and $n-k$’s equal to $A$. Then

$$S_a(k) := \frac{1}{n} \sum_{i=1}^{n} a_i^2 - \frac{(\sum_{i=1}^{n} a_i)^2}{n^2} = \frac{ka^2 + (n-k)A^2}{n} - \frac{(ka + (n-k)A)^2}{n^2}$$

This quadratic function of $k$ attains its maximum at $k = n/2$, and it is increasing for $k \in [0, n/2]$ and decreasing for $k \in [n/2, n]$. So the maximum value of $S_a$ on $[0, n]$ is

$$S_a\left(\left\lfloor \frac{n}{2} \right\rfloor \right) = \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor\right)(A-a)^2.$$

Thus, the maximum of the expression in (7) is

$$\frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor\right)(A-a)^2 \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor\right)(B-b)^2,$$

which implies (6).

In the same spirit, using the best approximation $E_{0,\infty}$, we can restate Corollary 2, whose proof is similar to that of Corollary 6.

**Corollary 8** (Corollary 2 revised). Suppose that $f \in L_{\infty}(\mu)$. If $\int_X f(x) d\mu = 0$, then

$$\int_X |f(x)| d\mu \leq E_0(f).$$

Furthermore, the equality holds if and only if

$$f(x) = a[\pm \chi_A(x) \mp \chi_{X\setminus A}(x)] + b$$

for some constants $a$ and $b$, and a measurable set $A \subset X$ satisfying $\mu(A) = 1/2$.

The Grüss-type inequalities as given in Theorem 5 and Corollary 6 give the real meaning of the constant on the right-hand side in the original Grüss inequality. We also remark that the above form of Grüss inequality has the advantage over the original Grüss inequality in that if we know more about functions $f$ and $g$, then we can derive better bound automatically. For example, if we assume that both $f$ and $g$ are in $\text{Lip}_1$ class on $X = [0, 1]$, then

$$E_{0,2}(f) \leq \left\{ \int_0^1 \left| f(x) - \int_0^1 f(y) dy \right|^2 dx \right\}^{1/2}$$

$$= \left\{ \int_0^1 f(x)^2 dx - \left( \int_0^1 f(y) dy \right)^2 \right\}^{1/2}.$$
(In fact, the equality holds in the first inequality.) But
\[
\int_0^1 f(x)^2 \, dx - \left( \int_0^1 f(y) \, dy \right)^2 \leq \frac{1}{2} \int_0^1 \int_0^1 (f(x) - f(y))^2 \, dx \, dy
\]
\[
\leq \frac{L_f}{2} \int_0^1 \int_0^1 (x - y)^2 \, dx \, dy = \frac{L_f^2}{12},
\]
where \(L_f\) denotes the Lip1 constant of \(f\). So
\[
E_{0,2}(f) \leq \sqrt{\frac{L_f^2}{12}} = \frac{L_f}{2\sqrt{3}}.
\]
Therefore, Theorem 5 yields a better bound in this case,
\[
\left| \int f \, g \, d\mu - \int f \, d\mu \int g \, d\mu \right| \leq E_{0,2}(f) E_{0,2}(g) \leq \frac{L_f L_g}{12}, \tag{8}
\]
while the original Grüss inequality in general gives only \(L_f L_g/4\) as the right-hand side, since
\[
M_f - m_f = \max_{x, y \in [0, 1]} |f(x) - f(y)| \leq L_f \max_{x, y \in [0, 1]} |x - y| \leq L_f
\]
and, similarly, \(M_g - m_g \leq L_g\). In this direction of specializing the function classes in Grüss-type inequalities to obtain different upper bounds on the right-hand side, we mention [1]. But we caution our readers that for a particular function, the concrete upper bounds obtained for special function classes are not always better than those given in the original Grüss inequality. For example, take \(f\) and \(g\) from the Lip1 such that, for some \(\epsilon \in (0, 1/2)\),
\[
f(x) = g(x) = \begin{cases} -1, & 0 \leq x < \frac{1}{2} - \epsilon \\ \frac{1}{\epsilon}(x - \frac{1}{2}), & \frac{1}{2} - \epsilon \leq x < \frac{1}{2} + \epsilon \\ 1, & \frac{1}{2} + \epsilon \leq x \leq 1. \end{cases}
\]
Then both \(f\) and \(g\) are indeed from Lip1 with Lip1 constant \(L_f = L_g = 1/\epsilon\). Note that \(M_f = M_g = 1\) and \(m_f = m_g = -1\). Thus, the original Grüss inequality gives an estimate \(|\int_0^1 f \, g - \int_0^1 f \int_0^1 g| \leq 1\). On the other hand, the estimate in (8) gives an estimate \(1/(16\epsilon^2)\). The latter estimate goes to infinity as \(\epsilon\) approaches to 0. Thus the original Grüss inequality provides a better estimate when \(\epsilon\) is small.

To generalize Grüss inequality further, we first observe the following fact.

**Lemma 9.** Corollaries 6 and 8 are equivalent.
Proof. Corollary 6 implies Corollary 8 by taking \( g(x) = \text{sgn}(f(x) - \int_X f d\mu) \).

To derive Corollary 6 from Corollary 8, we again use the duality theorem. Since \( L_1^*(\mu) = L_\infty(\mu) \), we have

\[
\sup_{\|g\|_{L_\infty(\mu)}=1} \int_X (f - \int_X f d\mu) g d\mu = \int_X \left| f - \int_X f d\mu \right| d\mu,
\]

which is less than \( E_{0,\infty}(f) \) by Corollary 8. So

\[
\sup_{\|g\|_{L_\infty(\mu)}=1} \int_X (f - \int_X f d\mu) g d\mu \leq E_{0,\infty}(f)
\]
or, equivalently,

\[
\left| \int_X (f - \int_X f d\mu) g d\mu \right| \leq E_{0,\infty}(f) \|g\|_{L_\infty(\mu)}.
\]

Now, replacing \( g(x) \) by \( g(x) - d \) for any constant \( d \) in the foregoing inequality gives us

\[
\left| \int_X (f - \int_X f d\mu) g d\mu \right| \leq E_{0,\infty}(f) \|g - d\|_{L_\infty(\mu)}
\]

Taking infimum among all possible choices of the constant \( d \) yields the desired inequality in Corollary 6.

The case of equality can be handled by a straightforward calculation.

We now consider a generalization of Corollary 8.

**Theorem 10.** Suppose that \( f \in L_\infty(\mu) \). If

\[
\int_X x^k f(x) d\mu(x) = 0, \quad k = 0, 1, \ldots, n,
\]

then

\[
\int_X |f(x)| d\mu \leq E_{n,\infty}(f).
\]

**Proof.** Let \( P \in \mathcal{P}_n \). We first note that the vanishing-moments condition implies that the best \( L_2(\mu) \) approximation to \( f \) out of \( \mathcal{P}_n \) is the zero function. In fact, from

\[
\int_X f^2 d\mu = \left( \int_X f^2 d\mu \right)^{1/2} \left( \int_X (f - P)^2 d\mu \right)^{1/2},
\]

we get

\[
\|f\|_{L_2(\mu)} \leq \|f - P\|_{L_2(\mu)}.
\]

So \( \|f\|_{L_2(\mu)} = E_{n,2}(f) \). Using \( E_{n,2}(f) \leq E_{n,\infty}(f) \), we obtain

\[
\|f\|_{L_2(\mu)} \leq E_{n,\infty}(f).
\]

Now (9) follows from the fact that \( \|f\|_{L_1(\mu)} \leq \|f\|_{L_2(\mu)} \).
Finally, we generalize Corollary 6, the Grüss inequality. For every integer $n \geq 0$, define an operator $\Delta_n: L_\infty(\mu) \to L_\infty(\mu)$ as follows: For every $f \in L_\infty(\mu)$, the function $\Delta_n(f)$ is given by

$$
\Delta_n(f)(x) = \begin{vmatrix}
\int_X f(x) \, d\mu & 1 & x & \cdots & x^n \\
\int_X f(x) \, d\mu & 1 & \int_X x \, d\mu & \cdots & \int_X x^n \, d\mu \\
\int_X xf(x) \, d\mu & \int_X x \, d\mu & \int_X x^2 \, d\mu & \cdots & \int_X x^{n+1} \, d\mu \\
\vdots & \int_X x^n f(x) \, d\mu & \int_X x^n \, d\mu & \int_X x^{n+1} \, d\mu & \cdots & \int_X x^{2n} \, d\mu \\
\int_X x^n f(x) \, d\mu & \int_X x^n \, d\mu & \int_X x^{n+1} \, d\mu & \cdots & \int_X x^{2n} \, d\mu 
\end{vmatrix}.
$$

The desired property of function $\Delta_n(f)$ is the vanishing moments of order $n$:

$$
\int_X x^k \Delta_n(f)(x) \, d\mu = 0, \quad \text{for } k = 0, 1, \ldots, n.
$$

Note that Grüss inequality (5) can be written as

$$
\left| \int_X \left( f - \int_X f \, d\mu \right) \left( g - \int_X g \, d\mu \right) \, d\mu \right| \leq E_{0, \infty}(f) E_{0, \infty}(g),
$$

and

$$
\Delta_0(f)(x) = f(x) - \int_X f \, d\mu.
$$

So we can rewrite (5) as

$$
\left| \int_X \Delta_0(f) \Delta_0(g) \, d\mu \right| \leq E_{0, \infty}(f) E_{0, \infty}(g).
$$

Another Grüss-type inequality for functions of vanishing moments of order $n$ is as follows.

**Theorem 11.** Suppose $f, g \in L_\infty(\mu)$. For every integer $n \geq 0$, we have

$$
\left| \int_X \Delta_n(f) \Delta_n(g) \, d\mu \right| \leq E_{n, \infty}(f) E_{n, \infty}(g).
$$

**Proof.** By Theorem 10, together with the vanishing moments property of $\Delta_n(f)$, we have

$$
\int_X |\Delta_n(f)| \, d\mu \leq E_{n, \infty}(\Delta_n(f)).
$$

Note that $\Delta_n(f) = f + P^*$ for some $P^* \in \mathcal{P}_n$. So $E_{n, \infty}(\Delta_n(f)) = E_{n, \infty}(f)$. Thus we have

$$
\int_X |\Delta_n(f)| \, d\mu \leq E_{n, \infty}(f).
$$
Now, as before, the foregoing inequality coupled with the duality $L_1(\mu)^* = L_\infty(\mu)$ implies
\[
\left| \int_X \Delta_n(f)gd\mu \right| \leq E_{n, \infty}(f)\|g\|_{L_\infty(\mu)}.
\]
Using the vanishing moment property of $\Delta_n(f)$ again, we can replace $g$ by $g - P$ for any $P \in \mathcal{P}_n$ to obtain
\[
\left| \int_X \Delta_n(f)gd\mu \right| \leq E_{n, \infty}(f)\|g - P\|_{L_\infty(\mu)}.
\]
Taking the infimum among all $P \in \mathcal{P}_n$, we get
\[
\left| \int_X \Delta_n(f)gd\mu \right| \leq E_{n, \infty}(f)E_{n, \infty}(g).
\]
To finish the proof, we note that the left-hand side is unchanged if we replace $g$ by $\Delta_n(g)$ since $\Delta_n(g) - g \in \mathcal{P}_n$. This completes our proof.

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