J. Math. Anal. Appl. 345 (2008) 607–614

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

# A generalization of the Mulholland inequality for continuous Archimedean t-norms

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ARTICLE INFO

*Article history:*
Received 19 July 2007
Available online 28 March 2008
Submitted by S. Kaijser

*Keywords:*
Mulholland inequality
Minkowski inequality
Triangular norm (t-norm)
Dominance relation

ABSTRACT

It is well known that dominance between strict t-norms is closely related to the Mulholland inequality, which can be seen as a generalization of the Minkowski inequality. However, strict t-norms constitute only one part of the class of continuous Archimedean t-norms, the basic elements from which all continuous t-norms are composed. In this paper, dominance between continuous Archimedean t-norms is shown to be related to a generalization of the Mulholland inequality. We provide sufficient and necessary conditions for its fulfillment.



## 1. Introduction

In 1950, Mulholland presented a generalization of the Minkowski inequality, which later on became known as the *Mulholland inequality* [13]. In the same contribution, he provided a sufficient condition for its fulfillment by a continuous function that is strictly increasing on its domain. In 1984, Tardiff demonstrated that this inequality plays an essential role in the investigation of dominance between strict *triangular norms* (t-norms for short) and provided a different sufficient condition [24]. In 2002, Jarczyk and Matkowski clarified the relationship between the two sufficient conditions, showing that Tardiff's condition implies that of Mulholland [5].

On the other hand, the dominance relation was originally introduced in the framework of probabilistic metric spaces [22] and was soon abstracted to operations on a partially ordered set (see, e.g. [20]). The dominance relation, in particular between t-norms, plays a profound role in various topics, such as the construction of Cartesian products of probabilistic metric and normed spaces [11,20,22], the construction of many-valued equivalence relations [2,3,25] and many-valued order relations [1], as well as in the preservation of various properties during (dis-)aggregation processes in flexible querying, preference modelling and computer-assisted assessment [2,4,14,16]. These applications instigated the study of the dominance relation in the broader context of aggregation operators [12,14,16].

The dominance relation is an interesting mathematical notion *per se*. As it constitutes a reflexive and antisymmetric relation on the class of t-norms, and counterexamples for its transitivity were not readily found, it remained an intriguing open problem [7,18,20,21,24] for more than 20 years whether or not it was an order relation. Only recently the question was answered to the negative [17,19]. However, due to its relevance in applications, it is still of interest to determine whether or not the dominance relation establishes an order relation on some subclasses of t-norms. Of particular importance are the continuous Archimedean t-norms, as they are the basic elements of which all continuous t-norms are composed. Therefore,

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0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2008.03.045

establishing sufficient conditions for dominance between continuous Archimedean t-norms is of interest and constitutes the main goal of our contribution.

After some brief preliminaries on t-norms, we demonstrate the close relationship between dominance between continuous Archimedean t-norms and a generalization of the Mulholland inequality. A short survey on sufficient conditions for continuous functions which are strictly increasing on the whole domain is followed by appropriate sufficient and necessary conditions in the more general case. This provides the basis for the investigation of dominance between continuous Archimedean t-norms in the last section.

## 2. Continuous Archimedean t-norms

We briefly summarize some basic properties of t-norms for a thorough understanding of this paper (see, e.g. [6–10]).

**Definition 1.** A *t-norm* $T : [0,1]^2 \to [0,1]$ is a binary operation on the unit interval that is commutative, associative, increasing and has 1 as neutral element.

Well-known examples of t-norms are the *minimum* $T_{\mathbf{M}}$, the *product* $T_{\mathbf{P}}$ and the *Łukasiewicz t-norm* $T_{\mathbf{L}}$ defined by $T_{\mathbf{M}}(u,v) = \min(u,v)$, $T_{\mathbf{P}}(u,v) = u \cdot v$ and $T_{\mathbf{L}}(u,v) = \max(u+v-1,0)$.

Since t-norms are just functions from the unit square to the unit interval, their comparison is done pointwisely: $T_1 \leqslant T_2$ if $T_1(u,v) \leqslant T_2(u,v)$ for all $u, v \in [0,1]$, expressed as "$T_1$ is *weaker* than $T_2$" or "$T_2$ is *stronger* than $T_1$." The minimum $T_{\mathbf{M}}$ is the strongest of all t-norms. Furthermore, it holds that $T_{\mathbf{P}} \geqslant T_{\mathbf{L}}$.

A continuous t-norm $T$ is *Archimedean* if and only if for all $u \in ]0,1[$ it holds that $T(u,u) < u$. The class of continuous Archimedean t-norms can be partitioned into two subclasses: the class of *strict t-norms*, which are continuous and strictly increasing, and the class of *nilpotent t-norms*, which are continuous and fulfill that for each $u \in ]0,1[$ there exists some $n \in \mathbb{N}$ such that

$$T(\underbrace{u,\dots,u}_{n\ \text{times}}) = 0.$$

The product $T_{\mathbf{P}}$ is strict, whereas the Łukasiewicz t-norm $T_{\mathbf{L}}$ is nilpotent.

Note that for a strict t-norm $T$ it holds that $T(u,v) > 0$ for all $u, v \in ]0,1]$, while for a nilpotent t-norm $T$ it holds that for any $u \in ]0,1[$ there exists some $v \in ]0,1[$ such that $T(u,v) = 0$ (each $u \in ]0,1[$ is a so-called *zero divisor*). Therefore, for a nilpotent t-norm $T_1$ and a strict t-norm $T_2$ it can never hold that $T_1 \geqslant T_2$.

Of particular interest in the discussion of continuous Archimedean t-norms is the notion of an *additive generator*.

**Definition 2.** An *additive generator* of a continuous Archimedean t-norm $T$ is a continuous, strictly decreasing function $t : [0,1] \to [0,\infty]$ which satisfies $t(1) = 0$ such that for all $u, v \in [0,1]$ it holds that

$$T(u,v) = t^{(-1)}\big(t(u) + t(v)\big) \tag{1}$$

with

$$t^{(-1)}(u) = t^{-1}\big(\min\big(t(0), u\big)\big) \tag{2}$$

the *pseudo-inverse* of the decreasing function $t$.

An additive generator is uniquely determined up to a positive multiplicative constant. Any additive generator of a strict t-norm satisfies $t(0) = \infty$, while that of a nilpotent t-norm satisfies $t(0) < \infty$. In the case of strict t-norms, the pseudo-inverse $t^{(-1)}$ of an additive generator $t$ coincides with its standard inverse $t^{-1}$. In any case, the following relationships between an additive generator $t$ and its pseudo-inverse $t^{(-1)}$ hold

$$t \circ t^{(-1)}\big|_{\mathrm{Ran}(t)} = \mathrm{id}_{\mathrm{Ran}(t)} \quad \text{and} \quad t^{(-1)} \circ t = \mathrm{id}_{[0,1]}. \tag{3}$$

## 3. Dominance and related inequalities

Just as triangular norms, the dominance relation finds its origin in the field of probabilistic metric spaces [20,22]. It was originally introduced for associative operations (with common neutral element) on a partially ordered set [20], and has been further investigated for t-norms [15,17–19,21,24] and aggregation operators [12,14,16]. We state the definition for t-norms only.

**Definition 3.** Consider two t-norms $T_1$ and $T_2$. We say that $T_1$ *dominates* $T_2$ (or $T_2$ is dominated by $T_1$), denoted by $T_1 \gg T_2$, if for all $x, y, u, v \in [0,1]$ it holds that

$$T_1\big(T_2(x,y), T_2(u,v)\big) \geqslant T_2\big(T_1(x,u), T_1(y,v)\big). \tag{4}$$

Note that any t-norm is dominated by itself and by $T_{\mathbf{M}}$. Since all t-norms have neutral element 1, dominance between two t-norms implies their comparability: $T_1 \gg T_2$ implies $T_1 \geqslant T_2$. The converse does not hold, not even for strict t-norms [24]. Since for a nilpotent t-norm $T_1$ and a strict t-norm $T_2$, it cannot hold that $T_1 \geqslant T_2$, it also cannot hold that $T_1 \gg T_2$. Therefore, for a continuous Archimedean t-norm $T_1$ and a strict t-norm $T_2$, $T_1 \gg T_2$ implies that also $T_1$ is strict.

The dominance relation between two continuous Archimedean t-norms can be expressed in terms of their generators. This was shown for strict t-norms in [24] and is generalized below.

**Theorem 1.** *Consider two continuous Archimedean t-norms $T_1$ and $T_2$ with additive generators $t_1$ and $t_2$. Then $T_1$ dominates $T_2$ if and only if the function $h = t_1 \circ t_2^{(-1)} : [0,\infty] \to [0,\infty]$ fulfills for all $a,b,c,d \in [0, t_2(0)]$*

$$h^{(-1)}\big(h(a)+h(c)\big) + h^{(-1)}\big(h(b)+h(d)\big) \geqslant h^{(-1)}\big(h(a+b)+h(c+d)\big) \tag{5}$$

*with $h^{(-1)} : [0,\infty] \to [0,\infty]$ the* pseudo-inverse *of the increasing function $h$, given by $h^{(-1)} = t_2 \circ t_1^{(-1)}$.*

**Proof.** The case of two strict t-norms $T_1$ and $T_2$ was treated by Tardiff [24]. Therefore, we suppose that at least one of the t-norms involved is nilpotent.

Note also that (4) is trivially fulfilled when $0 \in \{x,y,u,v\}$. Hence, the verification of (5) can be restricted to $a,b,c,d \in [0, t_2(0)[$ only.

(i) Suppose first that $T_1 \gg T_2$. Expressing (4) in terms of generators and applying the decreasing function $t_2$ to both sides leads to

$$h^{(-1)}\big[h\big(t_2(x)+t_2(y)\big) + h\big(t_2(u)+t_2(v)\big)\big] \leqslant t_2 \circ t_2^{(-1)}\big[h^{(-1)}\big(t_1(x)+t_1(u)\big) + h^{(-1)}\big(t_1(y)+t_1(v)\big)\big],$$

for all $x,y,u,v \in [0,1]$. Consider $a,b,c,d \in [0,t_2(0)]$, then the continuity of $t_2$ implies the existence of $x = t_2^{-1}(a) = t_2^{(-1)}(a)$, $y = t_2^{-1}(b) = t_2^{(-1)}(b)$, $u = t_2^{-1}(c) = t_2^{(-1)}(c)$, $v = t_2^{-1}(d) = t_2^{(-1)}(d)$. It then follows that

$$h^{(-1)}\big(h(a+b)+h(c+d)\big) \leqslant t_2 \circ t_2^{(-1)}\big[h^{(-1)}\big(h(a)+h(c)\big) + h^{(-1)}\big(h(b)+h(d)\big)\big].$$

Denote $K = h^{(-1)}(h(a)+h(c)) + h^{(-1)}(h(b)+h(d))$. If $K \geqslant t_2(0)$, then

$$h^{(-1)}\big(h(a+b)+h(c+d)\big) \leqslant t_2 \circ t_2^{(-1)}(K) = t_2 \circ t_2^{(-1)} \circ t_2(0) = t_2(0) \leqslant K.$$

Otherwise, it holds that

$$h^{(-1)}\big(h(a+b)+h(c+d)\big) \leqslant t_2 \circ t_2^{(-1)}(K) = t_2 \circ t_2^{-1}(K) = K.$$

This shows that (5) is fulfilled for all $a,b,c,d \in [0,t_2(0)]$.

(ii) Conversely, suppose that $h$ fulfills (5) for all $a,b,c,d \in [0,t_2(0)]$, then

$$\begin{aligned} &t_2 \circ t_1^{(-1)}\big(t_1 \circ t_2^{(-1)}(a) + t_1 \circ t_2^{(-1)}(c)\big) + t_2 \circ t_1^{(-1)}\big(t_1 \circ t_2^{(-1)}(b) + t_1 \circ t_2^{(-1)}(d)\big) \\ &\quad \geqslant t_2 \circ t_1^{(-1)}\big(t_1 \circ t_2^{(-1)}(a+b) + t_1 \circ t_2^{(-1)}(c+d)\big). \end{aligned}$$

Consider $x,y,u,v \in [0,1]$ and let $a = t_2(x)$, $b = t_2(y)$, $c = t_2(u)$ and $d = t_2(v)$. It then follows that

$$t_2 \circ t_1^{(-1)}\big(t_1(x)+t_1(u)\big) + t_2 \circ t_1^{(-1)}\big(t_1(y)+t_1(v)\big) \geqslant t_2 \circ t_1^{(-1)}\big(t_1 \circ t_2^{(-1)}\big(t_2(x)+t_2(y)\big) + t_1 \circ t_2^{(-1)}\big(t_2(u)+t_2(v)\big)\big).$$

Applying the decreasing function $t_2^{(-1)}$ to both sides leads to

$$T_2\big(T_1(x,u), T_1(y,v)\big) \leqslant T_1\big(T_2(x,y), T_2(u,v)\big).$$

Hence, $T_1$ dominates $T_2$. □

## 4. The Mulholland inequality

Using the notations of Theorem 1, if $T_1$ and $T_2$ are strict, then $t_2(0) = \infty$, $h$ is strictly increasing and thus $h^{(-1)} = h^{-1}$. Inequality (5) then simplifies to

$$h^{-1}\big(h(a)+h(c)\big) + h^{-1}\big(h(b)+h(d)\big) \geqslant h^{-1}\big(h(a+b)+h(c+d)\big), \tag{6}$$

for all $a,b,c,d \in [0,\infty[$ (the inequality is trivially fulfilled when $\infty \in \{a,b,c,d\}$). This inequality is known as the *Mulholland inequality* and is a generalization of the *Minkowski inequality* [13].

It is remarkable that functions $h$ fulfilling (6) have been investigated independently from the context of dominance [5,13,23,24]. A brief overview of the most important findings is given next.

**Proposition 2.** *(See [13].) Consider a continuous, strictly increasing function $h : [0,\infty[ \to [0,\infty[$ such that $h(0) = 0$. If $h$ fulfills the Mulholland inequality (6), then it is convex on $]0,\infty[$.*

**Proposition 3.** *(See [13].) Consider a continuous, strictly increasing function $h : [0, \infty[ \to [0, \infty[$ such that $h(0) = 0$. If $h$ is convex on $]0, \infty[$ and $\log \circ h \circ \exp$ is convex on $]-\infty, \infty[$, then $h$ fulfills the Mulholland inequality* (6)*.*

**Proposition 4.** *(See [24].) Consider a differentiable, strictly increasing function $h : [0, \infty[ \to [0, \infty[$ such that $h(0) = 0$. If $h$ is convex on $]0, \infty[$ and $\log \circ h' \circ \exp$ is convex on $]-\infty, \infty[$, then $h$ fulfills the Mulholland inequality* (6)*.*

It can be shown that for a continuous function $f : [0, \infty[ \to [0, \infty[$ such that $f(]0, \infty[) \subseteq ]0, \infty[$, it holds that $\log \circ f \circ \exp$ is convex on $]-\infty, \log(t)[$, with $t \in ]0, \infty[$, if and only if $f$ fulfills

$$f(\sqrt{xy}) \leqslant \sqrt{f(x)f(y)} \tag{7}$$

for all $x, y \in ]0, t[$. The latter condition is referred to as the *geometric convexity* of $f$ on $]0, t[$ (*geo-convexity* for short); if $f(0) = 0$, then the geo-convexity holds on $[0, t[$. Moreover, if $f$ is increasing, then the convexity of $\log \circ f$ on $]0, t[$, called *log-convexity* of $f$, implies its geo-convexity. Jarczyk and Matkowski [5] have investigated the relationship between the geo-convexity of a function and that of its derivative.

**Proposition 5.** *(See [5].) Consider a differentiable function $f : ]0, \infty[ \to ]0, \infty[$ such that $\lim_{x\to 0} f(x) = 0$ and $f'(x) > 0$ for all $x \in ]0, \infty[$. If $f'$ is geo-convex, then so is $f$.*

Combining the above results leads to the following relationships between the sufficient conditions on $h$ for the fulfillment of the Mulholland inequality:

$$\boxed{\begin{array}{lcl} h \text{ is convex, fulfills } h(0) = 0, \quad \text{and } \ldots & & \\ h' \text{ is geo-convex} & \Leftarrow & h' \text{ is log-convex} \\ \Downarrow & & \\ h \text{ is geo-convex} & \Leftarrow & h \text{ is log-convex} \\ \Downarrow & & \\ h \text{ fulfills (6)} & & \end{array}}$$

## 5. A generalization of the Mulholland inequality

In this section, we aim at a generalization of the results of Mulholland and Tardiff in order to guarantee their applicability to the investigation of dominance between two continuous Archimedean t-norms.

### 5.1. A first sufficient condition

**Theorem 6.** *Consider a function $h : [0, \infty] \to [0, \infty]$ and some fixed value $t \in ]0, \infty[$ such that*

(h1) *$h$ is continuous on $[0, t]$;*
(h2) *$h$ is strictly increasing on $[0, t]$ and $h(x) \geqslant h(t)$ whenever $x \geqslant t$;*
(h3) $h(0) = 0$;
(h4) *$h$ is convex on $]0, t[$;*
(h5) *$h$ is geo-convex on $]0, t[$.*

*Define the functions $g : [0, \infty] \to [0, \infty]$ and $H : [0, \infty]^2 \to [0, \infty]$ by*

$$g(x) := \begin{cases} h^{-1}(x), & \text{if } x \in [0, h(t)], \\ t, & \text{otherwise}, \end{cases} \tag{8}$$

$$H(x, y) := g\big(h(x) + h(y)\big). \tag{9}$$

*Then the following inequality holds for all $a, b, c, d \in [0, \infty]$*

$$H(a + b, c + d) \leqslant H(a, c) + H(b, d). \tag{10}$$

**Remark 1.** Clearly, $g$ is continuous and increasing. Also $H$ is continuous in each argument and increasing. Obviously, it holds that

$$H(t, x) = H(x, t) = t, \quad \text{for all } x \in [0, \infty], \tag{11}$$

$$H(0, x) = H(x, 0) = x, \quad \text{for all } x \in [0, t]. \tag{12}$$

Further, the convexity of $h$ on $]0,t[$ is equivalent to the concavity of $g$ on $]0,h(t)[$. Since $h$ is increasing and continuous on $[0,t]$, its convexity on $]0,t[$ implies its convexity on $[0,t]$. As argued before, the geo-convexity of $h$ on $]0,t[$ is equivalent to the convexity of $\log\circ h\circ \exp$ on $]-\infty,\log(t)[$, which in its turn is equivalent to the concavity of the function $\log\circ g\circ\exp$ on $]-\infty,\log(h(t))[$. It is easy to show that in these cases, it also holds that $g$ is concave on $]0,\infty[$ and $\log\circ g\circ\exp$ is concave on $]-\infty,\infty[$.

Inspired by Mulholland [13], we introduce another function that will be essential in our proof.

**Lemma 7.** *Under the assumptions of Theorem* 6*, define the function* $\psi:[0,t]\to[0,\infty]$ *by*

$$\psi(x) := \begin{cases} \frac{h(x)}{x}, & \text{if } x>0,\\ \lim_{y\to 0^+}\frac{h(y)}{y}, & \text{if } x=0. \end{cases} \tag{13}$$

*Then* $\psi$ *is increasing on* $[0,t]$.

**Proof.** Note that the function $\psi$ is strictly positive on $]0,t]$ and continuous on $[0,t]$. Consider $0<x<x+\epsilon<t$, then we need to show that $\psi(x)\leqslant\psi(x+\epsilon)$. Let $\alpha=\frac{\epsilon}{x+\epsilon}$ and $\beta=1-\alpha$, then the convexity of $h$ on $[0,x+\epsilon]$ implies that

$$h\big(\beta(x+\epsilon)\big)\leqslant \alpha h(0)+\beta h(x+\epsilon)=\beta h(x+\epsilon),$$

which can be rewritten as $h(x)\leqslant\frac{x}{x+\epsilon}h(x+\epsilon)$, and hence $\psi(x)\leqslant\psi(x+\epsilon)$. The continuity of $\psi$ then implies that it is increasing on $[0,t]$. □

We now turn to the proof of Theorem 6.

**Proof of Theorem 6.** The proof consists of several cases.

(1) *At least one of* $a,b,c,d$ *belongs to* $[t,\infty]$.

Since $H$ is increasing, it follows from (11) that $H(x,y)=t$ whenever $x\geqslant t$ or $y\geqslant t$. This implies that (10) trivially holds when one of the arguments is greater than or equal to $t$.

(2) *All of* $a,b,c,d$ *belong to* $[0,t[$ *and* $a+b<t$ *and* $c+d<t$.

If $a=b=0$ or $c=d=0$, then (10) holds due to (12). We therefore assume that $0<a+b$ as well as $0<c+d$. The proof of this case is based on the observation that (10) is a consequence of a more general inequality, namely

$$x\psi(a+b)+y\psi(c+d)\leqslant H(x,y)\frac{h(a+b)+h(c+d)}{H(a+b,c+d)}, \tag{14}$$

for all $x,y$ such that $0\leqslant x\leqslant a+b$ and $0\leqslant y\leqslant c+d$. Indeed, assume that (14) holds, then expressing it for both $(x,y)=(a,c)$ and $(x,y)=(b,d)$ and adding side by side leads to

$$h(a+b)+h(c+d)=a\psi(a+b)+c\psi(c+d)+b\psi(a+b)+d\psi(c+d)\leqslant\big(H(a,c)+H(b,d)\big)\frac{h(a+b)+h(c+d)}{H(a+b,c+d)},$$

which implies (10), since $h(a+b)+h(c+d)>0$ and $H(a+b,c+d)>0$. We therefore attempt to show (14).

(a) In case $x=y=0$, it is trivially fulfilled.

(b) In case $x=0$ and $y>0$, we need to show that

$$\psi(c+d)\leqslant\frac{h(a+b)+h(c+d)}{H(a+b,c+d)}.$$

In case $h(a+b)+h(c+d)\leqslant h(t)$, it holds that

$$\frac{h(a+b)+h(c+d)}{H(a+b,c+d)}=\frac{h(g(h(a+b)+h(c+d)))}{H(a+b,c+d)}=\psi\big(H(a+b,c+d)\big).$$

Since $\psi(c+d)=\psi(H(0,c+d))$, the increasingness of $H$ and $\psi$ (see Remark 1 and Lemma 7) imply that $\psi(c+d)\leqslant\psi(H(a+b,c+d))$ and hence

$$\psi(c+d)\leqslant\frac{h(a+b)+h(c+d)}{H(a+b,c+d)}.$$

In case $h(a+b)+h(c+d)>h(t)$, it holds that $H(a+b,c+d)=t=H(t,c+d)$ and the increasingness of $H$ and $\psi$ imply again that

$$\psi(c+d)=\psi\big(H(0,c+d)\big)\leqslant\psi\big(H(t,c+d)\big)=\frac{h(H(t,c+d))}{H(t,c+d)}=\frac{h(t)}{H(a+b,c+d)}<\frac{h(a+b)+h(c+d)}{H(a+b,c+d)}.$$

(c) The case $x>0$ and $y=0$ is similar to the previous one.

(d) If $x>0$, $y>0$, and both are such that $h(x)+h(y)\geqslant h(t)$, then (14) also trivially holds, since $H(x,y)=H(a+b,c+d)=t$, $x\leqslant a+b$, $y\leqslant c+d$ and $\psi$ is positive. If $h(x)+h(y)<h(t)$, then we can transform (14) into

$$\frac{x\psi(a+b)+y\psi(c+d)}{H(x,y)}\leqslant\frac{h(a+b)+h(c+d)}{H(a+b,c+d)}=\frac{(a+b)\psi(a+b)+(c+d)\psi(c+d)}{H(a+b,c+d)}.$$

It is therefore sufficient to show that the function $G:]0,a+b]\times]0,c+d]\to]0,\infty]$ defined by

$$G(x,y):=\frac{x\psi(a+b)+y\psi(c+d)}{H(x,y)} \tag{15}$$

attains its maximum at $(a+b,c+d)$. Since $h(x)+h(y)<h(t)$, it holds that $H(x,y)=h^{-1}(h(x)+h(y))$. This question is identical to the one positively answered by Mulholland on a subdomain $]0,a+b]\times]0,c+d]$ of $[0,\infty[^2$ [13]. Note that his way of proving this result initially relies on the existence of the derivative of $h$, a condition that is later on removed thanks to the other conditions on $h$, so that we can conclude that (5) holds whenever all $a,b,c,d$ belong to $[0,t[$ and $a+b<t$, $c+d<t$.

(3) *All of $a,b,c,d$ belong to $[0,t[$ and $a+b\geqslant t$ or $c+d\geqslant t$.*

We first assume that $a+b=t$ and consider the sequence $(b_n)_{n\in\mathbb{N}}$ with $b_n:=t-a-\frac{1}{n}$. It then holds that $a+b_n<t$, yet $\lim_{n\to\infty}a+b_n=a+b=t$. However, for any $n\in\mathbb{N}$, the previous case implies that

$$H(a+b_n,c+d)\leqslant H(a,c)+H(b_n,d).$$

Since $H$ is continuous in each argument, we can further conclude that

$$H(a+b,c+d)=\lim_{n\to\infty}H(a+b_n,c+d)\leqslant H(a,c)+\lim_{n\to\infty}H(b_n,d)=H(a,c)+H(b,d).$$

Next we assume that $a+b>t$. As a consequence, it holds that

$$H(a+b,c+d)=H(t,c+d)=H\big(a+(t-a),c+d\big)=t$$

and the increasingness of $H$ implies that

$$H(a+b,c+d)=H\big(a+(t-a),c+d\big)\leqslant H(a,c)+H(t-a,d)\leqslant H(a,c)+H(b,d).$$

The case $c+d\geqslant t$ is completely analogous. □

### 5.2. *A second sufficient condition*

A careful inspection of the proof of Proposition 5 as provided in [5] shows that it can be generalized as follows.

**Lemma 8.** *Consider a function $f:]0,\infty[\to]0,\infty[$ with $\lim_{x\to 0}f(x)=0$ and such that $f$ is differentiable on $[0,t[$ with $t\in]0,\infty[$ and $f'(x)>0$ for all $x\in[0,t]$. If $f'$ is geo-convex on $]0,t[$, then so is $f$.*

Based on this result we can immediately generalize the result of Tardiff [23,24].

**Proposition 9.** *Consider a function $h:[0,\infty]\to[0,\infty]$ and some fixed value $t\in]0,\infty[$ such that*

(h1) *$h$ is continuous on $[0,t]$;*
(h2) *$h$ is strictly increasing on $[0,t]$ and $h(x)\geqslant h(t)$ whenever $x\geqslant t$;*
(h3) *$h(0)=0$;*
(h4) *$h$ is convex on $]0,t[$;*
(h6) *$h$ is differentiable on $]0,t[$ and $h'$ is geo-convex on $]0,t[$.*

*Define the function $g:[0,\infty]\to[0,\infty]$ by (8) and the function $H:[0,\infty]^2\to[0,\infty]$ by (9). Then the following inequality holds for all $a,b,c,d\in[0,\infty]$,*

$$H(a+b,c+d)\leqslant H(a,c)+H(b,d).$$

### 5.3. A necessary condition

The convexity of $h$ on $]0, \infty[$ is a necessary condition for the classical Mulholland inequality to hold, and as such it is part of each of the known sets of sufficient conditions. A similar observation holds for the generalized Mulholland inequality, but now the convexity of $h$ on $]0, t[$ is a necessary condition.

**Proposition 10.** *Consider a function $h : [0, \infty] \to [0, \infty]$ and some fixed value $t \in\, ]0, \infty[$ such that*

(h1) *$h$ is continuous on $[0, t]$;*
(h2) *$h$ is strictly increasing on $[0, t]$ and $h(x) \geqslant h(t)$ whenever $x \geqslant t$;*
(h3) *$h(0) = 0$.*

*Define the function $g : [0, \infty] \to [0, \infty]$ by (8) and the function $H : [0, \infty]^2 \to [0, \infty]$ by (9). If $H$ fulfills (10) for all $a, b, c, d \in [0, \infty]$, then $h$ is convex on $]0, t[$.*

**Proof.** As the convexity of $h$ on $]0, t[$ is equivalent to the concavity of $g$ on $]0, h(t)[$, and $g$ is continuous, it suffices to show that

$$g\left(\frac{x+y}{2}\right) \geqslant \frac{1}{2} g(x) + \frac{1}{2} g(y),$$

for all $x, y \in\, ]0, h(t)[$. Choose arbitrary $x, y \in\, ]0, h(t)[$ such that $x < y$ and put $a = g(x)$, $b = g(\frac{x+y}{2}) - g(x)$, $c = g(\frac{y-x}{2})$ and $d = 0$. Note that in each of these cases $g$ coincides with $h^{-1}$ and that $a, b, c, d \in\, ]0, t[$. We can therefore compute

$$\begin{aligned}
h(a) + h(c) &= \frac{x+y}{2}, \\
h(b) + h(d) &= h\left(g\left(\frac{x+y}{2}\right) - g(x)\right), \\
h(a+b) &= \frac{x+y}{2}, \\
h(c+d) &= \frac{y-x}{2}.
\end{aligned}$$

Since $H$ fulfills (10) it holds that $H(a+b, c+d) \leqslant H(a, c) + H(b, d)$, or explicitly

$$g(y) = g\left(\frac{x+y}{2} + \frac{y-x}{2}\right) \leqslant g\left(\frac{x+y}{2}\right) + g\left(\frac{x+y}{2}\right) - g(x) = 2g\left(\frac{x+y}{2}\right) - g(x). \qquad \square$$

## 6. Dominance between continuous Archimedean t-norms

Consider two continuous Archimedean t-norms $T_1$ and $T_2$ with additive generators $t_1$ and $t_2$ and the corresponding function $h = t_1 \circ t_2^{(-1)} : [0, \infty] \to [0, \infty]$. As mentioned in Section 4, if $T_1$ and $T_2$ are strict, then $t_2(0) = \infty$, $h$ is strictly increasing, $h^{(-1)} = h^{-1}$ and dominance between $T_1$ and $T_2$ is equivalent to the Mulholland inequality for $h$. Recall that if $T_2$ is strict, then $T_1 \gg T_2$ implies that $T_1$ is strict as well. In case $T_2$ is a nilpotent t-norm, $T_1$ might be a strict or nilpotent t-norm and the parameters of Theorem 6 and Proposition 9 are given by:

(1) If $T_1$ is strict, then $h = t_1 \circ t_2^{(-1)}$, $g = t_2 \circ t_1^{-1} = h^{(-1)}$, $t = t_2(0)$, and $h(t) = \infty$.
(2) If $T_1$ is nilpotent, then $h = t_1 \circ t_2^{(-1)}$, $g = t_2 \circ t_1^{(-1)} = h^{(-1)}$, $t = t_2(0)$, and $h(t) = t_1(0)$.

Note that in any case, $h$ is continuous, strictly increasing on $[0, t_2(0)]$ and fulfills $h(0) = 0$ as well as $h(x) = h(t_2(0)) = t_1(0)$ for all $x \geqslant t_2(0)$. Moreover, it holds that $H(x, y) = h^{(-1)}(h(x) + h(y))$, in accordance with Theorem 1. As such we can rephrase Theorem 6 and Proposition 9 as well as Proposition 10 for the dominance between continuous Archimedean t-norms.

**Proposition 11.** *Consider two continuous Archimedean t-norms $T_1$ and $T_2$ with additive generators $t_1$ and $t_2$. If the function $h = t_1 \circ t_2^{(-1)} : [0, \infty] \to [0, \infty]$ is convex and geo-convex on $]0, t[$, then $T_1$ dominates $T_2$.*

**Proposition 12.** *Consider two continuous Archimedean t-norms $T_1$ and $T_2$ with additive generators $t_1$ and $t_2$. If the function $h = t_1 \circ t_2^{(-1)} : [0, \infty] \to [0, \infty]$ is differentiable and convex on $]0, t_2(0)[$, and $h'$ is geo-convex on $]0, t_2(0)[$, then $T_1$ dominates $T_2$.*

**Proposition 13.** *Consider two continuous Archimedean t-norms $T_1$ and $T_2$ with additive generators $t_1$ and $t_2$. If $T_1$ dominates $T_2$, then the function $h = t_1 \circ t_2^{(-1)} : [0, \infty] \to [0, \infty]$ is convex on $]0, t_2(0)[$.*

## Acknowledgments

Susanne Saminger-Platz has been on a sabbatical year at the Dipartimento di Matematica "Ennio De Giorgi", Università del Salento (Italy), when writing and investigating large parts of the contents of this contribution. She therefore gratefully acknowledges the support by the Austrian Science Fund (FWF) in the framework of the Erwin Schrödinger Fellowship J 2636-N15 "The Property of Dominance – From Products of Probabilistic Metric Spaces to (Quasi-)Copulas and Applications."

## References

[1] U. Bodenhofer, A Similarity-Based Generalization of Fuzzy Orderings, Schriftenreihe der Johannes Kepler Universität Linz C, vol. 26, Universitätsverlag Rudolf Trauner, 1999.
[2] B. De Baets, R. Mesiar, $T$-partitions, Fuzzy Sets and Systems 97 (1998) 211–223.
[3] B. De Baets, R. Mesiar, Metrics and $T$-equalities, J. Math. Anal. Appl. 267 (2002) 531–547.
[4] S. Díaz, S. Montes, B. De Baets, Transitivity bounds in additive fuzzy preference structures, IEEE Trans. Fuzzy Syst. 15 (2007) 275–286.
[5] W. Jarczyk, J. Matkowski, On Mulholland's inequality, Proc. Amer. Math. Soc. 130 (11) (2002) 3243–3247.
[6] E.P. Klement, R. Mesiar (Eds.), Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms, Elsevier, Amsterdam, 2005.
[7] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Trends Log. Stud. Log. Libr., vol. 8, Kluwer Academic, Dordrecht, 2000.
[8] E.P. Klement, R. Mesiar, E. Pap, Triangular norms. Position paper I: Basic analytical and algebraic properties, Fuzzy Sets and Systems 143 (2004) 5–26.
[9] E.P. Klement, R. Mesiar, E. Pap, Triangular norms. Position paper II: General constructions and parameterized families, Fuzzy Sets and Systems 145 (2004) 411–438.
[10] E.P. Klement, R. Mesiar, E. Pap, Triangular norms. Position paper III: Continuous t-norms, Fuzzy Sets and Systems 145 (2004) 439–454.
[11] B. Lafuerza Guillén, Finite products of probabilistic normed spaces, Rad. Mat. 13 (1) (2004) 111–117.
[12] R. Mesiar, S. Saminger, Domination of ordered weighted averaging operators over t-norms, Soft Computing 8 (2004) 562–570.
[13] H.P. Mulholland, On generalizations of Minkowski's inequality in the form of a triangle inequality, Proc. London Math. Soc. (2) 51 (1950) 294–307.
[14] S. Saminger, Aggregation in Evaluation of Computer-Assisted Assessment, Schriftenreihe der Johannes Kepler Universität Linz C, vol. 44, Universitätsverlag Rudolf Trauner, 2005.
[15] S. Saminger, B. De Baets, H. De Meyer, On the dominance relation between ordinal sums of conjunctors, Kybernetika 42 (3) (2006) 337–350.
[16] S. Saminger, R. Mesiar, U. Bodenhofer, Domination of aggregation operators and preservation of transitivity, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 10 (2002) 11–35.
[17] S. Saminger, P. Sarkoci, B. De Baets, The dominance relation on the class of continuous t-norms from an ordinal sum point of view, Lecture Notes in Artificial Intelligence 4342 (2006) 334–354.
[18] P. Sarkoci, Domination in the families of Frank and Hamacher t-norms, Kybernetika 41 (2005) 345–356.
[19] P. Sarkoci, Dominance is not transitive on continuous triangular norms, Aequationes Math., in press.
[20] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, North-Holland, New York, 1983.
[21] H. Sherwood, Characterizing dominates on a family of triangular norms, Aequationes Math. 27 (1984) 255–273.
[22] R.M. Tardiff, Topologies for probabilistic metric spaces, Pacific J. Math. 65 (1976) 233–251.
[23] R.M. Tardiff, On a functional inequality arising in the construction of the product of several metric spaces, Aequationes Math. 20 (1980) 51–58.
[24] R.M. Tardiff, On a generalized Minkowski inequality and its relation to dominates for t-norms, Aequationes Math. 27 (1984) 308–316.
[25] L. Valverde, On the structure of $F$-indistinguishability operators, Fuzzy Sets and Systems 17 (1985) 313–328.