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On the Total Irregularity Strength of Fan, Wheel, Triangular Book, and Friendship Graphs

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Abstract

A totally irregular total k-labeling $\lambda : V \cup E \rightarrow \{1, 2, \cdots, k\}$ of a graph G is a total labeling such that G has a total edge irregular labeling and a total vertex irregular labeling at the same time. The minimum k for which a graph G has a totally irregular total k-labeling is called the total irregularity strength of G, denoted by $ts(G)$. In this paper, we investigate some graphs whose total irregularity strength equals to the lower bound.

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1. Introduction

Let G be a finite, simple, and undirected graph with the vertex set V and the edge set E. A *labeling* of a graph is a mapping that sends some set of graph elements to a set of numbers (usually to positive or non-negative integer). If the domain is the vertex-set, or the edge-set, or the union of the vertex-set and the edge-set, the labelings are called, respectively, a *vertex labeling*, or an *edge labeling*, or a *total labeling*.

The *corona product* of G with H, denoted by $G \odot H$, is a graph obtained by taking one copy of an n-vertex graph G and n copies $H_1, H_2, \cdots, H_n$ of H and then joining the i^{th} vertex of G to every vertex in H_i.

[1]Bača *et al.* introduced an edge irregular total labeling and a vertex irregular total labeling. They determined the total edge irregular strength (tes) and total vertex irregular strength (tvs) of some certain graphs. They proved that for every graph G with the vertex set V and the edge set E,

$$\left\lceil \frac{|E| + 2}{3} \right\rceil \leq tes(G) \leq |E|. \tag{1}$$

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They also determined the total edge irregular strength of a wheel graph W_n and a friendship graph F_n, respectively as follows:

$$\begin{aligned} tes(W_n) &= \lceil \frac{2n+2}{3} \rceil; \\ tes(F_n) &= \lceil \frac{3n+2}{3} \rceil. \end{aligned} \tag{2}$$

[4]Nurdin et al. determined the total edge irregular strength of the corona product of a path with a path, a cycle, and a star as follows:

$$\begin{aligned} tes(P_m \odot P_n) &= \lceil \frac{2mn+1}{3} \rceil; \\ tes(P_m \odot C_n) &= \lceil \frac{(2n+1)m+1}{3} \rceil; \\ tes(P_m \odot S_n) &= \lceil \frac{2m(n+1)}{3} \rceil. \end{aligned} \tag{3}$$

It can be checked that the given labeling did not provide the distinct weight among vertices at the same time. [7]Wijaya and Slamin determined the total vertex irregularity strength of a fan graph f_n, a wheel graph W_n, and a friendship graph F_n as follows:

$$\begin{aligned} tvs(f_n) &= \lceil \frac{n+2}{4} \rceil; \\ tvs(W_n) &= \lceil \frac{n+3}{4} \rceil; \\ tvs(F_n) &= \lceil \frac{2n+2}{3} \rceil. \end{aligned} \tag{4}$$

For further result of tes and tvs , one can refer [2]. [3]Marzuki *et al.* introduced a new irregular total k-labeling called totally irregular total k-labeling which is the combining of both edge irregular total labeling and vertex irregular total labeling. For a graph G with the vertex-set V and the edge-set E, a *totally irregular total k-labeling* $\lambda : V \cup E \rightarrow \{1, 2, \cdots, k\}$ of G is a total labeling such that for every two distinct edges x_1y_1 and x_2y_2 in $E(G)$ satisfies $w(x_1y_1) \neq w(x_2y_2)$ and every two distinct vertices x and y in $V(G)$ satisfies $w(x) \neq w(y)$. The minimum k for which G has a totally irregular total k-labeling is called the *total irregularity strength* of G, denoted by $ts(G)$. They proved that for any graph G,

$$ts(G) \geq \max\{\text{tes(G)}, \text{tvs(G)}\}; \tag{5}$$

and determined the ts of a cycle and a path. [5]Ramdani and Salman gave the ts of the cartesian product of P_2 and a path, a star, a cycle, and a fan graph. [6]Ramdani *et al.* also estimated the upper bound of ts of any graph and determined the ts of a gear graph, the ts of a fungus graph and the ts of a disjoint union of stars.

In this paper, we investigate some graphs whose total irregularity strength equals to the lower bound. We show that those graphs have totally irregular total k- labeling and determine the exact value of their ts.

2. Results

In this section, we determine the total irregularity strength of a fan graph f_n for $n \geq 3$, a wheel graph W_n for $n \geq 3$, a triangular book graph $P_1 \odot S_n$ for $n \geq 2$, and a friendship graph F_n for $n \geq 3$.

2.1. Total irregularity strength of a fan graph

Theorem 1. *Let $n \geq 3$ and f_n be a fan graph with $n+1$ vertices and $2n-1$ edges. Then*

$$ts(f_n) = \left\lceil \frac{2n+1}{3} \right\rceil.$$

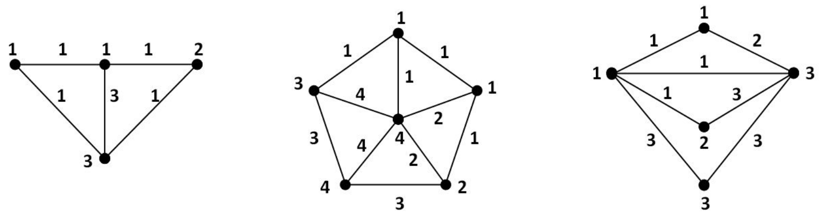

Fig. 1. Totally irregular total labeling of: (a) f_3; (b) W_5; (c) $P_1 \odot S_3$.

Proof. Since $|V(f_n)| = n + 1$ and $|E(f_n)| = 2n - 1$, by (1), (4), and (5), we have $ts(f_n) \geq \left\lceil \frac{2n+1}{3} \right\rceil$. Let $t = \left\lceil \frac{2n+1}{3} \right\rceil$. For the reverse inequality, we construct an irregular total labeling $\lambda : V \cup E \to \{1, 2, \cdots, t\}$. Let $V(f_n) = \{v\} \cup \{v_i | 1 \leq i \leq n\}$ and $E(f_n) = \{vv_i, v_j v_{j+1} | 1 \leq i \leq n, 1 \leq j \leq n-1\}$. We divide proof into 4 cases as follows:

Case 1. $n = 3$

The result is obvious as shown in Figure 1(a).

Case 2. $n \notin \{3, 16, 20, 21\}$ and $n \neq 9a + b + 25$ for two nonnegative integers a and b, with $0 \leq b \leq 2$

Define λ as follows:

$$\lambda(v) = t;$$

$$\lambda(v_i) = \begin{cases} \left\lceil \frac{i}{2} \right\rceil, & \text{for } 1 \leq i \leq t; \\ t, & \text{for } t+1 \leq i \leq n; \end{cases}$$

$$\lambda(vv_i) = \begin{cases} \left\lceil \frac{i+1}{2} \right\rceil, & \text{for } 1 \leq i \leq t; \\ i - t + 2, & \text{for } t+1 \leq i \leq n; \end{cases}$$

$$\lambda(v_i v_{i+1}) = \begin{cases} 1, & \text{for } 1 \leq \mathrm{i} \leq \mathrm{t} - 1; \\ \left\lceil \frac{t+3}{2} \right\rceil, & \text{for } i = t; \\ n - 2t + 2 + i, & \text{for } t+1 \leq i \leq n-1. \end{cases}$$

It is easy to check that the largest label is t.

Next, we have

$$w(vv_i) = \begin{cases} t + \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil, & \text{for } 1 \leq i \leq t; \\ t + 2 + i, & \text{for } t+1 \leq i \leq n; \end{cases}$$

$$w(v_i v_{i+1}) = \begin{cases} \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil + 1, & \text{for } 1 \leq i \leq t-1; \\ t + 1 + \left\lceil \frac{t}{2} \right\rceil + \left\lceil \frac{t+1}{2} \right\rceil, & \text{for } i = t; \\ n + 2 + i, & \text{for } t+1 \leq i \leq n-1; \end{cases}$$

and

$$w(v) = \begin{cases} \frac{1}{2}n(n + 5 - 2t) + \frac{1}{4}t(3t - 2), & \text{for even } t; \\ \frac{1}{2}n(n + 5 - 2t) + \frac{1}{4}t(3t - 2) + \frac{1}{4}, & \text{for odd } t; \end{cases}$$

$$w(v_i) = \begin{cases} 3, & \text{for } i = 1; \\ \left\lceil \frac{i}{2} \right\rceil + \left\lceil \frac{i+1}{2} \right\rceil + 2, & \text{for } 2 \leq i \leq t-1; \\ \left\lceil \frac{t}{2} \right\rceil + t + 3, & \text{for } i = t; \\ \left\lceil \frac{t+1}{2} \right\rceil + n + 7, & \text{for } i = t+1; \\ 2n - 4t + 5 + 3i, & \text{for } t+2 \leq i \leq n-1; \\ 3n - 2t + 3 & \text{for } i = n. \end{cases}$$

It can be checked that the edge-weights under λ form a consecutive sequence $3, 4, \cdots, 2n + 1$ and the vertex-weights $w(v_i)$ for $1 \leq i \leq n - 1$ are pairwise distinct because of:

$$\begin{aligned}
\{w(v_1)\} &= \{3\};\\
\{w(v_i) \mid 2 \leq i \leq t-1\} &= \{5, 6, \cdots, t+2\};\\
\{w(v_t)\} &= \left\{\left\lceil \frac{t}{2} \right\rceil + t + 3\right\};\\
\{w(v_{t+1})\} &= \left\{\left\lceil \frac{t+1}{2} \right\rceil + n + 7\right\};\\
\{w(v_i) \mid t+2 \leq i \leq n-1\} &= \{2n - t + 11, 2n - t + 14, \cdots, 5n - 4t + 2\}.
\end{aligned}$$

Next, we verify $w(v) \neq w(v_i), w(v) \neq w(v_n)$, and $w(v_i) \neq w(v_n)$. Consider that whenever n is increase, the weight of v is strictly increase. Hence, just by checking on the lowest value of n, we have $w(v) > w(v_{n-1})$ and $w(v) > w(v_n)$. Suppose that $w(v_i) = w(v_n)$. Then $n = \frac{1}{2}\left\lceil \frac{t+1}{2} \right\rceil + t + 2$ or $n = 3i - 2t + 2$ for $t + 2 \leq i \leq n - 1$. It can be checked that all integers n which satisfied this condition are $n \in \{16, 20, 21\}$ or $n = 9a + b + 25$ for two nonnegative integer a and b, where $0 \leq b \leq 2$. Thus, the vertex-weights are pairwise distinct.

Case 3. $n \in \{16, 20, 21\}$

By applying λ, we have $w(v_{t+1}) = w(v_n)$. Hence, we modified λ by defining a new labeling λ' where $\lambda'(v_{t+1}v_{t+2}) = \lambda(vv_{t+2})$ and $\lambda'(vv_{t+2}) = \lambda(v_{t+1}v_{t+2})$. We have $w(v_{t+1}) = \left\lceil \frac{t+1}{2} \right\rceil + t + 8$. It can be checked that the modification just change $w(v_{t+1})$. Thus, the vertex-weights (and the edge-weights) are pairwise distinct.

Case 4. $n = 9a + b + 25$ for two nonnegative integers a and b, with $0 \leq b \leq 2$,

By applying λ, we have $w(v_{t+2+a}) = w(v_n)$. Hence, we modified with $\lambda'(v_{t+2+a}v_{t+3+a}) = \lambda(vv_{t+3+a})$ and $\lambda'(vv_{t+3+a}) = \lambda(v_{t+2+a}v_{t+3+a})$. We have $w(v_{t+2+a}) = n + 3a + 12$. It can be checked that the modification just change $w(v_{t+1})$. Thus, the vertex-weights (and the edge-weights) are pairwise distinct. It complete the proof. □

2.2. Total irregularity strength of a wheel graph

Theorem 2. *Let $n \geq 3$ and W_n be a wheel graph with $n + 1$ vertices and $2n$ edges. Then*

$$ts(W_n) = \left\lceil \frac{2n+2}{3} \right\rceil.$$

Proof. Since $|V(W_n)| = n + 1$ and $|E(W_n)| = 2n$, by (2), (4), and (5), we have $ts(W_n) \geq \left\lceil \frac{2n+2}{3} \right\rceil$. Let $t = \left\lceil \frac{2n+2}{3} \right\rceil$. For the reverse inequality, we construct a total labeling $\lambda : V \cup E \to \{1, 2, \cdots, t\}$. Let $V(W_n) = \{v\} \cup \{v_i \mid 1 \leq i \leq n\}$ and $E(W_n) = \{vv_i, v_jv_{j+1}, v_nv_1 \mid 1 \leq i \leq n, 1 \leq j \leq n-1\}$. We divide proof into 2 cases as follows:

Case 1. $n = 5$

The result is obvious as shown in Figure 1(b).

Case 2. $n \neq 5$

Define λ as follows:

$$\begin{aligned}
\lambda(v) &= t - 1;\\
\lambda(v_i) &= \begin{cases} \left\lceil \frac{i}{2} \right\rceil, & \text{for } 1 \leq i \leq t-1;\\ t, & \text{for } t \leq i \leq n; \end{cases}\\
\lambda(vv_i) &= \begin{cases} \left\lceil \frac{i+1}{2} \right\rceil, & \text{for } 1 \leq i \leq t-1;\\ i - t + 3, & \text{for } t \leq i \leq n; \end{cases}\\
\lambda(v_nv_1) &= t - 1;\\
\lambda(v_iv_{i+1}) &= \begin{cases} 1, & \text{for } 1 \leq i \leq t-2;\\ \left\lceil \frac{t}{2} \right\rceil + 1, & \text{for } i = t-1;\\ n - 2t + 3 + i, & \text{for } t \leq i \leq n-1. \end{cases}
\end{aligned}$$

It is easy to check that the largest label is t.
Next, we have

$$w(vv_i) = \begin{cases} t+i, & \text{for } 1 \le i \le t-1; \\ t+2+i, & \text{for } t \le i \le n; \end{cases}$$

$$w(v_iv_{i+1}) = \begin{cases} i+2, & \text{for } 1 \le i \le t-2; \\ 2t+1, & \text{for } i = t-1; \\ n+3+i, & \text{for } t \le i \le n-1; \end{cases}$$

$$w(v_nv_1) = 2t;$$

and

$$w(v) = \begin{cases} \frac{1}{2}n(n+7-2t) + \frac{1}{4}t(3t-6) - 1, & \text{for even } t; \\ \frac{1}{2}n(n+7-2t) + \frac{1}{4}t(3t-6) - \frac{5}{4}, & \text{for odd } t; \end{cases}$$

$$w(v_i) = \begin{cases} t+2 & \text{for } i = 1; \\ i+3, & \text{for } 2 \le i \le t-2; \\ \left\lceil \frac{t}{2} \right\rceil + t + 2, & \text{for } i = t-1; \\ \left\lceil \frac{t}{2} \right\rceil + n + 7, & \text{for } i = t; \\ 2n - 4t + 3i + 8, & \text{for } t+1 \le i \le n-1; \\ 3n - t + 4 & \text{for } i = n. \end{cases}$$

It can be checked that the edge-weights under λ form a consecutive sequence $3, 4, \cdots, 2n+2$ and the vertex-weigths $w(v_i)$ for $1 \le i \le n-1$ are pairwise distinct because of:

$$\begin{aligned} \{w(v_1)\} &= \{t+2\}; \\ \{w(v_i) \mid 2 \le i \le t-2\} &= \{5, 6, \cdots, t+1\}; \\ \{w(v_{t-1})\} &= \left\{\left\lceil \frac{t}{2} \right\rceil + t + 2\right\}; \\ \{w(v_t)\} &= \left\{\left\lceil \frac{t}{2} \right\rceil + n + 7\right\}; \\ \{w(v_i) \mid t+1 \le i \le n-1\} &= \{2n-t+11, 2n-t+14, \cdots, 5n-4t+5\}. \end{aligned}$$

Next, we verify $w(v) \neq w(v_i), w(v) \neq w(v_n)$, and $w(v_i) \neq w(v_n)$. It is easy to check on $n < 5$. For $n > 5$, we consider that whenever n is increase, the weight of v is strictly increase. Hence, just by checking on the lowest value $n = 6$, we have $w(v) > w(v_{n-1})$ and $w(v) > w(v_n)$. Since $w(v_n) > w(v_i)$, for $i \le t$, we suppose that $w(v_n) = w(v_i)$, for $t+1 \le i \le n-1$, then $3i = n + 3t - 4$. There is no integer n which satisfy this condition. Thus, the vertex-weights are pairwise distinct. It complete the proof. □

2.3. Total irregularity strength of a triangular book graph

Theorem 3. *Let $n \ge 3$ and $P_1 \odot S_n$ be a book graph with n triangular pages with $n+1$ vertices and $2n-1$ edges. Then*

$$ts(P_1 \odot S_n) = \left\lceil \frac{2n+3}{3} \right\rceil.$$

Proof. Since $|V(P_1 \odot S_n)| = n+2$ and $|E(P_1 \odot S_n)| = 2n+1$, by (1), (3), and (5), we have $ts(P_1 \odot S_n) \ge \left\lceil \frac{2n+3}{3} \right\rceil$. Let $t = \left\lceil \frac{2n+3}{3} \right\rceil$. For the reverse inequality, we construct an irregular total labeling $\lambda : V \cup E \to \{1, 2, \cdots, t\}$.
Let $V(P_1 \odot S_n) = \{u, v, v_1, v_2, \cdots, v_n\}$ and $E(P_1 \odot S_n) = \{uv, uv_i, vv_i \mid 1 \le i \le n\}$. Let $n \equiv m \bmod 3$ for $m = 0, 1, 2$. We divide proof into 2 cases as follows:
Case 1. $n = 3$
The result is obvious as shown in Figure 1(c).

Case 2. $n \neq 3$
Define λ as follows:

$$\lambda(u) = 1;$$
$$\lambda(v) = t;$$
$$\lambda(v_i) = \begin{cases} i, & \text{for } 1 \leq i \leq t; \\ t, & \text{for } t+1 \leq i \leq n; \end{cases}$$
$$\lambda(uv) = t;$$
$$\lambda(uv_i) = \begin{cases} 1, & \text{for } 1 \leq i \leq t; \\ i-t+1, & \text{for } t+1 \leq i \leq n; \end{cases}$$
$$\lambda(vv_i) = \begin{cases} n-t+2, & \text{for } 1 \leq i \leq \frac{1}{2}(t+m-1); \\ n-t+3, & \text{for } \frac{1}{2}(t+m+1) \leq i \leq t; \\ n-2t+3+i, & \text{for } t+1 \leq i \leq n. \end{cases}$$

It is easy to check that the largest label is t.
Next, we have

$$w(uv) = 2t+1;$$
$$w(uv_i) = i+2, \qquad \text{for } 1 \leq i \leq n;$$
$$w(vv_i) = \begin{cases} n+2+i, & \text{for } 1 \leq i \leq \frac{1}{2}(t+m-1); \\ n+3+i, & \text{for } \frac{1}{2}(t+m+1) \leq i \leq n; \end{cases}$$

and

$$w(u) = \frac{1}{2}n\,(n-2t+3) + \frac{1}{2}t\,(t+1) + 1;$$
$$w(v) = \frac{1}{2}n\,(3n-4t+7) + \frac{1}{2}t\,(t+2) - \frac{1}{2}(m-1);$$
$$w(v_i) = \begin{cases} n-t+3+i & \text{for } 1 \leq i \leq \frac{1}{2}(t+m-1); \\ n-t+4+i, & \text{for } \frac{1}{2}(t+m+1) \leq i \leq t; \\ n-2t+4+2i & \text{for } t+1 \leq i \leq n. \end{cases}$$

It can be checked that the edge- weights under λ form a consecutive sequence $3, 4, \cdots, 2n+3$ and the vertex-weights $w(v_i)$ for $1 \leq i \leq n$ are pairwise distinct because of:

$$\left\{w(v_i) \mid 1 \leq i \leq \frac{1}{2}(t+m-1)\right\} = \left\{n-t+4,\ n-t+5,\ \cdots,\ \frac{1}{2}(t+m-1)+n-t+3\right\};$$
$$\left\{w(v_i) \mid \frac{1}{2}(t+m+1) \leq i \leq t\right\} = \left\{\frac{1}{2}(t+m-1)+n-t+5,\ \frac{1}{2}(t+m-1)+n-t+6,\ \cdots,\ n+4\right\};$$
$$\{w(v_i) \mid t+1 \leq i \leq n\} = \{n+6, n+8, \cdots, 3n-2t+4\}.$$

Next, since $w(u) < w(v)$, we verify $w(u) \neq w(v_n)$. Since the weight of u is strictly increase whenever n is increase,we check on the lowest value $n = 4$, we have $w(u) > w(v_n)$. Thus, the vertex-weights are pairwise distinct. It complete the proof. □

2.4. Total irregularity strength of a friendship graph

A friendship graph F_n is a set of n-copies of a triangle whose a common vertex as a center and the other mutually disjoint vertices. For the i^{th} triangle , let v be the center and the other two vertices as x_i and y_i, respectively.

Theorem 4. *Let $n \geq 2$ and F_n be a friendship graph with $2n+1$ vertices and $3n$ edges. Then*

$$ts(F_n) = n + 1.$$

Proof. Let $V(F_n) = \{v, x_i, y_i | 1 \leq i \leq n\}$ and $E(F_n) = \{vx_i, vy_i, x_iy_i \mid 1 \leq i \leq n$. By (2), (4) and (5), $ts(F_n) \geq \left\lceil \frac{3n+2}{3} \right\rceil$. Let $t = \left\lceil \frac{3n+2}{3} \right\rceil$ and $r = \left\lfloor \frac{n-1}{2} \right\rfloor$. For the reverse inequality, we divide all triangles into 3 different part, say r first triangles $vx_{i_1}y_{i_1}v$, triangle $vx_{r+1}y_{r+1}v$, and $n-r-1$ triangles $vx_{i_2}y_{i_2}v$, where $i = 1, 2, \cdots, s$. We construct an irregular total labeling $\lambda : V \cup E \rightarrow \{1, 2, \cdots, t\}$ of F_n as follows:

$$\begin{aligned}
\lambda(v) &= r+1; & \\
\lambda(x_{i_1}) &= 1, & &\text{for } 1 \leq i \leq r;\\
\lambda(y_{i_1}) &= 1, & &\text{for } 1 \leq i \leq r;\\
\lambda(x_{r+1}) &= r+1; & \\
\lambda(y_{r+1}) &= r+1; & \\
\lambda(x_{i_2}) &= t, & &\text{for } 1 \leq i \leq n-r-1;\\
\lambda(y_{i_2}) &= t, & &\text{for } 1 \leq i \leq n-r-1;\\
\lambda(x_{i_1}y_{i_1}) &= i, & &\text{for } 1 \leq i \leq r;\\
\lambda(vx_{i_1}) &= 2i-1, & &\text{for } 1 \leq i \leq r;\\
\lambda(vy_{i_1}) &= 2i, & &\text{for } 1 \leq i \leq r;\\
\lambda(x_{r+1}y_{r+1}) &= r+1; & \\
\lambda(vx_{r+1}) &= r+2; & \\
\lambda(vy_{r+1}) &= r+3; & \\
\lambda(x_{i_2}y_{i_2}) &= r+i+1, & &\text{for } 1 \leq i \leq n-r-1;\\
\lambda(vx_{i_2}) &= \begin{cases} 2i, & \text{for even } n \text{ with } 1 \leq i \leq n-r-1;\\ 2i+1, & \text{for odd } n \text{ with } 1 \leq i \leq n-r-1; \end{cases}\\
\lambda(vy_{i_2}) &= \begin{cases} 2i+1, & \text{for even } n \text{ with } 1 \leq i \leq n-r-1;\\ 2i+2, & \text{for odd } n \text{ with } 1 \leq i \leq n-r-1. \end{cases}
\end{aligned}$$

It is easy to check that the largest label is t.
Next, we have

$$\begin{aligned}
w(x_{i_1}y_{i_1}) &= i+2, & &\text{for } 1 \leq i \leq r;\\
w(vx_{i_1}) &= r+2i+1, & &\text{for } 1 \leq i \leq r;\\
w(vy_{i_1}) &= r+2i+2, & &\text{for } 1 \leq i \leq r;\\
w(x_{r+1}y_{r+1}) &= 3r+3; & \\
w(vx_{r+1}) &= 3r+4; & \\
w(vy_{r+1}) &= 3r+5; & \\
w(x_{i_2}y_{i_2}) &= 2t+r+i+1, & &\text{for } 1 \leq i \leq n-r-1;\\
w(vx_{i_2}) &= \begin{cases} r+t+2i+1, & \text{for even } n \text{ with } 1 \leq i \leq n-r-1;\\ r+t+2i+2, & \text{for odd } n \text{ with } 1 \leq i \leq n-r-1,; \end{cases}\\
w(vy_{i_2}) &= \begin{cases} r+t+2i+2, & \text{for even } n \text{ with } 1 \leq i \leq n-r-1;\\ r+t+2i+3, & \text{for odd } n \text{ with } 1 \leq i \leq n-r-1; \end{cases}
\end{aligned}$$

and

$$w(v) = \begin{cases} 4r\,(r+1) + n\,(2n-4r-1) + 4, & \text{for even } n; \\ 4r\,(r+1) + n\,(2n-4r+1) + 2, & \text{for odd } n; \end{cases}$$

$$w(x_{i_1}) = 3i, \qquad \text{for } 1 \le i \le r;$$

$$w(y_{i_1}) = 3i + 1, \qquad \text{for } 1 \le i \le r;$$

$$w(x_{r+1}) = 3r + 4;$$

$$w(y_{r+1}) = 3r + 5;$$

$$w(x_{i_2}) = \begin{cases} t + r + 3i + 1, & \text{for even } n \text{ with } 1 \le i \le n-r-1; \\ t + r + 3i + 2, & \text{for odd } n \text{ with } 1 \le i \le n-r-1; \end{cases}$$

$$w(y_{i_2}) = \begin{cases} t + r + 3i + 2, & \text{for even } n \text{ with } 1 \le i \le n-r-1; \\ t + r + 3i + 3, & \text{for odd } n \text{ with } 1 \le i \le n-r-1. \end{cases}$$

It can be checked that the edge-weights under λ form a consecutive sequence $3, 4, \cdots, 2n+2$.
For even n, we have

$$\{w(v)\} = \{4r\,(r+1) + n\,(2n-4r-1) + 4\};$$

$$\{w(x_{i_1}) \mid 1 \le i \le r\} = \{3, 6, \cdots, 3r\};$$

$$\{w(y_{i_1}) \mid 1 \le i \le r\} = \{4, 7, \cdots, 3r+1\};$$

$$\{w(x_{r+1})\} = \{3r+4\};$$

$$\{w(y_{r+1})\} = \{3r+5\};$$

$$\{w(x_{i_2}) \mid 1 \le i \le n-r-1\} = \{t+r+4, t+r+7, \cdots, 3n+t-2r-2\};$$

$$\{w(y_{i_2}) \mid 1 \le i \le n-r-1\} = \{t+r+5, t+r+8, \cdots, 3n+t-2r-1\}.$$

Next, we verify $w(v) \neq w(y_{i_2})$. Since the weight of v is strictly increase whenever n is increase, we check on the lowest value $n = 2$, we have $w(v) > w(y_{n-r-1})$. Thus, the vertex-weights are pairwise distinct. It is similar for odd n. It complete the proof. □

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