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On q-deformed infinite-dimensional n-algebra

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Abstract

The q-deformation of the infinite-dimensional n-algebras is investigated. Based on the structure of the q-deformed Virasoro–Witt algebra, we derive a nontrivial q-deformed Virasoro–Witt n-algebra which is nothing but a sh-n-Lie algebra. Furthermore in terms of the pseud-differential operators, we construct the (co)sine n-algebra and the q-deformed $SDiff(T^2)$ n-algebra. We find that they are the sh-n-Lie algebras for the n even case. In terms of the magnetic translation operators, an explicit physical realization of the (co)sine n-algebra is given.

1. Introduction

Infinite dimensional Lie algebras have played a crucial role in physics. Much interest has been attributed to their q-deformed versions. It is well-known that the Virasoro algebra is an important infinite dimensional Lie algebra. Its q-deformation has been widely studied in the literature [1–10]. A q-deformation of the centerless Virasoro or Virasoro–Witt (V–W) algebra was

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first obtained by Curtright and Zachos [1]. Its central extension was later furnished by Aizawa and Sato [2]. Chaichian and Prešnajder [3] proposed a different version of the q-deformed Virasoro algebra by carrying out a Sugawara construction on a q-analogue of an infinite dimensional Heisenberg algebra. Shiraishi et al. [4] presented a q-Virasoro algebra $Vir_{q,t}$, where q and t are two complex parameters. They constructed a free boson realization of this q-Virasoro algebra and showed that singular vectors can be expressed by the Macdonald symmetric functions. It is similar to the case of the ordinary Virasoro algebra whose singular vectors are given by the Jack symmetric functions. It is well-known that there is a remarkable connection between the Virasoro algebra and the Korteweg–de Vries (KdV) equation [11,12]. For the q-deformed Virasoro algebra, Chaichian et al. [13] showed that it generates the symplectic structure which can be used for a description of the discretization of the KdV equation. Furthermore the quantum KdV equations associated with the algebraic symmetry have been investigated in Refs. [14,15]. The integrable one-dimensional quantum spin chains have attracted much interest from physical and mathematical points of view. One noted that the deformed Virasoro algebra plays an important role in the study of the XYZ model [16,17].

The W_N algebras are extensions of the Virasoro algebra which contain generators of all conformal spins from $s = 2$ up to $s = N$. The W_∞ algebra may arise in an appropriate large N limit of W_N. The q-deformations of W_N and W_∞ algebras have been well investigated [6,18–21]. Recently Taki [22] proposed the generalized Alday–Gaiotto–Tachikawa–Wyllard (AGT-W) correspondence between $5d$ uplift of $4d$ $N = 2$ $SU(N)$ asymptotically-free gauge theories and the q-deformed W_N algebra. It was found that the Nekrasov partition function of a $5d$ gauge theory is equal to the scalar product of the corresponding Whittaker vectors of the q-deformed W_N algebra. The contraction of the W_∞ algebra leads to the so-called w_∞ algebra which is equivalent to the algebra of smooth area-preserving diffeomorphisms of the cylinder $S^1 \times R^1$ [23]. It is worth to emphasize that the algebra of the area-preserving diffeomorphisms of the torus T^2, i.e., the so-called $SDiff(T^2)$ algebra [24,25] is also an important infinite-dimensional algebra. In terms of the Gauss derivatives on the quantum plane, Kinani et al. [26] presented the q-deformed $SDiff(T^2)$ algebra. It should be noted that the sine algebra arises as the unique Lie algebra deformation of $SDiff(T^2)$ in some suitable basis. There has been considerable interest in the (super) sine algebra [27–30].

The Nambu 3-algebra was introduced in [31,32] as a natural generalization of a Lie algebra for higher-order algebraic operations. Recently Bagger and Lambert [33,34], and Gustavsson [35] (BLG) found that 3-algebras play an important role in the world-volume description of the multiple M2-branes. Due to BLG theory, there has been considerable interest in the 3-algebra and its application. More recently there has been the progress in constructing the infinite-dimensional 3-algebras, such as V–W [36,37], Kac–Moody [38] and w_∞ 3-algebras [39,40]. It is well-known that the infinite-dimensional algebras have a deep intrinsic connection to the integrable systems. Recently the relation between the infinite-dimensional 3-algebras and the integrable systems has also been studied in the framework of Nambu mechanics [41,42].

Recently Curtright et al. [36], constructed a V–W algebra through the use of $su(1,1)$ enveloping algebra techniques. It is worthwhile to mention that this ternary algebra depends on a parameter z and is only a Nambu–Lie algebra when $z = \pm 2\mathbf{i}$. Ammar et al. [43] presented a q-deformation of this 3-algebra and noted it carrying the structure of ternary Hom–Nambu–Lie algebra. We know that the deep insights into the q-deformed algebra have been achieved. However for the q-deformed infinite-dimensional 3-algebra, much less is still known about its structure and property. As to the q-deformed infinite-dimensional n-algebra, to our best knowledge, it has not been reported so far in the existing literature. The goal of this paper is to construct the q-deformed infinite-dimensional n-algebras and explore their intriguing features.

This paper is organized as follows. In section 2, we introduce the definitions of n-Lie algebra and sh-n-Lie algebra. In section 3, we construct the q-deformed V–W n-algebra. In section 4, in terms of the pseud-differential operators on the q-plane, we construct the (co)sine n-algebra and the q-deformed $SDiff(T^2)$ n-algebra. An explicit physical realization of the (co)sine n-algebra is given in section 5. We end this paper with the concluding remarks in section 6.

2. n-Lie algebra and sh-n-Lie algebra

For later convenience, we shall recall the definitions of n-Lie algebra and sh-n-Lie algebra in this section. For a more detailed description we refer the reader to Refs. [44–46].

The notion of n-Lie algebra or Filippov n-algebra was introduced by Filippov [44]. It is a natural generalization of Lie algebra.

Definition 1. (See [44].) An n-Lie algebra structure is a linear space V endowed with a multilinear map called Nambu bracket $[\cdot, \cdots, \cdot]$: $V^{\otimes n} \to V$ satisfying the following properties:
(1). Skew-symmetry

$$[X_{\sigma(1)}, \cdots, X_{\sigma(n)}] = (-1)^{\pi(\sigma)}[X_1, \cdots, X_n]. \tag{1}$$

(2). Fundamental identity (FI) or Filippov condition

$$[Y_1, \cdots, Y_{n-1}, [X_1, \cdots, X_n]] = \sum_{k=1}^{n}[X_1, \cdots, X_{k-1}, [Y_1, \cdots, Y_{n-1}, X_k], X_{k+1}, \cdots, X_n], \tag{2}$$

for any $X_1, \cdots, X_n, Y_1, \cdots, Y_{n-1}$ in V, where σ is any permutation of the indices $(1, 2, \cdots, n)$, $\pi(\sigma)$ is the parity of the permutation σ.

For the case of 3-Lie algebra, the corresponding FI is

$$\begin{aligned}[Y_1, Y_2, [X_1, X_2, X_3]] = {} & [[Y_1, Y_2, X_1], X_2, X_3] + [X_1, [Y_1, Y_2, X_2], X_3] \\ & + [X_1, X_2, [Y_1, Y_2, X_3]].\end{aligned} \tag{3}$$

We have already seen that an n-Lie algebra V is a vector space V endowed with an n-ary skew-symmetric multiplication satisfying the FI (2). We now turn to the notion of sh-n-Lie algebra.

Definition 2. (See [45].) Let $[\cdot, \cdots, \cdot]$ be an n-ary skewsymmetric product on a vector space V. We say that $(V, [\cdot, \cdots, \cdot])$ is a sh-n-Lie algebra if $[\cdot, \cdots, \cdot]$ satisfies the sh-Jacobi identity

$$\sum_{\sigma \in Sh(n,n-1)} (-1)^{\pi(\sigma)} \left[[X_{\sigma(1)}, \cdots, X_{\sigma(n)}], X_{\sigma(n+1)}, \cdots, X_{\sigma(2n-1)}\right] = 0, \tag{4}$$

for any $X_i \in V$, where $Sh(n, n-1)$ is the subset of the permutation group Σ_{2n-1} of the indices $(1, 2, \cdots, 2n-1)$ defined by

$$Sh(n, n-1) = \{\sigma \in \Sigma_{2n-1} | \sigma(1) < \cdots < \sigma(n), \sigma(n+1) < \cdots < \sigma(2n-1)\}.$$

By means of the skewsymmetry of the n-bracket, it is known that the sh-Jacobi identity (4) is equivalent to the following generalized Jacobi identity [46]:

$$\begin{aligned}&\sum_{\sigma\in\Sigma_{2n-1}}(-1)^{\pi(\sigma)}\big[[X_{\sigma(1)},\cdots,X_{\sigma(n)}],X_{\sigma(n+1)},\cdots,X_{\sigma(2n-1)}\big]\\&\quad=\sum_{\sigma\in Sh(n,n-1)}n!(n-1)!(-1)^{\pi(\sigma)}\big[[X_{\sigma(1)},\cdots,X_{\sigma(n)}],X_{\sigma(n+1)},\cdots,X_{\sigma(2n-1)}\big]=0.\end{aligned}\tag{5}$$

In terms of the Lévi-Cività symbol, i.e.,

$$\epsilon^{i_1\cdots i_p}_{j_1\cdots j_p}=\det\begin{pmatrix}\delta^{i_1}_{j_1}&\cdots&\delta^{i_1}_{j_p}\\ \vdots&&\vdots\\ \delta^{i_p}_{j_1}&\cdots&\delta^{i_p}_{j_p}\end{pmatrix},\tag{6}$$

the generalized Jacobi identity (or sh-Jacobi identity) (5) can also be expressed as

$$\epsilon^{i_1\cdots i_{2n-1}}_{1\cdots 2n-1}\big[[X_{i_1},\cdots,X_{i_n}],X_{i_{n+1}},\cdots,X_{i_{2n-1}}\big]=0.\tag{7}$$

When $n=2$, both the FI (2) and sh-Jacobi identity (4) become the well-known Jacobi identity. When $n=3$, we see that the FI is (3). The corresponding sh-Jacobi identity (4) is

$$\begin{aligned}&[[X_1,X_2,X_3],X_4,X_5]-[[X_1,X_2,X_4],X_3,X_5]+[[X_1,X_2,X_5],X_3,X_4]\\&\quad+[[X_1,X_3,X_4],X_2,X_5]-[[X_1,X_3,X_5],X_2,X_4]+[[X_1,X_4,X_5],X_2,X_3]\\&\quad-[[X_2,X_3,X_4],X_1,X_5]+[[X_2,X_3,X_5],X_1,X_4]-[[X_2,X_4,X_5],X_1,X_3]\\&\quad+[[X_3,X_4,X_5],X_1,X_2]=0.\end{aligned}\tag{8}$$

We have briefly introduced the n-Lie algebra and sh-n-Lie algebra. It should be noted that any n-Lie algebra is a sh-n-Lie algebra, but a sh-n-Lie algebra is an n-Lie algebra if and only if any adjoint operator is a derivation. It is worth also to emphasize that the sh-n-Lie algebra in Definition 2 does indeed correspond to a particular case of the SH Lie algebra [46–48].

3. q-deformed V–W n-algebra

3.1. q-deformed V–W 3-algebra

As a start before investigating the q-deformed 3-algebra, let us recall the case of q-deformed algebra. The deformation of the commutator is defined by

$$[A,B]_{(p,q)}=pAB-qBA.\tag{9}$$

It possesses the following properties [5,10]:

$$\begin{aligned}&[A,B]_{(p,q)}=-[B,A]_{(q,p)},\\&[A+B,C]_{(p,q)}=[A,C]_{(p,q)}+[B,C]_{(p,q)},\\&[AB,C]_{(p,q)}=A[B,C]_{(p,r)}+[A,C]_{(r,q)}B,\\&[A,BC]_{(p,q)}=B[A,C]_{(r,q)}+[A,B]_{(p,r)}C,\end{aligned}\tag{10}$$

and the q-Jacobi identity

$$\begin{aligned}&[A,[B,C]_{(q_1,q_1^{-1})}]_{(q_3/q_2,q_2/q_3)}+[B,[C,A]_{(q_2,q_2^{-1})}]_{(q_1/q_3,q_3/q_1)}\\&\quad+[C,[A,B]_{(q_3,q_3^{-1})}]_{(q_2/q_1,q_1/q_2)}=0.\end{aligned}\tag{11}$$

The Virasoro algebra is an infinite dimensional Lie algebra and plays important roles in physics. The V–W algebra is indeed the centerless Virasoro algebra. It is given by

$$[L_m,L_n]=(m-n)L_{m+n}.\tag{12}$$

To construct the deformed V–W algebra, let us take the q-deformed generators

$$L_m=-q^N(a^\dagger)^{m+1}a,\tag{13}$$

where the q-deformed oscillator is deformed by the following relations [49–51]:

$$\begin{aligned}&aa^\dagger-qa^\dagger a=q^{-N},\ \ a^\dagger a=[N],\\&[N,a]=-a,\ \ [N,a^\dagger]=a^\dagger.\end{aligned}\tag{14}$$

Substituting the q-generators (13) into the commutator (9) and using the q-deformed oscillator (14), it leads to the so-called q-deformed V–W algebra [1]

$$[L_m,L_n]_{(q^{m-n},q^{n-m})}=q^{m-n}L_mL_n-q^{n-m}L_nL_m=[m-n]L_{m+n},\tag{15}$$

where $[k]=\frac{q^k-q^{-k}}{q-q^{-1}}$. In the limit $q\to 1$, (15) reduces to the V–W algebra (12).

Let us introduce

$$\begin{aligned}&[L_m,[L_n,L_k]_{(q^{n-k},q^{k-n})}]_{(q^{2m-n-k},q^{n+k-2m})}\\&\quad=q^{2m-n-k}L_m[L_n,L_k]_{(q^{n-k},q^{k-n})}-q^{n+k-2m}[L_n,L_k]_{(q^{n-k},q^{k-n})}L_m.\end{aligned}\tag{16}$$

In terms of (16), one can confirm the following q-Jacobi identity [5] satisfied by the q-deformed V–W algebra (15):

$$[L_m,[L_n,L_k]_{(q^{n-k},q^{k-n})}]_{(q^{2m-n-k},q^{n+k-2m})}+cycl.perms.=0.\tag{17}$$

Let us now turn our attention to the case of 3-algebra. The operator Nambu 3-bracket is defined by [31,46]

$$\begin{aligned}[A,B,C]&=A[B,C]+B[C,A]+C[A,B]\\&=[B,C]A+[C,A]B+[A,B]C,\end{aligned}\tag{18}$$

where $[A,B]=AB-BA$.

Summing the two right hand side lines in (18), we may rewrite the operator Nambu 3-bracket as [52]

$$\begin{aligned}[A,B,C]&=\frac{1}{2}(A[B,C]+[B,C]A+B[C,A]+[C,A]B+C[A,B]+[A,B]C)\\&=\frac{1}{2}(\{A,[B,C]\}+\{B,[C,A]\}+\{C,[A,B]\},\end{aligned}\tag{19}$$

where $\{A,B\}=AB+BA$.

For the q-deformed V–W algebra (15), we have already seen that the q-Jacobi identity (17) is guaranteed to hold. We note that the expression (18) is to be contrasted to the Jacobi identity obtained by taking the difference of the right hand side lines in (18) [52],

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0.$$

Based on (15) and (19), let us define the q-3-bracket as follows:

$$\begin{aligned}
[\![L_m,L_n,L_k]\!] &= \frac{1}{2}\{L_m,[L_n,L_k]_{(q^{n-k},q^{k-n})}\}_{(q^{2m-n-k},q^{n+k-2m})} \\
&\quad+\frac{1}{2}\{L_n,[L_k,L_m]_{(q^{k-m},q^{m-k})}\}_{(q^{2n-k-m},q^{k+m-2n})} \\
&\quad+\frac{1}{2}\{L_k,[L_m,L_n]_{(q^{m-n},q^{n-m})}\}_{(q^{2k-m-n},q^{m+n-2k})} \\
&=\frac{1}{2}\Big(q^{2m-n-k}L_m[L_n,L_k]_{(q^{n-k},q^{k-n})}+q^{n+k-2m}[L_n,L_k]_{(q^{n-k},q^{k-n})}L_m \\
&\quad+q^{2n-k-m}L_n[L_k,L_m]_{(q^{k-m},q^{m-k})}+q^{k+m-2n}[L_k,L_m]_{(q^{k-m},q^{m-k})}L_n \\
&\quad+q^{2k-m-n}L_k[L_m,L_n]_{(q^{m-n},q^{n-m})}+q^{m+n-2k}[L_m,L_n]_{(q^{m-n},q^{n-m})}L_k\Big). \qquad (20)
\end{aligned}$$

By means of (15), we may derive the following q-deformed 3-algebra from (20):

$$\begin{aligned}
[\![L_m,L_n,L_k]\!] &= \frac{1}{q-q^{-1}}([2m-2k]+[2k-2n]+[2n-2m])L_{m+n+k} \\
&=\left(q-q^{-1}\right)([m-n][m-k][n-k])L_{m+n+k} \\
&=-\frac{q^{-2(m+n+k)}}{\left(q-q^{-1}\right)^2}\det\begin{pmatrix}1&1&1\\ q^{2m}&q^{2n}&q^{2k}\\ q^{4m}&q^{4n}&q^{4k}\end{pmatrix}L_{m+n+k}. \qquad (21)
\end{aligned}$$

Performing lengthy but straightforward calculations, we find that (21) satisfies the sh-Jacobi identity (8), but the FI (3) does not hold. It is easy to verify that the skew-symmetry holds for this ternary algebra

$$[\![L_m,L_n,L_k]\!]=-[\![L_n,L_m,L_k]\!]=-[\![L_k,L_n,L_m]\!]. \qquad (22)$$

Therefore the q-deformed V–W 3-algebra (21) is indeed a sh-3-Lie algebra. In the limit $q\to 1$, (21) reduces to the null 3-algebra derived in [37],

$$[L_m,L_n,L_k]=0. \qquad (23)$$

The FI (3) is trivially satisfied for this null 3-algebra.

3.2. *q-Deformed V–W n-algebra*

Now encouraged by the possibility of constructing the nontrivial sh-3-Lie algebra (21), it would be interesting to study further and see whether one could construct the q-deformed V–W n-algebra with a genuine sh-n-Lie algebra structure. Let us now turn our attention to the q-deformed V–W n-algebra.

The n-bracket is defined by [37,46]

$$\begin{aligned}
[L_{i_1},L_{i_2},\cdots,L_{i_n}] &= \sum_{s=1}^{n}(-1)^{s+1}L_{i_s}[L_{i_1},L_{i_2},\cdots,\widehat{L_{i_s}},\cdots,L_{i_n}] \\
&=\sum_{s=1}^{n}(-1)^{s+n}[L_{i_1},L_{i_2},\cdots,\widehat{L_{i_s}},\cdots,L_{i_n}]L_{i_s}. \qquad (24)
\end{aligned}$$

Here we denote a notational convention used frequently in the rest of this paper. Namely for any arbitrary symbol Z, the hat symbol $\hat{Z}$ stands for the term that is omitted.

It is obvious that (24) can be expressed as

$$[L_{i_1}, L_{i_2}, \cdots, L_{i_n}] = \frac{1}{2}\sum_{s=1}^{n}(-1)^{s+1}(L_{i_s}[L_{i_1}, L_{i_2}, \cdots, \widehat{L_{i_s}}, \cdots, L_{i_n}] + (-1)^{n-1}[L_{i_1}, L_{i_2}, \cdots, \widehat{L_{i_s}}, \cdots, L_{i_n}]L_{i_s}). \tag{25}$$

When n is odd and even, we may rewrite the n-bracket (25) as

$$[L_{i_1}, L_{i_2}, \cdots, L_{i_n}] = \frac{1}{2}\sum_{s=1}^{n}(-1)^{s+1}\{L_{i_s}, [L_{i_1}, L_{i_2}, \cdots, \widehat{L_{i_s}}, \cdots, L_{i_n}]\} \tag{26}$$

and

$$[L_{i_1}, L_{i_2}, \cdots, L_{i_n}] = \frac{1}{2}\sum_{s=1}^{n}(-1)^{s+1}[L_{i_s}, [L_{i_1}, L_{i_2}, \cdots, \widehat{L_{i_s}}, \cdots, L_{i_n}]], \tag{27}$$

respectively.

For examples, when $n = 3$, (26) gives the 3-bracket (19). When $n = 4$, (27) leads to the 4-bracket [52]

$$[A, B, C, D] = \frac{1}{2}([A, [B, C, D]] - [B, [A, C, D]] + [C, [A, B, D]] - [D, [A, B, C]]). \tag{28}$$

Based on the 4-bracket (28) and the q-deformed V–W 3-algebra (21), let us define the following q–4-bracket:

$$\begin{aligned}[\![L_{i_1}, L_{i_2}, L_{i_3}, L_{i_4}]\!] &= \frac{1}{2}([L_{i_1}, [\![L_{i_2}, L_{i_3}, L_{i_4}]\!]]_{(q^{4i_1}, q^{-4i_1})} - [L_{i_2}, [\![L_{i_1}, L_{i_3}, L_{i_4}]\!]]_{(q^{4i_2}, q^{-4i_2})} \\ &\quad + [L_{i_3}, [\![L_{i_1}, L_{i_2}, L_{i_4}]\!]]_{(q^{4i_3}, q^{-4i_3})} - [L_{i_4}, [\![L_{i_1}, L_{i_2}, L_{i_3}]\!]]_{(q^{4i_4}, q^{-4i_4})}) \\ &= \frac{1}{2}\Big(q^{4i_1}L_{i_1}[\![L_{i_2}, L_{i_3}, L_{i_4}]\!] - q^{-4i_1}[\![L_{i_2}, L_{i_3}, L_{i_4}]\!]L_{i_1} \\ &\quad - q^{4i_2}L_{i_2}[\![L_{i_1}, L_{i_3}, L_{i_4}]\!] + q^{-4i_2}[\![L_{i_1}, L_{i_3}, L_{i_4}]\!]L_{i_2} \\ &\quad + q^{4i_3}L_{i_3}[\![L_{i_1}, L_{i_2}, L_{i_4}]\!] - q^{-4i_3}[\![L_{i_1}, L_{i_2}, L_{i_4}]\!]L_{i_3} \\ &\quad - q^{4i_4}L_{i_4}[\![L_{i_1}, L_{i_2}, L_{i_3}]\!] + q^{-4i_4}[\![L_{i_1}, L_{i_2}, L_{i_3}]\!]L_{i_4}\Big).\end{aligned} \tag{29}$$

Substituting (21) into (29), we may derive the following q-deformed 4-algebra:

$$\begin{aligned}&[\![L_{i_1}, L_{i_2}, L_{i_3}, L_{i_4}]\!] \\ &= \frac{q^{-4\Sigma_{l=1}^4 i_l} + q^{-2\Sigma_{l=1}^4 i_l}}{2\left(q - q^{-1}\right)^3}\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ q^{2i_1} & q^{2i_2} & q^{2i_3} & q^{2i_4} \\ q^{4i_1} & q^{4i_2} & q^{4i_3} & q^{4i_4} \\ q^{6i_1} & q^{6i_2} & q^{6i_3} & q^{6i_4} \end{pmatrix} L_{\Sigma_{l=1}^4 i_l} \\ &= \frac{1}{2}\left(q - q^{-1}\right)^3\left(q^{-\Sigma_{l=1}^4 i_l} + q^{\Sigma_{l=1}^4 i_l}\right)\prod_{1\le m<n\le 4}[i_n - i_m]L_{\Sigma_{l=1}^4 i_l}.\end{aligned} \tag{30}$$

As the case of (21), we find that (30) satisfies the sh-Jacobi identity (7), but the FI (2) does not hold. Since the skew-symmetry also holds, the q-deformed V–W 4-algebra (30) is a sh-4-Lie algebra.

For the n-bracket (25), let us define a q–n-bracket as follows:

$$\begin{aligned}
[\![L_{i_1}, L_{i_2}, \cdots, L_{i_n}]\!] = \frac{1}{2}\sum_{s=1}^{n}(-1)^{s+1}\Big(& q^{xi_s - y\Sigma_{j\neq s} i_j} L_{i_s}[\![L_{i_1}, L_{i_2}, \cdots, \widehat{L_{i_s}}, \cdots, L_{i_n}]\!] \\
&+ (-1)^{n-1} q^{y\Sigma_{j\neq s} i_j - xi_s}[\![L_{i_1}, L_{i_2}, \cdots, \widehat{L_{i_s}}, \cdots, L_{i_n}]\!] L_{i_s}\Big),
\end{aligned} \tag{31}$$

where $(x = n-1,\ y = 1)$ for odd $n \geq 3$ and $(x = n,\ y = 0)$ for even $n \geq 4$. In the limit $q \to 1$, q–n-bracket (31) reduces to the n-bracket (25). When $n = 3$ and 4, (31) gives the q–3-bracket (20) and the q–4-bracket (29), respectively.

Theorem 3. *When $n \geq 3$, the q-generators (13) satisfy the following closed algebraic structure relation:*

$$[\![L_{i_1}, L_{i_2}, \cdots, L_{i_n}]\!]
= \begin{cases}
sign\,(n)\left(q - q^{-1}\right)^{\frac{(n-1)(n-2)}{2}} \displaystyle\prod_{1\leq m<k\leq n} [i_k - i_m] L_{\Sigma_{l=1}^n i_l} \\
\quad = \frac{sign(n) q^{(-n+1)\Sigma_{l=1}^n i_l}}{(q-q^{-1})^{n-1}} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ q^{2i_1} & q^{2i_2} & \cdots & q^{2i_n} \\ \vdots & \vdots & \vdots & \vdots \\ q^{2(n-1)i_1} & q^{2(n-1)i_2} & \cdots & q^{2(n-1)i_n} \end{pmatrix} \\
\quad \times L_{\Sigma_{l=1}^n i_l},\ \textit{for } n \textit{ odd}, \\
\frac{1}{2} sign\,(n)\left(q - q^{-1}\right)^{\frac{(n-1)(n-2)}{2}} \left(q^{-\Sigma_{l=1}^n i_l} + q^{\Sigma_{l=1}^n i_l}\right) \displaystyle\prod_{1\leq m<k\leq n} [i_k - i_m] L_{\Sigma_{l=1}^n i_l} \\
\quad = \frac{sign(n)\left(q^{-n\Sigma_{l=1}^n i_l} + q^{(-n+2)\Sigma_{l=1}^n i_l}\right)}{2(q-q^{-1})^{n-1}} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ q^{2i_1} & q^{2i_2} & \cdots & q^{2i_n} \\ \vdots & \vdots & \vdots & \vdots \\ q^{2(n-1)i_1} & q^{2(n-1)i_2} & \cdots & q^{2(n-1)i_n} \end{pmatrix} \\
\quad \times L_{\Sigma_{l=1}^n i_l},\ \textit{for } n \textit{ even},
\end{cases} \tag{32}$$

where $sign\,(n)$ is the signature function, i.e., $sign\,(n) = \begin{cases} 1, & \textit{for } n \bmod 4 = 0, 1 \\ -1, & \textit{for } n \bmod 4 = 2, 3 \end{cases}$.

Proof. Let us confirm this by the mathematical induction for n. From (21) and (30), we know that (32) is satisfied for $n = 3$. Let us first suppose (32) is satisfied for odd $n = 2k - 1$. By means of (31) and (32), we obtain

$$\begin{aligned}
[\![L_{i_1}, L_{i_2}, \cdots, L_{i_{2k}}]\!] = \frac{1}{2}\sum_{s=1}^{2k}(-1)^{s+1}\Big(& q^{2ki_s} L_{i_s}[\![L_{i_1}, L_{i_2}, \cdots, \widehat{L_{i_s}}, \cdots, L_{i_{2k}}]\!] \\
&+ (-1)^{2k-1} q^{-2ki_s}[\![L_{i_1}, L_{i_2}, \cdots, \widehat{L_{i_s}}, \cdots, L_{i_{2k}}]\!] L_{i_s}\Big)
\end{aligned}$$

$$
=\frac{sign\,(2k-1)}{2\left(q-q^{-1}\right)^{2k-2}}\det\begin{pmatrix} q^{2ki_1+2\Sigma_{j\neq 1}i_j}-q^{(2-2k)i_1} & \cdots & q^{2ki_{2k}+2\Sigma_{j\neq 2k}i_j}-q^{(2-2k)i_{2k}} \\ q^{(-2k+2)i_1} & \cdots & q^{(-2k+2)i_{2k}} \\ \vdots & \vdots & \vdots \\ q^{(2k-4)i_1} & \cdots & q^{(2k-4)i_{2k}} \\ q^{(2k-2)i_1} & \cdots & q^{(2k-2)i_{2k}} \end{pmatrix}
$$

$$
\cdot\left[q^{-\Sigma_{j=1}^{2k}i_j+1}q^{2N}\left(a^{\dagger}\right)^{\Sigma_{j=1}^{2k}i_j+2}a^2-\frac{q^2}{\left(q-q^{-1}\right)}L_{\Sigma_{j=1}^{2k}i_j}\right]
$$

$$
+\frac{sign\,(2k-1)}{2\left(q-q^{-1}\right)^{2k-1}}\det\begin{pmatrix} q^{2ki_1}-q^{-2ki_1} & \cdots & q^{2ki_{2k}}-q^{-2ki_{2k}} \\ q^{(-2k+2)i_1} & \cdots & q^{(-2k+2)i_{2k}} \\ \vdots & \vdots & \vdots \\ q^{(2k-4)i_1} & \cdots & q^{(2k-4)i_{2k}} \\ q^{(2k-2)i_1} & \cdots & q^{(2k-2)i_{2k}} \end{pmatrix}L_{\Sigma_{l=1}^{2k}i_l}
$$

$$
=\frac{sign\,(2k)}{2\left(q-q^{-1}\right)^{2k-1}}\det\begin{pmatrix} q^{-2ki_1}-q^{2ki_1} & \cdots & q^{-2ki_{2k}}-q^{2ki_{2k}} \\ q^{(-2k+2)i_1} & \cdots & q^{(-2k+2)i_{2k}} \\ \vdots & \vdots & \vdots \\ q^{(2k-4)i_1} & \cdots & q^{(2k-4)i_{2k}} \\ q^{(2k-2)i_1} & \cdots & q^{(2k-2)i_{2k}} \end{pmatrix}L_{\Sigma_{l=1}^{2k}i_l}
$$

$$
=\frac{1}{2}sign\,(2k)\left(q-q^{-1}\right)^{\frac{(2k-1)(2k-2)}{2}}\left(q^{-\Sigma_{l=1}^{2k}i_l}+q^{\Sigma_{l=1}^{2k}i_l}\right)\prod_{1\leq m<j\leq 2k}[i_j-i_m]L_{\Sigma_{l=1}^{2k}i_l}, \tag{33}
$$

which shows that (32) is satisfied for $n=2k$.

By the similar method, we can show that when (32) is satisfied for $n=2k$, then it is also true for odd $(n+1)$. Now we complete the proof. □

For the q-3-bracket (21), we already recognize that it satisfies the sh-Jacobi identity (8), but the FI (3) does not hold. Let us consider the case of the q–n-bracket (32). Taking $Y_i=L_{-i-1}, i=1,2,\cdots,n-2$, $Y_{n-1}=L_{\frac{(n-1)n}{2}}$ and $X_j=L_{j-1}$, $j=1,2\cdots,n$ in (2), straightforward calculation shows that the left-hand side of (2) equals zero, but its right-hand side does not. It indicates that the FI (2) does not hold for (32). Therefore the q–n-bracket relation (32) is not an n-Lie algebra. In spite of this negative result, it is instructive to pursue the analysis of the q–n-bracket (32).

Proposition 4. *When $n\geq 3$, the q–n-bracket relation (32) is a sh-n-Lie algebra.*

Proof. Since the structure constants are determined by the determinant, the n-bracket (32) is anticommutative.

Note that the sh-Jacobi identity is equivalent to (7), so our next goal is to show that (7) is satisfied. Let us take $X_j=L_{m_j}$, $j=1,\cdots,2n-1$ in (7), then (7) becomes

$$\epsilon^{i_1\cdots i_{2n-1}}_{m_1\cdots m_{2n-1}}[\![[\![L_{i_1},\cdots,L_{i_n}]\!],L_{i_{n+1}},\cdots,L_{i_{2n-1}}]\!]=0. \tag{34}$$

Let us first prove that the n-bracket (32) with n odd satisfies (34). In terms of the Lévi-Cività symbol (6), we can rewrite n-bracket (32) as

$$[\![L_{i_1},\cdots,L_{i_n}]\!]=\frac{sign\,(n)}{\left(q-q^{-1}\right)^{n-1}}\epsilon^{j_1\cdots j_n}_{i_1\cdots i_n}q^{(-n+1)j_1+(-n+3)j_2+\cdots+(n-1)j_n}L_{\Sigma_{l=1}^{n}i_l}. \tag{35}$$

By using the expression (35) to calculate left-hand side of (34), we obtain

$$\begin{aligned}&\epsilon^{i_1\cdots i_{2n-1}}_{m_1\cdots m_{2n-1}}[\![[\![L_{i_1},\cdots,L_{i_n}]\!],L_{i_{n+1}},\cdots,L_{i_{2n-1}}]\!]\\&=\frac{n!\,(n-1)!}{\left(q-q^{-1}\right)^{2n-2}}\sum_{k=2}^{n+1}(-1)^k\,\epsilon^{j_1\cdots j_{2n-1}}_{m_1\cdots m_{2n-1}}q^{\rho}L_{\Sigma_{l=1}^{2n-1}m_l},\end{aligned} \tag{36}$$

where the power ρ of q is given by

$$\begin{aligned}\rho=&\,2\,(-n+k-1)\,j_1+2\,(-n+k)\,j_2+\cdots+2\,(k-2)\,j_n\\&+(-n+1)\,j_{n+1}+(-n+3)\,j_{n+3}+\cdots+(-n+2k-5)\,j_{n+k-2}\\&+(-n+2k-1)\,j_{n+k-1}+\cdots+(n-1)\,j_{2n-1},\end{aligned} \tag{37}$$

and the following formula is useful in simplifying expression:

$$\epsilon^{i_1\cdots i_n}_{m_1\cdots m_n}\epsilon^{j_1\cdots j_k}_{i_1\cdots i_k}=k!\epsilon^{j_1\cdots j_k i_{k+1}\cdots i_n}_{m_1\cdots m_n}. \tag{38}$$

From the expression (37) of ρ, we observe that the coefficients of two different j_μ should be equal. Since $\epsilon^{j_1\cdots j_{2n-1}}_{m_1\cdots m_{2n-1}}$ is completely antisymmetric, it is easy to see that (36) equals zero. It indicates that the sh-Jacobi identity (4) is satisfied by (32) for the n odd case.

When n is even, we can rewrite n-bracket (32) as

$$\begin{aligned}[\![L_{i_1},\cdots,L_{i_n}]\!]=&\frac{sign\,(n)}{2\left(q-q^{-1}\right)^{n-1}}(\epsilon^{j_1\cdots j_n}_{i_1\cdots i_n}q^{-nj_1+(-n+2)j_2+\cdots+(n-2)j_n}\\&-(-1)^{n-1}\,\epsilon^{j_1\cdots j_n}_{i_1\cdots i_n}q^{(-n+2)j_1-nj_2+\cdots+(n-2)j_{n-1}+nj_n})L_{\Sigma_{l=1}^{n}i_l}.\end{aligned} \tag{39}$$

By using the expression (39) to calculate left-hand side of (34), we obtain

$$\begin{aligned}&\epsilon^{i_1\cdots i_{2n-1}}_{m_1\cdots m_{2n-1}}[\![[\![L_{i_1},\cdots,L_{i_n}]\!],L_{i_{n+1}},\cdots,L_{i_{2n-1}}]\!]\\&=\frac{n!\,(n-1)!}{4\left(q-q^{-1}\right)^{2n-2}}\sum_{k=1}^{n}(-1)^{k-1}\,\epsilon^{j_1\cdots j_{2n-1}}_{m_1\cdots m_{2n-1}}\left(q^{\rho_1}+q^{\rho_2}+q^{\rho_3}+q^{\rho_4}\right)L_{\Sigma_{l=1}^{2n-1}m_l},\end{aligned} \tag{40}$$

where the powers ρ_i, $i=1,2,3,4$ of q are given by

$$\begin{aligned}\rho_1=&\,(-2n+2k-2)\,j_1+(-2n+2k)\,j_2+\cdots+(2k-4)\,j_n-nj_{n+1}+\cdots\\&+(-n+2k-4)\,j_{n+k-1}+(-n+2k)\,j_{n+k}+\cdots+(n-2)\,j_{2n-1},\\\rho_2=&\,(-2n+2k+2)\,j_1+(-2n+2k+4)\,j_2+\cdots+2kj_n+(-n+2)\,j_{n+1}+\cdots\\&+(-n+2k-2)\,j_{n+k-1}+(-n+2k+2)\,j_{n+k}+\cdots+nj_{2n-1},\\\rho_3=&\,(-2n+2k)\,j_1+(-2n+2k+2)\,j_2+\cdots+(2k-2)\,j_n+(-n+2)\,j_{n+1}+\cdots\\&+(-n+2k-2)\,j_{n+k-1}+(-n+2k+2)\,j_{n+k}+\cdots+nj_{2n-1},\\\rho_4=&\,(-2n+2k)\,j_1+(-2n+2k+2)\,j_2+\cdots+(2k-2)\,j_n-nj_{n+1}+\cdots\\&+(-n+2k-4)\,j_{n+k-1}+(-n+2k)\,j_{n+k}+\cdots+(n-2)\,j_{2n-1}.\end{aligned} \tag{41}$$

From the expressions (41) of ρ_i, we observe that the coefficients of two different j_μ should be equal. Due to the anti-symmetry of $\epsilon_{m_1\cdots m_{2n-1}}^{i_1\cdots i_{2n-1}}$, it is easy to see that (40) equals zero. Thus the sh-Jacobi identity (4) is also satisfied by (32) for the n even case.

Based on the above analysis, it is clear that the q-deformed V–W n-algebra with $n \geq 3$ is indeed a sh-n-Lie algebra. □

We have constructed the q-deformed V–W n-algebra (32). When $n = 3$ and 4, the corresponding q-deformed n-algebra are (21) and (30), respectively. Let us list next few q-deformed V–W n-algebras as follows:

- $[\![L_{i_1}, L_{i_2}, L_{i_3}, L_{i_4}, L_{i_5}]\!]$

$$
\begin{aligned}
&= \left(q-q^{-1}\right)^{-4} q^{-4\sum_{k=1}^{5} i_k} \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ q^{2i_1} & q^{2i_2} & q^{2i_3} & q^{2i_4} & q^{2i_5} \\ q^{4i_1} & q^{4i_2} & q^{4i_3} & q^{4i_4} & q^{4i_5} \\ q^{6i_1} & q^{6i_2} & q^{6i_3} & q^{6i_4} & q^{6i_5} \\ q^{8i_1} & q^{8i_2} & q^{8i_3} & q^{8i_4} & q^{8i_5} \end{pmatrix} L_{\sum_{k=1}^{5} i_k} \\
&= \left(q-q^{-1}\right)^{6} \prod_{1\leq m<n\leq 5} [i_n - i_m] L_{\sum_{k=1}^{5} i_k}.
\end{aligned}
\tag{42}
$$

- $[\![L_{i_1}, L_{i_2}, L_{i_3}, L_{i_4}, L_{i_5}, L_{i_6}]\!]$

$$
\begin{aligned}
&= -\frac{\left(q^{-6\sum_{k=1}^{6} i_k} + q^{-4\sum_{k=1}^{6} i_k}\right)}{2\left(q-q^{-1}\right)^{5}} \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ q^{2i_1} & q^{2i_2} & q^{2i_3} & q^{2i_4} & q^{2i_5} & q^{2i_6} \\ q^{4i_1} & q^{4i_2} & q^{4i_3} & q^{4i_4} & q^{4i_5} & q^{4i_6} \\ q^{6i_1} & q^{6i_2} & q^{6i_3} & q^{6i_4} & q^{6i_5} & q^{6i_6} \\ q^{8i_1} & q^{8i_2} & q^{8i_3} & q^{8i_4} & q^{8i_5} & q^{8i_6} \\ q^{10i_1} & q^{10i_2} & q^{10i_3} & q^{10i_4} & q^{10i_5} & q^{10i_6} \end{pmatrix} \\
&\quad \times L_{\sum_{k=1}^{6} i_k} \\
&= -\frac{1}{2}\left(q-q^{-1}\right)^{10} \left(q^{-\sum_{k=1}^{6} i_k} + q^{\sum_{k=1}^{6} i_k}\right) \prod_{1\leq m<n\leq 6} [i_n - i_m] L_{\sum_{k=1}^{6} i_k}.
\end{aligned}
\tag{43}
$$

It should be noted that the structure constant of the q-deformed infinite-dimensional n-algebra (32) is determined by the Vandermonde determinant. In the limit $q \to 1$, it is easy to see that (32) reduces to the null n-algebra.

4. q-deformed $SDiff(T^2)$ n-algebra

4.1. Sine 3-algebra and q-deformed $SDiff(T^2)$ 3-algebra

The q-differential calculus on the q-plane $\mathbf{C}_q[x, y]$ have been well investigated [53]. For the q-plane $\mathbf{C}_q[x, y]$, each of its elements is a finite linear combination of the monomes $y^n x^m$, satisfying

$$
x^m y^n = q^{nm} y^n x^m, \quad m, n \in \mathbf{N}. \tag{44}
$$

The Gauss derivatives on $\mathbf{C}_q[x, y]$ can be extended to be formal pseud-differential operators D_x, D_y which can be defined on the set $\mathbf{C}_q[[x, y]]$ and satisfy

$$D_x^n D_y^m = q^{nm} D_y^m D_x^n, \quad m, n \in \mathbf{Z}, \tag{45}$$

where $\mathbf{C}_q[[x, y]]$ is a set of all Laurent series in x, y such that (44) is valid for $n, m \in \mathbf{Z}$.

In terms of the pseud-differential operators D_x and D_y, Kinani et al. [26] introduced the following generators:

$$T_n = q^{n_1 \cdot n_2/2} \cdot D_y^{n_1} D_x^{n_2}, \tag{46}$$

where $n = (n_1, n_2) \in \mathbf{Z}^2$. In the rest of this paper, we denote the subscript l on T_l being a two-dimensional vector with integer components.

By means of (45), it is easy to verify that the generators T_n (46) satisfy

$$T_n T_m = q^{\frac{1}{2} m \wedge n} T_{n+m}, \tag{47}$$

where $m \wedge n = m_1 n_2 - m_2 n_1$.

Thus we have the algebra

$$[T_m, T_n] = T_m T_n - T_n T_m = (q^{\frac{1}{2} n \wedge m} - q^{\frac{1}{2} m \wedge n}) T_{m+n}. \tag{48}$$

When $q = exp(-2\pi \mathbf{i}\alpha)$, (48) becomes the sine algebra [27]

$$[T_m, T_n] = 2\mathbf{i} \sin(\pi\alpha m \wedge n) T_{m+n}, \tag{49}$$

where $\mathbf{i} = \sqrt{-\mathbf{1}}$ and α is an arbitrary constant.

Taking the rescaled generators $\bar{T}_m = -\frac{\mathbf{i}}{2\pi\alpha} T_m$, we note that in the limit $\alpha \to 0$, (49) leads to the $SDiff(T^2)$ algebra [24,25]

$$[\bar{T}_m, \bar{T}_n] = (m \wedge n) \bar{T}_{m+n}. \tag{50}$$

The q-deformation of the $SDiff(T^2)$ algebra (50) is given by [26]

$$\begin{aligned} [\bar{T}_m, \bar{T}_n]_{(q^{\frac{3}{2} m \wedge n}, q^{\frac{3}{2} n \wedge m})} &= q^{\frac{3}{2} m \wedge n} \bar{T}_m \bar{T}_n - q^{\frac{3}{2} n \wedge m} \bar{T}_n \bar{T}_m \\ &= [m \wedge n] \bar{T}_{m+n}, \end{aligned} \tag{51}$$

where $\bar{T}_m = \frac{1}{q - q^{-1}} T_m$.

Let us turn to the case of the 3-algebra. Substituting the generators (46) into the operator Nambu 3-bracket (19) and using (47) and (48), by direct calculation, we may derive the following 3-algebra:

$$\begin{aligned} [T_m, T_n, T_k] = (&-q^{\frac{1}{2}(m \wedge n - n \wedge k + k \wedge m)} + q^{-\frac{1}{2}(m \wedge n - n \wedge k + k \wedge m)} \\ &- q^{\frac{1}{2}(m \wedge n + n \wedge k - k \wedge m)} + q^{-\frac{1}{2}(m \wedge n + n \wedge k - k \wedge m)} \\ &- q^{\frac{1}{2}(-m \wedge n + n \wedge k + k \wedge m)} + q^{-\frac{1}{2}(-m \wedge n + n \wedge k + k \wedge m)}) T_{m+n+k}. \end{aligned} \tag{52}$$

Performing straightforward calculations, we find that the 3-algebra with the general q parameter (52) does not satisfy the FI (3) and the sh-Jacobi identity (8).

An interesting case is for the special value of q parameter. When particularized to the $q = e^{-\pi \mathbf{i}}$ case, we may rewrite (52) as

$$[T_m, T_n, T_k] = 2\mathbf{i}(\sin(\frac{\pi}{2}(m\wedge n - n\wedge k + k\wedge m)) + \sin(\frac{\pi}{2}(m\wedge n + n\wedge k - k\wedge m)) \\ + \sin(\frac{\pi}{2}(-m\wedge n + n\wedge k + k\wedge m)))T_{m+n+k}. \tag{53}$$

Not as the case of (52), an intriguing property of (53) is that it does satisfy the FI (3). Since the skew symmetry also holds, the sine 3-algebra (53) is indeed a Fillipov 3-algebra.

Let us take the rescaled generators $\bar{T}_n = \frac{1}{(q-q^{-1})^{1/2}} T_n$ and define the q-3-bracket

$$\begin{aligned} &[\![\bar{T}_m, \bar{T}_n, \bar{T}_k]\!] \\ &= \frac{1}{2}(q^{\frac{3}{2}m\wedge(n+k)}\bar{T}_m[\bar{T}_n, \bar{T}_k]_{(q^{\frac{3}{2}n\wedge k}, q^{\frac{3}{2}k\wedge n})} + q^{\frac{3}{2}(n+k)\wedge m}[\bar{T}_n, \bar{T}_k]_{(q^{\frac{3}{2}n\wedge k}, q^{\frac{3}{2}k\wedge n})}\bar{T}_m \\ &+ q^{\frac{3}{2}n\wedge(k+m)}\bar{T}_n[\bar{T}_k, \bar{T}_m]_{(q^{\frac{3}{2}k\wedge m}, q^{\frac{3}{2}m\wedge k})} + q^{\frac{3}{2}(k+m)\wedge n}[\bar{T}_k, \bar{T}_m]_{(q^{\frac{3}{2}k\wedge m}, q^{\frac{3}{2}m\wedge k})}\bar{T}_n \\ &+ q^{\frac{3}{2}k\wedge(m+n)}\bar{T}_k[\bar{T}_m, \bar{T}_n]_{(q^{\frac{3}{2}m\wedge n}, q^{\frac{3}{2}n\wedge m})} + q^{\frac{3}{2}(m+n)\wedge k}[\bar{T}_m, \bar{T}_n]_{(q^{\frac{3}{2}m\wedge n}, q^{\frac{3}{2}n\wedge m})}\bar{T}_k). \end{aligned} \tag{54}$$

Then we have the q-deformed $SDiff(T^2)$ 3-algebra

$$\begin{aligned} [\![\bar{T}_m, \bar{T}_n, \bar{T}_k]\!] &= ([m\wedge n - n\wedge k + k\wedge m] + [m\wedge n + n\wedge k - k\wedge m] \\ &+ [-m\wedge n + n\wedge k + k\wedge m])\bar{T}_{m+n+k} \\ &= \left(\left[\det\begin{pmatrix} m_1 & n_1 & k_1 \\ m_2 & n_2 & k_2 \\ -1 & 1 & 1 \end{pmatrix}\right] + \left[\det\begin{pmatrix} m_1 & n_1 & k_1 \\ m_2 & n_2 & k_2 \\ 1 & -1 & 1 \end{pmatrix}\right]\right. \\ &\left. + \left[\det\begin{pmatrix} m_1 & n_1 & k_1 \\ m_2 & n_2 & k_2 \\ 1 & 1 & -1 \end{pmatrix}\right]\right)\bar{T}_{m+n+k}. \end{aligned} \tag{55}$$

As the case of (52), the infinite-dimensional q-deformed 3-algebra (55) does not satisfy the FI (3) and the sh-Jacobi identity (8).

In the limit $q \to 1$, (55) reduces to the $SDiff(T^2)$ 3-algebra [41]

$$[\bar{T}_m, \bar{T}_n, \bar{T}_k] = (m\wedge n + n\wedge k + k\wedge m)\,\bar{T}_{m+n+k}, \tag{56}$$

which satisfies the FI (3). Taking $\bar{T}_k = \bar{T}_0$ in (56), (56) can be regarded as the parametrized bracket relation $[\bar{T}_m, \bar{T}_n]_{\bar{T}_0}$. This parametrized bracket relation gives rise to the $SDiff(T^2)$ algebra (50).

4.2. (co)Sine n-algebra

For the generators T_n (46), we note that they are the associative operators with the product (47). According to the definition of the n-bracket (25), we get the following result.

Theorem 5. *The generators (46) satisfy the following closed algebraic structure relation:*

$$[T_{i_1}, \cdots, T_{i_n}] = \epsilon^{j_1\cdots j_n}_{i_1\cdots i_n} q^{\frac{1}{2}\Sigma_{k>s} j_k\wedge j_s} T_{\Sigma_{l=1}^n i_l}. \tag{57}$$

Proof. The n-bracket (57) will follow from (47) if we can show that

$$[T_{i_1}, \cdots, T_{i_n}] = \epsilon^{j_1 \cdots j_n}_{i_1 \cdots i_n} T_{j_1} T_{j_1} \cdots T_{j_n}. \tag{58}$$

First, let us prove (58) by the mathematical induction for n. By (48), it is obvious that (58) holds for $n = 2$. We suppose (58) is satisfied for n-bracket. Note that the generators T_i are the associative operators under the product (47), we obtain

$$\begin{aligned}
[T_{i_1}, \cdots, T_{i_{n+1}}] &= \sum_{l=1}^{n+1} (-1)^{l-1} T_{i_l} [T_{i_1}, \cdots, \hat{T}_{i_l}, \cdots, T_{i_{n+1}}] \\
&= \sum_{l=1}^{n+1} (-1)^{l-1} \epsilon^{j_2 \cdots j_{n+1}}_{i_1 \cdots \hat{i}_l \cdots i_{n+1}} T_{i_l} \left(T_{j_2} \cdots T_{j_{n+1}} \right) \\
&= \sum_{l=1}^{n+1} (-1)^{l-1} \epsilon^{j_2 \cdots j_{n+1}}_{i_1 \cdots \hat{i}_l \cdots i_{n+1}} \left(\delta^{j_1}_{i_l} T_{j_1} \right) \left(T_{j_2} \cdots T_{j_{n+1}} \right) \\
&= \left(\sum_{l=1}^{n+1} (-1)^{l-1} \epsilon^{j_2 \cdots j_{n+1}}_{i_1 \cdots \hat{i}_l \cdots i_{n+1}} \delta^{j_1}_{i_l} \right) T_{j_1} T_{j_2} \cdots T_{j_{n+1}} \\
&= \epsilon^{j_1 \cdots j_{n+1}}_{i_1 \cdots i_{n+1}} T_{j_1} T_{j_2} \cdots T_{j_{n+1}},
\end{aligned} \tag{59}$$

which shows that (58) is also satisfied for $(n + 1)$-bracket.

Substituting (47) into (58), we immediately obtain (57). The proof is completed. □

When $n = 3$ in (57), we have known that the corresponding 3-algebra (52) does not satisfy the FI (3) and the sh-Jacobi identity (8). Let us now analyze the property of the n-algebra (57) for $n \geq 4$. In Ref. [46], it has been proved that the following n-bracket with arbitrary associative operators

$$\left[X_{i_1}, \cdots, X_{i_n} \right] := \epsilon^{j_1 \cdots j_n}_{i_1 \cdots i_n} X_{j_1} X_{j_1} \cdots X_{j_n}, \tag{60}$$

satisfies the generalized Jacobi identity (5) when n is even. Note that the generalized Jacobi identity and the sh-Jacobi identity are equivalent. Due to the associative operators T_n (46) and the expression (58), we can confirm that the n-algebra (57) is a sh-n-Lie algebra for the n even case.

We finally remark that when n is odd, the n-algebra (57) is not a sh-n-Lie algebra. When n odd, the left side of the generalized Jacobi identity is $(n!)^2$ times the larger bracket $[T_{i_1}, \cdots, T_{i_{2n-1}}]$ for odd n [46]. By (58), we get that the coefficient of $T_{\Sigma_{m=1}^{2n-1} k_m}$ in the left side of sh-Jacobi identity (2) is

$$n \epsilon^{j_1 \cdots j_{2n-1}}_{k_1 \cdots k_{2n-1}} q^{\frac{1}{2} \Sigma_{1 \leq s < k \leq 2n-1} j_k \wedge j_s}. \tag{61}$$

Let us choose $k_l = (l, 1)$, we observe that the coefficient of the monomial with the maximal power is

$$n \epsilon^{k_1 \cdots k_{2n-1}}_{k_1 \cdots k_{2n-1}} = n \tag{62}$$

but not zero. Therefore the sh-Jacobi identity does not hold for this case. Moreover by the similar way, we can also prove that the n-algebra (57) with any n does not satisfy the FI (2).

We have presented the n-algebra (57) with the general q parameter. Let us now focus on the case of the special q parameter. Taking $q = \exp(-2\pi\mathbf{i}\alpha)$, $\alpha \in R$, we can express the n-bracket (57) as

$$\begin{aligned}
&[T_{i_1}, \cdots, T_{i_n}] \\
&= \frac{1}{2}\left(\epsilon^{j_1\cdots j_n}_{i_1\cdots i_n} \exp(\pi\mathbf{i}\alpha\Sigma_{k<s} j_k \wedge j_s) + \epsilon^{j_n\cdots j_1}_{i_1\cdots i_n} \exp(\pi\mathbf{i}\alpha\Sigma_{k>s} j_k \wedge j_s)\right) T_{\Sigma^n_{l=1} i_l} \\
&= \frac{1}{2}\epsilon^{j_1\cdots j_n}_{i_1\cdots i_n} \cos(\pi\alpha\Sigma_{k<s} j_k \wedge j_s)\left(1 + (-1)^{\frac{n(n-1)}{2}}\right) T_{\Sigma^n_{l=1} i_l} \\
&\quad + \frac{\mathbf{i}}{2}\epsilon^{j_1\cdots j_n}_{i_1\cdots i_n} \sin(\pi\alpha\Sigma_{k<s} j_k \wedge j_s)\left(1 - (-1)^{\frac{n(n-1)}{2}}\right) T_{\Sigma^n_{l=1} i_l}. \qquad (63)
\end{aligned}$$

When n is even, (63) is a sh-n-Lie algebra. However not as the case of (57) with n odd, we find that when $n = 3$, for the special value $\alpha = \frac{1}{2}$, (63) gives a Fillipov 3-algebra (53).

Let us consider the $n = 5$ case. In this case, (63) gives

$$[T_{i_1}, \cdots, T_{i_5}] = \epsilon^{j_1\cdots j_5}_{i_1\cdots i_5} \cos(\pi\alpha\Sigma_{k<l} j_k \wedge j_l)\, T_{i_1+\cdots+i_5}. \qquad (64)$$

Taking $\alpha = \frac{1}{3}$ in (64), it is interesting to note that the sh-Jacobi identity (4) holds, but the FI (2) fails in this example. Thus for this special α, the cosine 5-algebra (64) gives a sh-5-Lie algebra.

4.3. q-Deformed $SDiff(T^2)$ n-algebra

To construct the q-deformed $SDiff(T^2)$ n-algebra, let us define a q–n-bracket as follows:

$$\begin{aligned}
[\![\bar{T}_{i_1}, \bar{T}_{i_2}, \cdots, \bar{T}_{i_n}]\!] &= \frac{1}{2}\sum_{s=1}^{n} (-1)^{s+1} \left(q^{\frac{3}{2} i_s \wedge \Sigma_{j\neq s} i_j}\, \bar{T}_{i_s} [\![\bar{T}_{i_1}, \bar{T}_{i_2}, \cdots, \hat{\bar{T}}_{i_s}, \cdots, \bar{T}_{i_n}]\!]\right. \\
&\quad \left. + (-1)^{n-1} q^{\frac{3}{2}\Sigma_{j\neq s} i_j \wedge i_s} [\![\bar{T}_{i_1}, \bar{T}_{i_2}, \cdots, \hat{\bar{T}}_{i_s}, \cdots, \bar{T}_{i_n}]\!]\bar{T}_{i_s}\right). \qquad (65)
\end{aligned}$$

According to the definition of the n-bracket (65), in similarity with the case of (57), we may derive the following q-deformed $SDiff(T^2)$ n-algebra:

$$[\![\bar{T}_{i_1}, \cdots, \bar{T}_{i_n}]\!] = \frac{\epsilon^{j_1\cdots j_n}_{i_1\cdots i_n} q^{\Sigma_{k<s} j_k \wedge j_s}}{q - q^{-1}} \bar{T}_{\Sigma^n_{l=1} i_l}. \qquad (66)$$

When $n = 3$, (66) gives the q-deformed $SDiff(T^2)$ 3-algebra (55). In the limit $q \to 1$, it is not hard to verify that (66) reduces to the null n-algebra for $n \geq 4$,

$$[\bar{T}_{i_1}, \cdots, \bar{T}_{i_n}] = 0. \qquad (67)$$

Proposition 6. *When n is even, the n-algebra (66) is a sh-n-Lie algebra.*

Proof. Due to the skew-symmetry of $\epsilon^{j_1\cdots j_n}_{i_1\cdots i_n}$ in (66), it is obvious that the skew-symmetry holds for the n-bracket (66). We are left to show that the sh-Jacobi identity is satisfied.

Taking $X_j = T_{k_j}$, $j = 1, \cdots, 2n-1$ in (7) and substituting (66) into the left-hand side of (7), we obtain

$$\frac{\epsilon^{i_1\cdots i_{2n-1}}_{k_1\cdots k_{2n-1}}\epsilon^{j_1\cdots j_n}_{i_1\cdots i_n}q^{\Sigma_{1\le k<l\le n}j_k\wedge j_l}}{q-q^{-1}}[\![\bar{T}_{\Sigma_{k=1}^n j_k},\bar{T}_{i_{n+1}},\cdots,\bar{T}_{i_{2n-1}}]\!]$$
$$=\frac{1}{\left(q-q^{-1}\right)^2}\sum_{s=0}^{n-1}(-1)^s\,\epsilon^{i_1\cdots i_{2n-1}}_{k_1\cdots k_{2n-1}}\epsilon^{j_1\cdots j_n}_{i_1\cdots i_n}\epsilon^{j_{n+1}\cdots j_{2n-1}}_{i_{n+1}\cdots i_{2n-1}}q^{\Sigma_{1\le k<l\le n}(j_k\wedge j_l)}q^{\Sigma_{n+1\le k<l\le 2n-1}(j_k\wedge j_l)}$$
$$\cdot\, q^{\Sigma_{n+1\le l\le n+s}j_l\wedge\left(\Sigma_{k=1}^n j_k\right)}q^{\Sigma_{n+s+1\le l\le 2n-1}\left(\Sigma_{k=1}^n j_k\right)\wedge j_l}\bar{T}_{\Sigma_{k=1}^{2n-1}i_k}$$
$$=\frac{1}{\left(q-q^{-1}\right)^2}\sum_{s=0}^{n-1}(-1)^s\,n!\,(n-1)!\epsilon^{j_1\cdots j_{2n-1}}_{k_1\cdots k_{2n-1}}q^{\Sigma_{1\le k<l\le n}(j_k\wedge j_l)}q^{\Sigma_{n+1\le k<l\le 2n-1}(j_k\wedge j_l)}$$
$$\cdot\, q^{\Sigma_{n+1\le l\le n+s}j_l\wedge\left(\Sigma_{k=1}^n j_k\right)}q^{\Sigma_{n+s+1\le l\le 2n-1}\left(\Sigma_{k=1}^n j_k\right)\wedge j_l}\bar{T}_{\Sigma_{k=1}^{2n-1}i_k}. \tag{68}$$

Let us change the indices $(j_1,\cdots,j_{2n-1})$ to be $(j_{s+1},\cdots,j_{s+n},j_1,\cdots j_s,j_{n+s+1},\cdots,j_{2n-1})$, thus the right-hand side of (68) can be rewritten as

$$\frac{1}{\left(q-q^{-1}\right)^2}\sum_{s=0}^{n-1}(-1)^s\,n!\,(n-1)!\epsilon^{j_{s+1}\cdots j_{s+n}j_1\cdots j_s j_{n+s+1}\cdots j_{2n-1}}_{k_1\cdots k_{2n-1}}q^{\Sigma_{1\le k<l\le 2n-1}j_k\wedge j_l}T_{\Sigma_{k=1}^{2n-1}i_k}$$
$$=\frac{1}{\left(q-q^{-1}\right)^2}n!\,(n-1)!\epsilon^{j_1\cdots j_{2n-1}}_{k_1\cdots k_{2n-1}}q^{\Sigma_{1\le k<l\le 2n-1}j_k\wedge j_l}\sum_{s=0}^{n-1}(-1)^{s(n+1)}\,T_{\Sigma_{k=1}^{2n-1}i_k}. \tag{69}$$

When n is even, we have $\sum_{s=0}^{n-1}(-1)^{s(n+1)}=0$. It indicates that (68) equals zero. Therefore the sh-Jacobi identity holds. The proof is completed. □

As the case of (57), it is not difficult to prove that the n-algebra (66) with n odd is not a sh-n-Lie algebra. Moreover it is not a n-Lie algebra for any n case.

5. A physical realization of the (co)sine n-algebra

Recently one has made an attempt of studying the application of the quantum Nambu n-algebra in condensed matter physics [54–57]. The quantum Hall effect appears in two-dimensional systems of electrons in the presence of a strong perpendicular uniform magnetic field. The higher dimensional quantum Hall effect can be considered as a realization of the A-class topological insulator with Landau levels. More recently Estienne et al. [54] investigated the D-dimensional topological insulators and presented the corresponding D-algebra structure. It was found that there are the close relations between quantum Nambu bracket in even dimensions and A-class topological insulator. Moreover Hasebe [57] performed a detail study of the higher dimensional quantum Hall effects and the A-class topological insulators and discussed the physical realization of the quantum Nambu geometry in the context of the A-class topological insulator.

For the sine algebra, there has been considerable interest. It was found that the problem of Bloch electrons in a constant uniform magnetic field admits the sine algebra as the symmetry algebra [29]. Moreover the realization of the supersymmetric extension of the sine algebra was also given in terms of the symmetry operators of a spin-$\frac{1}{2}$ Bloch electron.

To give the realization of the (co)sine n-algebra (63), let us now consider a spinless non-relativistic electron moving on a Bravais lattice in the xy-plane under the influence of a constant uniform magnetic field $\mathbf{B}=B\mathbf{e}_z$. The Hamiltonian is [29]

$$H = \frac{1}{2\mu}(\pi_x^2 + \pi_y^2) + V(x, y), \tag{70}$$

where the substrate potential $V(x, y)$ is periodic in x and y, i.e., $V(x + a_1, y) = V(x, y + a_2) = V(x, y)$, with a_1 and a_2 being the unit lattice spacing, the kinetic momentum operators are define by

$$\pi_x = p_x - \frac{e}{c}A_x, \quad \pi_y = p_y - \frac{e}{c}A_y, \tag{71}$$

in which $p_x = -\mathbf{i}\hbar\frac{\partial}{\partial x}$ and $p_y = -\mathbf{i}\hbar\frac{\partial}{\partial y}$ are the canonical momentum operators, $\mathbf{A} = (A_x, A_y)$ is the vector potential and can be given by

$$A_x = -\frac{B}{2}y + \frac{\partial\Lambda}{\partial x}, \quad A_y = -\frac{B}{2}x + \frac{\partial\Lambda}{\partial y}. \tag{72}$$

Here Λ is an arbitrary scalar function determining the gauge. For simplicity, we choose $\Lambda = \frac{1}{2}Bxy$.

We take an arbitrary Bravais lattice vector as follows:

$$\boldsymbol{R}_m = m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2, \tag{73}$$

where $m = (m_1, m_2) \in \mathbf{Z}^2$, $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are two given vectors in the directions $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$, respectively.

Let $\boldsymbol{\beta} = (\beta_1, \beta_2)$ be a vector which is classically connected with the cyclotron center given by

$$\beta_1 = \pi_x - \mu\omega y, \quad \beta_2 = \pi_y + \mu\omega x, \tag{74}$$

where $\omega = eB/\mu c$ is the Larmor frequency.

Let us take the following magnetic translation operators [29]:

$$T_m = \exp(\sqrt{2\pi}\,\mathbf{i}\boldsymbol{R}_m \cdot \boldsymbol{\beta}/\hbar). \tag{75}$$

By a straightforward calculation, we obtain the product relation of (75) as follows:

$$T_m T_n = T_{m+n} \exp(\pi\mathbf{i}\alpha m \wedge n), \tag{76}$$

where α is the number of fluxons passing through the unit cell, it is given by

$$\alpha = \phi_1/\phi_0, \tag{77}$$

in which $\phi_1 = (\boldsymbol{a}_1 \times \boldsymbol{a}_2) \cdot \boldsymbol{B}$ and $\phi_0 = hc/e$ are the magnetic flux through the unit cell $\boldsymbol{a}_1 \times \boldsymbol{a}_2$ and the magnetic flux quantum, respectively.

For the magnetic translation operators (75), it was found that they generate the infinite-dimensional sine algebra (49) [29]. Taking the rescaled generators $\bar{T}_m = -\frac{\mathbf{i}}{2\pi\alpha}T_m$, then the $SDiff(T^2)$ algebra (50) is recovered in the limit $\alpha \to 0$.

Since the product of the magnetic translation operators (75) satisfies (76), according to Theorem 5, it is known that for the generators (75), the corresponding n-algebra is nothing but the (co)sine n-algebra (63). Thus in terms of the magnetic translation operators (75), we give an explicit physical realization of the (co)sine n-algebra (63). A remarkable feature of (63) is that when n is even, it is a sh-n-Lie algebra.

Let us now discuss first few n-algebras.

- 3-algebra

$$[T_{i_1}, T_{i_2}, T_{i_3}] = 2\mathbf{i}[\sin(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_2 \wedge i_3)) - \sin(\pi\alpha(-i_1 \wedge i_2 + i_1 \wedge i_3 + i_2 \wedge i_3)) - \sin(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 - i_2 \wedge i_3))]T_{i_1+i_2+i_3}. \tag{78}$$

For the case of the general parameter α (77), it is known that the sine 3-algebra (78) does not satisfy the FI (3) and the sh-Jacobi identity (8). Comparing (78) with (53), it is of interest to note that when particularized to the $\alpha = \frac{1}{2}$ case, (78) becomes a Fillipov 3-algebra which satisfies the FI (3).

Let us take the rescaled generators $\bar{T}_{i_j} = \sqrt{\frac{-\mathbf{i}}{2\pi\alpha}} T_{i_j}$, $j = 1, 2, 3$, in (78), then the limit $\alpha \to 0$ in (78) reproduces the $SDiff(T^2)$ 3-algebra (56).

• 4-algebra

$$\begin{aligned}
&[T_{i_1}, T_{i_2}, T_{i_3}, T_{i_4}] \\
&\quad = 2(\cos(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 + i_3 \wedge i_4)) \\
&\qquad - \cos(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 - i_3 \wedge i_4)) \\
&\qquad - \cos(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 - i_2 \wedge i_3 + i_2 \wedge i_4 + i_3 \wedge i_4)) \\
&\qquad + \cos(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 - i_2 \wedge i_3 - i_2 \wedge i_4 + i_3 \wedge i_4)) \\
&\qquad + \cos(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 + i_2 \wedge i_3 - i_2 \wedge i_4 - i_3 \wedge i_4)) \\
&\qquad - \cos(\pi\alpha(i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 - i_2 \wedge i_3 - i_2 \wedge i_4 - i_3 \wedge i_4)) \\
&\qquad - \cos(\pi\alpha(-i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 + i_3 \wedge i_4)) \\
&\qquad + \cos(\pi\alpha(-i_1 \wedge i_2 + i_1 \wedge i_3 + i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 - i_3 \wedge i_4)) \\
&\qquad + \cos(\pi\alpha(-i_1 \wedge i_2 - i_1 \wedge i_3 + i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 + i_3 \wedge i_4)) \\
&\qquad - \cos(\pi\alpha(-i_1 \wedge i_2 - i_1 \wedge i_3 - i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 + i_3 \wedge i_4)) \\
&\qquad - \cos(\pi\alpha(-i_1 \wedge i_2 + i_1 \wedge i_3 - i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 - i_3 \wedge i_4)) \\
&\qquad + \cos(\pi\alpha(-i_1 \wedge i_2 - i_1 \wedge i_3 - i_1 \wedge i_4 + i_2 \wedge i_3 + i_2 \wedge i_4 - i_3 \wedge i_4)))\, T_{i_1+i_2+i_3+i_4}.
\end{aligned} \tag{79}$$

For the cosine 4-algebra (79) with the arbitrary value $\alpha \in \boldsymbol{R}$, we note that it is a sh-4-Lie algebra. Taking the rescaled generators $\bar{T}_{i_j} = \sqrt[3]{\frac{-\mathbf{i}}{2\pi\alpha}} T_{i_j}$, $j = 1, 2, 3, 4$, in (79), then when $\alpha \to 0$, (79) becomes the null 4-algebra

$$[\bar{T}_{i_1}, \bar{T}_{i_2}, \bar{T}_{i_3}, \bar{T}_{i_4}] = 0. \tag{80}$$

• 5-algebra

When $n = 5$, the corresponding 5-algebra is given by (64). Taking $\alpha = \frac{1}{3}$ in (64), it immediately gives a sh-5-Lie algebra. Let us take rescaled generators $\bar{T}_{i_j} = \sqrt[4]{\frac{-\mathbf{i}}{2\pi\alpha}} T_{i_j}$, $j = 1, 2, \cdots, 5$, in (64), then we have the null 5-algebra in the limit $\alpha \to 0$,

$$[\bar{T}_{i_1}, \bar{T}_{i_2}, \bar{T}_{i_3}, \bar{T}_{i_4}, \bar{T}_{i_5}] = 0. \tag{81}$$

Not as the case of the sine 3-algebra (78), we note that when $\alpha \to 0$, the cosine 4-algebra (79) and 5-algebra (64) with the rescaled generators become the null 4 and 5-algebras, respectively. For the (co)sine n-algebra (63), $n \geq 4$, taking the rescaled generators $\bar{T}_{i_j} = \sqrt[n-1]{\frac{-\mathbf{i}}{2\pi\alpha}} T_{i_j}$, $j = 1, 2, \cdots, n$, it is easy to verify that in the limit $\alpha \to 0$, (63) gives the null n-algebra (67).

6. Concluding remarks

We have investigated the q-deformation of the infinite-dimensional n-algebras. The V–W algebra is the centerless Virasoro algebra. Its q-deformation has been well investigated in the literature. One has already known that in the usual way, the V–W n-algebra is null [37]. In this paper, we first investigated the q-deformation of the null V–W n-algebra and constructed the nontrivial q-deformed V–W n-algebra. As one of intriguing results in this paper, we found that the q-deformed V–W n-algebra is indeed a sh-n-Lie algebra. Furthermore in terms of the pseud-differential operators on the q-plane, we constructed the q-deformed $SDiff(T^2)$ n-algebra and proved that when n is even, it satisfies the sh-Jacobi identity. In this case, this q-deformed n-algebra is a sh-n-Lie algebra. Moreover we presented the (co)sine n-algebra which is also the sh-n-Lie algebra for the n even case. The emphasis will be on the $n = 3$ and 5 cases. We found that there exists a sine 3-algebra which is indeed a Fillipov 3-algebra. When $n = 5$, we derived a cosine 5-algebras which is a sh-5-Lie algebras. An interesting open question is whether there exists the special parameter values such that the (co)sine n-algebra (63) with any odd n is the Fillipov n-algebra or sh-n-Lie algebra.

Our work has brought to light a connection between the q-deformed infinite-dimensional n-algebra and the sh-n-Lie algebra. Due to the rich structures of the sh-n-Lie algebra, it provides new insight into the q-deformation of n-algebra. It would be interesting to study further and see whether there exist the central extension terms for the sh-n-Lie algebras derived in this paper. Furthermore it should be mentioned that not as the case of the q-deformed V–W n-algebra, the sh-Jacobi identity only holds for the q-deformed $SDiff(T^2)$ n-algebra with n even. How to construct the appropriate q–n-brackets such that this q-deformed infinite-dimensional n-algebra with any n is the sh-n-Lie algebra still deserves further study. Finally, it is worth to emphasize that we reinvestigate the motion of a Bloch electron in the constant uniform magnetic field and present an explicit physical realization of the (co)sine n-algebra in terms of the so-called magnetic translation operators. This symmetry structure will provide new insight into this well-known physical model. For the q-deformed V–W and $SDiff(T^2)$ n-algebras, due to their special algebraic property, we believe that their applications in physics should also be of interest.

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