ELSEVIER
Linear Algebra and its Applications 357 (2002) 273–289
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

# Patterns, linesums, and symmetry ☆

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Received 14 March 2001; accepted 5 May 2002

Submitted by R.A. Brualdi

**Abstract**

Two questions about the existence of structured matrices with given linesums and given zero–nonzero patterns are settled: when is there a symmetric matrix and when is there a skew-symmetric matrix? The solutions use some ideas from prior treatment of the analogous problems without structural constraints, but the current results require new conditions and new methodology.


*Keywords:* Pattern; Linesum; Symmetry; Skew-symmetry

## 1. Introduction

An (unsigned) *pattern* is an $m$-by-$n$ array $P$ with entries from $\{0, *\}$. A real $m$-by-$n$ matrix $A$ belongs to $P$ if the nonzero entries of $A$ appear precisely in the positions of the $*$'s in $P$.

In [3], a complete solution to the following general problem was given.

(G): *Given the $m$-by-$n$ pattern $P$, for which pairs of vectors $r \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ does there exist a real matrix $A$, belonging to $P$, whose row sum vector is $r$ and whose column sum vector is $c$?*

☆ Research supported in part by NSF REU Grant DMS-99-87803.
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0024-3795/02/$ - see front matter 
PII: S0024-3795(02)00417-2

We note that analogous problems involving sign patterns, rather than zero–nonzero patterns, have been studied explicitly in [2], and "weak" versions have appeared among more general results in [1]. The methods used in those studies differ significantly from those in [3] and herein.

If the matrix $A$ of (G) exists, we say that $P$ *strongly allows* row sums $r$ and column sums $c$. An $m$-by-$n$ pattern $\tilde{P} = (\tilde{p}_{ij})$ is *subordinate* to the $m$-by-$n$ pattern $P = (p_{ij})$ if $p_{ij} = 0$ implies $\tilde{p}_{ij} = 0$. We say that the pattern $P$ *weakly allows* row sums $r$ and column sums $c$ if there is a pattern $\tilde{P}$, subordinate to $P$, that strongly allows row sums $r$ and column sums $c$. The (much easier) weak version (G′) of problem (G) was also resolved in [3]. While [1] likely implies the solution of (G′), it seems that it encompasses neither the strong problem (G), which was the focus of [3], nor the strong, structured problems that are the focus of the present work.

It is natural to ask questions analogous to (G) and (G′) when additional structural conditions are imposed on the realizing matrix $A$. For example, consider the problem

(S): *Given a symmetric $n$-by-$n$ pattern $P$, for which $n$-by-1 vectors $r$ does there exist a* <u>*symmetric*</u> *$n$-by-$n$ matrix $A$ belonging to $P$ whose row sum vector (and hence column sum vector) is $r$?* There is, of course, a weak version (S′) of (S) in which $A$ is only required to belong to a pattern subordinate to $P$.

If a structured version of (G) or (G′) has a solution, it is, of course, a solution to the unstructured version. It may happen, as we shall see, that the unstructured version has a solution when the structured version does not. In this case, we seek additional conditions on the data that guarantee, and are guaranteed by, the existence of a structured solution. One structure of interest in this paper is symmetry, and we observe the following. It is not difficult to see that if the problem (G′) has a solution for the data: symmetric $P$, $r$, and $c = r$, then (S′) has a solution for the same $P$ and $r$; the converse is, of course, trivial. If $A$ is a solution of (G′), then $\frac{1}{2}(A + A^{\mathrm{T}})$ is a solution to (S′). However, because of the possibility of cancellation, it is much less clear that (G) is so related to (S). In fact, the existence of a solution to (G) for symmetric $P$, $r$, and $c = r$, does <u>not</u> imply the solvability of (S) for the same $P$ and $r$, as shown by the following example.

**Example.** Let

$$P = \begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$$

and $r = [1, 1, 2]^{\mathrm{T}}$. Then, according to [3], problem (G) with $c = r$ has a solution; in particular,

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

is a solution for these data. However, because $a_{12} = -a_{21}$, $\frac{1}{2}(A + A^{\mathrm{T}})$ is not a solution to (S) for these data, and, in fact, it is an easy calculation to see that (S) has no solution for these data. In every solution $A$ to (G), $a_{12} = -a_{21}$. The solvability of (S), then, must entail conditions on the pair $P$ and $r$ in addition to the conditions given in [3] for the solvability of (G). It is a main purpose here to discover and prove the additional conditions that ensure solvability of (S).

Another natural structural variant upon (G) is the problem (K) in which $A$ is asked to be skew-symmetric; thus $P$ must again be symmetric (and with a zero diagonal), $c = -r$, and the sum of the entries of $r$ must be 0. This variant is also treated in the present work. In both cases we are able to use some of the techniques developed in [3], but some new ones are required as well; in particular, while the bipartite graph of $P$ facilitated solution of (G), viewing the problem via the usual undirected graph of $P$ becomes natural for (S) and (K). This, however, requires some special care in the case of certain instances of the problem that were previously "reducible".

Finally, similar to [3], we note that, for given totally nonzero $x$ and $y$ in $\mathbb{R}^n$, the seemingly more general question of when there is a symmetric or skew-symmetric $A$ belonging to $P$ such that $Ax = y$ is actually a special case of our results here.

## 2. Background and preliminaries

Let $P = (p_{ij})$ be an $m$-by-$n$ pattern. The *bipartite graph of P* is the bipartite graph $G$ with $m$ "row" vertices $r_1, r_2, \ldots, r_m$, $n$ "column" vertices $c_1, c_2, \ldots, c_n$, and an edge joining $r_i$ to $c_j$ if and only if $p_{ij} = *$. The pattern $P$ is *connected* provided the graph $G$ is connected.

The *variable pattern* of $P$ is the $m$-by-$n$ array $X$ obtained from $P$ by replacing each * with a subscripted variable $x_j$. The $j$'s run from 1 to $k$, no subscript is used twice, and the numbering proceeds from top to bottom, and within each row from left to right.

For any positive integer $q$, we denote by $\overline{q}$ the set $\{1, 2, \ldots, q\}$. If $M$ is any $m$-by-$n$ array (matrix, pattern, variable pattern) and $\alpha \subset \overline{m}$ and $\beta \subset \overline{n}$, then $M(\alpha, \beta)$ denotes the subarray lying in the rows indexed by $\alpha$ and the columns indexed by $\beta$. The symbol $\alpha^{\mathrm{c}}$ will denote the complement of $\alpha$ in $\overline{m}$, and $\beta^{\mathrm{c}}$ will be the complement of $\beta$ in $\overline{n}$. $M^{(j)}$ will denote the $j$th column of $M$, and $M_{(j)}$ will denote the $j$th row of $M$.

**Definition.** The variable $x_i$ (or, alternatively, the * in $P$ that corresponds to $x_i$) is a single star of $X$ (or $P$) with respect to $\alpha$ and $\beta$ provided $\alpha \subset \overline{m}$, $\beta \subset \overline{n}$, $x_i$ is the only variable appearing in $X(\alpha, \beta^{\mathrm{c}})$ and no variables appear in $X(\alpha^{\mathrm{c}}, \beta)$.

The following theorem [3, Theorem 9] is the solution to the general problem (G) of Section 1.

**Theorem.** *Let P be an m-by-n connected pattern and suppose $r \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, with $\sum_{j=1}^{m} r_j = \sum_{k=1}^{n} c_k$. The following are equivalent*:
(1) *P strongly allows a matrix with row sums r and column sums $c$.*
(2) *If P has a single star with respect to $\alpha$ and $\beta$, then $\sum_{j\in\alpha} r_j \neq \sum_{j\in\beta} c_j$.*

Here and throughout the paper, we agree that the sum on the right is 0 if $\beta$ is empty.

For the problem (S), with which this paper deals, the pattern $P$ is $n$-by-$n$ and symmetric.

**Definition.** If $P$ is a symmetric $n$-by-$n$ pattern, the undirected graph of $P$ is the graph $H$ with vertices $1, 2, \ldots, n$ and an (undirected) edge joining $i$ to $j$ if and only if $p_{ij} = p_{ji} = *$.

It can easily be checked that $P$ is connected (i.e. the bipartite graph $G$ of $P$ is connected) if and only if $H$ is connected and contains a cycle of odd edge length (i.e., an "odd cycle").

If the *m*-by-*n* pattern *P* has bipartite graph *G* that is not connected, then *P* is permutation *equivalent* ($Q = SPT$ with *S* and *T* permutation matrices) to the direct sum of connected principal subpatterns. The existence of a solution to (G) is invariant under permutation equivalence (with the sum vectors *r* and *c* properly permuted). Thus if (G) is solved for connected patterns, it is easy to solve (G) for *P* by considering problems on connected subpatterns. For this reason, results in [3] are stated only for connected patterns.

The existence of a solution to the symmetric problem (S) is not invariant under permutation equivalence (symmetry is lost), so we may lose generality by assuming a nonconnected *P* to be a direct sum. The existence of a solution to (S) is, however, invariant under permutation *similarity* ($Q = S^{\mathrm{T}}PS$ with *S* a permutation matrix). Suppose, then, that *P* is square and symmetric with bipartite graph *G* and undirected graph *H*. If *H* is not connected, then *G* is not connected and, furthermore, *P* is permutation similar to a direct sum of principal subpatterns. Thus we may without loss of generality always assume *H* is connected. In the next section, Theorem 2 deals with the case that *P* is connected; that is, *H* is connected and contains an odd cycle. Theorem 11 considers the case that *P* is not connected, but *H* is, so that *P* is not permutation similar to a direct sum. We have found no simple statement that combines the two results.

## 3. The symmetric problem

In considering the symmetric problem (S), it is convenient to define the variable pattern differently, and to define a "coefficient matrix" of *P* differently than was done in [3]. For purposes of this paper, we make the following definitions.

Let $P$ be a symmetric $n$-by-$n$ pattern. The *symmetric variable pattern of P* is the $n$-by-$n$ array $X$ obtained from $P$ by replacing all *'s that are on or above the main diagonal with variables $x_i$, numbering from top to bottom and within each row from left to right, and then replacing each * that is below the main diagonal with the variable that appears already in the symmetrically placed position.

For example, if

$$P = \begin{bmatrix} 0 & * & * & 0 \\ * & 0 & * & 0 \\ * & * & 0 & * \\ 0 & 0 & * & * \end{bmatrix},$$

then

$$X = \begin{bmatrix} 0 & x_1 & x_2 & 0 \\ x_1 & 0 & x_3 & 0 \\ x_2 & x_3 & 0 & x_4 \\ 0 & 0 & x_4 & x_5 \end{bmatrix}.$$

Setting the row sums of $X$ equal to the entries in $r$ yields a linear system that has a totally nonzero solution (i.e., a solution in which no $x_i$ is 0) if and only if $P$ allows a symmetric matrix with row sum vector $r$. If $k$ is the number of distinct variables in $X$, the $n$-by-$k$ coefficient matrix $C = (c_{ij})$ of this system is called the *symmetric coefficient matrix of P*. We note that $c_{ij} = 1$ if $x_j$ appears in the $i$th row of $X$, and $c_{ij} = 0$ otherwise. If $x_s$ appears in rows $i$ and $j$ of $X$, then $C^{(s)} = e_i + e_j$; if $x_s$ appears as the $i$th diagonal entry in $X$, then $C^{(s)} = e_i$. In any case, we label the edge in $H$ joining vertex $i$ and vertex $j$ with $x_s$. We observe that no two columns of $C$ are equal.

If there is a symmetric matrix $A \in P$ with row sum vector $r$, then by [3, Theorem 9] it is necessary that $\sum_{j\in\alpha} r_j \neq \sum_{j\in\beta} r_j$ whenever $P$ has a single star with respect to $\alpha \subset \overline{n}$ and $\beta \subset \overline{n}$. An additional necessary condition is illustrated by the following example:

Let

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & 0 \end{bmatrix},$$

in which $P_{11}$ is square, has diagonal entries 0, and contains exactly two stars (necessarily symmetrically placed). Suppose that $P$ allows a symmetric matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix},$$

in which $A_{11}$ has the nonzero number $a$ in the two star positions of $P_{11}$, and suppose the row sum vector of $A$ is $r$. Let $S$ denote the sum of the entries in $A_{12}$, which is equal to the sum of the entries in $A_{21}$. If $A_{11}$ has $k$ rows, then

$$\sum_{i=1}^{k} r_i = 2a + S \neq S = \sum_{i=k+1}^{n} r_i.$$

This example shows the necessity of condition (ii) in Theorem 2.

**Definition 1.** Let $\sigma \subset \overline{n}$. We say that $x_i$ is a double star of $X$ (or of $P$) with respect to $\sigma$ provided $X(\sigma, \sigma)$ contains the variable $x_i$ in two positions and contains no other variables, and $X(\sigma^c, \sigma^c)$ contains no variables at all.

**Theorem 2.** *Suppose $P$ is an $n$-by-$n$ connected symmetric pattern and $r$ is in $\mathbb{R}^n$. Then there is a symmetric matrix $A$ in $P$ with row sum vector $r$ if and only if the following two conditions hold*:
(i) *If $P$ has a single star with respect to $\alpha \subset \overline{n}$ and $\beta \subset \overline{n}$, then $\sum_{j\in\alpha} r_j \neq \sum_{j\in\beta} r_j$.*
(ii) *If $P$ has a double star with respect to $\sigma \subset \overline{n}$, then $\sum_{j\in\sigma} r_j \neq \sum_{j\in\sigma^c} r_j$.*

The proof of Theorem 2 requires several lemmas. Throughout the rest of this section, we assume that $P$ is a connected symmetric $n$-by-$n$ pattern, $X$ is its symmetric variable pattern, $H$ is its undirected graph, and $C$ is its $n$-by-$k$ symmetric coefficient matrix.

**Lemma 3.** *Suppose $x_s$ is an edge in $H$ that lies on a closed path in $H$ that traverses $x_s$ only once, and that has even edge length. If $C'$ is obtained from the coefficient matrix $C$ of $P$ by deleting the $s$th column, then $\operatorname{rank}(C') = \operatorname{rank}(C)$.*

**Proof.** Let $s_1 = s$, and let the closed path be $\{v_{t_1}, x_{s_1}, v_{t_2}, \ldots, v_{t_{2w}}, x_{s_{2w}}, v_{t_1}\}$, so that $s_j \neq s_1$ for $j = 2, 3, \ldots, 2w$. We have

$$\begin{aligned} C^{(s_1)} &= e_{t_1} + e_{t_2} \\ &= (e_{t_2} + e_{t_3}) - (e_{t_3} + e_{t_4}) + (e_{t_4} + e_{t_5}) - \cdots + (e_{t_{2w}} + e_{t_1}) \\ &= C^{(s_2)} - C^{(s_3)} + C^{(s_4)} - \cdots + C^{(s_{2w})}. \end{aligned}$$

Note that the subscript on the first $t_i$ in each parenthesis is even if and only if that parenthesis is preceded by a $+$. Hence the last parenthesis is preceded by a $+$. Hence $C^{(s_1)}$ is a linear combination of the other columns in $C$, and so $\operatorname{rank}(C') = \operatorname{rank}(C)$. □

**Lemma 4.** *Suppose $M$ is an odd cycle in $H$, and $x_s$ is an edge in $H$ that lies on a cycle $D$ in $H$, but does not lie on the cycle $M$. If $C'$ is obtained from the symmetric coefficient matrix $C$ by deleting the $s$th column, then $\operatorname{rank}(C') = \operatorname{rank}(C)$.*

**Proof.** If $D$ has even length, the conclusion follows by Lemma 3. Suppose $D$ has odd length.

If $M$ and $D$ have a common vertex $v_t$, consider the closed path that starts at $v_t$, traverses the cycle $D$, then traverses the cycle $M$, ending at $v_t$. This closed path has even edge length and traverses $x_s$ exactly once, so the conclusion follows from Lemma 3.

If $M$ and $D$ do not have a common vertex, there are vertices $v_{t_1}$ of $M$ and $v_{t_2}$ of $D$ and a path $p$ from $v_{t_1}$ to $v_{t_2}$ that shares no edges with either $M$ or $D$. In this case, the path starting at $v_{t_1}$ and following $p$, $D$, $p^-$, and $M$ is a closed path of even edge length that traverses $x_s$ exactly once, so the conclusion follows from Lemma 3. □

**Lemma 5.** *Suppose $x_s$ is an edge in $H$ incident with a vertex $v$ of degree one. If $C'$ is obtained from $C$ by deleting the $s$th column, then* $\operatorname{rank}(C') = \operatorname{rank}(C) - 1$.

**Proof.** Row $v$ of $C$ has 1 in the $s$th position and 0's elsewhere, so $C^{(s)}$ is not a linear combination of the other columns of $C$. □

**Lemma 6.** *The rank of the symmetric coefficient matrix $C$ of $P$ is $n$.*

**Proof.** Since $P$ is connected, $H$ is connected and contains a cycle $M$ of odd length $s$. Remove from $H$ an edge that lies on a cycle in $H$ but does not lie on $M$ (if any). Then remove an edge from the resulting graph that lies on a cycle, but does not lie on $M$. Continue this process until $M$ is the only cycle. The resulting graph, $H'$, is connected, and consists of the $s$-cycle $M$, $n - s$ vertices not in $M$, and $n - s$ edges not in $M$ (more edges would imply another cycle; fewer would violate connectedness). Let $C'$ be the matrix obtained from $C$ by deleting the columns corresponding to the edges deleted from $H$. By Lemma 4, $\operatorname{rank}(C') = \operatorname{rank}(C)$. Now further reduce $H'$ by removing consecutively edges incident to vertices of degree one, until $M$, together with $n - s$ isolated vertices, is obtained. Each edge removal corresonds to a deletion of a column from $C'$ which, by Lemma 5, reduces the rank by 1. Call the resulting matrix $C_M$. Since $n - s$ edges are removed, $\operatorname{rank}(C_M) = \operatorname{rank}(C') - (n - s) = \operatorname{rank}(C) - (n - s)$. It remains to show that $\operatorname{rank}(C_M) = s$, so that $s = \operatorname{rank}(C) - (n - s)$; that is, $\operatorname{rank}(C) = n$.

$C_M$ is the symmetric coefficient matrix of an $n$-by-$n$ symmetric pattern $P_M$ (obtained from $P$ by replacing stars with 0's as edges of $H$ are removed) whose graph is the $s$-cycle $M$ together with $n - s$ isolated vertices. We may assume the symmetric variable pattern for $P_M$ is of the form $\begin{bmatrix} X_M \\ 0 \end{bmatrix}$, where

$$X_M = \begin{bmatrix} 0 & x_1 & 0 & 0 & \cdots & 0 & x_s \\ x_1 & 0 & x_2 & 0 & \cdots & 0 & 0 \\ 0 & x_2 & 0 & x_3 & \cdots & 0 & 0 \\ & & & & \cdots & & \\ & & & & \cdots & & \\ & & & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & x_{s-1} \\ x_s & 0 & 0 & 0 & \cdots & x_{s-1} & 0 \end{bmatrix}.$$

Thus the nonzero rows of $C_M$ form the $s$-by-$s$ matrix

$$C'_M = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ & & & & \cdots & & \\ & & & & \cdots & & \\ & & & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}.$$

Expanding $\det(C'_M)$ by the first row (remember that $s$ is odd), we find $\det(C'_M) = 2$, so $\operatorname{rank}(C_M) = \operatorname{rank}(C'_M) = s$, and the lemma is established. □

**Lemma 7.** *If $e_i^{\mathrm{T}} \in RS(C)$, then there is a positive number $z \in \{1, 2\}$ and subsets $\kappa$ and $\lambda$ of $\overline{n}$ such that $\kappa \cap \lambda = \phi$ and*

$$ze_i^{\mathrm{T}} = \sum_{j \in \kappa} C_{(j)} - \sum_{j \in \lambda} C_{(j)}.$$

**Proof.** Recall that $C$ is a 0,1 matrix and each column has either one or two 1's. Also, no two columns of $C$ are equal, and $\operatorname{rank}(C) = n$.

Suppose that the unique representation of $e_i^{\mathrm{T}}$ as a linear combination of the rows of $C$ is

$$e_i^{\mathrm{T}} = \sum_{j=1}^{n} y_j C_{(j)}, \quad y_j\text{'s in } \mathbb{R}.$$

We first suppose that $\alpha$ is the only value of $j$ for which $c_{ji} = 1$. We construct the linear combination, and note that, by its uniqueness, all of the choices in the following construction are forced.

The coefficient $y_\alpha$ must be 1; that is, the sequential construction starts with $1C_{(\alpha)}$. If there are any 1's in this vector other than the one in position $i$, each must be "removed" by adding $-1$ times the only other row that has a 1 in the same column. This creates a 0 in that position, and the addition of multiples of any new rows will not change that 0. When all of the 1's other than that in the $i$th position have been removed, there may be $-1$'s in positions that previously had 0. At this point each nonzero entry other than the $i$th is either $-1$ or $-2$. But note that $-2$ cannot occur, since it could not be removed by the addition of more terms to the combination. Each $-1$ must be removed by adding 1 times the only other row with a 1 in that position. This other row will not be one that has appeared before in the construction, and no positions that have been changed from 1 to 0 will be affected. As before, 2 cannot occur in the vector, so all nonzero entries are now 1. If the combination thus far constructed is not $e_i^{\mathrm{T}}$, it is because there are some 1's in positions other than the $i$th, and we continue the construction. Since all these choices have been forced, and since the desired linear combination must exist, we will at some point arrive at $e_i^{\mathrm{T}}$, and all coefficients used will have been 1 or $-1$, so the result is obtained with $z = 1$.

Now suppose there are two values of $j$, $\alpha$ and $\beta$, for which $c_{ji} = 1$. If either of $y_\alpha$ or $y_\beta$ is 0, we proceed as before to construct the unique linear combination and find $z = 1$. We are left with the case $y_\alpha y_\beta \neq 0$ and $y_\alpha + y_\beta = 1$. Our construction of the linear combination begins with $y_\alpha C_{(\alpha)} + y_\beta C_{(\beta)}$. This vector has a 1 in the $i$th position, but is not $e_i^{\mathrm{T}}$, since if it were, rows $\alpha$ and $\beta$ of $C$ would be multiples of each other and we would have $\mathrm{rank}(C) < n$, which is impossible by Lemma 6. If, for some $s \neq i$, the $s$th entry were $y_\alpha + y_\beta$, the $i$th and $s$th columns of $C$ would be equal. Thus each nonzero entry in $y_\alpha C_{(\alpha)} + y_\beta C_{(\beta)}$, other than the 1 in the $i$th position, is either $y_\alpha$ or $y_\beta$. We continue the construction by removing each $y_\alpha$ by adding $-y_\alpha$ times a new row to the combination, and then removing any $-y_\alpha$'s, and so forth as before. There results a linear combination with 1 in the $i$th position and each other nonzero position containing either $\pm y_\beta$ or $\pm(y_\alpha - y_\beta)$. Here we have used the fact that the absence of duplicate columns in $C$ guarantees that no position other than the $i$th can contain $\pm(y_\alpha + y_\beta)$. We continue by removing any $\pm y_\beta$'s, leaving only 0's and, possibly, $\pm(y_\alpha - y_\beta)$'s. There are now two possiblities:

(i) If now some entry is $\pm(y_\alpha - y_\beta)$, then, since all such entries must be 0, we have $y_\alpha = y_\beta = \frac{1}{2}$, and our construction is complete. In this case, the lemma is proved with $z = 2$.

(ii) If, on the other hand, all entries other than the $i$th are 0, we have

$$e_i^{\mathrm{T}} = \sum_{j\in A} y_\alpha C_{(j)} + \sum_{j\in B} y_\beta C_{(j)} + \sum_{j\in D} (-y_\alpha) C_{(j)} + \sum_{j\in E} (-y_\beta) C_{(j)},$$

where $A$, $B$, $D$, and $E$ are pairwise disjoint subsets of $N$. We can now reverse the roles of $\alpha$ and $\beta$ to obtain

$$e_i^{\mathrm{T}} = \sum_{j\in A} y_\beta C_{(j)} + \sum_{j\in B} y_\alpha C_{(j)} + \sum_{j\in D} (-y_\beta) C_{(j)} + \sum_{j\in E} (-y_\alpha) C_{(j)}.$$

Since $y_\alpha C_{(\alpha)} + y_\beta C_{(\beta)} \neq e_i^{\mathrm{T}}$, one of $A$, $B$, $D$, and $E$ contains an index other than $\alpha$ and $\beta$. If $\gamma$ is such an index, the uniqueness of the coefficient of $C_{(\gamma)}$ (because $\mathrm{rank}(C) = n$) in the expression for $e_i^{\mathrm{T}}$ implies that $y_\alpha = y_\beta = \frac{1}{2}$, so again $z = 2$ and the lemma is proved. □

Possibility (i) in the proof is illustrated by the symmetric variable pattern

$$X = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & x_4 & x_5 \\ x_2 & x_4 & 0 & 0 \\ x_3 & x_5 & 0 & 0 \end{bmatrix}.$$

whose undirected graph is connected and contains two 3-cycles. The symmetric coefficient matrix is

$$C = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

and applying the construction given in the proof, we obtain

$$\begin{aligned} e_1^{\mathrm{T}} &= y_\alpha C_{(1)} + y_\beta C_{(2)} - y_\alpha C_{(3)} - y_\beta C_{(4)} \\ &= [y_\alpha + y_\beta \quad 0 \quad y_\alpha - y_\beta \quad y_\beta - y_\alpha \quad 0]. \end{aligned}$$

We have not found a connected symmetric variable pattern that illustrates possibility (ii), and we suspect that none exists.

**Lemma 8.** *If $e_i^{\mathrm{T}} \in RS(C)$, then $x_i$ is either a single star or a double star for P.*

**Proof.** By Lemma 7, $ze_i^{\mathrm{T}} = \sum_{j\in\kappa} C_{(j)} - \sum_{j\in\lambda} C_{(j)}$ with $z \in \{1, 2\}$, where $\kappa \cap \lambda = \phi$ and $\kappa \cup \lambda \subset \overline{n}$. Thus for $1 \leqslant s \leqslant k$,

$$d_s \equiv \sum_{j\in\kappa} c_{js} - \sum_{j\in\lambda} c_{js} = \begin{cases} z > 0 & \text{if } s = i, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for each $s$, there are either one or two values of $j$ such that $c_{js} = 1$, and $c_{js} = 0$ otherwise. It follows that $d_i$ is either 1 or 2, so the variable $x_i$ appears in one or two rows of $X$, each of which is indexed in $\kappa$. Hence $x_i$ appears in one or two positions in $X(\kappa, \kappa)$. If in one position, it is on the diagonal of $X$, and if in two positions, they are not on the diagonal.

If $1 \leqslant s \leqslant k$ and $s \neq i$, then $d_s = 0$, so one of the following is true:

(i) $x_s$ lies in two rows of $X$, one indexed in $\kappa$ and one in $\lambda$. In this case, $x_s$ appears once in $X(\kappa, \lambda)$ and once in $X(\lambda, \kappa)$.

(ii) Any appearance of $x_s$ in $X$ occurs in a row indexed in neither $\kappa$ nor $\lambda$, so each $x_s$ appears in $X((\kappa \cup \lambda)^{\mathrm{c}}, (\kappa \cup \lambda)^{\mathrm{c}})$.

Thus, up to permutation similarity, we have

$$X = \begin{bmatrix} Y & U & 0 \\ U^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & Z \end{bmatrix},$$

where $Y = X(\kappa, \kappa)$, $U = X(\kappa, \lambda)$, and $Z = X((\kappa \cup \lambda)^{\mathrm{c}}, (\kappa \cup \lambda)^{\mathrm{c}})$. If $\kappa \cup \lambda \neq \overline{n}$, then $X$ is a direct sum and is not connected. Hence

$$X = \begin{bmatrix} Y & U \\ U^{\mathrm{T}} & 0 \end{bmatrix}.$$

If $x_i$ appears only once (on the diagonal of $Y$), $x_i$ is a single star of $P$ with respect to $\alpha = \kappa$ and $\beta = \lambda$. If $x_i$ appears twice (in nondiagonal positions in $Y$), then $x_i$ is a double star of $P$ with respect to $\sigma = \kappa$. □

We note that the following converse of Lemma 8 can be established. We state it here without proof, since it is not needed in our exposition.

**Lemma 9.** *If $x_i$ is either a single star or a double star of $P$, then $e_i^{\mathrm{T}} \in RS(C)$.*

**Proof of Theorem 2.** Suppose first that there is a symmetric matrix in $P$ with row sum vector $r$. Then (i) holds by [3, Theorem 9], and (ii) holds by an easy extension of the argument preceding Definition 1.

Now suppose that (i) and (ii) hold. Let $C$ be the symmetric coefficient matrix of $P$. We must show that the equation $Cx = r$ has a totally nonzero solution $x \in \mathbb{R}^k$. Let $\mathscr{E} = \{i : 1 \leqslant i \leqslant k, e_i^{\mathrm{T}} \in RS(C)\}$. By [3, Lemma 6], there is a $u \in \mathbb{R}^k$ such that $Cu = 0$ and $u_i = 0$ if and only if $i \in \mathscr{E}$. By our Lemma 6, the equation $Cx = r$ has a solution $v$.

We claim now that $v_i \neq 0$ for $i \in \mathscr{E}$. let $A$ be the real matrix obtained from $X$ by replacing each $x_j$ with $v_j$, $j = 1, 2, \ldots, k$. Then $A$ has a pattern subordinate to $P$, and $A$ has row and column sums $r$. If $i \in \mathscr{E}$, then by Lemma 8, $x_i$ is either a single star with respect to some $\alpha, \beta \subset \overline{n}$, or $x_i$ is a double star of $P$ with respect to some $\sigma \subset \overline{n}$.

If $x_i$ is a single star with respect to $\alpha$ and $\beta$, then $a_{pq} = 0$ for all but one choice of $(p, q)$ with $p \in \alpha$ and $q \in \beta^{\mathrm{c}}$, and for that one choice, $a_{pq} = v_i$. Furthermore, $a_{pq} = 0$ for all $p \in \alpha^{\mathrm{c}}$ and $q \in \beta$. Hence

$$\begin{aligned}\textstyle\sum_{j\in\alpha} r_j &= \sum_{j\in\alpha}\sum_{q=1}^{n} a_{jq}\\ &= \sum_{j\in\alpha}\sum_{q\in\beta} a_{jq} + v_i\\ &= \sum_{q\in\beta}\sum_{j\in\alpha} a_{jq} + v_i\\ &= \sum_{q\in\beta}\sum_{j=1}^{n} a_{jq} + v_i\\ &= \sum_{q\in\beta} r_q + v_i.\end{aligned}$$

Thus $v_i = \sum_{j\in\alpha} r_j - \sum_{j\in\beta} r_j \neq 0$.

If $x_i$ is a double star with respect to $\sigma$, then $a_{pq} = 0$ for all but two choices of $(p, q)$ with $p$ and $q$ in $\sigma$, and for those two choices, $a_{pq} = v_i$. Furthermore, $a_{pq} = 0$ for all $p$ and $q$ in $\sigma^{\mathrm{c}}$. Hence

$$\begin{aligned}\textstyle\sum_{j\in\sigma} r_j &= \sum_{j\in\sigma}\sum_{q=1}^{n} a_{jq}\\ &= \sum_{j\in\sigma}\sum_{q\in\sigma} a_{jq} + \sum_{j\in\sigma}\sum_{q\in\sigma^{\mathrm{c}}} a_{jq}\\ &= 2v_i + \sum_{j\in\sigma}\sum_{q\in\sigma^{\mathrm{c}}} a_{jq}\end{aligned}$$

$$= 2v_i + \sum_{q\in\sigma^{\mathrm{c}}}\sum_{j\in\sigma} a_{jq}$$

$$= 2v_i + \sum_{q\in\sigma^{\mathrm{c}}}\sum_{j=1}^{n} a_{jq}$$

$$= 2v_i + \sum_{q\in\sigma^{\mathrm{c}}} r_q.$$

Thus

$$v_i = \frac{1}{2}\left(\sum_{j\in\sigma} r_j - \sum_{j\in\sigma^{\mathrm{c}}} r_j\right) \neq 0,$$

and the claim is established.

We now choose $t \in \mathbb{R}$ so that $tu_i + v_i \neq 0$ for $1 \leqslant i \leqslant k$ and $i \notin \mathscr{E}$, and the vector $tu + v$ is a totally nonzero solution of $Cx = r$, so Theorem 2 is established. □

We observe that statements (i) and (ii) in Theorem 2 are independent. The connected pattern

$$P = \begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix} \quad \text{with } r = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

satisfies (i) but not (ii). The connected pattern

$$P = \begin{bmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 \\ 0 & * & 0 & * & 0 \\ * & * & * & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{with } r = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

satisfies (ii) but not (i).

We consider now the case that $H$ is connected, but does not contain an odd cycle. Then $P$ is not connected, and is not permutation similar to a direct sum. In this case, $H$ is bipartite, and there is a unique partition $\{\pi, \pi^{\mathrm{c}}\}$ of the vertices $\{1, 2, \ldots, n\}$ of $H$ such that each edge in $H$ connects a vertex in $\pi$ to one in $\pi^{\mathrm{c}}$, we will call $\{\pi, \pi^{\mathrm{c}}\}$ the *bipartition of P* . Clearly, $P(\pi, \pi) = P(\pi^{\mathrm{c}}, \pi^{\mathrm{c}}) = 0$. For Theorem 11, we need the following additional lemma.

**Lemma 10.** *Suppose H is connected, but does not contain an odd cycle. Let* $\{\pi, \pi^{\mathrm{c}}\}$ *be the bipartition of P, and* $\sigma \subset \overline{n}$. *Then P has a double star with respect to* $\sigma$ *if and only if* $P(\pi, \pi^{\mathrm{c}})$ *has a single star with respect to* $\sigma \cap \pi$ *and* $\sigma^{\mathrm{c}} \cap \pi^{\mathrm{c}}$. (*Here the rows and columns of* $P(\pi, \pi^{\mathrm{c}})$ *retain the indexing from P.*)

**Proof.** By permutation similarity, we may assume that $\pi = \{1, 2, \ldots, t\}$ with $t < n$. A further permutation similarity will bring the rows and columns originally indexed by $\sigma$ into contiguous positions. Since $P(\sigma, \sigma) \neq 0$, $\sigma$ intersects both $\pi$ and $\pi^{\mathrm{c}}$. Thus we may assume without loss of generality that

$$P = \begin{bmatrix} 0 & 0 & \| & U & S \\ 0 & 0 & \| & Q & V \\ = & = & = & = & = \\ U^{\mathrm{T}} & Q^{\mathrm{T}} & \| & 0 & 0 \\ S^{\mathrm{T}} & V^{\mathrm{T}} & \| & 0 & 0 \end{bmatrix}$$

in which the indicated partition pictures $\pi$ with $P(\pi, \pi) = P(\pi^{\mathrm{c}}, \pi^{\mathrm{c}}) = 0$ and

$$P(\pi, \pi^{\mathrm{c}}) = \begin{bmatrix} U & S \\ Q & V \end{bmatrix},$$

and $P(\sigma, \sigma)$ is the central block

$$\begin{bmatrix} 0 & Q \\ Q^{\mathrm{T}} & 0 \end{bmatrix}.$$

If $P$ has a double star with respect to $\sigma$, the blocks labelled $Q$ and $Q^{\mathrm{T}}$ each contain exactly one star, and the blocks labelled $S$ and $S^{\mathrm{T}}$ are 0. Clearly, then, the star in $Q$ is a single star of $P(\pi, \pi^{\mathrm{c}})$ with respect to $\sigma \cap \pi$ and $\sigma^{\mathrm{c}} \cap \pi^{\mathrm{c}}$.

If $P(\pi, \pi^{\mathrm{c}})$ has a single star with respect to $\sigma \cap \pi$ and $\sigma^{\mathrm{c}} \cap \pi^{\mathrm{c}}$, then $Q$ contains exactly one star and $S$ is 0. By the symmetry of $P$, $Q^{\mathrm{T}}$ contains exactly one star and $S^{\mathrm{T}}$ is 0, so $P$ has a double star with respect to $\sigma$. □

**Theorem 11.** *Suppose $P$ is an $n$-by-$n$ symmetric pattern and $r$ is in $\mathbb{R}^n$. Suppose further that the undirected graph $H$ of $P$ is connected, but does not contain an odd cycle. Let $\{\pi, \pi^{\mathrm{c}}\}$ be the bipartition of $P$. Then there is a symmetric matrix $A$ in $P$ with row sum vector $r$ if and only if the following two conditions hold*:

(i) $\sum_{j\in\pi} r_j = \sum_{j\in\pi^{\mathrm{c}}} r_j$.

(ii) *If $P$ has a double star with respect to $\sigma \subset \overline{n}$, then* $\sum_{j\in\sigma\cap\pi} r_j \neq \sum_{j\in\sigma^{\mathrm{c}}\cap\pi^{\mathrm{c}}} r_j$.

**Proof.** Let $r_\pi$ denote the vector of entries in $r$ indexed by $\pi$, and $r_{\pi^{\mathrm{c}}}$ the vector of entries in $r$ indexed by $\pi^{\mathrm{c}}$. Since symmetry of a matrix is invariant under permutation similarity, we may assume $P$ has the structure shown in the proof of the Lemma 10. It is then clear that there is a symmetric $A \in P$ with row sums $r$ if and only if there is a matrix

$$B \in P(\pi, \pi^{\mathrm{c}}) = \begin{bmatrix} U & S \\ Q & V \end{bmatrix}$$

with row sums $r_\pi$ and column sums $r_{\pi^{\mathrm{c}}}$. We note that the bipartite graph of $P(\pi, \pi^{\mathrm{c}})$ is the undirected graph $H$ of $P$, and as such is connected. Hence by [3, Theorem 9] and Lemma 10, there is such a matrix $B$ if and only if (i) and (ii) hold. □

We observe that, under the hypotheses of Theorem 11, (i) and (ii) imply that $\sum_{j\in\alpha} r_j \neq \sum_{j\in\beta} r_j$ whenever $P$ has a single star with respect to $\alpha \subset \overline{n}$ and $\beta \subset \overline{n}$. Thus, (see Theorem 2), the necessity of the latter condition does not depend on the absence of an odd cycle in $H$.

## 4. The skew-symmetric problem

We are given an $n$-by-$n$ symmetric pattern $P$ with zero diagonal, and a vector $r$ in $\mathbb{R}^n$, the sum of whose entries is 0. We seek necessary and sufficient conditions on $P$ and $r$ that there exist a skew-symmetric matrix in $P$ with row sum vector $r$ (and consequently column sum vector $-r$). Of course if such a matrix exists, then by [3, Theorem 9] it is necessary that $\sum_{j\in\alpha} r_j \neq \sum_{j\in\beta}(-r_j)$ whenever $P$ has a single star for $\alpha \subset \overline{n}$ and $\beta \subset \overline{n}$. That an additional condition is necessary is illustrated by the following example:

$$P = \begin{bmatrix} 0 & * & * & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & * & * \\ 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & * & 0 \end{bmatrix}, \quad r = \begin{bmatrix} 2 \\ -4 \\ 2 \\ 2 \\ -4 \\ 2 \end{bmatrix}.$$

The matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -3 & 0 & -1 & 0 & 0 & 0 \\ 1 & 3 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 3 & 1 \\ 0 & 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

belongs to $P$ and has row sums $r$ and column sums $-r$. However, if $B$ is any matrix in $P$ with row sums $r$ and column sums $-r$, we compare the sum of the first three rows of $B$ with the sum of its first three columns, and see that the 3,4 entry and the 4,3 entry must be equal and nonzero. Thus $P$ allows no skew-symmetric matrix with row sum vector $r$.

We note that, in the undirected graph $H$ of $P$, the edge joining 3 and 4 is a “cut edge” separating vertices 1, 2, and 3 from vertices 4, 5, and 6. We also note that the failure of $P$ to contain a skew-symmetric matrix with row sums $r$ follows from the fact that $r_1 + r_2 + r_3 = 0$ (and hence $r_4 + r_5 + r_6 = 0$). These observations motivate the following solution to the skew-symmetric problem. If $H$ has a cut edge $w$ adjacent to a vertex $p$, then $V_p$ will denote the set of vertices in the component of $H - w$ that contains $p$.

**Theorem 12.** *Suppose P is an n-by-n symmetric pattern with zero diagonal, r is a vector in* $\mathbb{R}^n$ *the sum of whose entries is* 0, *and the undirected graph H of P is connected. Then there is a skew-symmetric matrix in P with row sum vector r if and only if the following two conditions hold*:

(i) *If P has a single star with respect to* $\alpha \subset \overline{n}$ *and* $\beta \subset \overline{n}$, *then* $\sum_{j \in \alpha} r_j \neq \sum_{j \in \beta} (-r_j)$.

(ii) *If H has a cut edge that is adjacent to vertex p, then* $\sum_{t \in V_p} r_t \neq 0$.

Our proof strategy is to find a matrix $A$ in $P$ with row sums $r$ and column sums $-r$ such that $\frac{1}{2}(A - A^{\mathrm{T}})$ is in $P$. We need the following lemmas. We assume throughout this section that $P$ is a symmetric $n$-by-$n$ pattern with a zero diagonal, $r$ is in $\mathbb{R}^n$, the sum of the entries in $r$ is 0, and the undirected graph $H$ of $P$ is connected.

**Lemma 13.** *Suppose H has a cut edge joining vertices p and q, and* $A = (a_{ij})$ *is any matrix in P with row sum vector r and column sum vector* $-r$. *Then* $a_{pq} - a_{qp} = 2 \sum_{t \in V_p} r_t$.

**Proof.** Since $H$ has a cut edge joining vertices $p$ and $q$, we may use permutation similarity to assume without loss of generality that $A$ has the form

$$A = \begin{bmatrix} B & C \\ C^{\mathrm{T}} & D \end{bmatrix},$$

where $C$ has exactly one nonzero entry, which lies in row $p$ and column $q$ of $A$, $B$ is $s$-by-$s$, and $V_p = \{1, 2, \ldots, s\}$. Let $S$ denote the sum of the entries in $B$. Adding all the entries in $B$ and $C$, we obtain $\sum_{t \in V_p} r_t = S + a_{pq}$. Adding all the entries in $B$ and $C^{\mathrm{T}}$, we obtain $\sum_{t \in V_p} (-r_t) = S + a_{qp}$. Subtracting these equations yields the result. □

**Lemma 14.** *Suppose A is in P and has row sum vector r and column sum vector* $-r$. *Suppose also that* $a_{pq} = a_{qp} \neq 0$, *and that the edge in H joining p and q is not a cut edge of H. Then there is a B in P with row sum vector r and column sum vector* $-r$ *such that* $b_{pq} \neq b_{qp}$ *and* $a_{ij} \neq a_{ji}$ *implies* $b_{ij} \neq b_{ji}$.

**Proof.** Since the edge joining $p$ and $q$ is not a cut edge, it lies on a cycle in $H$. Let the vertices of such a cycle be $p_1, p_2, \ldots, p_s$, where $p_1 = p_s = p$, $p_2 = q$, and $p_i \neq p_j$ unless $\{i, j\} = \{1, s\}$. Let $C = (c_{ij})$ be the $n$-by-$n$ matrix defined by

$$c_{ij} = \begin{cases} 1 & \text{if } i = p_t \text{ and } j = p_{t+1} \text{ with } 1 \leqslant t < s, \\ -1 & \text{if } i = p_{t+1} \text{ and } j = p_t \text{ with } 1 \leqslant t < s, \\ 0 & \text{otherwise.} \end{cases}$$

Then all linesums of $C$ are 0, so for any number $t$, the matrix $B = A + tC$ has row sum vector $r$ and column sum vector $-r$. If $t \neq 0$, then $b_{pq} \neq b_{qp}$. Furthermore, there are only finitely many choices of $t$ that would produce a 0 in $B$ where $A$ is nonzero, and only finitely many choices of $t$ that would produce $b_{ij} = b_{ji}$ where $a_{ij} \neq a_{ji}$. Hence we may select $t \neq 0$ so that $B$ establishes the lemma. □

**Proof of Theorem 12.** Suppose there is a skew-symmetric $A$ in $P$ with row sum vector $r$. Then (i) follows from [3, Theorem 9], and (ii) follows from Lemma 13.

Now suppose (i) and (ii) hold. (i) implies the existence of a matrix $A = (a_{ij})$ in $P$ with row sum vector $r$ and column sum vector $-r$. Let $1 \leqslant p, q \leqslant n$, and suppose $a_{pq} \neq 0$. Let $w$ be the edge in $H$ joining $p$ to $q$. If $w$ is a cut edge in $H$, (ii) together with Lemma 13 implies that $a_{pq} - a_{qp} \neq 0$. If $w$ is not a cut edge and $a_{pq} - a_{qp} = 0$, then by Lemma 14 we may perturb $A$ to make $a_{pq} - a_{qp} \neq 0$ without leaving $P$ and without creating any new instances of $a_{ij} - a_{ji} = 0$. We repeat this perturbation process for all $p, q$ for which $a_{pq} \neq 0$ and $a_{pq} - a_{qp} = 0$, and arrive eventually at a matrix $A$ in $P$ with row sum vector $r$, column sum vector $-r$, and $a_{ij} - a_{ji} \neq 0$ whenever $a_{ij} \neq 0$. It is then easy to check that the skew-symmetric matrix $\frac{1}{2}(A - A^{\mathrm{T}})$ is in $P$ and has row sum vector $r$. □

## 5. Corollaries involving the equation $Ax = y$

Theorems 2, 11, and 12 give conditions under which the equation $Ae = r$ has a solution $A$ in a symmetric pattern $P$, where $e$ denotes the vector of all 1's in $\mathbb{R}^n$. Easy corollaries solve the seemingly more general problem $Ax = y$ as follows. We retain the notation of the previous sections.

**Theorem 15.** *Suppose $P$ is an $n$-by-$n$ connected symmetric pattern. Let $x$ and $y$ be vectors in $\mathbb{R}^n$ with $x$ totally nonzero. Then there is a symmetric matrix $A$ in $P$ satisfying $Ax = y$ if and only if the following two conditions hold:*

(i) *If $P$ has a single star with respect to $\alpha \subset \overline{n}$ and $\beta \subset \overline{n}$, then $\sum_{j\in\alpha} x_j y_j \neq \sum_{j\in\beta} x_j y_j$.*

(ii) *If $P$ has a double star with respect to $\sigma \subset \overline{n}$, then $\sum_{j\in\sigma} x_j y_j \neq \sum_{j\in\sigma^{\mathrm{c}}} x_j y_j$.*

**Theorem 16.** *Suppose $P$ is a symmetric $n$-by-$n$ pattern whose undirected graph $H$ is connected but does not contain an odd cycle, so that $H$ is bipartite. Let $\{\pi, \pi^{\mathrm{c}}\}$ be the bipartition of $P$. Suppose $x$ and $y$ are vectors in $\mathbb{R}^n$ with $x$ totally nonzero. Then there is a symmetric matrix $A$ in $P$ satisfying $Ax = y$ if and only if the following two conditions hold:*

(i) $\sum_{i\in\pi} x_i y_i = \sum_{j\in\pi^{\mathrm{c}}} x_j y_j$.

(ii) *If $P$ has a double star with respect to $\sigma \subset \overline{n}$, then $\sum_{i\in\sigma\cap\pi} x_i y_i \neq \sum_{j\in\sigma^{\mathrm{c}}\cap\pi^{\mathrm{c}}} x_j y_j$.*

**Theorem 17.** *Suppose $P$ is a symmetric $n$-by-$n$ pattern with zero diagonal and $x$ and $y$ are vectors in $\mathbb{R}^n$ with $x$ totally nonzero, and the undirected graph $H$ of $P$ is* *connected. Then there is a skew-symmetric matrix A in P satisfying $Ax = y$ if and only if the following three conditions hold*:

(i) $\sum_{j=1}^{n} x_j y_j = 0$.

(ii) *If P has a single star with respect to* $\alpha \subset \overline{n}$ *and* $\beta \subset \overline{n}$, *then* $\sum_{j\in\alpha} x_j y_j \neq \sum_{j\in\beta}(-x_j y_j)$.

(iii) *If the undirected graph H of P has a cut edge that is adjacent to vertex p*, *then* $\sum_{t\in V_p} x_t y_t \neq 0$.

Theorems 15–17 are easily proved by applying Theorems 2, 11, and 12 (resp.) to the matrix $D_x A D_x$, where $D_x$ denotes the diagonal matrix with diagonal *x*.

## Acknowledgements

We wish to point out that there is a typographical error in the statement of Theorem 11 in [3]. The inequality in part (2) of that statement should read:

$$\sum_{j\in\alpha} u_j y_j \neq \sum_{k\in\beta} v_k x_k.$$

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