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On a generalization of the van der Waerden Theorem

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ABSTRACT

For a given length and a given degree and an arbitrary partition of the positive integers, there is always a cell containing a polynomial progression of that length and that degree; moreover, the coefficients of the generating polynomial can be chosen from a given subsemigroup and one can prescribe the occurring powers. A multidimensional version is included.

1. Introduction

A sequence in $\mathbb{R}$ will be called a *polynomial progression* if it is of the form $\{P(1), P(2), P(3), \ldots\}$ for some polynomial $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$. This progression is said to be of *degree d* if P has degree equal to d and not less.

Theorem 1. *Given two positive integers d and l, if the set of the positive integers is split up into finitely many non-overlapping parts, there exists a polynomial progression of length l and of degree d that belongs to precisely one of these parts.*

For $d = 1$ the polynomials look like $P(x) = a + bx$ and the first l-segment of the polynomial progression takes the form

$$(a + b, a + 2b, a + 3b, \ldots, a + lb):$$

the theorem boils down to the well-known van der Waerden Theorem on monochromatic arithmetic progressions.[1]

It is fun to write down the $d = 2$ case.

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[1] As it stands Theorem 1 is a direct consequence of van der Waerden's Theorem. In fact, given d and l, the van der Waerden Theorem guarantees the existence of a cell A in any given partition of $\mathbb{N}$, that contains a "long" arithmetic progression $\{b + a, b + 2a, \ldots, b + l^d a\}$. Write the generating polynomial as $P_1(x) = ax + b$ and let $P_d = ax^d + b$. Then range $P_d \subset$ range P_1, whence $\{P_d(1), P_d(2), \ldots, P_d(l) \subset A\}$. Different polynomials will appear in what follows.

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Corollary 1. *Given any $l \in \mathbb{N}$ and any finite coloring of $\mathbb{N}$, there exist three positive integers a, b and c for which all terms in $(a + b + c, a + 2b + 4c, a + 3b + 9c, \ldots, a + lb + l^2)$ have the same color.*

The 1927 proof of van der Waerden's Theorem is quite complicated, involving a double induction argument. The 1927 issue of the journal, [5], is difficult to access nowadays, but a very clear exposition is found in Graham, Rothschild and Spencer [3, pp. 29–34]. As B.L. van der Waerden once remarked, around 1927 he was not aware of the impact of his result as a prototypical Ramsey Theorem – after all, Ramsey's famous paper stems from 1930 – and merely considered it as a clever exercise. A proof of the above theorem by means of induction seems a Sisyphean task. We rather use some ideal theory in the semigroup $\beta\mathbb{N}$. As a matter of fact, the argument in the Hindman–Strauss treatise [4] for the van der Waerden Theorem (see 14.1 *l.c.*) is readily adapted to the present situation. By preferring the smooth $\beta\mathbb{N}$-argument to a complicated induction proof we ignore the calvinistic concern (see [4, p. 280]) that it "is enough to make someone raised on the work ethic feel guilty".[2]

Polynomial extensions of van der Waerden's Theorem are by no means a new topic. They have been studied extensively. Whereas the present paper remains within the realm of ideal theory for the semigroup $\beta\mathbb{N}$, beautiful results have been obtained by means of ergodic theory. We refer to Bergelson [1] for a survey.

The more restrictions one puts on the polynomials, the smaller the reservoir of admissible polynomials one has at his/her disposal and the more difficult it seems to force a polynomial progression belonging to one and the same cell. The polynomials $P(x) = \sum_{k=0}^{d} a_k x^k$ we admit here are subjected to the following restrictions:

- the admissible coefficients a_k belong to one and the same subsemigroup $\mathbb{S} \neq \{0\}$ of $(\omega, +)$, where $\omega = \mathbb{N} \cup 0$;
- the admissible exponents in the powers x^k belong to a subset $\mathbb{D} \subset \{0, 1, 2, \ldots, d\}$ containing 0 and d.

Such polynomials will be called $(\mathbb{S}, \mathbb{D})$-polynomials for short.

The sharpened theorem reads

Theorem 2. *Given two positive integers d and l, if the set of the positive integers is split up into finitely many non-overlapping parts, there exists a polynomial progression of length l and of degree d, generated by a $(\mathbb{S}, \mathbb{D})$-polynomial, that belongs to precisely one of these parts.*

Proof. Since Theorem 1 concerns the special case where $\mathbb{S} = \omega$ and $\mathbb{D} = \{0, 1, 2, \ldots, d\}$, we only need to prove Theorem 2.

Fix d and l in $\mathbb{N} = \{1, 2, 3 \ldots\}$. Without loss of generality we may assume that $l > d$. In fact, once the theorem has been proved for "long" progressions (that is $l > d$), then the pertinent cell certainly contains shorter segments ($l \leq d$). We consider polynomials $P(x) = \sum_{i=0}^{d} a_i x^i$ in one indeterminate x of degree $\leq d$ with coefficients in ω^{d+1}. Consider the following sets S_o and I_o in ω^l consisting of l consecutive polynomial values

$$S_o = \left\{ (P(1), P(2), \ldots, P(l)) \in \omega^l : \; P(x) = \sum_{k \in \mathbb{D}} a_k x^k, \text{ with } (a_0, a_1, \ldots, a_d) \in \mathbb{S}^{d+1} \right\}$$

$$I_o = \left\{ (P(1), P(2), \ldots, P(l)) \in \mathbb{N}^l : \; P(x) = \sum_{k \in \mathbb{D}} a_k x^k, \text{ with } (a_0, a_1, \ldots, a_d) \in (\mathbb{S} \cap \mathbb{N})^{d+1} \right\}.$$

The impact of the assumption that $l > d$ is that each element in S_o corresponds to a *unique* polynomial. In fact, if such an l-tuple would be generated by two different polynomials, the difference of these polynomials would have more zeros (*viz.* at the l points $1, 2, \ldots, l$ in $\mathbb{C}$) than its degree $d < l$ permits.

S_o is a subsemigroup of $\mathbb{S}^{d+1}$ under coordinatewise addition, the restrictions $k \in \mathbb{D}$ meaning that only addition of coordinates k from $\mathbb{D}$ matters. In fact, the sum of two l-tuples in S_o corresponds to the

[2] As usual, a finite partition of $\mathbb{N}$ is called a *coloring* and a subset belonging to one and the same part is *monochromatic*.

sum of their unique polynomials and the latter is again a polynomial of degree $\leq d$ with coefficients in the semigroup $\mathbb{S}$.

The progressions $(P(1), P(2), \ldots, P(l))$ in I_o all have degree $= d$, since $a_d \geq 1$. It follows that I_o is a proper subset of S_o. Obviously, I_o is also a semigroup. Moreover, I_o is a ideal in S_o. In fact, upon adding any point in S_o to an arbitrary element of I_o, all coefficients of the sum polynomial are again ≥ 1 and this polynomial is of exact degree d. Although trivial, we notice that S_o contains constant $\mathbb{S}$-valued polynomials, but I_o contains none of these. This will be instrumental shortly.

Consider the Stone–Čech compactification $\beta\omega$. We are going to use a few facts about $\beta\omega$ that are found in Hindman and Strauss [4]. We find it convenient to ignore the slight differences in the ideal theory between the two *semigroups* (see [4, Chap. 4]) $\beta\omega$ and $\beta\mathbb{N}$, writing $\beta\mathbb{N}$ where $\beta\omega$ would sometimes be more appropriate. From this point onwards we can follow the proof of the van der Waerden theorem in [4, Theorem 14.1], almost *verbatim*.

Take the compact product space $Y = (\beta\mathbb{N})^l$ and the closures $S = cl_Y(S_o)$ and $I = cl_Y(I_o)$. The semigroup $\beta\mathbb{N}$ has a *smallest* ideal $K(\beta\mathbb{N}) \neq \emptyset$ (see [4, Chap. 4],), which will be our main tool.

Take any point $p \in K(\beta\mathbb{N})$ and consider the constant l-tuple $\vec{p} = (p, p, \ldots, p)$. The crucial step is to show that $\vec{p}$ belongs to S.

The closures $cl_{\beta\mathbb{N}}B$ of the members $B \in p$ form a neighborhood basis in $\beta\mathbb{N}$ around p. It follows that for the product topology in Y there exist members $B_1, B_2, \ldots, B_r \in p$ for which the box $U = \prod_{1\leq i\leq r} cl_{\beta\mathbb{N}}(B_i)$ is a Y-neighborhood of $\vec{p}$. The intersection $\cap_{1\leq i\leq r} cl_{\beta\mathbb{N}}(B_i)$ is a $\beta\mathbb{N}$-neighborhood of p. The set $\mathbb{N}$ lying dense in $\beta\mathbb{N}$, is intersected by this neighborhood. Select $a \in \mathbb{N} \cap (\cap_{1\leq i\leq r} cl_{\beta\mathbb{N}}(B_i))$. The constant l-string $\vec{a} = (a, a, \ldots, a)$ thus belongs to U. Also, S_o containing all constant l-tuples, we have $\vec{a} \in S_o$. Consequently, we have $\vec{a} \in S_o \cap U$. This shows that $\vec{p}$ belongs to the closure of S_o in Y, and so $\vec{p} \in S$, indeed.

Next we use the fact that by [4, Theorem 2.23], the K-functor preserves products. From $p \in K(\beta\mathbb{N})$ we infer $\vec{p} \in (K(\beta\mathbb{N}))^l = K((\beta\mathbb{N})^l) = K(Y)$. Conclusion: $\vec{p} \in S \cap K(Y)$.

Having shown that $S \cap K(Y) \neq \emptyset$, we can invoke [4, Theorem 1.65], to determine the smallest ideal of the semigroup S: it is simply $K(S) = S \cap K(Y)$. This leads to

$$\vec{p} \in K(S). \tag{1.1}$$

Obviously, I is an ideal in S. The smallest ideal in S is contained in I: $K(S) \subset I$. It follows from (1.1) that $\vec{p} \in I$.

Finally, let $\mathbb{N} = \bigcup_i A_i$ be a finite partition. The closures $\bar{A}_i = cl_{\beta\mathbb{N}}A_i$ are open and form a partition of $\beta\mathbb{N}$. Hence, precisely one of them, $\bar{A}_j$ say, is a $\beta\mathbb{N}$-neighborhood of our point $p \in K(\beta\mathbb{N})$. Then $V = (\bar{A}_j)^l$ is a Y-neighborhood of $\vec{p}$. Since $\vec{p} \in I$, V must meet the dense subset I_o of I we can select a polynomial P in such a manner that $(P(1), P(2), \ldots, P(l))$ belongs to V. But the $P(1), P(2), \ldots, P(l)$ still are integers in $\mathbb{N}$. For this reason

$$\{P(1), P(2), \ldots, P(l)\} \subset \bar{A}_j \cap \mathbb{N} = A_j$$

and the segment $(P(1), P(2), \ldots, P(l)\}$ has the color of A_j. $\square$

2. Free gifts

The essential property of the set A_j used in the last part of the above proof is the fact that $\bar{A}_j$ contains a point p belonging to $K(\beta\mathbb{N})$, or $A_j \in p$. Sets $A \subset \mathbb{N}$ belonging to some $p \in K(\beta\mathbb{N})$ are called *piecewise syndetic* sets. We recall that in terms of $\mathbb{N}$ itself, A is piecewise syndetic if and only if the *gaps* between its intervals of consecutive elements remain bounded in lengths (see [4, Theorem 4.40]). It follows that A_j may be replaced by any infinite piecewise syndetic set A and we get as a

Bonus 3. *Given a piecewise syndetic set $A \subset \mathbb{N}$, a length l and a degree d, there exists a polynomial progression of degree d for which the first l terms belong to A.*

Finally we consider a *multidimensional* version of the theorem, dealing with m polynomial progressions of varying lengths and degrees simultaneously.

Bonus 4. *Pick the following items in $\mathbb{N}$: a dimension parameter m, degrees $d_1, d_2, \ldots, d_m$, and lengths $l_1, l_2, \ldots, l_m$. If the set $\mathbb{N}$ is split up into finitely many non-overlapping parts, there exist m polynomial progressions of length l_i and of degree d_i each, $1 \le i \le m$, that simultaneously belong to one of these parts. Also, any given piecewise syndetic set contains such a collection of polynomial progressions.*

Remark 5. There is an obvious $(\mathbb{S}, \mathbb{D})$ version.

Proof. We introduce arrays

$$\mathcal{P} = \begin{pmatrix} P_1(1) & P_1(2) \cdots P_1(l_1) \\ P_2(1) & P_2(2) \cdots P_2(l_2) \\ \cdots & \cdots \\ P_m(1) & P_m(2) \cdots P_m(l_m) \end{pmatrix}$$

generated by polynomials $P_1, P_2, \ldots, P_m$ with coefficients from ω.

These arrays $\mathcal{P}$ need not have the customary rectangular form, the ith row having l_i entries. Extending these rows by putting zeros in the empty places until they all get $\max\{l_i : i = 1, 2, \ldots, m\}$ entries would unnecessarily complicate the definition of $\mathcal{I}_o$ *infra*.

We have avoided calling these $\mathcal{P}$ matrices since they are not intended to act as transformations in some vector space. In order to describe the set these arrays belong to we write $e_i = \{0, \ldots, 1, 0, \ldots, 0\} = \delta_{ij}$ for the usual unit vectors in $\mathbb{R}^m$. These unit vectors are customarily envisaged as rows; upon transposition we get the unit columns e_i^{T}. The direct sum decomposition

$$\omega^m = e_1^{\mathrm{T}}\omega \oplus e_2^{\mathrm{T}}\omega \oplus \cdots \oplus e_m^{\mathrm{T}}\omega$$

divides ω^m, and thereby $\mathbb{N}^m$, into m horizontal layers, each equal to $\mathbb{N}$ and each row is an additive semigroup on its own.

$$e_i^{\mathrm{T}}\mathbb{N} = \begin{pmatrix} 0 \\ \cdots \\ 1\ 2\ 3\ 4\ 5\ \ldots \\ 0 \\ \cdots \\ 0 \end{pmatrix}.$$

Picture: the ith row of an array $\mathcal{P}$ is contained in the ith layer.

Upon replacing the l-tuples in the definitions of S_o and I_o by the arrays $\mathcal{P}$ we get

$$\mathcal{S}_o = \left\{ \mathcal{P} \in \bigoplus_{i=1}^{m} e_i^{\mathrm{T}}\mathbb{N} : \ P_i(x) = \sum_{k=0}^{d_i} a_{ki}x^k, \text{ with } (a_0, a_1, \ldots, a_{d_i}) \in \omega^{d_i+1} \text{ for } 1 = 1, 2, \ldots, m \right\},$$

$$\mathcal{I}_o = \left\{ \mathcal{P} \in \bigoplus_{i=1}^{m} e_i^{\mathrm{T}}\mathbb{N} : \ P_i(x) = \sum_{k=0}^{d_i} a_{ki}x^k, \text{ with } (a_0, a_1, \ldots, a_{d_i}) \in \mathbb{N}^{d_i+1} \text{ for } 1 = 1, 2, \ldots, m \right\}.$$

These are subsemigroups of the $\bigoplus_{i=1}^{m} e_i^{\mathrm{T}}\mathbb{N}$ and $\mathcal{I}_o$ is a proper ideal in $\mathcal{S}_o$.

We refrain from repeating all the details the above proof for the $m = 1$ case.

For a start, we may assume without loss of generality that $l_1 > d_1, l_2 > d_2, \ldots, l_m > d_m$. Define $l = \max_{1 \le i \le m} l_i$. This time we have to deal with the compact space Y^m, one $Y = (\beta\mathbb{N})^l$ for each layer, so that $Y^m = (\beta\mathbb{N})^{lm}$. The closure $\mathcal{I} = cl_{Y^m}(\mathcal{I}_o)$ is an ideal in the semigroup $\mathcal{S} = cl_{Y^m}(\mathcal{S}_o)$

To every $p \in K(\beta\mathbb{N})$ we assign the constant $m \times l$ array

$$\overrightarrow{\mathbf{p}} = \begin{pmatrix} p & p \cdots p \\ p & p \cdots p \\ \cdots & \cdots \\ p & p \cdots p \end{pmatrix}.$$

After a little twist the above argument leads to $\overrightarrow{\mathbf{p}} \in K(\mathcal{S}) \subset \mathcal{I}$. For a piecewise syndetic set $A \in p$ the product $V = (\bar{A})^{lm}$ is a Y^m-neighborhood of $\overrightarrow{\mathbf{p}}$ which intersects the dense subset $\mathcal{I}_o$ of $\mathcal{I}$ in at least one point. This point is an array $\mathcal{P}$, say. It follows that the entries $P_i(j)$ of $\mathcal{P}$ belong to $cl_{Y^m}A$ and thus to $cl_{\beta\mathbb{N}}A$. All $P_i(j)$ being positive integers, we may write

$$\bigcup\{P_i(j) : i = 1, \ldots, m; j = 1, \ldots l_i\} \subset \bar{A} \cap \mathbb{N}.$$

Conclusion: these m polynomial progressions do lie in A itself. □

Concluding remarks. The above proof of Theorem 2 employs only few elementary facts from the semigroup theory: $\beta\mathbb{N}$, ideals and $K(\beta\mathbb{N})$. More sophisticated approaches, employing heavier machinery, are possible. Here are two examples:

(i) Neil Hindman kindly pointed out (email) that a much shorter proof can be obtained by the Central Sets Theorem, and its Corollary 14.13 in [4].

(ii) We owe to Bernard Hoste the observation that Theorem 2 is an immediate consequence of a main result by Furstenberg [2] in dynamical topology.

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