Bicolored graph partitioning, or: gerrymandering at its worst

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\begin{abstract}
This study is motivated by an electoral application where we look into the following question: how much biased can the assignment of parliament seats be in a majority system under the effect of vicious gerrymandering when the two competing parties have the same electoral strength? To give a first theoretical answer to this question, we introduce a stylized combinatorial model, where the territory is represented by a rectangular grid graph, the vote outcome by a “balanced” red/blue node bicoloring and a district map by a connected partition of the grid whose components all have the same size. We constructively prove the existence in cycles and grid graphs of a balanced bicoloring and of two antagonist “partisan” district maps such that the discrepancy between their number of “red” (or “blue”) districts for that bicoloring is extremely large, in fact as large as allowed by color balance.
\end{abstract}

1. Introduction

Not long after the dawn of modern democracies, in which the lawmaking power is delegated by citizens to elected representatives, insidious practices started to creep in, aimed to favor a certain candidate or party through the artful design of electoral district boundaries. These malpractices, which came to be known under the name of gerrymandering,\textsuperscript{1} have occurred numerous times throughout the modern history of elections (see [5]) and pose a dangerous threat even nowadays [2]. In order to oppose gerrymandering practices, some districting criteria are commonly adopted: integrity (no unit may be split between two or more districts); contiguity (the units within the same district should be geographically contiguous); population equality (the district populations should be equal or nearly equal, especially in majoritarian systems); compactness (each district should be compact, that is, according to the Oxford Dictionary, “closely and neatly packed together”); conformity to administrative boundaries (the electoral district boundaries should not cross other administrative boundaries, such as those of regions, provinces, local or minority communities).

The aim of the present paper is to give a theoretical answer to the question: “how bad can the outcome of gerrymandering be?” Basically, our answer will be: “as bad as materially possible” (we are going to give a precise meaning to this statement later). Our worst-case analysis will be performed on a stylized combinatorial model of elections, which generalizes a classical example of Dixon and Plischke [3], showing how gerrymandering can dramatically reverse the election outcome.

We recall here Dixon and Plischke’s example.

Suppose that only two parties P and C compete under a first-past-the-post system and that, as in Fig. 1, the territory is divided into elementary units having the same population with an homogeneous electoral behavior, that is, the whole

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\textsuperscript{1} In 1810 Elbridge Gerry, governor of Massachusetts, enacted a salamander-shaped district so as to enhance the probability of being re-elected. Hence the term “Gerrymander” (a contraction of Gerry+Salamander).

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population of an elementary unit votes for the same party. If the district map of Fig. 1(a) is adopted, party C wins in 8 districts out of 9; however, if the alternative district map of Fig. 1(b) is adopted, party C wins only in 2 districts out of 9, so the outcome is drastically reversed.

Notice that in this example the two parties feature nearly equal overall electoral strengths: 24 units vote for party C and 21 units vote for party P. Likewise, in our analysis we shall assume that the total number of votes is equally, or nearly equally, split between the two parties.

A careful look at Fig. 1 gives us a clue about an effective strategy for maximizing the number of districts won by either party: the districts should be designed so that every win should be close and every loss should be sweeping.

2. Problem statement and paper outline

In this section we shall consider an idealized graph-theoretic formulation that captures the essence of the example by Dixon and Plischke. Given a territory composed by territorial units, define the following integers:

- $n$ is the number of territorial units;
- $p$ is the number of districts;
- $s$ is the common district size (number of territorial units in each district).

We shall make the following assumptions on the integers $n$, $s$, and $p$:

- $s$ is odd and $s \geq 3$: these assumptions forbid trivial cases and ties between the two parties;
- the relation $n = ps$ holds.

We model the territory as an undirected graph $G = (V, E)$ with $|V| = n$, where the vertices represent territorial units and the edges represent adjacency between territorial units.

A vote outcome is a bicolored of the vertices that assigns to each vertex either the color blue or the color red: this means that all voters in the corresponding unit vote for the same party, blue or red, respectively. A vote outcome is balanced if the total number $n_b$ of blue vertices and that $n_r$ of red ones satisfy the relation $|n_b - n_r| \leq 1$; that is, $n_b = n_r$ when $n$ is even and, without loss of generality, $n_b = n_r + 1$ when $n$ is odd. A balanced vote outcome corresponds to a situation in which the electoral population is split as equally as possible between two parties. From now on we shall consider only balanced vote outcomes.

A connected partition of $G$ is a partition of its set of vertices $V$ such that each component induces a connected subgraph of $G$. A (connected) s-equipartition of $G$ is a connected partition such that each component has $s$ vertices. We say that a graph $G$ is s-equipartitionable if there exists some connected s-equipartition of $G$. Notice that under the above assumption $n = ps$, a connected s-equipartition has $p$ components. In the following we will refer to connected s-equipartitions also as district maps and to their components also as districts. This definition takes into account the criteria of integrity, contiguity and population equality.

We will denote by $\Omega$ the set of all possible balanced vote outcomes and by $\Pi$ the set of all district maps. We define an electoral competition to be a pair $(\omega, \pi)$ such that $\omega \in \Omega$ and $\pi \in \Pi$.

In the literature there exist other types of related models that can be used for political districting. In particular one could refer to discrete geometry on red and blue points in the plane (see e.g. [7]) to represent territorial units as bicolored points in the plane and districts as subdivisions of the plane into internally disjoint convex polygons.

Given an electoral competition $(\omega, \pi)$, if in a district $D \in \pi$ the number of blue vertices is greater than the number of red ones, we will say that $D$ is a blue district. In a similar way we define a red district. Notice that a blue (red) district has at least $(s + 1)/2$ blue (red) vertices and at most $(s - 1)/2$ red (blue) vertices. We will refer to the partition $\pi$ as a blue partition if the number of blue districts is greater than the number of red ones. In a similar way we define a red partition. The functions $b(\omega, \pi)$ and $r(\omega, \pi)$ count the number of blue and red districts, respectively, resulting from the electoral competition $(\omega, \pi)$. Let

$$B(G) = \max_{\omega \in \Omega, \pi \in \Pi} b(\omega, \pi)$$
be the maximum number of blue districts for all the electoral competitions \((\omega, \pi) \in \Omega \times \Pi\). In a similar way we can define \(R(G)\).

Under the vote balance condition, whatever the district map, neither party can win in all districts, since the excess of blue votes in the blue districts must be compensated by a surplus of red votes in the red districts. On these grounds, in Section 3 we derive the following upper bounds on the maximum number of districts that can be won by either party:

If \(n\) is even then
\[
b(\omega, \pi), r(\omega, \pi) \leq \lceil n/(s + 1) \rceil,\]
if \(n\) is odd then
\[
b(\omega, \pi) \leq \lfloor (n + 1)/(s + 1) \rfloor,\]
\[
r(\omega, \pi) \leq \lfloor (n - 1)/(s + 1) \rfloor.\]

For a given bicoloring \(\omega \in \Omega\) a partition \(\pi\) will be called \((\text{blue})\) extremal w.r.t. \(\omega\) if the number \(b(\omega, \pi)\) of blue districts in \(\pi\) attains its upper bound. A similar concept can be introduced for the red party. It is not hard to prove that the above upper bounds are sharp. An explicit formula for \(B(G)\) and \(R(G)\) ensues (see Section 3). A more challenging problem consists of finding, for a given \(\omega \in \Omega\), the range of all possible values for the number \(b(\omega, \pi)\) of blue districts when \(\pi \in \Pi\). Formally we define the gap of \(\omega\) to be:
\[
\text{gap}(\omega) = \max_{\pi \in \Pi} b(\omega, \pi) - \min_{\pi \in \Pi} b(\omega, \pi) = \max_{\pi \in \Pi} b(\omega, \pi) + \max_{\pi \in \Pi} r(\omega, \pi) - p.
\]

Broadly speaking, the gap quantifies, for a given vote outcome, the maximum reversal of the number of seats that can be obtained by two opposite partisan district maps; in a certain sense, it measures the maximum "bias" that can be produced by gerrymandering on both sides for that vote outcome. Our aim being a worst-case analysis of the bias induced by gerrymandering, we ask whether there are any vote outcomes for which the bias turns out to be extremely large. Having this in mind, we formally introduce the following optimization problem:

\[
\text{GAP}(G) = \max_{\omega \in \Omega} \text{gap}(\omega).
\]

For a given graph \(G\) the function \(\text{GAP}(G)\) is a measure of the maximum bias of an electoral outcome in terms of number of seats in single member majority districts. Our main results imply that any grid graph has the following property: with respect to the same balanced vote outcome, there exist both a blue extremal partition and a red extremal partition. Graphs having this property, and the corresponding balanced vote outcome (or bicoloring), will be called two-faced. In a two-faced graph, we can obtain a simple explicit formula for the gap. In this case, gerrymandering has the ability to reverse, as much as permitted by sheer vote balance, the outcome of an election in terms of parliament seats.

The following result relates the function \(\text{GAP}(G)\) to \(B(G)\) and \(R(G)\).

**Proposition 2.1.** \(\text{GAP}(G) \leq B(G) + R(G) - p\).

**Proof.** Since \(b(\omega, \pi) + r(\omega, \pi) = p\), then
\[
\text{GAP}(G) = \max_{\omega \in \Omega} \left( \max_{\pi \in \Pi} b(\omega, \pi) + \max_{\pi \in \Pi} r(\omega, \pi) \right) - p \\
\leq \max_{\omega \in \Omega} \max_{\pi \in \Pi} b(\omega, \pi) + \max_{\omega \in \Omega} \max_{\pi \in \Pi} r(\omega, \pi) - p = B(G) + R(G) - p. \quad \Box
\]

**Corollary 2.2.** We have
\[
\text{GAP}(G) = B(G) + R(G) - p
\]
if and only if \(G\) is two-faced.

**Proof.** Follows from (1). \(\Box\)

It is worth mentioning that there are alternative ways to look at a worst-case analysis. For example, if one assumes that the vote outcome \(\omega\) is fixed once and for all, what is the value of the gap, or at least some bounds on it? A partial experimental answer to this question has been given in [1], where we report that gerrymandering can almost fully reverse the outcome of the elections in all the regions of our sample, except for one of them where the vote is quite unbalanced. So, instead of asking oneself about the worst damage that gerrymandering can do for some vote outcomes, one can ask whether gerrymandering can do at least a certain amount of damage in all the possible vote outcomes. In other words, one wants to minimize, over all possible vote outcomes, the gap of the vote outcome. This is also a legitimate worst-case analysis of gerrymandering, and a very interesting topic for future research. Another way of looking at the same question is: are there any vote outcomes that are gerrymandering-proof? For example, at first sight a "chessboard" bicoloring of a grid – that is, a bicoloring of the grid as...
a bipartite graph – would appear to be such, but one can provide examples showing that even such a bicoloring is prone to a considerable bias due to gerrymandering.

We now deal with the question of the existence of (connected) $s$-equipartitions in a graph. It is easy to show that, if $s$ is any positive divisor of $n$, any graph with a hamiltonian path or cycle is $s$-equipartitionable. Thus paths, cycles and grid graphs are $s$-equipartitionable. This positive result on grid graphs may lead one to hope that the property of being $s$-equipartitionable can be easily recognized, at least in bipartite graphs. The following negative result, by Dyer and Frieze [4], defeats this hope.

**Theorem 2.3.** Let $G$ be a graph with $n$ vertices, and let $s$ be a positive integer divisor of $n$. Deciding whether $G$ is $s$-equipartitionable is NP-complete even when $G$ is bipartite.

However, when $G$ is sufficiently connected, $G$ turns out to be $s$-equipartitionable, as Corollary 2.5 below shows.

**Theorem 2.4.** If $G$ is $p$-connected and $s_1, \ldots, s_p$ are any $p$ positive integers such that $s_1 + \cdots + s_p = n$, then there always exists a connected partition of $G$ into $p$ components with sizes $s_1, \ldots, s_p$, respectively.

**Proof.** See Győri [6] and Lovász [8].

**Corollary 2.5.** Every $p$-connected graph with $n = ps$ vertices is $s$-equipartitionable.

In the present paper we only deal with cycles and grid graphs; for these graphs, as mentioned above, the condition $n = ps$ is both necessary and sufficient for the existence of an $s$-equipartition.

Here is an outline of our paper. After providing the electoral motivation of our study (Section 1) and formally defining the graph-theoretic problems under investigation together with the appropriate notation (Section 2), in Section 3 we present some useful arithmetic properties of extremal partitions in an arbitrary graph. Section 4 includes our main results: all cycles and all grid graphs are two-faced. In fact, the result for cycles implies that every hamiltonian graph is two-faced; in particular, even grid graphs are such (the result for odd grid graphs is trickier to prove). Finally, in Section 5 we exhibit some simple and not so simple examples of graphs that are not two-faced.

Some of our results were presented in a previous paper of ours [1], where, however, only the case of even $n$ was dealt with and different constructions were employed.

3. Structure and arithmetic properties of extremal partitions in general graphs

We start this section with some upper bounds on the number of blue and red districts for a given electoral competition. We say that a district is (blue) edgy if it contains $(s + 1)/2$ blue vertices and $(s − 1)/2$ red vertices. Similarly a (red) edgy district contains $(s + 1)/2$ red vertices and $(s − 1)/2$ blue vertices.

**Proposition 3.1.** If $G$ is $s$-equipartitionable, then:

$$B(G) = \left\lfloor \frac{(n + \delta)}{(s + 1)} \right\rfloor$$

and

$$R(G) = \left\lfloor \frac{(n - \delta)}{(s + 1)} \right\rfloor,$$

where $\delta = 0, 1$ according to $n$ even or odd, respectively.

**Proof.** Obviously, it is optimal for the blue party to have $(s + 1)/2$ voters (or vertices) in as many districts as possible. Since the blue party has $(n + \delta)/2$ voters we have:

$$\frac{n + \delta}{2} = B(G) \frac{s + 1}{2} + k_B$$

where $0 \leq k_B < \frac{S + 1}{2}$ is the remainder of the division of $\frac{n + \delta}{2}$ by $\frac{s + 1}{2}$. By similar arguments, the maximum number of red districts can be obtained by the following relation:

$$\frac{n - \delta}{2} = R(G) \frac{s + 1}{2} + k_R$$

where $0 \leq k_R < \frac{s + 1}{2}$ is the remainder of the division of $\frac{n - \delta}{2}$ by $\frac{s + 1}{2}$. \hfill \□

**Remark 3.1.** Notice that $k_B$ is the maximum number of blue vertices that belong to the red districts of a blue extremal partition, for all the electoral competitions $(\omega, \pi) \in \Omega \times \Pi$. Similarly $k_R$ is the maximum number of red vertices that belong to the blue districts of a red extremal partition, for all the electoral competitions $(\omega, \pi) \in \Omega \times \Pi$. Moreover, if $k_B = 0 \ (k_R = 0)$ the bound $B(G) (R(G))$ is attained only when all blue (red) districts are edgy.

**Corollary 3.2.** Given an $s$-equipartitionable graph $G$, for any $(\omega, \pi) \in \Omega \times \Pi$ the following inequalities hold:

$$b(\omega, \pi) \leq \left\lfloor \frac{(n + \delta)}{(s + 1)} \right\rfloor,$$

$$r(\omega, \pi) \leq \left\lfloor \frac{(n - \delta)}{(s + 1)} \right\rfloor.$$

**Proof.** Follows from Proposition 3.1 \hfill \□
Corollary 3.3. If \( G \) is \( s \)-equipartitionable, and \( p = q(s + 1) + r \) with \( 1 \leq r \leq s + 1 \) then\(^2\):

\[
B(G) = \begin{cases} 
qs + r - 1 & \text{if } r \geq 2 \\
qs + r & \text{if } r = 1 
\end{cases}
\]

and

\[
R(G) = qs + r - 1.
\]

Hence \( B(G) = R(G) \), unless \( r = 1 \), in which case \( B(G) = R(G) + 1 \).

**Proof.** Since \( s \) is odd, \( n, p, \delta \) and \( r \) have all the same parity. From *Proposition 3.1* one has:

\[
B(G) = \left\lfloor \frac{n + \delta}{s + 1} \right\rfloor = qs + \left\lfloor \frac{rs + \delta}{s + 1} \right\rfloor.
\]

Since \( r - \delta \leq s + 1 \) and \( r \) and \( \delta \) have the same parity, one has

\[
\left\lfloor \frac{rs + \delta}{s + 1} \right\rfloor = \left\lfloor \frac{r - r - \delta}{s + 1} \right\rfloor = \begin{cases} 
q - 1 & \text{if } r \geq 2 \\
q & \text{if } r = 1 
\end{cases}
\]

hence

\[
B(G) = \begin{cases} 
qs + r - 1 & \text{if } r \geq 2 \\
qs + r & \text{if } r = 1 
\end{cases}.
\]

Similarly, from *Proposition 3.1* one has:

\[
R(G) = \left\lfloor \frac{n - \delta}{s + 1} \right\rfloor = qs + \left\lfloor \frac{rs - \delta}{s + 1} \right\rfloor.
\]

Since \( r \) and \( \delta \) have the same parity one gets \( r + \delta \leq s + 1 \), hence

\[
\left\lfloor \frac{rs - \delta}{s + 1} \right\rfloor = \left\lfloor \frac{r + \delta}{s + 1} \right\rfloor = r - 1.
\]

It follows:

\[
R(G) = qs + r - 1. \quad \square
\]

Let \( \gamma \) be defined as follows:

\[
\gamma = \begin{cases} 
0 & \text{if } r \geq 2 \\
1 & \text{if } r = 1 
\end{cases}.
\]

We can write:

\[
B(G) = qs + r - 1 + \gamma.
\]

Two-faced graphs are those for which gerrymandering exhibits its worst case bias. There is an absolute threshold for the maximum number of seats that a party can obtain when the vote outcome is balanced. In two-faced graphs, for a suitable balanced vote, both parties can achieve this threshold by artful gerrymandering. By *Corollary 3.3* this threshold is equal for the red and the blue party except when \( r = 1 \).

We say that a blue extremal partition is \emph{blue edgy} if it has a red district containing \( k_B \) blue vertices. Similarly we say that a red extremal partition is \emph{red edgy} if it has a blue district containing \( k_B \) red vertices. In a blue (red) edgy extremal partition all blue (red) districts are edgy, and at most one red (blue) district contains blue (red) vertices. We will use edgy extremal partitions in the next section, where we will show that \( s \)-equipartitionable cycles and grid graphs are two-faced.

The following definitions are useful to describe edgy extremal partitions. Given an electoral competition \((\omega, \pi) \in \Omega \times \Pi\), we say that a district of \( \pi \) is:

- \emph{(blue) sweeping} if all its vertices are blue;
- \emph{(red) sweeping} if all its vertices are red;
- \emph{(blue) quasi sweeping} if it contains \( k_B \) red vertices and \( s - k_B \) blue vertices;
- \emph{(red) quasi sweeping} if it contains \( k_B \) blue vertices and \( s - k_B \) red vertices.

*Proposition 3.4* yields the actual values of \( k_B \) and \( k_R \); as expected, they are smaller than \( (s + 1)/2 \).

\(^2\) Notice that \( q \) and \( r \) might not coincide with the quotient and the remainder, respectively, of the division of \( p \) by \( s + 1 \).
Table 1
Arithmetic characteristics of edgy extremal partitions.

<table>
<thead>
<tr>
<th></th>
<th>Blue edgy extr. part.</th>
<th>Red edgy extr. part.</th>
</tr>
</thead>
<tbody>
<tr>
<td>N. of edgy districts</td>
<td>(qs + r - 1 + \gamma)</td>
<td>(qs + r - 1)</td>
</tr>
<tr>
<td>N. of sweeping districts</td>
<td>(q)</td>
<td>(q)</td>
</tr>
<tr>
<td>N. of quasi sweeping districts</td>
<td>(1 - \gamma)</td>
<td>(1)</td>
</tr>
<tr>
<td>(k_B, k_R)</td>
<td>(-\gamma(\frac{s+1}{2}))</td>
<td>(-\gamma(\frac{s+1}{2}))</td>
</tr>
</tbody>
</table>

Proposition 3.4. We have:

\[
k_B = \frac{s - r + 1 + \delta}{2} - \gamma \left(\frac{s + 1}{2}\right),
\]

\[
k_R = \frac{s - r + 1 - \delta}{2}.
\]

Proof. Follows from the proof of Proposition 3.1 and from Corollary 3.3.

Notice that if \(k_B = 0 (k_R = 0)\) then blue (red) quasi sweeping districts are sweeping. Moreover, by Proposition 3.4, \(k_B = 0\) if \(r = 1\) or \(r = s + 1\) and \(k_R = 0\) if \(r = s + 1\). By Remark 3.1 in these cases any blue (red) extremal partition is edgy since all blue (red) districts are edgy and all red (blue) districts are sweeping.

Table 1 summarizes some information related to edgy extremal partitions that will be useful in the next section.

4. Two-facedness of cycles and grid graphs

4.1. Gerrymandering on cycles

In this section we will show that, under the hypothesis that \(s\) is odd, any cycle \(H = (V_H, E_H)\) having \(n = ps\) vertices is two-faced. Since \(H\) is a cycle, any partition into \(p\) connected components can be obtained by cutting \(p\) edges. Moreover any \(s\)-equipartition is uniquely determined by one of its cuts and can be obtained from any other \(s\)-equipartition by a shifting of all cuts by \(t\) edges in the same direction, for a given \(t \in \{1, \ldots, s - 1\}\). In the following we will fix a shifting direction, say clockwise, and, given an \(s\)-equipartition, we will refer to a shifting of all cuts by \(t\) edges in this direction as a \(t\)-rotation, \(t \in \{1, \ldots, s - 1\}\).

We will show that any \(s\)-equipartitionable cycle admits a two-faced bicoloring of the vertices such that the red extremal partition can be obtained from the blue one by an \((s - 1)/2\)-rotation; then any \(s\)-equipartitionable cycle is two-faced. For this reason we will say that \(\rho = (s - 1)/2\) is the rotation number. Let a block be a subpath of \(H\) and a \(k\)-block be a block having \(k\) vertices. Two blocks \(A\) and \(B\) are adjacent if in \(H\) there exist two adjacent vertices \(v\) and \(u\) such that \(v \in A\) and \(u \in B\). Notice that a \(\rho\)-block and an adjacent \((\rho + 1)\)-block form a district.

Let us consider the bicolorized cycle of Fig. 2 (a) where \(n = 30\) and \(s = 5\). By Table 1 a blue edgy extremal partition has five blue edgy districts and one red sweeping district and symmetrically a red edgy extremal partition has five red edgy districts and one blue sweeping district. Here and in the following figures we will represent blue vertices in black and red vertices in white. The red edgy extremal partition can be obtained from the blue one by a \(\rho\)-rotation (with \(\rho = 2\)).

Fig. 2. Bicolored cycles and extremal partitions: (a) \(n = 30, s = 5\); (b) \(n = 35, s = 7\).
In Fig. 2(b) we consider a bicolor graph with \( n = 35 \) and \( s = 7 \). In this case a blue edgy extremal partition has four blue edgy districts and one red quasi sweeping district with \( k_8 = 2 \) blue vertices and a red edgy extremal partition has four red edgy districts and one blue quasi sweeping district with \( k_B = 1 \) red vertices. Also in this case the red extremal partition is obtained from the blue one by a \( \rho \)-rotation (with \( \rho = 3 \)). In both the examples, the cycle is partitioned into a sequence of \( \rho \)-blocks and \((\rho + 1)\)-blocks.

The bicolourings of Fig. 2 are based on the following idea: the cycle is visited clockwise and the colors are assigned alternately to the vertices of a \( \rho \)-block and of a \((\rho + 1)\)-block in such a way that each \( \rho \)-block forms, together with the previous \((\rho + 1)\)-block, a district of an edgy red extremal partition and, at the same time, with the next \((\rho + 1)\)-block, it forms a district of an edgy blue extremal partition. As an example, consider the cycle of Fig. 2(b). Starting from vertex 1, the vertices of the first \( \rho \)-block \([1, 2, 3]\) are colored in blue; then in order to obtain a blue edgy district, the \((\rho + 1)\)-block \([4, 5, 6, 7]\) must have one blue vertex and three red vertices. After that, the \( \rho \)-block \([8, 9, 10]\) must have one red vertex and two blue vertices so that together with \([4, 5, 6, 7]\) it forms a red edgy district. The same criterion is used in order to color the blocks \([11, 12, 13, 14]\), \([15, 16, 17]\) and \([18, 19, 20, 21]\). This last \((\rho + 1)\)-block has three blue vertices and then we can obtain a blue quasi sweeping district by assigning the color blue to all the vertices of \([22, 23, 24]\). Then the block \([25, 26, 27, 28]\) has one blue vertex and three red vertices so that it forms a blue edgy district together with \([22, 23, 24]\). Next the block \([29, 30, 31]\) has two blue vertices and one red vertex so that it forms together with block \([25, 26, 27, 28]\) a red edgy district. Finally, the block \([32, 33, 34, 35]\) has four red vertices so that district \([29, 30, 31, 32, 33, 34, 35]\) is red quasi sweeping and district \([32, 33, 34, 35, 1, 2, 3]\) is red edgy. Summarizing, we colored eighteen vertices in blue and seventeen vertices in red; by cutting the edge \((35, 1)\) we obtain a blue edgy extremal partition and by cutting the edge \((3, 4)\) we obtain a red edgy extremal partition.

We start formalizing these ideas for \( 3 \leq \rho = \tau \leq s + 1 \). As we will see, the general construction that will be provided later on in the sequence \((6)\) holds also when \( \rho = 1, 2 \). We denote by \( S(h) \) a \( \rho \)-block containing \( h \) red vertices and \( \rho - h \) blue vertices, and by \( L(h) \) a \((\rho + 1)\)-block containing \( h \) red vertices and \( \rho + 1 - h \) blue vertices.

A cobra \( C(i, j) \) is a sequence of blocks defined as follows:

\[
C(i, j) = S(i)L(\rho - i)S(i + 1)L(\rho - (i + 1)) \ldots S(j)L(\rho - j)
\]

with \( 0 \leq i \leq j \leq \rho \). If \( i > j \) we define \( C(i, j) \) to be the empty sequence. Notice that a cobra \( C(i, j) \) contains \( \max\{j - i + 1, 0\} \) blue edgy districts.

A shifted cobra \( \overline{C}(i, j) \) is the sequence:

\[
\overline{C}(i, j) = L(\rho + 1 - i)S(i)L(\rho - i)S(i + 1) \ldots L(\rho + 1 - j)S(j)
\]

with \( 0 \leq i \leq j \leq \rho \). As for cobras, if \( i > j \) we define \( \overline{C}(i, j) \) to be the empty sequence. A shifted cobra contains \( \max\{j - i + 1, 0\} \) red edgy districts. In Fig. 3 we show a cobra and a shifted cobra for \( \rho = 3 \).

The following relations hold:

\[
\begin{align*}
C(i, j) &= S(i)\overline{C}(i + 1, j)L(\rho - j) \\
C(i, j)L(j + 1) &= S(i)\overline{C}(i + 1, j + 1) \\
L(\rho + 1 - i)C(i, j) &= \overline{C}(i, j)L(\rho - j) \\
\end{align*}
\]

In the sequence:

\[
C(0, \rho - k_8)C(0, \rho - k_B - 1)S(\rho - k_B)L(\rho + 1) \quad (4)
\]

the cobra \( C(0, \rho - k_B) \) contains \( \rho - k_B + 1 \) blue edgy districts and the cobra \( C(0, \rho - k_B - 1) \) contains \( \rho - k_B \) blue edgy districts. Remember that \( k_B, k_8 \leq \rho \), so \( \rho - k_B, \rho - k_B \geq 0 \). Moreover, the last two blocks in \((4)\) form a red quasi sweeping district. Hence the above sequence admits a unique equipartition that is blue edgy extremal.

On the other hand, the sequence:

\[
\overline{C}(1, \rho - k_B)L(k_B)S(0)\overline{C}(1, \rho - k_B)L(\rho + 1)S(0) \quad (5)
\]

admits a unique equipartition that is red edgy extremal. Since both the sequences \((4)\) and \((5)\) involve the same number of vertices, either of them could be used to color a cycle formed by such vertices. Since, by relations \((3)\), the two sequences can be obtained from each other by a \( \rho \)-rotation, we conclude that \((4)\) and \((5)\) provide a bicoloring of the cycle for which there exists both a blue and a red extremal partition.

As an example, we consider again the cycles of Fig. 2. In Fig. 4 we show that the sequences

\[
C(0, 2)C(0, 1)S(2)L(3)
\]
and
\[ C(0, 2)C(0, 0)S(1)L(4) \]
provide the bicolorings for the cycles of Fig. 2(a) and (b), respectively, and that by a \( \rho \)-rotation we obtain the following sequences:
\[ \overline{C}(1, 2)L(0)S(0)\overline{C}(1, 2)L(3)S(0) \]
and
\[ \overline{C}(1, 2)L(1)S(0)\overline{C}(1, 1)L(4)S(0). \]

Notice that in example (a) we have \( k_R = k_B = 0 \), while in example (b) we have \( k_R = 1 \) and \( k_B = 2 \).

Let us consider now the general case. As in Section 3, we write \( p = q(s + 1) + r \) where \( q \geq 0 \) and \( 1 \leq r \leq s + 1 \). As shown in Table 1, a blue edgy extremal partition has \( qs + r - 1 + \gamma \) blue edgy districts and \( q + 1 - \gamma \) red sweeping or quasi sweeping districts containing an overall number \( k_B \) of blue vertices. A red edgy extremal partition has \( qs + r - 1 \) red edgy districts and \( q + 1 \) blue sweeping or quasi sweeping districts containing an overall number \( k_R \) of red vertices. One can imagine the cycle with \( n = ps = qs(s + 1) + rs \) vertices partitioned into \( q \) paths having \( (s + 1)s \) vertices each, and one more path having \( rs \) vertices. In a (blue or red) edgy extremal partition, for any of the first \( q \) paths there must be one sweeping district and \( s \) edgy districts, while in the last path there must be one quasi sweeping district and \( r - 1 \) edgy districts if \( r \geq 2 \), or one blue edgy (blue quasi sweeping in a red extremal partition) district if \( r = 1 \). Hence using the sequence
\[
q \text{ times: } C(0, \rho)C(0, \rho - 1)S(\rho)L(\rho + 1)
\]
\[
C(0, \rho - k_R) \quad \text{if } r \geq 2
\]
\[
C(0, \rho - k_B - 1)S(\rho - k_B)L(\rho + 1),
\]
which contains \( q + 1 \) sequences of the type \( (4) \), we can show that any \( s \)-equipartitionable cycle is two-faced.

We call the attention of the reader on the fact that the case \( r = 1 \) is inherently different from the case \( r \geq 2 \), since, by Corollary 3.3, it is the only case where the number of red districts in a red extremal partition does not match the number of blue districts in a blue extremal partition.

**Remark 4.1.** If \( r = 2 \) then \( k_B = \rho \); hence the cobra \( C(0, \rho - k_B - 1) \) is empty and the sequence \( C(0, \rho - k_B)C(0, \rho - k_B - 1)S(\rho - k_B)L(\rho + 1) \) contains two districts.

**Lemma 4.1.** Sequence \((6)\) contains \( n = ps \) vertices.

**Proof.** The \( q \) sequences \( C(0, \rho)C(0, \rho - 1)S(\rho)L(\rho + 1) \) contain \( q(s + 1) \) vertices.

**Case** \( r \geq 2 \)

\( k_B, k_R \leq \rho \), so the last two cobras of \( (6) \) contain \( 2, \rho - k_R - k_B + 1 = r - 1 \) districts. Then, adding the district \( S(\rho - k_B)L(\rho + 1) \), we have \( q(s + 1) + r = p \) districts, that is, \( n = ps \) vertices.

**Case** \( r = 1 \)

\( k_B = \rho \), so the last cobra of \( (6) \) contains one district. Hence we have \( q(s + 1) + r = p \) districts, that is, \( n = ps \) vertices. \( \square \)

**Lemma 4.2.** The unique \( s \)-equipartition of sequence \((6)\) is blue extremal.
Proof. The q pairs of cobras $C(0, \rho), C(0, \rho - 1)$ contain qs blue edgy districts.

Case $r \geq 2$
The two cobras $C(0, \rho - k_0)$ and $C(0, \rho - k_0 - 1)$ contain $2\rho - k_0 - k_0 + 1 = r - 1$ blue edgy districts.

Case $r = 1$
The cobra $C(0, \rho - k_0)$ contains one blue edgy district if $r = 1$. Hence in any case the upper bound on the number of blue edgy districts is attained. □

By relations (3), after a $\rho$-rotation on sequence (6) we obtain the sequence:

$$
q \text{ times: } \overline{C}(1, \rho)L(0)S(0)\overline{C}(1, \rho)L(\rho + 1)S(0)
$$

$$
\overline{C}(1, \rho - k_0)L(k_0)S(0)
$$

$$
\overline{C}(1, \rho - k_0)L(\rho + 1)S(0) \text{ if } r \geq 2.
$$

(7)

Remark 4.2. If $r = 1$ then $\overline{C}(1, \rho - k_0)$ is empty; if $r = 2$ then both $\overline{C}(1, \rho - k_0)$ and $\overline{C}(1, \rho - k_0)$ are empty.

Lemma 4.3. The unique $s$-equipartition of sequence (7) is red extremal.

Proof. The q sequences $\overline{C}(1, \rho), L(0), S(0), \overline{C}(1, \rho), L(\rho + 1), S(0)$ contain qs red edgy districts.

Case $r \geq 2$
The sequence $\overline{C}(1, \rho - k_0)L(k_0)S(0)\overline{C}(1, \rho - k_0)L(\rho + 1)S(0)$ has $2\rho - k_0 - k_0 + 1 = r - 1$ red edgy districts. Hence the total number of red edgy districts is $qs + r - 1$. 

Case $r = 1$
The cobra $\overline{C}(1, \rho - k_0)$ is empty, then the total number of red edgy districts is $qs$. Hence in each case the upper bound on the number of red edgy districts is attained. □

Theorem 4.4. Any cycle with $n = ps$ vertices is two-faced.

Proof. Follows from Lemmas 4.1–4.3. □

Corollary 4.5. Any hamiltonian graph with $n = ps$ vertices is two-faced.

Proof. Follows from Theorem 4.4. □

Corollary 4.6. Let $H_{s(s+1)}$ be a cycle with $n = s(s + 1)$ vertices, to be partitioned into $p = s + 1$ districts, each of size $s$. Then

$$
\lim_{s \to \infty} \frac{\text{GAP}(H_{s(s+1)})}{s + 1} = 1.
$$

Proof. After Corollary 2.2 and Theorem 4.4, since $B(G), R(G) \leq p - 1$, one has

$$
\frac{\text{GAP}(H_{s(s+1)})}{s + 1} = \frac{B(G) + R(G) - s - 1}{s + 1} = \frac{2s - s - 1}{s + 1} = \frac{s - 1}{s + 1}.
$$

When $s$ odd $\to \infty$, the thesis follows. □

Corollary 4.6 is really stunning: it means that, for certain infinite families of cycles, as the number and size of the districts grow, vicious gerrymandering can make the percentages of blue districts and red ones both arbitrarily close to 1 even under the assumptions that the vote outcome is the same and that the blue party and the red one get the same total number of votes.

4.2. Gerrymandering on grid graphs

In this section we will show that any s-equipartitionable grid graph with $M$ rows and $N$ columns, $M, N \geq 2$, is two-faced. Notice that, if $M = 1$ or $N = 1$, the graph is a path, then it cannot be two-faced since it admits a unique s-equipartition.

Theorem 4.7. Any s-equipartitionable grid graph with $M, N \geq 2$ and with an even number of vertices is two-faced.
Fig. 5. A Hamiltonian cycle in a grid graph with an even number of rows.

Fig. 6. Decomposition of a grid graph with $M$ rows and $N$ columns into $p$ grid subgraphs with $M_i$ rows and $N_i$ columns each.

Proof. If $n$ is even, at least one between $M$ and $N$ is even, then it is well known and easy to show that $G$ is Hamiltonian (see Fig. 5). By Theorem 4.4, any $s$-equipartitionable cycle is two-faced, and hence it follows that any $s$-equipartitionable grid graph having an even number of vertices is two-faced. □

Theorem 4.8. Every grid graph with $n = ps$ vertices admits an $s$-equipartition in which all the components are grid graphs with the same number of rows and the same number of columns.

Proof. Let $M$ be the number of rows and $N$ be the number of columns of a given grid graph. Since $MN = ps$, there exist four natural numbers $M_1, M_2, N_1$ and $N_2$ such that:

$$M = M_1M_2, \quad N = N_1N_2, \quad M_1N_1 = s, \quad M_2N_2 = p.$$

As shown in Fig. 6, by partitioning the columns of $G$ into $N_2$ subsets having $N_1$ columns each and the rows of $G$ into $M_2$ subsets having $M_1$ rows each, one can decompose $G$ into $p$ grid subgraphs having $M_1$ rows and $N_1$ columns each. Notice that, when $s$ is odd, also $M_1$ and $N_1$ must be odd. □

Let us consider now the case $n = MN$ odd. Recall that, since $n = ps$, $p = q(s + 1) + r$ and $s$ is odd, also $p$ and $r$ are odd. We suppose $p \geq 3$, since the case $p = 1$ is trivial. On the basis of the equipartition provided by Theorem 4.8, for a given $s$, we can decompose $G$ into a grid subgraph $G_s = (V_s, E_s)$ having $s$ vertices and $M_i$ rows and $N_i$ columns and a subgraph $\tilde{G}_s$ induced by the vertices in $V - V_s$ (see Fig. 7). We can suppose that $G_s$ contains one of the vertices of $G$ having degree 2, that is, one of the vertices on a corner of $G$. Since $M$, $N$ and $s$ are odd, $M_i$ and $N_i$ are odd and $M - M_i$ and $N - N_i$ are even. Hence $G_s$ is either a grid graph with an even number of rows or columns, or it can be decomposed into two grid graphs, one with an even number of rows equal to $M - M_i$, and the other with an even number of columns equal to $N - N_i$.

As shown in the example of Fig. 7, $G_s$ has a Hamiltonian cycle that can be obtained by appropriately joining Hamiltonian cycles of these grid subgraphs. Moreover $G_s$ is $s$-equipartitionable since it contains $n - s$ vertices, hence, by Theorem 4.4, it is two-faced.

Suppose, without loss of generality, that $M_i \leq N_i$ and $G_s$ is the top left corner of $G$ (as in Fig. 7). Consider the unique row of $G_s$ such that all its vertices are adjacent to vertices of $G_s$. Since $s \geq 3$, this row contains at least three vertices. Let $u$ and $v$ be two adjacent vertices of this row such that $u$ is on the bottom left corner of $G_s$ (see Fig. 7). Notice that $u$ is not an articulation vertex of $G_s$.

Remark 4.3. There exists a Hamiltonian path of $G_s$ having two adjacent vertices $\overline{u}$ and $\overline{v}$ that are adjacent to $u$ and $v$ in $G$, respectively.

The bicoloring of $G$ provided by the following algorithm is two-faced.
Algorithm **OddGridBicoloring**

decompose $G$ into $G_s$ and $G_{\bar{s}}$ (see Fig. 8 (a));
let $u$, $v$, $\overline{u}$ and $\overline{v}$ be as defined above;
color in red $\rho$ vertices of $G_s$ and in blue $\rho + 1$ vertices of $G_{\bar{s}}$ in such a way that $u$ is blue;
let $H$ be a hamiltonian cycle of $G_s$ such that $\overline{u}$ and $v$ are adjacent;
color $H$ using the sequence (4) in such a way that, if $r \geq 2$,
$\overline{v}$ and $v$ belong to the blue quasi sweeping district of the red extremal partition and $\overline{v}$ is red and is not an articulation vertex of the district.

**Lemma 4.9.** The bicoloring provided by Algorithm **OddGridBicoloring** is balanced.

**Proof.** Since $n$ and $s$ are odd, $n - s$ is even; hence, by construction, $H$ has an even number of vertices, $(n - s)/2$ red and $(n - s)/2$ blue. It follows that $G$ has $(n - 1)/2$ red vertices and $(n + 1)/2$ blue vertices. □

**Lemma 4.10.** Any $s$-equipartitionable grid graph with an odd number of vertices colored by Algorithm **OddGridBicoloring** has a blue extremal partition.

**Proof.** Let $\pi_B$ be the blue extremal partition of $H$, which has $qs$ blue edgy districts if $r = 1$ and $qs + r - 2$ blue edgy districts if $r \geq 3$. Since the vertices in $V_s$ form a blue edgy district, the partition $\pi_B = \pi_B \cup V_s$ of $G$ has $qs + r$ blue edgy districts if $r = 1$ and $qs + r - 1$ blue edgy districts if $r \geq 3$. Hence $\pi_B$ is blue extremal. □

**Lemma 4.11.** Any $s$-equipartitionable grid graph with an odd number of vertices colored by Algorithm **OddGridBicoloring** has a red extremal partition.

**Proof.** Let $\pi_R$ be the red extremal partition of $H$, which has $qs$ red edgy districts if $r = 1$ and $qs + r - 2$ red edgy districts if $r \geq 3$. If $r = 1$, since the vertices in $V_s$ form a blue edgy district, the partition $\pi_R = \pi_R \cup V_s$ of $G$ has $qs$ red edgy districts and so it is red extremal. If $r \geq 3$, $\pi_R$ has a blue quasi sweeping district $W'$ with $(s - r + 2)/2$ red vertices, one more than in a blue quasi sweeping district of $G$. Moreover, $V_s$ has one red vertex less than a red edgy district. Let $W' = W - \{\overline{v}\} \cup \{u\}$ and $V'_s = V_s - \{u\} \cup \{\overline{v}\}$. $W'$ is a blue quasi sweeping district of $G$ and $V'_s$ is a red edgy district. The partition $\pi_R = \pi_R - W' \cup W' \cup V'_s$ of $G$ has $qs + r - 1$ red edgy districts (see Fig. 8 (b)). Hence $\pi_R$ is red extremal. □

**Theorem 4.12.** Any $s$-equipartitionable grid graph with at least two rows and two columns is two-faced.

**Proof.** Follows from Theorem 4.7 and Lemmas 4.9–4.11. □

In conclusion, we have shown that for all hamiltonian graphs and grid graphs one can construct Dixon–Plischke-like examples where gerrymandering can heavily reverse the electoral result in terms of Parliament seats.

5. **Examples of non two-faced graphs**

In the previous section we have shown that all hamiltonian graphs and all grid graphs are two-faced. But do non-two-faced graphs exist? An immediate example is given by trees since they admit a unique $s$-equipartition. Looking for more significant examples we notice that both even cycles and grid graphs are 2-connected, bipartite and planar. Here is an example of a graph having all these properties, but not two-faced.
Consider the graph $G$ on 18 vertices in Fig. 9 and let $s = 3$. It is easy to see that in every connected 3-equipartition of $G$ vertices 1, 2, and 3 must belong to the same component and the same must hold for vertices 10, 11, and 12. Thus, a connected 3-equipartition of $G$ is always given by the two components $\{1, 2, 3\}$, $\{10, 11, 12\}$ and four additional components obtained by splitting each of the two hexagons into two parts. Hence, every pair of connected 3-equipartitions of $G$ has at least two components in common. Since for every possible bicoloring of the vertices of $G$ the common components of the two partitions will be always colored in the same way, no pair of connected 3-equipartitions, one blue extremal and one red extremal, can be found in $G$.

The graph $G$ shown in Fig. 9 is not two-faced for $s = 3$, due to the fact that it is not possible to find a pair of connected 3-equipartitions without common components. In our third and last example this is indeed possible but there is a subtler reason for which $G$ is not two-faced.

Consider the graph $G$ shown in Fig. 10, with $n = 30$ and $s = 5$. Let $C$ be the cycle induced in $G$ by $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $P$ the path given by the vertices from 10 to 30.

**Claim 5.1.** In every connected 5-equipartition of $G$ the vertices 1, 29 and 30 never belong to the same component.

**Proof.** Suppose that the claim is not true, that is, there exists at least a 5-equipartition of $G$ in which the component containing vertex 1 (and both 29 and 30) contains also a number $m$, $m = 0, 1, 2$, of vertices of $C$. In any case, the remaining vertices of $C$ form a path with a number of vertices ranging from 6 to 8. These vertices must belong to at least two components, $K$ and $K'$, but only one of them, $K$, say, contains vertex 10, implying that $K'$, being connected and containing five vertices but not vertex 1, is entirely contained in $C$. Thus, $K'$ must contain both vertices 5 and 6, preventing 10 from belonging to $K$, a contradiction. $\square$

By the claim, in a connected 5-equipartition of $G$ either:

1. vertex 29 and vertex 30 belong to the same component;
2. vertex 1 and vertex 30 belong to the same component.

A connected 5-equipartition matching condition (1) will be referred to as a *partition of type I*, while one that satisfies condition (2) will be called a *partition of type II*.

In a partition of type I there is always a component consisting of a path of five vertices in $C$ including vertex 1. Then, a second component is forced to be formed by the remaining four vertices of $C$ together with vertex 10. The other components are uniquely generated by partitioning the path from vertex 11 to vertex 30 into the four consecutive subpaths $\{11, 12, 13, 14, 15\}$, $\{16, 17, 18, 19, 20\}$, $\{21, 22, 23, 24, 25\}$, $\{26, 27, 28, 29, 30\}$.

A partition of type II is characterized by a component given by a path of 4 vertices in $C$, including vertex 1, attached to vertex 30. The rest of the vertices in $C$ form another component, while the additional components of the partition
are uniquely determined by partitioning the path from vertex 10 to vertex 29 into the four consecutive subpaths 
\{10, 11, 12, 13, 14\}, \{15, 16, 17, 18, 19\}, \{20, 21, 22, 23, 24\}, \{25, 26, 27, 28, 29\}.

On the basis of the above results, we know the structure of all the possible connected 5-equipartitions of G, namely those of type I and II. We also notice that all the partitions of type I share at least one common component (for example, \{26, 27, 28, 29, 30\}), and the same holds for the partitions of type II (for example, they share component \{25, 26, 27, 28, 29\}). In addition, along the path from vertex 10 to vertex 30, one obtains the components of a partition of type I by shifting each cut of a partition of type II to the next edge. Then we have the following result.

**Claim 5.2.** The graph G shown in Fig. 10 is not two-faced for s = 5.

**Proof.** Suppose that the claim is not true, that is, G is two-faced. In this case, on the basis of the above considerations, the blue and red extremal partitions (connected 5-equipartitions) must be of different types. Without loss of generality, suppose that the blue one is of type I and the red one is of type II. Given any two vertices i and j, with 10 ≤ i ≤ 30, we denote by \(\Delta(i, j)\) the difference between the number of blue vertices and the number of red vertices belonging to the unique path from i to j (in the sense of increasing node labels). Notice that the function \(\Delta(i, j)\) is additive w.r.t. the concatenation of consecutive paths. It is impossible that the red sweeping component of the blue extremal partition and the blue sweeping component of the red extremal one are both contained in the cycle C. For, in this case, all the four components of the blue (red) extremal partition along P would be blue-edgy (red-edgy), implying both \(\Delta(11, 30) = 4\) and \(\Delta(10, 29) = -4\), which is impossible. Thus, at least one of the two sweeping components is entirely contained in the path P. Suppose that it is red sweeping, then in P there should be a red component with at least 4 red vertices; on the other hand, suppose that it is blue sweeping, then in P there is a blue component with at least 4 blue vertices. Both cases lead to a contradiction, showing that G is not two-faced. □

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**References**


