On the stability of multi-Jensen mappings in $\beta$-normed spaces

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**Abstract**

In this work, we prove the generalized Hyers–Ulam stability of the multi-Jensen mappings in $\beta$-normed spaces.

1. Introduction and preliminaries

The stability of the Jensen functional equation $2f((x + y)/2) = f(x) + f(y)$ ($f$ satisfying this equation is called a Jensen mapping) has been studied by a number of mathematicians (see for instance [1–3]), whereas the stability of the bi-Jensen equation was investigated by Bae and Park (see [4]) and Jun et al. (see [5]).


Throughout this work, we fix a real number $\beta$ with $0 < \beta \leq 1$, $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$, and $n \geq 1$ is an integer. Moreover, $\mathbb{N}$ stands for the set of all positive integers. Let $\mathcal{X}$ be a linear space over $\mathbb{K}$. A function $\| \cdot \|_\beta : \mathcal{X} \to [0, \infty)$ is called a $\beta$-norm on $\mathcal{X}$ if and only if it satisfies:

1. $\|x\|_\beta = 0$ if and only if $x = 0$;
2. $\|\lambda x\|_\beta = |\lambda|^\beta \|x\|_\beta$ for all $\lambda \in \mathbb{K}$ and all $x \in \mathcal{X}$;
3. $\|x + y\|_\beta \leq \|x\|_\beta + \|y\|_\beta$ for all $x, y \in \mathcal{X}$.

The pair $(\mathcal{X}, \| \cdot \|_\beta)$ is called a $\beta$-normed space (see [17]). A $\beta$-Banach space is a complete $\beta$-normed space.

Let $\mathcal{X}$ be a linear space over a field of characteristic different from 2 and $\mathcal{Y}$ be a linear space. A function $f : \mathcal{X}^n \to \mathcal{Y}$ is called a multi-Jensen mapping if it satisfies Jensen’s equation in each of its $n$ arguments, that is

$$2f(x_1, \ldots, x_{i-1}, (x_i + x'_i)/2, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$$

for all $i = 1, \ldots, n$ and all $x_1, \ldots, x_{i-1}, x_i, x'_i, x_{i+1}, \ldots, x_n \in \mathcal{X}$.

Denote by $|S|$ the cardinality of a set $S$ and put $\mathbf{n} := \{1, \ldots, n\}$. For a subset $S = \{j_1, \ldots, j_l\}$ of $\mathbf{n}$ with $1 \leq j_1 < \cdots < j_l \leq n$ and $x = (x_1, \ldots, x_n) \in \mathcal{X}^n$, $x_S := (0, \ldots, 0, x_{j_1}, 0, \ldots, 0, x_{j_l}, 0, \ldots, 0) \in \mathcal{X}^n$ denotes the vector which coincides with $x$ in exactly those components, which are indexed by the elements of $S$ and whose other components are set equal to zero. Note that $x_0 = 0, x_n = x$ and $(x_T)_S = (x_T)_S = x_{S\cap T}$ for $S, T \subseteq \mathbf{n}$.

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It is known (see [7, Lemma 1.1]) that a function \( f : \mathcal{X}^n \to \mathcal{Y} \) is multi-Jensen if and only if

\[
f \left( \frac{1}{2} (x + y) \right) = \frac{1}{2^n} \sum_{S \subseteq \mathbb{N}} f(x_S + y_{n,S}), \quad x, y \in \mathcal{X}^n.
\]

(1.2)

The present work deals with the generalized Hyers-Ulam stability of Eq. (1.2) in \( \beta \)-normed spaces without referring to any knowledge of solutions.

2. The main results

For a given mapping \( f : \mathcal{X}^n \to \mathcal{Y} \), we define the difference operator

\[
Df(x, y) := 2^n f \left( \frac{1}{2} (x + y) \right) - \sum_{S \subseteq \mathbb{N}} f(x_S + y_{n,S}), \quad x, y \in \mathcal{X}^n.
\]

Theorem 2.1. Let \( \mathcal{X} \) be a linear space over a field of characteristic different from 2 and \( \mathcal{Y} \) be a \( \beta \)-Banach space. Let for any \( k \in \mathbb{N} \), \( \psi_k : \mathcal{X}^2 \to [0, \infty) \) satisfy \( \psi_k(0, 0) = 0 \) and

\[
\sum_{i=0}^\infty \left( \frac{1}{2^{n-1}} \right)^\beta \sum_{k=1}^n \psi_k(2^i x_k, 2^i y_k) < \infty
\]

(2.1)

for all \( x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{X} \) and \( i \in \{0, \ldots, n-1 \} \). If \( f : \mathcal{X}^n \to \mathcal{Y} \) is a mapping satisfying

\[
\|Df(x, y)\|_\beta \leq \sum_{k=1}^n \psi_k(x_k, y_k)
\]

(2.2)

for all \( x, y \in \mathcal{X}^n \), then there exists a multi-Jensen function \( F : \mathcal{X}^n \to \mathcal{Y} \) such that

\[
\|f(x) - F(x)\|_\beta \leq \sum_{T \subseteq \mathbb{N}} \frac{1}{2 \left( \beta \right)} \sum_{i=1}^\infty \frac{\left( 2^{n-|T|} \right)^\beta}{\left( 2^{n-|T|} \right) \beta} \sum_{k \in \mathbb{N} \setminus T} \psi_k(2^i x_k, 0)
\]

(2.3)

for all \( x \in \mathcal{X}^n \). The mapping F is given by

\[
F(x) = \sum_{T \subseteq \mathbb{N}} F_T(x), \quad x \in \mathcal{X}^n,
\]

(2.4)

where \( F_n(x) := f(0) \) for all \( x \in \mathcal{X}^n \), and for any \( T \subseteq \mathbb{N} \),

\[
F_T(x) := \lim_{i \to \infty} \frac{1}{\left( \beta \right)} \sum_{\emptyset \not= S \subseteq \mathbb{N} \setminus T} (-1)^{n-|T|} f(2^i x_{S}) - \sum_{S \subseteq \mathbb{N} \setminus \{1, \ldots, j\}} (-1)^{n-|T| - |S|} f(2^{i+1} x_{S})
\]

(2.5)

Proof. Fix \( x \in \mathcal{X}^n \), \( l \in \mathbb{N} \setminus \{0\} \), \( i \in \{0, \ldots, n-1\} \) and a subset \( T = \{j_1, \ldots, j_l\} \) of \( \mathbb{N} \) with \( 1 \leq j_1 < \cdots < j_l \leq n \). Using [9, Lemma 1], we have

\[
\frac{1}{2 \left( \beta \right)} \sum_{S \subseteq \mathbb{N} \setminus \{1, \ldots, j\}} (-1)^{n-|T| - |S|} f(2^i x_{S}) = \frac{1}{\left( \beta \right)} \sum_{S \subseteq \mathbb{N} \setminus \{1, \ldots, j\}} (-1)^{n-|T| - |S|} f(2^{i+1} x_{S})
\]

(2.6)

By (2.2) and (2.6) we obtain

\[
\left\| \frac{1}{\left( \beta \right)} \sum_{S \subseteq \mathbb{N} \setminus \{1, \ldots, j\}} (-1)^{n-|T| - |S|} f(2^i x_{S}) - \frac{1}{\left( \beta \right)} \sum_{S \subseteq \mathbb{N} \setminus \{1, \ldots, j\}} (-1)^{n-|T| - |S|} f(2^{i+1} x_{S}) \right\|_\beta
\]

\[
\leq \left( \frac{2 \left( \beta \right)}{\left( \beta \right)} \right)^\beta \sum_{\emptyset \not= S \subseteq \mathbb{N} \setminus \{1, \ldots, j\}} \left\|Df(2^{i+1} x_{S}, 0)\right\|_\beta
\]

\[
\leq \left( \frac{2 \left( \beta \right)}{\left( \beta \right)} \right)^\beta \sum_{\emptyset \not= S \subseteq \mathbb{N} \setminus \{1, \ldots, j\}} \sum_{k \in S} \psi_k(2^{i+1} x_{k}, 0)
\]

\[
= \frac{2^{n-|T| - 1}}{\left( \beta \right)^\beta} \left( 2^{n-\beta(\beta+1)} \right) \sum_{k \in \mathbb{N} \setminus \{1, \ldots, j\}} \psi_k(2^{i+1} x_{k}, 0).
\]

(2.7)
For any non-negative integers $l$ and $m$ with $l < m$, using (2.7) we obtain
\[
\left\| \frac{1}{(2^{m-1})} \sum_{j \in \mathbf{n} \setminus \{j_1, \ldots, j_m\}} (-1)^{n-i-|S|} f(2^j x_S) - \frac{1}{(2^{m-1})} \sum_{j \in \mathbf{n} \setminus \{j_1, \ldots, j_m\}} (-1)^{n-i-|S|} f(2^m x_S) \right\|_{\beta} \\
\leq \sum_{j=1}^{m-1} \frac{2^{n-i-1}}{2^{2j}} \cdot (2^{n-i})^{\beta(i+1)} \sum_{k \in \mathbf{n} \setminus \{j_1, \ldots, j_m\}} \varphi_k(2^{j+1} x_k, 0). \quad (2.8)
\]

Therefore, from (2.1) and (2.8) it follows that \( \{ \frac{1}{(2^{m-1})} \sum_{j \in \mathbf{n} \setminus \{j_1, \ldots, j_m\}} (-1)^{n-i-|S|} f(2^j x_S) \}_{\beta \in \mathfrak{B}} \) is a Cauchy sequence in the $\beta$-Banach space $\mathcal{X}$. Thus this sequence is convergent and we define $F_T = F_{j_1, \ldots, j_m} : \mathcal{X}^n \to \mathcal{Y}$ by (2.5) and $F_n(x) = f(0)$. Putting $l = 0$ and letting $m \to \infty$ in (2.8), we get
\[
\left\| \sum_{j \in \mathbf{n} \setminus \{j_1, \ldots, j_m\}} (-1)^{n-i-|S|} f(x_S) - F_T(x) \right\|_{\beta} \leq \sum_{j=0}^{\infty} \frac{2^{n-i-1}}{2^{2j}} \cdot (2^{n-i})^{\beta(i+1)} \sum_{k \in \mathbf{n} \setminus \{j_1, \ldots, j_m\}} \varphi_k(2^{j+1} x_k, 0). \quad (2.9)
\]

As in the proof of Theorem 3 in [9], for any $x, y \in \mathcal{X}^n$, $r \in \mathbf{n}$ and $j_1, \ldots, j_{n-r} \in \mathbf{n}$ with $1 \leq j_1 < \cdots < j_{n-r} \leq n$ we have
\[
F_{j_1, \ldots, j_{n-r}}(x) := \lim_{l \to \infty} \frac{1}{(2^{r})^l} f(2^l x_{\mathbf{n} \setminus \{j_1, \ldots, j_{n-r}\}}) \quad (2.10)
\]
and
\[
D F_{j_1, \ldots, j_{n-r}}(x, y) = \lim_{l \to \infty} \frac{1}{(2^{r})^l} D f(2^l x_{\mathbf{n} \setminus \{j_1, \ldots, j_{n-r}\}}, 2^l y_{\mathbf{n} \setminus \{j_1, \ldots, j_{n-r}\}}).
\]

This together with (2.1) and (2.2) gives
\[
\|D F_{j_1, \ldots, j_{n-r}}(x, y)\|_{\beta} = \lim_{l \to \infty} \frac{1}{(2^{r})^l} \|D f(2^l x_{\mathbf{n} \setminus \{j_1, \ldots, j_{n-r}\}}, 2^l y_{\mathbf{n} \setminus \{j_1, \ldots, j_{n-r}\}})\|_{\beta} \leq \lim_{l \to \infty} \left( \frac{1}{2^l} \right) \sum_{j=1}^{n} \varphi_k(2^j x_k, 2^j y_k) = 0. \quad (2.11)
\]

Hence, the mapping $F_{j_1, \ldots, j_{n-r}}$ is multi-Jensen. Thus, for any $T \subseteq \mathbf{n}$ the mapping $F_T$ is multi-Jensen, and so is the mapping given by (2.4). For any $x \in \mathcal{X}^n$. Since (see the proof of Theorem 3 in [9])
\[
f(x) = \sum_{T \subseteq \mathbf{n}} \sum_{\mathbf{s} \subseteq \mathbf{n} \setminus T} (-1)^{n-|T|-|S|} f(x_S), \quad (2.12)
\]
(2.9) finally gives
\[
\|f(x) - F(x)\|_{\beta} \leq \sum_{T \subseteq \mathbf{n}} \left( \sum_{\mathbf{s} \subseteq \mathbf{n} \setminus T} (-1)^{n-|T|-|S|} f(x_S) - F_T(x) \right) \|_{\beta} \leq \sum_{T \subseteq \mathbf{n}} \sum_{\mathbf{s} \subseteq \mathbf{n} \setminus T} (-1)^{n-|T|-|S|} f(x_S) \|_{\beta} \leq \sum_{T \subseteq \mathbf{n}} \sum_{j=1}^{\infty} \frac{2^{n-(\beta+1)|T|-1}}{(2^{n-|T|})^{\beta j}} \sum_{k \in T} \varphi_k(2^j x_k, 0). \quad (2.11)
\]

**Corollary 2.2.** Let $\mathcal{X}$ be a $\beta$-normed space and $\mathcal{Y}$ be a $\beta$-Banach space. If $\beta > 0$, $0 < p < 1$, and $f : \mathcal{X}^n \to \mathcal{X}$ is a mapping such that
\[
\|D f(x, y)\|_{\beta} \leq \theta \sum_{k=1}^{n} \left( \|x_k\|_{\beta}^p + \|y_k\|_{\beta}^p \right) \quad (2.13)
\]
for all $x, y \in \mathcal{X}^n$, then there exists a multi-Jensen mapping $F : \mathcal{X}^n \to \mathcal{Y}$ such that
\[
\|f(x) - F(x)\|_{\beta} \leq \theta \sum_{T \subseteq \mathbf{n}} \frac{2^{\beta p+n-(\beta+1)|T|-1}}{(2^{\beta(n-|T|)})^{\beta j}} \sum_{k \in T} \|x_k\|_{\beta}^p
\]
for all $x \in \mathcal{X}^n$. The mapping $F$ is given by (2.4), where $F_n(x) := f(0)$ for all $x \in \mathcal{X}^n$, and for any $T \subseteq \mathbf{n}$, $F_T$ is given by (2.5).
Proof. Let \( \varphi_k(x_k, y_k) = \theta(\|x_k\|_p^p + \|y_k\|_p^p) \) for all \( x, y \in \mathbb{X}^n \) and \( k \in \mathbb{n} \). In the case of \( 0 < p < 1 \), we obtain the mapping \( F \) using Theorem 2.1.

Remark 2.3. For \( \beta = 1 \), Corollary 2.2 yields Theorem 3 in [9].

**Theorem 2.4.** Let \( \mathbb{X} \) be a linear space over a field of characteristic different from 2 and \( \mathbb{Y} \) be a \( \beta \)-Banach space. Let for any \( k \in \mathbb{n} \), \( \varphi_k : \mathbb{X} - \rightarrow [0, \infty) \) satisfy \( \varphi_k(0, 0) = 0 \) and

\[
\sum_{i=0}^{\infty} (2^{n-i})^l \sum_{k=1}^{n} \varphi_k \left( \frac{x_k}{2^n}, \frac{y_k}{2^n} \right) < \infty
\]

(2.14)

for all \( x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{X} \) and \( i \in \{0, \ldots, n-1\} \). If \( f : \mathbb{X}^n \rightarrow \mathbb{Y} \) is a mapping satisfying (2.2) for all \( x, y \in \mathbb{X}^n \), then there exists a multi-Jensen function \( F : \mathbb{X}^n \rightarrow \mathbb{Y} \) such that

\[
\|f(x) - F(x)\| \leq \sum_{T \subseteq \mathbb{n}} \sum_{j=0}^{\infty} 2^{n-(\beta+1)|T|} (2^n)^l \|f\| \sum_{k \in \mathbb{n} \setminus T} \varphi_k \left( \frac{x_k}{2^n}, 0 \right)
\]

(2.15)

for all \( x \in \mathbb{X}^n \). The mapping \( F \) is given by (2.4), where \( F_t(x) := f(0) \) for all \( x \in \mathbb{X}^n \), and for any \( T \subseteq \mathbb{n} \),

\[
F_T(x) := \lim_{l \to \infty} (2^{n-|T|})^l \sum_{s \subseteq \mathbb{n} \setminus T} (-1)^{n-|T|} f \left( \frac{X_s}{2^n} \right), \quad x \in \mathbb{X}^n.
\]

(2.16)

Proof. Fix \( x \in \mathbb{X}^n, l \in \mathbb{N} \cup \{0\}, i \in \{0, \ldots, n-1\} \) and a subset \( T = \{j_1, \ldots, j_l\} \) of \( n \) with \( 1 \leq j_1 < \cdots < j_l \leq n \). Using [9, Lemma 1], we have

\[
(2^{n-i+1}) \sum_{S \subseteq \mathbb{n} \setminus \{j_1, \ldots, j_l\}} (-1)^{n-|S|} f \left( \frac{X_S}{2^{2l+1}} \right) - (2^{n-i}) \sum_{S \subseteq \mathbb{n} \setminus \{j_1, \ldots, j_l\}} (-1)^{n-|S|} f \left( \frac{X_S}{2^n} \right)
\]

(2.17)

By (2.2) and (2.17) we obtain

\[
\left\| (2^{n-i})^l \sum_{S \subseteq \mathbb{n} \setminus \{j_1, \ldots, j_l\}} (-1)^{n-|S|} f \left( \frac{X_S}{2^n} \right) - (2^{n-i})^l \sum_{S \subseteq \mathbb{n} \setminus \{j_1, \ldots, j_l\}} (-1)^{n-|S|} f \left( \frac{X_S}{2^{2l+1}} \right) \right\| \leq (2^{n-i})^l \sum_{S \subseteq \mathbb{n} \setminus \{j_1, \ldots, j_l\}} \varphi_k \left( \frac{x_k}{2^n}, 0 \right)
\]

(2.18)

For any non-negative integers \( l \) and \( m \) with \( l < m \), using (2.18) we obtain

\[
\left\| (2^{n-i})^l \sum_{S \subseteq \mathbb{n} \setminus \{j_1, \ldots, j_l\}} (-1)^{n-|S|} f \left( \frac{X_S}{2^n} \right) - (2^{n-i})^m \sum_{S \subseteq \mathbb{n} \setminus \{j_1, \ldots, j_l\}} (-1)^{n-|S|} f \left( \frac{X_S}{2^m} \right) \right\| \leq (2^{n-i})^l \sum_{S \subseteq \mathbb{n} \setminus \{j_1, \ldots, j_l\}} \varphi_k \left( \frac{x_k}{2^n}, 0 \right).
\]

(2.19)

From (2.14) and (2.19) it follows that \( \{(2^{n-i})^l \sum_{S \subseteq \mathbb{n} \setminus \{j_1, \ldots, j_l\}} (-1)^{n-|S|} f \left( \frac{X_S}{2^n} \right)\}_{l \in \mathbb{N} \cup \{0\}} \) is a Cauchy sequence and we define \( F_T = T_{j_1, \ldots, j_l} : \mathbb{X}^n \rightarrow \mathbb{Y} \) by (2.16) and \( F_n(x) = f(0) \). Putting \( l = 0 \) and letting \( m \to \infty \) in (2.19), we get

\[
\left\| \sum_{S \subseteq \mathbb{n} \setminus \{j_1, \ldots, j_l\}} (-1)^{n-|S|} f(x_S) - F_T(x) \right\| \leq \sum_{j=0}^{\infty} (2^{n-i})^l \cdot 2^{n-i-1} \sum_{k \in \mathbb{n} \setminus \{j_1, \ldots, j_l\}} \varphi_k \left( \frac{x_k}{2^n}, 0 \right).
\]

As in the proof of Theorem 4 in [9], for any \( x, y \in \mathbb{X}^n, r \in \mathbb{n} \) and \( j_1, \ldots, j_{n-r} \in \mathbb{n} \) with \( 1 \leq j_1 < \cdots < j_{n-r} \leq n \) we get

\[
DF_{j_1,\ldots,j_{n-r}}(x, y) = \lim_{l \to \infty} (2^l)^l \sum_{T \subseteq \mathbb{n} \setminus \{j_1, \ldots, j_{n-r}\}} DF \left( \frac{x_T}{2^n}, \frac{y_T}{2^n} \right).
\]

The rest of the proof runs as before. \( \square \)
As an application of Theorem 2.4 we get the following corollary.

**Corollary 2.5.** Let \( \mathcal{X} \) be a \( \beta \)-normed space and \( \mathcal{Y} \) be a \( \beta \)-Banach space. If \( \theta > 0 \), \( p > n \), and \( f : \mathcal{X}^n \rightarrow \mathcal{Y} \) is a mapping such that (2.13) holds for all \( x, y \in \mathcal{X}^n \), then there exists a multi-Jensen mapping \( F : \mathcal{X}^n \rightarrow \mathcal{Y} \) such that

\[
\| f(x) - F(x) \|_\beta \leq \theta \sum_{T \subseteq n} 2^{\beta p + n - (\beta + 1)|T| - 1} \sum_{k \in n \setminus T} \| x_k \|_\beta^p
\]

for all \( x \in \mathcal{X}^n \). The mapping \( F \) is given by (2.4), where \( F_n(x) := f(0) \) for all \( x \in \mathcal{X}^n \), and for any \( T \subseteq n \), \( F_T \) is given by (2.16).

In the same manner as Corollaries 2.2 and 2.5 one can also prove the following result.

**Corollary 2.6.** Let \( \mathcal{X} \) be a \( \beta \)-normed space and \( \mathcal{Y} \) be a \( \beta \)-Banach space. If \( \theta > 0 \), \( p \in (1, n) \setminus \mathbb{N} \), and \( f : \mathcal{X}^n \rightarrow \mathcal{Y} \) is a mapping such that (2.13) holds for all \( x, y \in \mathcal{X}^n \), then there exists a multi-Jensen mapping \( F : \mathcal{X}^n \rightarrow \mathcal{Y} \) such that

\[
\| f(x) - F(x) \|_\beta \leq \theta \sum_{T \subseteq n} 2^{\beta p + n - (\beta + 1)|T| - 1} \sum_{k \in n \setminus T} \| x_k \|_\beta^p
\]

for all \( x \in \mathcal{X}^n \). The mapping \( F \) is given by (2.4), where \( F_n(x) := f(0) \) for all \( x \in \mathcal{X}^n \), and for any \( T \subseteq n \), \( F_T \) is given by (2.5) if \( |T| < n - p \) and by (2.16) if \( |T| > n - p \).

**Remark 2.7.** For \( \beta = 1 \), Corollaries 2.5 and 2.6 yield Theorems 4 and 5 in [9].

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**References**


