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The set of types of a finitely generated variety

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Abstract

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The paper presents an algorithm of polynomial time complexity to compute the type set of a finite algebraic system $\mathbf{A}$, as defined in the monograph of McKenzie and Hobby: 'The Structure of Finite Algebras'. To do so, it introduces the concept of a subtrace, and uses subtraces to characterize the type set of $\mathbf{A}$. It is also shown that to calculate the type set of the variety generated by $\mathbf{A}$ is more difficult, by presenting various examples, in which a given type occurs only in subalgebras of high powers of $\mathbf{A}$.

1. Introduction

McKenzie and Hobby in their monograph 'The Structure of Finite Algebras' provide important new structural invariants for algebras and for locally finite varieties. Their work shows that in a finite algebra $\mathbf{A}$ each covering pair of congruence relations of $\mathbf{A}$ can be assigned one of five types: **(1)** unary, **(2)** affine, **(3)** Boolean, **(4)** lattice, and **(5)** semilattice. The set of types that appear among all the covering

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pairs of congruences is denoted typ$\{A\}$. For a class of algebras $\mathcal{K}$, typ$\{\mathcal{K}\}$ denotes the union of the sets typ$\{A\}$ where A ranges over the finite algebras in $\mathcal{K}$. Much of the book [4] is devoted to showing how for an algebra A and a variety (equational class) $\mathcal{V}$, the sets typ$\{A\}$ and typ$\{\mathcal{V}\}$ are strongly linked to diverse algebraic properties. For example, if **1** or **5** appears in typ$\{\mathcal{V}\}$, then $\mathcal{V}$ satisfies no nontrivial congruence identity. If **3** or **4** occurs in typ$\{A\}$ then the cardinality of the free algebras on n free generators in the variety generated by A grows as a doubly exponential function of n. If **4** or **5** are in typ$\{\mathcal{V}\}$ then $\mathcal{V}$ is hereditarily undecidable. Other results relate varietal properties to a particular type not appearing in typ$\{\mathcal{V}\}$. Mal'cev conditions for a variety $\mathcal{V}$ that are equivalent to typ$\{\mathcal{V}\}$ omitting certain sets of types are presented and utilized in many ways.

Therefore, given a finite algebra A it is of considerable interest to compute typ$\{A\}$ and to know what types appear in $V(A)$, the variety generated by A. In Section 2 we provide two algorithms for computing typ$\{A\}$. The first one is a straightforward application of the definition of typ$\{A\}$ as given in [4]. We discuss this algorithm and briefly describe a computer implementation that we have for it. This algorithm is of exponential time complexity. The second algorithm we present is more subtle. We introduce the notion of a subtrace of an algebra and we use two-element subtraces in order to compute typ$\{A\}$. The use of two-element subtraces allows us to compute typ$\{A\}$ without having to compute the congruence lattice and the unary polynomials of the algebra A. This second algorithm is of polynomial time complexity.

The final section deals with typ$\{V(A)\}$ for a finite algebra A. Basically, our results show how types not present in A or its subalgebras can appear in $V(A)$. Our main examples are given in Theorems 3.4 and 3.5, where algebras C with at most five elements are constructed for every type in $\{\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ and positive integer m such that the first occurrence of the given type in the variety generated by C is in a subalgebra of C^m. The subdirectly irreducible algebras in these varieties show an interesting distribution. Two-element subtraces also figure in our proofs in this section. The results in Section 3 complement some of the results in [6] in which conditions are given that guarantee that no new types appear in $V(A)$ that are not already present in the subalgebras of A.

We have tried to write this paper so that it is accessible to someone not familiar with [4]. The basic notation and the definitions are developed in this introductory section and elsewhere through the paper as needed. The notation and the few facts from universal algebra that we use are found in the dozen pages that comprise Chapter 0 of [4].

For an algebra A with universe A, Pol A is the set of all polynomial operations of A and $\mathrm{Pol}_n\, A$ is the set of n-ary polynomials of A. If S is a nonvoid subset of A and if f is a polynomial of A such that S is closed under f, then $f|_S$ denotes the restriction of f to S. The set $(\mathrm{Pol}\, A)|_S$ is the set of all such $f|_S$ for which f preserves S and $A|_S$ is the algebra $\langle S, (\mathrm{Pol}\, A)|_S\rangle$. A unary polynomial e is called idempotent if $e(e(a)) = e(a)$ for all $a \in A$ and $E(A)$ denotes the set of all idempotent unary polynomials of A.

A nontrivial finite algebra $\boldsymbol{M}$ is called minimal if every unary polynomial of $\boldsymbol{M}$ is either a permutation or a constant. In [7], Pálfy proved that if $\boldsymbol{M}$ is a minimal algebra then up to polynomial equivalence exactly one of the following is true:

(1) $\boldsymbol{M}$ is a unary algebra in which each basic operation is a permutation.
(2) $\boldsymbol{M}$ is a vector space.
(3) $\boldsymbol{M}$ is a two-element Boolean algebra.
(4) $\boldsymbol{M}$ is a two-element lattice.
(5) $\boldsymbol{M}$ is a two-element semilattice.

Then, depending on which of these five cases hold, the type of a minimal algebra, denoted by $\mathrm{typ}(\boldsymbol{M})$, is **1**, **2**, **3**, **4**, or **5** respectively.

For any algebra $\boldsymbol{A}$, Con $\boldsymbol{A}$ denotes the set of congruence relations of $\boldsymbol{A}$ and **Con** $\boldsymbol{A}$ is the congruence lattice of $\boldsymbol{A}$. For $\alpha, \beta \in \mathrm{Con}\, \boldsymbol{A}$, β covers α if $\alpha < \beta$ and for no $\gamma \in \mathrm{Con}\, \boldsymbol{A}$ does $\alpha < \gamma < \beta$ hold. If β covers α, then we write $\alpha \prec \beta$ and the ordered pair $\langle \alpha, \beta \rangle$ is called a covering pair.

Let $\boldsymbol{A}$ be a finite algebra and let α be covered by β in Con $\boldsymbol{A}$. Let $e \in E(\boldsymbol{A})$ be such that $e(A)$ is minimal (under inclusion) for $e(\beta) \nsubseteq \alpha$. Then $U = e(A)$ is called an $\langle \alpha, \beta \rangle$-*minimal set of* $\boldsymbol{A}$. If N is of the form $U \cap x/\beta$ and if $N^2 \nsubseteq \alpha$, then a series of results in [4] culminating in Corollary 5.2 show that the algebra $\boldsymbol{M} = (\boldsymbol{A}|_N)/(\alpha|_N)$ is a minimal algebra and that $\mathrm{typ}(\boldsymbol{M})$ is the same regardless of how e and N are chosen. The set N is called an $\langle \alpha, \beta \rangle$-trace in $\boldsymbol{A}$. The type of the covering pair $\langle \alpha, \beta \rangle$ is assigned the value **i** if $\mathrm{typ}(\boldsymbol{M}) = \mathbf{i}$, and this is written $\mathrm{typ}(\alpha, \beta) = \mathbf{i}$. The set $\mathrm{typ}\{\boldsymbol{A}\}$ is defined to be $\{\mathrm{typ}(\alpha, \beta)$: $\alpha \prec \beta$ in **Con** $\boldsymbol{A}\}$. If $\mathscr{K}$ is a class of algebras, then $\mathrm{typ}\{\mathscr{K}\} = \bigcup \{\mathrm{typ}\{\boldsymbol{A}\}$: $\boldsymbol{A}$ is a finite member of $\mathscr{K}\}$.

2. Determining typ$\{A\}$

In this section we present two algorithms for determining $\mathrm{typ}\{\boldsymbol{A}\}$ for a finite algebra $\boldsymbol{A}$. We also introduce the notion of a subtrace. Two-element subtraces are used for the second of our two algorithms and they also appear in the proofs in Section 3.

We begin by explaining how to represent the n-ary polynomials of an algebra. Take a finite algebra $\boldsymbol{A} = \langle A, F \rangle$ with $|A| = k$, and consider $\boldsymbol{A}^{A^n}$. This can be viewed as the set of all n-ary functions defined on A. Let x_j be the jth projection function, which assigns a_j to $(a_1, \ldots, a_n)$. Then it is easy to see that all k^n members of A^n appear exactly once among $(x_1(i), x_2(i), \ldots, x_n(i))$, where i runs over the elements of A^n. Moreover, $\mathrm{Pol}_n \boldsymbol{A}$ is just the universe of the subalgebra of $\boldsymbol{A}^{A^n}$ generated by $x_1, \ldots, x_n$ and the k constant functions from A^n to A (the 'diagonal' of A^{A^n}). In particular, $\mathrm{Pol}_1 \boldsymbol{A}$ corresponds to the subuniverse of the subalgebra of $\boldsymbol{A}^A$ generated by the k constants and the identity map.

For a non-empty subset S of A, the set $(\mathrm{Pol}_n\, \boldsymbol{A})|_S$ can be characterized similarly. To get it, take the projections $y_1, \ldots, y_n$ from S^n to S and the k constant functions from S^n to A, and intersect the universe of the subalgebra generated by these elements of $\boldsymbol{A}^{S^n}$ with S^{S^n}. For $a \neq b$ in A, we will have occasion to view $(\mathrm{Pol}_2 \boldsymbol{A})|_{\{a, b\}}$ as the intersection of $\{a, b\}^4$ with the subuniverse of the subalgebra of A^4 generated by the $|A|$ constants and the two 4-tuples (a, a, b, b) and (a, b, a, b).

So to calculate induced algebras we have to generate subalgebras of the powers of A. The details of a computer implementation of this subuniverse generation procedure, which uses hashing to recognize distinct elements, is given in [3].

The following lemma provides a method of distinguishing between the five types of minimal algebras.

Lemma 2.1. *Let* $\boldsymbol{M}$ *be a finite algebra that is minimal and suppose* $|M| = m$, $|\mathrm{Pol}_1 \boldsymbol{M}| = p_1$ *and* $|\mathrm{Pol}_2 \boldsymbol{M}| = p_2$. *Then* $\mathrm{typ}(\boldsymbol{M})$ *is completely determined by the following chart.*

m	p_1	p_2	$\mathrm{typ}(\boldsymbol{M})$	
2	3	5	**5**	
2	3	6	**4**	
2	4	16	**3**	
2	4	8	**2**	
2	3	4	**1**	
2	4	6	**1**	
>2			**1**	*if* $p_2 = 2p_1 - m$
>2			**2**	*if* $p_2 > 2p_1 - m$

Proof. The $m = 2$ cases exhaust the possibilities of p_1 and p_2 for 2-element algebras. For $m > 2$, $\boldsymbol{M}$ has an essentially binary polynomial if and only if $p_2 > 2p_1 - m$. $\square$

The description of $\mathrm{typ}\{A\}$ given in the final paragraph of Section 1 provides the following.

Algorithm for determining typ$\{A\}$ for a finite algebra A.

The input is all operation tables for the basic operations of A.

The output is a list of all covering pairs $\alpha \prec \beta$ in Con A and for each the value of $\mathrm{typ}(\alpha, \beta)$.

(1) Compute $\mathrm{Pol}_1 A$.

(2) Compute $E(A)$.

(3) Compute Con A and determine all covering pairs $\langle \alpha, \beta \rangle$ in **Con** A.

(4) For each covering pair $\langle \alpha, \beta \rangle$ in **Con** A do

(5) Find an $e \in E(A)$, $e(\beta) \not\subseteq \alpha$, with $|e(A)|$ minimal.

(6) Find an $\langle \alpha, \beta \rangle$-trace N in $e(A)$.

(7) Reduce modulo α to find a minimal algebra $\boldsymbol{M}$.

(8) Compute $|\mathrm{Pol}_1 \boldsymbol{M}|$ and $|\mathrm{Pol}_2 \boldsymbol{M}|$.

(9) Use Lemma 2.1 to compute $\mathrm{typ}(\boldsymbol{M})$. This is $\mathrm{typ}(\alpha, \beta)$.

We briefly discuss some aspects of each step of this algorithm.

Suppose the universe of $\mathbf{A}$ is $A=\{0, 1,\ldots, k-1\}$. Elements of $\mathrm{Pol}_1 \mathbf{A}$ computed in step (1) are represented as k-tuples of integers in A. $\mathrm{Pol}_1 \mathbf{A}$ is generated by the elements $(0, \ldots, 0), \ldots, (k-1, \ldots, k-1)$ and $(0, 1, \ldots, k-1)$. The basic operations of $\mathbf{A}$ are applied coordinatewise. If f is a basic operation that is m-ary and if $|\mathrm{Pol}_1(\mathbf{A})|=p$, then f will be applied p^m times.

The computation in step (2) can be done with one pass through the list of $\mathrm{Pol}_1 \mathbf{A}$.

One way to find Con $\mathbf{A}$ is to first find all principal congruence relations $\Theta(i,j)$ for $0\leqslant i<j<k$. Since $\mathrm{Pol}_1 \mathbf{A}$ has been found in step (1), $\Theta(i,j)$ may be computed as the transitive closure of $\{(c, d)\colon \{c, d\}=\{g(i), g(j)\},\ g\in \mathrm{Pol}_1 \mathbf{A}\}$. Once all principal congruences have been found, Con $\mathbf{A}$ can be found by taking the transitive closure of sets of these principal congruences. For small $\mathbf{A}$ an alternate approach is to generate all equivalence relations on A and test each one to see if it is a congruence relation of $\mathbf{A}$.

The computation in step (8) can be handled in the following manner. Suppose $\alpha|_N$ has m classes. Let $S=\{a_1, \ldots, a_m\}\subseteq N$ be a family of representatives, one from each $\alpha|_N$ class. Then $p_1=|\mathrm{Pol}_1 \mathbf{M}|$ is the number of distinct m-tuples $(s_1,\ldots, s_m)\in S^m$ for which there exists $f\in \mathrm{Pol}_1 \mathbf{A}$ with $(f(a_i), s_i)\in\alpha$ for all i, $1\leqslant i\leqslant m$. The value of $p_2=|\mathrm{Pol}_2 \mathbf{M}|$ is computed in an analogous fashion. Note that if $m\geqslant 3$, then the computation of p_2 can be halted once it is determined that $p_2>2p_1-m$.

An alternate approach in step (5) is to scan $\mathrm{Pol}_1 \mathbf{A}$ for a polynomial g with $g(\beta)\nsubseteq\alpha$ and such that $|g(A)|$ is minimal with this property. Then by Lemma 2.8(2) of [4], the trace needed in step (6) will be any set $N=g(A)\cap a/\beta$ for some $a\in g(A)$ with $a/\beta\cap g(A)\nsubseteq a/\alpha$.

A computer program implementing this algorithm has been written by the first author. For algebras having a small universe, say, up to ten elements, and having only binary and unary operations, this program provides a practical way of computing $\mathrm{typ}\{\mathbf{A}\}$ and in so doing it gives a complete labeling of the covering pairs in the congruence lattice of the algebra.

Although the algorithm works satisfactorily for small algebras $\mathbf{A}$, for larger algebras it has some serious drawbacks. Firstly, $\mathrm{Pol}_1 \mathbf{A}$ may be intractably large. For example if $\mathbf{A}$ is functionally complete, i.e. $\mathrm{Pol}_n \mathbf{A}$ consists of all n-ary operations on A, then $\mathbf{A}$ is a simple algebra and $\mathrm{typ}(\mathbf{A})=\mathbf{3}$. However, in this case $|\mathrm{Pol}_1 \mathbf{A}|=|A|^{|A|}$ and for $|A|$ larger than 10, say, step (1) becomes intractable. Another serious problem with the algorithm is that **Con** $\mathbf{A}$ may become very large even for relatively small algebras. For example, if $\mathbf{A}$ has only the projections and constants as its basic operations, then **Con** $\mathbf{A}$ is the full partition lattice and every covering pair of congruences has type **1**. However, in this case the cardinality of Con $\mathbf{A}$ grows exponentially fast. For $n=12$ this is already 4,213,597 congruence relations.

We present an alternate algorithm for computing $\mathrm{typ}\{\mathbf{A}\}$; an algorithm in which neither $\mathrm{Pol}_1 \mathbf{A}$ nor **Con** $\mathbf{A}$ need be computed. This algorithm uses a local analysis of the 2-element subsets of the algebra.

We first need to define *polynomial isomorphism*. Let $\mathbf{A}$ be an algebra and let B and C be nonvoid subsets of A. The sets B and C are called *polynomially isomorphic*, if there exist $f, g \in \mathrm{Pol}_1 A$ such that (i) $f(B)=C$, (ii) $g(C)=B$, (iii) $gf|_B = \mathrm{id}_B$, and (iv) $fg|_C = \mathrm{id}_C$. If B and C are polynomially isomorphic, then this is denoted by $B \simeq C$, and if the role of f is to be emphasized, $f: B \simeq C$ is written.

If B and C are finite nonvoid subsets of A and if there exist $f, g \in \mathrm{Pol}_1 A$ for which $f(B)=C$ and $g(C)=B$ hold, then $f: B \simeq C$. To prove this we note that $f|_B$ and $g|_C$ are one-to-one, and since B and C are finite, there exists an integer n such that $(gf)^n|_B = \mathrm{id}_B$. Let h denote the polynomial $(gf)^{n-1}g$. Then $h(C)=B$ and $hf|_B = \mathrm{id}_B$. Also $fh|_C = \mathrm{id}_C$, since for every $c \in C$, $g(c)=(gf)^n g(c) = g(fh(c))$, and so $c = fh(c)$, since g is one-to-one on C.

Therefore, for a nonvoid subset S of a finite algebra A, if $f \in \mathrm{Pol}_1 A$, then $f: S \simeq f(S)$ if and only if $S \simeq f(S)$. Also, if $S \not\simeq f(S)$, then for no $g \in \mathrm{Pol}_1 A$ does $gf(S) \simeq S$ hold.

Definition 2.2. Let A be a finite algebra with $\alpha \prec \beta$ in **Con** A. A subset S of A is called an $\langle \alpha, \beta \rangle$-subtrace of A if $S^2 \nsubseteq \alpha$ and S is a subset of an $\langle \alpha, \beta \rangle$-trace of A. The set S is a subtrace of A if there exist $\alpha \prec \beta$ in **Con** A for which S is an $\langle \alpha, \beta \rangle$-subtrace.

For an algebra A the relation $\simeq$ of polynomial isomorphism is an equivalence relation on the set of subsets of A. For $\alpha \prec \beta$ in Con A all $\langle \alpha, \beta \rangle$-traces of A are in the same $\simeq$ equivalence class (e.g. 5.2.2 in [4]). For $\langle \alpha, \beta \rangle$-subtraces we have the following.

Lemma 2.3. *Let S be an $\langle \alpha, \beta \rangle$-subtrace of A.*
(i) *If $S \simeq T$, then T is an $\langle \alpha, \beta \rangle$-subtrace.*
(ii) *If $f \in \mathrm{Pol}_1 A$ is such that $f(S)^2 \nsubseteq \alpha$, then $S \simeq f(S)$.*

Proof. Both of these claims follow by an application of Theorem 2.8 (3) of [4], or more precisely, the version of this theorem relativized to traces in Exercise 5.11 (3). □

Definition 2.4. For a nonempty subset S of an algebra A let $\sigma(S)$ denote the transitive closure of $\{f(S)^2 : f \in \mathrm{Pol}_1 A, f(S) \not\simeq S\}$.

Lemma 2.5. *Let S with $|S| \geqslant 2$ be a subset of a finite algebra A. Then:*
(i) *$\sigma(S)$ is a congruence relation on A and $\sigma(S) \leqslant \Theta(S)$;*
(ii) *if $S \simeq T$, then $\sigma(S) = \sigma(T)$.*

Proof. The relation $\sigma(S)$ is reflexive since $|S| \geqslant 2$ and the constant operations are in $\mathrm{Pol}_1 A$. If $g \in \mathrm{Pol}_1 A$ is arbitrary and if $f(S) \not\simeq S$, then $gf(S) \not\simeq S$ as well. So if $(a, b) \in f(S)^2$ with $f(S) \not\simeq S$, then $(g(a), g(b)) \in \sigma(S)$. This shows $\sigma(S)$ is a congruence relation. For any $f \in \mathrm{Pol}_1 A, f(S)^2 \subseteq \Theta(S)$. To prove (ii), we let $g: T \simeq S$ and we suppose $f \in \mathrm{Pol}_1 A$ is such that $f(S) \not\simeq S$. We wish to show that $(f(a), f(b)) \in \sigma(T)$ for all $a, b \in S$. Let $c, d \in T$ be such that $g(c)=a$ and $g(d)=b$. Then $fg(T) \not\simeq T$ and $(f(a), f(b)) = (fg(c), fg(d)) \in \sigma(T)$. □

Lemma 2.6. *Let A be a finite algebra and let S be a nontrivial subset of A. Then S is a subtrace of A if and only if $S^2 \nsubseteq \sigma(S)$.*

Proof. Let S be an $\langle \alpha, \beta \rangle$-subtrace for $\alpha \prec \beta$ in **Con** A. If $f \in \mathrm{Pol}_1 A$ and $f(S)^2 \nsubseteq \alpha$, then $S \simeq f(S)$ by Lemma 2.3. The contrapositive of this statement shows $\sigma(S) \leqslant \alpha$ and since $S^2 \nsubseteq \alpha$ we are done. Conversely, suppose $S^2 \nsubseteq \sigma(S)$. Let β denote $\Theta(S)$ and let α be any congruence with $\sigma(S) \leqslant \alpha \prec \beta$ in **Con** A. Let $(a, b) \in S^2 - \alpha$ be arbitrary. By 2.8 (4) of [4] there exists an $f \in \mathrm{Pol}_1 A$ such that $(f(a), f(b)) \in \beta - \alpha$ and $f(S)$ is an $\langle \alpha, \beta \rangle$-subtrace. It must be the case that $S \simeq f(S)$ for otherwise $f(S)^2 \subseteq \sigma(S) \leqslant \alpha$. An application of Lemma 2.3 yields that S is a subtrace. □

We focus on subtraces that consist of two elements. For a finite algebra A and for $a \neq b$ in A we write $\sigma(a, b)$ in place of $\sigma(\{a, b\})$. For $S = \{a, b\}$, Lemma 2.6 reduces to

$$\{a, b\} \text{ is a subtrace if and only if } (a, b) \notin \sigma(a, b).$$

It follows from Definition 2.4 and Lemma 2.5 that $(c, d) \in \sigma(a, b)$ if and only if there exist $n \geqslant 1$ and $z_0, z_1, \ldots, z_n \in A$ with $c = z_0$, $d = z_n$ and there are $f_1, \ldots, f_n \in \mathrm{Pol}_1 A$ such that for each i, $1 \leqslant i \leqslant n$, $\{z_i, z_{i-1}\} = \{f_i(a), f_i(b)\}$ and $\{a, b\} \not\simeq \{f_i(a), f_i(b)\}$. Two special instances of this description of $\sigma(a, b)$ are contained in the next lemma. They will be used in Section 3.

Lemma 2.7. *Let $a \neq b$ in a finite algebra A.*

(i) *If $a \in \{f(a), f(b)\}$ implies that $\{a, b\} \supseteq \{f(a), f(b)\}$ for every $f \in \mathrm{Pol}_1 A$, then $\{a, b\}$ is a subtrace.*

(ii) *$\{a, b\}$ is not a subtrace if there exist $f, g \in \mathrm{Pol}_1 A$ and $c \in A$ such that $\{a, c\} = \{f(a), f(b)\}$, $\{b, c\} = \{g(a), g(b)\}$ and $\{a, b\} \not\simeq \{f(a), f(b)\}$, $\{a, b\} \not\simeq \{g(a), g(b)\}$.*

Proof. If the condition in (i) holds then $a/\sigma(a, b) = \{a\}$. If the condition in (ii) holds then (a, c) and (c, b) are in $\sigma(a, b)$ so $(a, b) \in \sigma(a, b)$. □

Let A be an algebra and let $\langle a, b \rangle \in A^2$ with $a \neq b$. The ordered pair $\langle a, b \rangle$ is called a *1-snag* if there exists $f \in \mathrm{Pol}_2 A$ such that $f(b, b) = b$ and $f(a, b) = f(b, a) = a$. The pair $\langle a, b \rangle$ is called a *2-snag* if there exists $f \in \mathrm{Pol}_2 A$ such that $f(b, b) = b$ and $f(a, b) = f(b, a) = f(a, a) = a$. By Theorem 7.2 of [4] if $\alpha \prec \beta$ in Con A for a finite algebra A, then $\mathrm{typ}(\alpha, \beta) = \mathbf{1}$ if and only if $\beta - \alpha$ contains no 1-snags and $\mathrm{typ}(\alpha, \beta) = \mathbf{2}$ if and only if $\beta - \alpha$ contains no 2-snags and contains at least one 1-snag.

We now assign to every two-element subset of an algebra one of five types.

Definition 2.8. Let $S = \{a, b\}$, $a \neq b$, be a subset of an algebra A and let $\boldsymbol{S}$ denote the algebra $A|_S$. We define the type of S in the following way:

(i) If $\boldsymbol{S}$ is polynomially equivalent to a Boolean algebra, then S has type **3**.

(ii) If $\boldsymbol{S}$ is polynomially equivalent to a distributive lattice, then S has type **4**.

(iii) If $\boldsymbol{S}$ is polynomially equivalent to a semilattice, then S has type **5**.

(iv) If none of the previous cases hold but $\langle a, b\rangle$ or $\langle b, a\rangle$ is a 1-snag in A, then S has type **2**.

(v) In all other cases, S has type **1**.

We write typ$\{a, b\}=\mathbf{i}$ if the type of $\{a, b\}$ is **i**. Note that in case (iv) typ$\{a, b\}$ is determined by the induced *partial* algebra. We will frequently use the fact that typ$\{a, b\}=\mathbf{3}$, **4** or **5** if and only if at least one of $\langle a, b\rangle$ or $\langle b, a\rangle$ is a 2-snag.

Example 2.9. Let $A=\{0, a, b, 1\}$, consider the partial ordering on A given by $0<a, b<1$ and a, b incomparable, and let $\boldsymbol{A}$ be the algebra whose operations are all operations that are monotonic in this partial order. It is easily verified that $\boldsymbol{A}$ is simple, every trace in $\boldsymbol{A}$ has two elements, and typ$(\boldsymbol{A})=\mathbf{4}$. Except for $\{a, b\}$, every two-element subset of A is a subtrace of type **4**. Note that $\boldsymbol{A}|_{\{a,b\}}$ is polynomially equivalent to a Boolean algebra.

Example 2.10. Let $p>2$ be a prime number, let $\boldsymbol{Z}_p=\langle\{0, 1, \dots, p-1\},+\rangle$ with + addition modulo p, and let $\boldsymbol{U}_p=\langle\{0, 1, \dots, p-1\}, S_p\rangle$ with S_p the set of all permutations on $\{0, 1, \dots, p-1\}$. Then $\boldsymbol{Z}_p$ and $\boldsymbol{U}_p$ are minimal algebras, with typ$(\boldsymbol{Z}_p)=\mathbf{2}$, typ$(\boldsymbol{U}_p)=\mathbf{1}$. Any two-element subset of either algebra is a subtrace. If $0\leqslant i<j<p$, then $\boldsymbol{U}_p|_{\{i,j\}}$ is an essentially unary algebra and it can be argued that $\boldsymbol{Z}_p|_{\{i,j\}}$ is essentially unary as well. However, $\langle i, j\rangle$ in $\boldsymbol{Z}_p$ is a 1-snag, so the type of $\{i, j\}$ in $\boldsymbol{U}_p$ is **1** and in $\boldsymbol{Z}_p$ the type of $\{i, j\}$ is **2**.

Example 2.11. It is easily checked that if $\boldsymbol{M}$ is a minimal algebra and if S is a two-element subset of M, then S is a subtrace of $\boldsymbol{M}$ and the type of S is the same as typ$(\boldsymbol{M})$. This observation is crucial in the following result.

Theorem 2.12. *Let $\boldsymbol{A}$ be a finite algebra and let $\alpha\prec\beta$ in* Con $\boldsymbol{A}$. *Then every two-element $\langle\alpha, \beta\rangle$-subtrace has type* typ$(\alpha, \beta)$. *Moreover,* $\mathbf{i}\in$typ$\{\boldsymbol{A}\}$ *if and only if there is a two-element subtrace of $\boldsymbol{A}$ having type* **i**.

Proof. Let $\{a, b\}$ be an $\langle\alpha, \beta\rangle$-subtrace of $\boldsymbol{A}$. So $(a, b)\in\beta-\alpha$ and there exists $e\in E(\boldsymbol{A})$ with $a, b\in e(A)=U$ such that U is an $\langle\alpha, \beta\rangle$-minimal set. If N denotes the trace $U\cap a/\beta$, then the algebra $\boldsymbol{M}=(\boldsymbol{A}|_N)/\alpha|_N$ is minimal and typ$(\boldsymbol{M})=$typ(α, β). If $a'=a/\alpha|_N$ and $b'=b/\alpha|_N$, then, as noted in Example 2.11, $\{a', b'\}$ is a subtrace of $\boldsymbol{M}$ and typ$(\boldsymbol{M})$ is the same as the type of $\{a', b'\}$. We also note that for $f\in$Pol$\boldsymbol{A}$ with $f|_{\{a,b\}}\in(\text{Pol}\,\boldsymbol{A})|_{\{a,b\}}$, if g denotes the operation induced by ef on the quotient $N/\alpha|_N$, then $g\in$Pol$(\boldsymbol{M})$ and $g|_{\{a',b'\}}$ behaves as $f|_{\{a,b\}}$. Similarly if $\langle a, b\rangle$ is a 1-snag in $\boldsymbol{A}$, then $\langle a', b'\rangle$ is a 1-snag in $\boldsymbol{M}$. With these observations made we commence the proof.

If typ$(\alpha, \beta)=\mathbf{3}$, then by 4.17 of [4] $N=\{a, b\}$ and M is N, so the type of $\{a, b\}$ in $\boldsymbol{A}$ is the same as the type of $\{a, b\}$ in $\boldsymbol{M}$, and this is **3**. Conversely if the type of $\{a, b\}$ in $\boldsymbol{A}$ is **3**, then every member of $(\text{Pol}\,\boldsymbol{A})|_{\{a,b\}}$ induces an equivalent polynomial on

$(\mathrm{Pol}\, \boldsymbol{M})|_{\{a', b'\}}$. So the polynomial structure of $\{a', b'\}$ in $\boldsymbol{M}$ is at least as rich as that of Boolean algebras. So the type of $\{a', b'\}$ in $\boldsymbol{M}$ is **3**.

If $\mathrm{typ}(\alpha, \beta) = \mathbf{4}$, then as in the previous case $M = N = \{a, b\}$. So the type of $\{a, b\}$ in $\boldsymbol{A}$ is **3** or **4**. By the previous case this type cannot be **3**. Conversely if the type of $\{a, b\}$ in $\boldsymbol{A}$ is **4**, then the set $\{a', b'\}$ in $\boldsymbol{M}$ admits lattice polynomials, so the type of $\{a', b'\}$ in $\boldsymbol{M}$ is **3** or **4**. By the previous case, **3** is not possible, so we conclude $\mathrm{typ}(\alpha, \beta) = \mathbf{4}$.

If $\mathrm{typ}(\alpha, \beta) = \mathbf{5}$, then by 4.15 of [4] $\langle a, b\rangle$ or $\langle b, a\rangle$ is a 2-snag of $\boldsymbol{A}$. If both $\langle a, b\rangle$ and $\langle b, a\rangle$ were 2-snags, then $\langle a', b'\rangle$ and $\langle b', a'\rangle$ would be 2-snags in $\boldsymbol{M}$, contradicting $\mathrm{typ}(\alpha, \beta) = \mathbf{5}$. So $\{a, b\}$ has type **5**. Conversely, suppose $\mathrm{typ}\{a, b\} = \mathbf{5}$ with say $\langle a, b\rangle$ a 2-snag in $\boldsymbol{A}$. Then $\langle a', b'\rangle$ is a 2-snag in $\boldsymbol{M}$ and hence $\mathrm{typ}(\boldsymbol{M}) = \mathbf{3}, \mathbf{4}$ or **5**. By virtue of the previous cases, $\mathrm{typ}(\boldsymbol{M}) = \mathbf{5}$.

If $\mathrm{typ}(\alpha, \beta) = \mathbf{2}$, then by 7.2 and 4.20 of [4], neither $\langle a, b\rangle$ nor $\langle b, a\rangle$ are 2-snags and $\boldsymbol{A}|_N$ admits a Mal'cev term d. The polynomial $d(x, a, y)$ shows that $\langle b, a\rangle$ is a 1-snag. So the type of $\{a, b\}$ is **2**. Conversely if the type of $\{a, b\}$ in $\boldsymbol{A}$ is **2** and, say, $\langle a, b\rangle$ is a 1-snag of $\boldsymbol{A}$, then $\langle a', b'\rangle$ is a 1-snag in $\boldsymbol{M}$. So $\mathrm{typ}(\boldsymbol{M}) \neq \mathbf{1}$ and from the previous cases we conclude $\mathrm{typ}(\boldsymbol{M}) = 2$.

Finally, if $\mathrm{typ}(\alpha, \beta) = \mathbf{1}$, then by 7.2 of [4], neither $\langle a, b\rangle$ nor $\langle b, a\rangle$ are 1-snags (or 2-snags). So the type of $\{a, b\}$ is **1**. The converse follows from the previous four cases and from our definition of the type of $\{a, b\}$. $\square$

A development of tame congruence theory by means of two-element subtraces, i.e. $\{a, b\}$ such that $(a, b) \notin \sigma(a, b)$, is the subject of the paper [2].

An algorithm for computing $\mathrm{typ}\{\boldsymbol{A}\}$ based on two-element subtraces is now clear.

Algorithm for determining typ$\{\boldsymbol{A}\}$ for a finite algebra $\boldsymbol{A}$.

The input is all operation tables for the basic operations of $\boldsymbol{A}$.

The output is $\mathrm{typ}\{\boldsymbol{A}\}$.

(1) For each $a \neq b$ in A do

  (2) If $\{a, b\}$ is a subtrace do

    (3) Determine $\mathrm{typ}\{a, b\}$.

These computations can be done without computing $\mathrm{Con}\, \boldsymbol{A}$ or $\mathrm{Pol}_1 \boldsymbol{A}$. In fact, for finite algebras $\boldsymbol{A}$ of a fixed finite similarity type, we show that the time complexity of this computation is a polynomial function of the size of A.

We present some details for this algorithm. Let $\boldsymbol{A}$ be a finite algebra of cardinality k. Let $G(\boldsymbol{A})$ denote the directed graph whose nodes are all $\binom{k}{2}$ two-element subsets of A and in $G(\boldsymbol{A})$ there is an edge from $\{a, b\}$ to $\{c, d\}$ if and only if there is a unary polynomial f with $\{c, d\} = \{f(a), f(b)\}$. Note that $\langle \{a, b\}, \{c, d\}\rangle$ and $\langle \{c, d\}, \{a, b\}\rangle$ are both edges of $G(\boldsymbol{A})$ if and only if $\{a, b\}$ and $\{c, d\}$ are polynomially isomorphic in $\boldsymbol{A}$. The graph $G(\boldsymbol{A})$ is transitive and reflexive. Strongly connected components of $G(\boldsymbol{A})$ correspond to blocks of the equivalence relation $\simeq$ on the set of two-element subsets of A. For $\{a, b\} \subseteq A$, let $G_{a,b}(\boldsymbol{A})$ be the graph with vertex set A and having edge (c, d) if and only if $\langle \{a, b\}, \{c, d\}\rangle$ is an edge of $G(\boldsymbol{A})$ and $\{a, b\} \not\simeq \{c, d\}$. Then $\{a, b\}$ is a subtrace in A if and only if there is no path from a to b in the graph $G_{a,b}(A)$. There exist standard path algorithms of time complexity $O(|A|^2)$ that decide if such a path exists (see for example [1], Section 5.10).

In order to determine the edges of $G(A)$ that originate at node $\{a, b\}$, it suffices to consider the subalgebra of A^2 generated by (a, b) and the k elements (i, i) for $i \in A$. If this algebra has t elements and if g is a basic operation of arity m, then testing all t elements as the arguments of g contributes time complexity $O(t^m)$ to the computation of $G(A)$. Here we assume that a decision on whether a generated element is new or not can be made using a table look-up. This is realistic since $t \leqslant |A|^2$. We will assume that the similarity type is fixed, so if there are b basic operations in the similarity type and if m is the largest arity of a basic operation, then $G(A)$ has a worst case time complexity of $c \cdot b \cdot \binom{k}{2}(k^2)^m$, for some constant c. Thus for fixed similarity type a worst case time complexity for $G(A)$ is $O(k^{2m+2})$. Note that this dominates the path algorithm time complexity, so with the algorithm we have described $O(k^{2m+2})$ is the total time complexity for finding all two-element subtraces of A. The input is of size at least k^m so the complexity is bounded by a cubic function of the length of the input.

A certain amount of speed up in this algorithm is possible. If at some stage in the computation it is discovered that $\{a, b\} \simeq \{c, d\}$, then for all nodes $\{x, y\}$, $\langle \{a, b\}, \{x, y\} \rangle$ is an edge in $G(A)$ if and only if $\langle \{c, d\}, \{x, y\} \rangle$ is an edge and $\{a, b\}$ is a subtrace if and only if $\{c, d\}$ is a subtrace. Moreover, if $\{a, b\} \simeq \{c, d\}$ then $\operatorname{typ}\{a, b\} = \operatorname{typ}\{c, d\}$. So if the equivalence relation $\simeq$ is dynamically maintained during computation, only one representative from each $\simeq$ class is required for computing $\operatorname{typ}\{A\}$.

For another possible speed up, we note that if $\operatorname{Pol}_1 A$ is known to be not too large for computation, then it might be faster to compute $\operatorname{Pol}_1 A$ first and then use it to determine $G(A)$. If say $\operatorname{Pol}_1 A$ has p elements, $\operatorname{Pol}_1 A$ can be computed as a subalgebra of A^k in $O(p^m)$ applications of basic operations to k-tuples, and thus in time $O(kp^m)$ (again using table look-ups to decide if an element is new or not). If p is not much larger than k, then this approach might improve the time for computing $G(A)$ by a factor of k or more. E.g. if $m = 2$ and $p \leqslant k^2$ this gives $O(k^5)$ in comparison to $O(k^6)$.

The type of a two-element set $\{a, b\}$ can be found by an examination of the 4-tuples that appear when computing the subalgebra $\mathbf{C}$ of A^4 generated by $\langle a, b, a, b \rangle$; $\langle b, a, a, b \rangle$; and the k constants $\langle i, i, i, i \rangle$, $i \in A$. If the subalgebra $\mathbf{C}$ has t elements, then $t \leqslant k^4$, hence an m-ary basic operation will contribute $t^m \leqslant k^{4m}$ towards the total time complexity. This dominates the other steps, so the computation of the type of $\{a, b\}$ is bounded above by the fourth power of the size of the input. However, in many cases when $\mathbf{C}$ is large, for example when $\mathbf{C} = A^4$, the type of $\{a, b\}$ can be computed without computing all of $\mathbf{C}$. To this end, when computing $G(A)$ it is easy to flag those pairs $\{a, b\}$ for which $(\operatorname{Pol}_1 A)|_{\{a,b\}}$ contains an involution f, i.e. $f(a) = b$ and $f(b) = a$. When generating $\mathbf{C}$ for the pair $\{a, b\}$, if a 2-snag arises (i.e. $\langle b, b, a, b \rangle$ or $\langle a, a, a, b \rangle$) and if $\{a, b\}$ has been flagged as having an involution, then computation may be halted, the type is **3**. Conversely, if $C = A^4$, then the type of $\{a, b\}$ is **3** and in the computation , thc involution and 2-snag will be generated, so it is not necessary to generate all of $\mathbf{C}$.

Similarly, if $(\mathrm{Pol}_1 A)|_{\{a,b\}}$ has no involution and both 2-snags are generated, then the computation may be halted since $\mathrm{typ}\{a, b\} = \mathbf{4}$.

3. Determining typ$\{V(A)\}$

Let A be a finite algebra and let $V(A)$ denote the variety generated by A, i.e. $V(A) = HSP(A)$. What types can appear in $\mathrm{typ}\{V(A)\}$ that are not in $\mathrm{typ}\{A\}$? Is there a computationally feasible method to determine $\mathrm{typ}\{V(A)\}$ for a given A? This section deals with these questions.

We first note that for a class $\mathscr{K}$ of finite algebras, $\mathrm{typ}\{\mathscr{K}\} = \mathrm{typ}\{H\mathscr{K}\}$ so we may concentrate on the class $SP(A)$ when we investigate $\mathrm{typ}\{V(A)\}$. Although types not appearing in A can appear in subalgebras of A, the algorithms described in Section 2 may be applied to subalgebras of A. Hence our focus is on how types not in $S(A)$ can occur in $SP(A)$. Since we are interested in finite algebras, we can in fact focus on the class $SP_f(A)$ (subalgebras of finite products).

Two results contained in [6] are worth mentioning at this point. If A is finite and if $V(A)$ is congruence modular, then $\mathrm{typ}\{V(A)\} = \mathrm{typ}\{S(A)\}$. So in this situation $\mathrm{typ}\{S(A)\}$, which can be determined using the algorithms of Section 2, provides complete information about $\mathrm{typ}\{V(A)\}$. Note that if $V(A)$ is congruence modular, then $\mathrm{typ}\{V(A)\} \subseteq \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$. A general problem is this: Provide other general algebraic conditions on A or $V(A)$ that guarantee $\mathrm{typ}\{V(A)\} = \mathrm{typ}\{S(A)\}$. A recent related result of Kearnes [5] states that if $\mathscr{V}$ is a locally finite variety that has the Congruence Extension Property, then $\mathrm{typ}\{\mathscr{V}\} \subseteq \mathrm{typ}\{\boldsymbol{F}_{\mathscr{V}}(2)\} \cup \{\mathbf{3}\}$, and if $\mathbf{4} \notin \mathrm{typ}\{\boldsymbol{F}_{\mathscr{V}}(2)\}$, then $\mathrm{typ}\{\mathscr{V}\} = \mathrm{typ}\{\boldsymbol{F}_{\mathscr{V}}(2)\}$.

The second result of [6] that we mention is that if a type occurs in $P(A)$, then it already occurs in A^r for an $r \leqslant |A|^2$. Thus we have an a priori bound on the number of factors needed to produce new types in algebras in $P(A)$. The more general problem is to find a bound on the number of factors of A needed to determine $\mathrm{typ}\{SP_f(A)\}$.

Our main results in this section are negative in that they show there is no small bound m for which $\mathrm{typ}\{SP_f(A)\} = \bigcup_{n<m} \mathrm{typ}\{S(A^n)\}$.

Suppose $\mathbf{i} \in \mathrm{typ}\{\boldsymbol{B}\}$ for a finite algebra $\boldsymbol{B} \in S(A^n)$. Then there are $\bar{a}, \bar{b} \in B$ such that $\{\bar{a}, \bar{b}\}$ is a subtrace of $\boldsymbol{B}$ with $\mathrm{typ}\{\bar{a}, \bar{b}\} = \mathbf{i}$. Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ and let j be such that $a_j \neq b_j$ in A. It is immediate that if $\langle \bar{a}, \bar{b} \rangle$ is a 1-snag (or 2-snag) in $\boldsymbol{B}$, then $\langle a_j, b_j \rangle$ is a 1-snag (or 2-snag) in A. If $\mathrm{typ}\{\bar{a}, \bar{b}\} = \mathbf{3}$, then $\{a_j, b_j\}$ has type $\mathbf{3}$ as well. However, $\{a_j, b_j\}$ in A can have a richer polynomial structure than $\{\bar{a}, \bar{b}\}$ in $\boldsymbol{B}$, i.e. $\mathrm{typ}\{a_j, b_j\}$ can be higher than $\mathrm{typ}\{\bar{a}, \bar{b}\}$ in the partial order of types $\mathbf{1} < \mathbf{2} < \mathbf{3}$ and $\mathbf{1} < \mathbf{5} < \mathbf{4} < \mathbf{3}$. The other critical difference between $\{\bar{a}, \bar{b}\}$ and $\{a_j, b_j\}$ is that $\{a_j, b_j\}$ need not be a subtrace in A.

We cite the following result that is contained in Corollary 7.6 of [4].

Proposition 3.1. *Let A be a finite algebra.*
(i) *If* $\mathrm{typ}\{A\} = \{\mathbf{1}\}$, *then* $\mathrm{typ}\{V(A)\} = \mathbf{1}$.
(ii) *If* $\mathrm{typ}\{A\} \subseteq \{\mathbf{1}, \mathbf{2}\}$, *then* $\mathrm{typ}\{V(A)\} \subseteq \{\mathbf{1}, \mathbf{2}\}$.

Proof. If typ$\{A\}=\mathbf{1}$, then A has no 1-snags by 7.2 of [4]. So no finite algebra in $SP(A)$ has a 1-snag either. Thus typ$\{V(A)\}=\{\mathbf{1}\}$. If typ$\{A\}\subseteq\{\mathbf{1},\mathbf{2}\}$, then A has no 2-snags and again typ$\{V(A)\}\subseteq\{\mathbf{1},\mathbf{2}\}$. □

If typ$\{A\}=\{\mathbf{2}\}$, then it is possible for $\mathbf{1}$ to appear in typ$\{V(A)\}$. For example, Exercise 6.23.8 of [4] exhibits a simple algebra A with typ$(A)=\mathbf{2}$, yet $\mathbf{1}\in$typ$\{A^{|A|}\}$.

As an interesting example of typ$\{S(A)\}\neq$typ$\{SP(A)\}$ we first mention a result of McKenzie that is described in [6]. A is the algebra whose universe is the eight-element Tardos poset P (e.g. [8] or Exercise 10.5 of [4]) and the basic operations of A are all monotonic operations on P. A is simple and A has no proper subalgebras and typ$\{A\}=\mathbf{4}$, yet $\mathbf{3}\in$typ$\{S(A^6)\}$. This construction can be extended to other types by using reducts of A. Namely, for every $i\in\{4,5\}$ and for every $j\in\{1,2,3,4,5\}$ there is a set F_{ij} of operations monotonic on P such that if $A_{ij}=\langle P,F_{ij}\rangle$, then A_{ij} is simple of type $\mathbf{i}$, A_{ij} has no proper subalgebras, and there exists $Q\in S(A_{ij}^6)$ such that $\mathbf{j}\in$typ$\{Q\}$. In this construction a subtrace $\{\bar{a},\bar{b}\}$ of Q is found with $\bar{a}=(a_1,\ldots,a_6)$, $\bar{b}=(b_1,\ldots,b_6)$ such that if $a_i\neq b_i$ in A, then $\{a_i,b_i\}$ is the pair of incomparable middle-level elements of P. Since traces in A consist of two-element ordered sets, $\{a_i,b_i\}$ is not a subtrace of A.

Our first result of this section gives a uniform construction for all the non-Abelian types.

Theorem 3.2. *For every $i\in\{3,4,5\}$ and every $j\in\{1,2,3,4,5\}$ there exists a seven-element algebra C_{ij} which is simple, has no proper subalgebras, has type* $\mathbf{i}$, *and C_{ij}^2 contains a subalgebra D with* $\mathbf{j}\in$typ$\{D\}$.

Proof. The universe of C_{ij} is $C=\{a_1,a_2,b_1,b_2,s_1,s_2,e\}$. The set of basic operations of C_{ij} is denoted F_{ij}, so $C_{ij}=\langle C,F_{ij}\rangle$. We denote by U_1 the set of constant operations on C and by U_2 the set of all unary operations f such that $|f(C)|=2$ and $f(e)=e$. For all choices of i and j, F_{ij} includes U_1 and U_2. Thus C_{ij} is simple and C_{ij} has no proper subalgebras. For $j\in\{1,2,3,4,5\}$ let m_j be a ternary operation on C defined by $m_j(x,y,z)=e$ except if $\{x,y\}\subseteq\{a_k,b_k\}$, $z=s_k$, $k=1$ or 2, in which case $m_j(x,y,s_k)|_{\{a_k,b_k\}}$ is given by

	a_k	b_k
a_k	a_k	a_k
b_k	b_k	b_k

$j=1$

	a_k	b_k
a_k	a_k	b_k
b_k	b_k	a_k

$j=2$

	a_k	b_k
a_k	b_k	a_k
b_k	a_k	a_k

$j=3$

	a_k	b_k
a_k	a_k	a_k
b_k	a_k	b_k

$j=4$

	a_k	b_k
a_k	a_k	b_k
b_k	b_k	b_k

$j=5$

Let M denote the set of operations $\{m_1, m_2, m_3, m_4, m_5\}$. Note that the range of each $m \in M$ is $\{e, a_1, a_2, b_1, b_2\}$ and that the element e is an absorbing element for all the operations in U_2 and in M. Also, any nonconstant unary operation in the clone generated by $U_1 \cup U_2 \cup M$ has e as a fixed point. For $j \in \{1, 2, 3, 4, 5\}$ we include m_j in F_{ij} and for $j=4$ we include m_5 in F_{ij} as well. Thus for any choice of j, $\{a_1, e\}$ is a trace of $\boldsymbol{C}_{ij}$ and $\langle e, a_1 \rangle$ is a 2-snag.

To complete the description of F_{ij}, if $i=3$ we include the unary operation p_3 defined by $p_3(e)=a_1$ and $p_3(x)=e$ for all $x \neq e$ and if $i=4$ we include in F_{ij} the binary operation $p_4(x, y)$ such that p_4 induces a 2-snag on $\langle a_1, e \rangle$ and $p_4(x, y)=e$ for $\{x, y\} \not\subseteq \{a_1, e\}$. Thus $\mathrm{typ}(\boldsymbol{C}_{ij}) = \mathrm{typ}\{a_1, e\} = \mathbf{i}$, for $i=3$, 4 or 5.

We consider the subalgebra $\boldsymbol{D}$ of $\boldsymbol{C}_{ij}^2$ generated by the three elements $a=(a_1, a_2)$, $b=(b_1, b_2)$ and $s=(s_1, s_2)$. A series of claims will establish that $\{a, b\}$ is a trace in $\boldsymbol{D}$ and that the type of $\{a, b\}$ is $\mathbf{j}$. We will repeatedly use the fact that in $\boldsymbol{C}_{ij}$ no basic operation has both s_1 and s_2 in its range and that any basic operation that has more than two elements in its range is in M.

Claim 1. *D consists of the elements a, b and s, together with (c, c), (e, c), (c, e) for all $c \in C$.*

It is clear that each of these 22 elements is in D. Moreover, it is easy to check that the set of these 22 elements is closed under each of the operations in F_{ij}.

Claim 2. *If $p \in \mathrm{Pol}_1 \boldsymbol{D}$ and $p(a) \in \{a, b\}$, then $p(b) \in \{a, b\}$.*

To verify this claim we use induction on the complexity of p. If the polynomial p is a projection or a constant, we are done. Write $p(x) = q(r_1(x), \ldots, r_k(x))$ with q a k-ary basic operation of C_{ij} and $r_1, \ldots, r_k \in \mathrm{Pol}_1 \boldsymbol{D}$. Since both a_1 and a_2 or both $b_1$and b_2 are in the range of p, the basic operation q must be an $m \in M$. So $p(x) = m(r_1(x), r_2(x), r_3(x))$ where Claim 2 holds for the r_i. Now $p(a) \in \{a, b\}$ implies $r_3(a) = s$, but the only unary polynomial whose range contains s is a projection or constant, so $r_3(b) = s$ as well. If $p(a) \in \{a, b\}$, then $r_1(a)$ and $r_2(a)$ are in $\{a_1, b_1\} \times \{a_2, b_2\}$ and by Claim 1 we get $r_1(a)$, $r_2(a) \in \{a, b\}$. By the induction hypothesis $r_1(b), r_2(b) \in \{a, b\}$ as well, so $p(b) \in \{a, b\}$.

From Claim 2 and Lemma 2.7 it follows that $\{a, b\}$ is a subtrace of $\boldsymbol{D}$.

It remains to show that the type of $\{a, b\}$ is $\mathbf{j}$. We first note that if $g \in \mathrm{Pol}_2 \boldsymbol{D}$ induces a 1-snag on $\{a, b\}$, then by Claim 2, $g(\{a, b\}^2) \subseteq \{a, b\}$. Next we let g_j denote $m_j(x, y, s)|_{\{a,b\}}$ and if $j=4$ we define $\boldsymbol{G}_4 = \langle \{a, b\}, g_4, g_5 \rangle$ and if $j \neq 4$, $\boldsymbol{G}_j = \langle \{a, b\}, g_j \rangle$. Then $\boldsymbol{G}_j$ is a minimal algebra of type $\mathbf{j}$. An induction similar to that used in Claim 2 proves the following claim and thereby shows that the type of $\{a, b\}$ is $\mathbf{j}$ in the subalgebra $\boldsymbol{D}$ of $\boldsymbol{C}_{ij}^2$.

Claim 3. *If $p \in (\mathrm{Pol}\, \boldsymbol{D})|_{\{a,b\}}$ then $p \in \mathrm{Pol}\, \boldsymbol{G}_j$.*

Thus the proof of Theorem 3.2 is complete. □

Remark 3.3. With a little more effort it can be shown that in the construction in Theorem 3.2, $\{a, b\}$ is an $\langle \alpha, \beta \rangle$-trace in $\boldsymbol{D}$ in which the congruence relation α has only one nontrivial block, $D - \{a, b\}$, and β has two blocks, $D - \{a, b\}$ and $\{a, b\}$.

We have presented examples of algebras A and integers $m>1$ for which $\mathrm{typ}\{S(A^m)\}$ includes elements not in $\mathrm{typ}\{S(A)\}$. In Theorem 3.2, $m=2$; in the McKenzie example, $m=6$ with $|A|=8$; and in the example based on 6.23.8 of [4], $m=|A|$. These results suggest that perhaps there is a bound m_0, depending on the size of A, such that $\mathrm{typ}\{V(A)\}=\bigcup_{m\leqslant m_0}\mathrm{typ}\{S(A^m)\}$. The next two results show that for types other than **1** there is no such bound. In each case, statement (3) may be also of interest to those who work on the distribution of subdirectly irreducible algebras in finitely generated varieties. Readers unfamiliar with this topic may consult [4], especially Chapter 10 and Problem 12 (p. 192), while readers not interested in this topic should skip (3) and its proof.

Theorem 3.4. *For each* $k\in\{2,3,4\}$ *and for each integer* $m>1$ *there exists a five-element algebra* $\mathbf{C}$ *such that:*

(1) $\mathrm{typ}\{V(\mathbf{C})\}=\{\mathbf{1},\mathbf{k},\mathbf{5}\}$;

(2) $\mathbf{k}\in\mathrm{typ}\{S(\mathbf{C}^m)\}$ but $\mathrm{typ}\{S(\mathbf{C}^r)\}=\{\mathbf{1},\mathbf{5}\}$ *for all* r, $1\leqslant r\leqslant m$;

(3) *The variety generated by* $\mathbf{C}$ *contains arbitrarily large finite and arbitrarily large infinite subdirectly irreducible algebras of type* **k** *monolith, but the smallest such algebra has* $m+3$ *elements.*

Proof. Let $C=\{0,1,2,a,b\}$ and let Σ be the set of all sequences in $\{1,2\}^m$ that contain exactly one occurrence of 2. Define $f\colon C^{m+2}\to C$ as follows: for $\bar{c}\in C^m$ and $u,v\in C$

$$f(\bar{c},u,v)=\begin{cases} a & \text{if } \bar{c}\in\Sigma\setminus\{(2,1,1,\ldots,1)\}, \quad u=v=a;\\ b & \text{if } \bar{c}\in\Sigma\setminus\{(2,1,1,\ldots,1)\}, \quad u=v=b;\\ 0 & \text{otherwise.}\end{cases}$$

Let $H=\{h_+,h_\vee,h_\wedge,h_\downarrow\}$ be a set of $(m+2)$-ary operations on C such that for $h\in H$, $\bar{c}\in C^m$, and $u,v\in C$, the value of $h(\bar{c},u,v)=0$ except if $\bar{c}\in\Sigma$ and $\{u,v\}\subseteq\{a,b\}$ in which case the value of $h(\bar{c},u,v)$ is determined by the values of u and v according to the following scheme:

$$\begin{array}{c|cc} & a & b\\ \hline a & a & b\\ b & b & a \end{array}\qquad \begin{array}{c|cc} & a & b\\ \hline a & a & b\\ b & b & b \end{array}\qquad \begin{array}{c|cc} & a & b\\ \hline a & a & a\\ b & a & b \end{array}\qquad \begin{array}{c|cc} & a & b\\ \hline a & b & a\\ b & a & a \end{array}$$

$$h_+\qquad\qquad h_\wedge\qquad\qquad h_\vee\qquad\qquad h_\downarrow$$

If $k=2$ we let $\mathbf{C}=\langle C,f,h_+\rangle$; if $k=3$, $\mathbf{C}=\langle C,f,h_\downarrow\rangle$; and if $k=4$, $\mathbf{C}=\langle C,f,h,h_\wedge,h_\vee\rangle$. We shall prove that the algebra $\mathbf{C}$ defined this way satisfies the requirements of the theorem.

Let us outline the argument. We start investigating a subtrace $\{\bar{c},\bar{d}\}$ of type different from **1** and **5**, in a subalgebra $\mathbf{B}$ of $\mathbf{C}^r$. We show that the elements of $B\cap\{1,2\}^r$ must satisfy a certain condition. This condition will imply that $r\geqslant m$, and that the type of this subtrace is **k**. Then we show that every subdirectly irreducible factor of $\boldsymbol{B}$ having type **k** monolith must have at least $m+3$ elements.

In the second part of the proof we construct a particular subalgebra $\boldsymbol{B}_r$ of $\boldsymbol{C}^r$, which contains a subtrace of type **k**, and which has a large subdirectly irreducible factor of type **k** monolith.

First we note the following easily verified facts.

(i) Except for the projection operations and constants, every polynomial of $\boldsymbol{C}$ has range contained in $\{0, a, b\}$. Furthermore, every nontrivial polynomial of C^r has its range contained in $\{0, a, b\}^r$.

(ii) Every 1-snag $\langle c, d\rangle$ of $\boldsymbol{C}$ has $\{c, d\} \subseteq \{0, a, b\}$.

(iii) The element 0 is an absorbing element for $\boldsymbol{C}$, i.e. for every non-constant unary polynomial p of $\boldsymbol{C}$ we have $p(0)=0$.

(iv) Corresponding to $k=2, 3,$ and 4, respectively, the induced algebra $\boldsymbol{C}|_{\{a,b\}}$ is polynomially equivalent to a group (with $x+y=h_+(2, 1, \ldots, 1, x, y)$), to a Boolean algebra, and to a distributive lattice.

(v) $\mathbf{1}, \mathbf{5} \in \operatorname{typ}\{\boldsymbol{C}\}$, namely $\{1, 2\}$ and $\{a, 0\}$ are subtraces of type **1** and **5**, respectively.

To see (iv), use the polynomials $h(2, 1, 1, \ldots, 1, x, y)$ to construct the basic operations of this induced algebra. If $\mathbf{k}=\mathbf{4}$, then $C^2 \backslash \{(a, b)\}$ is a subalgebra of $\boldsymbol{C}^2$, hence the induced algebra on $\{a, b\}$ is not a Boolean algebra (it cannot contain an induced unary operation switching a and b). If $\mathbf{k}=\mathbf{2}$, then the subalgebra $C^4 \backslash \{(x, y, u, v) \in \{a, b\}^4 \mid x+y \neq u+v\}$ proves the same. Statement (v) follows from Lemma 2.7, using (i) and (iii) above.

Now let $1 \leqslant r$ and suppose $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{C}^r$ with $\bar{c}, \bar{d} \in B$ such that $\{\bar{c}, \bar{d}\}$ is a subtrace and the type of $\{\bar{c}, \bar{d}\}$ is not **1** or **5**. So we can assume that $\langle \bar{d}, \bar{c}\rangle$ is a 1-snag by virtue of a polynomial $p \in \operatorname{Pol}_2 \boldsymbol{B}$, that is, $p(\bar{c}, \bar{c})=\bar{c}$ and $p(\bar{c}, \bar{d})=p(\bar{d}, \bar{c})=\bar{d}$. The elements $\bar{c}$ and $\bar{d}$ are in the range of p so $\bar{c}$ and $\bar{d}$ are in $\{0, a, b\}^r$. Also, for each $1 \leqslant j \leqslant r$ we have either $\{c_j, d_j\} \subseteq \{a, b\}$, or $c_j=d_j=0$. Indeed, if, say $c_j=0$ but $d_j \neq 0$, then $\bar{c}$ would be an absorbing element of the induced algebra on $\{\bar{c}, \bar{d}\}$, which cannot happen in types **2**, **3**, or **4**. Let $J=\{j \mid 1 \leqslant j \leqslant r, \{c_j, d_j\} \subseteq \{a, b\}\}$.

The polynomial $p(x, y)$ may be written as either $h(t^1(x, y), \ldots, t^{m+2}(x, y))$ for $h \in H$ or as $f(t^1(x, y), \ldots, t^{m+2}(x, y))$. If the polynomial $t^i(x, y)$ is not constant for some $1 \leqslant i \leqslant m$, then $t^i(\bar{c}, \bar{c})$ and $t^i(\bar{c}, \bar{d})$ are contained in $\{0, a, b\}^r$, hence $p(\bar{c}, \bar{c})=p(\bar{c}, \bar{d})=\bar{0}$, which contradicts $\bar{c} \neq \bar{d}$. Therefore $t^i(x, y)$ is a constant $\bar{t}^i$ for $1 \leqslant i \leqslant m$. A similar argument works in each of the coordinates of $\boldsymbol{B}$, showing that for each $j \in J$, the sequence $(t_j^1, \ldots, t_j^m)$ must be in Σ, and if the outermost basic operation of p is f, then this sequence is not equal to $(2, 1, 1, \ldots, 1)$.

Suppose that one of the elements $\bar{t}^i$ $(1 \leqslant i \leqslant m)$ satisfies that for every $j \in J$, $t_j^i \neq 2$. Notice that if the outermost operation of p is f, then $\bar{t}^1$ is such an element. Since every operation $h \in H$ is totally symmetric in its first m variables, in the case when the outermost basic operation of p is h, we can also assume that $i=1$, by suitably rearranging $\bar{t}^1, \ldots, \bar{t}^m$. We shall get a contradiction from this assumption.

Let $g(x, y)=f(\bar{t}^1, \ldots, \bar{t}^m, x, y)$. By the properties of $\bar{t}^1,, \ldots, \bar{t}^m$ listed above, and the fact that $\bar{c}$ and $\bar{d}$ are contained in $\{a, b, 0\}^r$, we see that $g(\bar{c}, \bar{c})=\bar{c}$ and $g(\bar{d}, \bar{d})=\bar{d}$.

Furthermore, we have $g(\bar{c}, \bar{d})=g(\bar{d}, \bar{c})$, call this element $\bar{e}$. The polynomials $g(x, \bar{c})$ and $g(\bar{d}, y)$ send $\{\bar{c}, \bar{d}\}$ to $\{\bar{c}, \bar{e}\}$ and to $\{\bar{e}, \bar{d}\}$, respectively. Since we assumed that $\{\bar{c}, \bar{d}\}$ is a subtrace, by Lemma 2.7 either $\{\bar{c}, \bar{d}\} \simeq \{\bar{c}, \bar{e}\}$ or $\{\bar{c}, \bar{d}\} \simeq \{\bar{d}, \bar{e}\}$. Let j be such that $c_j \neq d_j$. Then $e_j=0$ by the definition of g. Hence either of the above polynomial isomorphisms imply, as 0 is absorbing, that $c_j=0$ or $d_j=0$, which is a contradiction, since $\{c_j, d_j\}=\{a, b\}$ for all such j.

So our assumption was wrong, hence for each $1 \leqslant i \leqslant m$ there exists a $j(i) \in J$ such that $t^i_{j(i)}=2$. We have also seen that for every $j \in J$ the sequence $(t^1_j, \ldots, t^m_j)$ must be in Σ, hence contains exactly one occurrence of 2. Therefore the numbers $j(1), \ldots, j(m)$ are all different, proving $m \leqslant |J| \leqslant r$. Thus we have proved that $\operatorname{typ}\{S(\boldsymbol{C}^r)\} \subseteq \{\mathbf{1}, \mathbf{5}\}$ for all $1 \leqslant r < m$. This, together with (v), gives the second part of statement (2) of the theorem.

In order to show that $\operatorname{typ}\{V(\boldsymbol{C})\} \subseteq \{\mathbf{1}, \mathbf{k}, \mathbf{5}\}$ we prove that $\operatorname{typ}\{\bar{c}, \bar{d}\}=\mathbf{k}$. Notice that for every $h \in H$ that is a basic operation of $\boldsymbol{C}$, the binary polynomial $h(\bar{t}^1, \ldots, \bar{t}^m, x, y)$ is in the induced algebra on $\{\bar{c}, \bar{d}\}$. On the other hand, take a j with $c_j \neq d_j$, By projecting $\boldsymbol{B}$ to its jth coordinate, observation (iv) above shows that the subtrace $\{\bar{c}, \bar{d}\}$ indeed has type $\mathbf{k}$.

So far, we have been chasing type $\mathbf{k}$ subtraces. The next task is to actually construct them, and this is easy, based on the knowledge above. Let $m \leqslant r$ and define the set S_r to consist of those elements of $\{1, 2\}^r$ which contain at least one occurrence of 2 and at least $m-1$ occurrences of 1. Let B_r be the union of S_r and the set of those sequences from $\{a, b, 0\}^r$, which are either constant or contain at least one 0. This is clearly the underlying set of a subalgebra of $\boldsymbol{C}^r$. Let $\bar{a}, \bar{b}$ be the corresponding constant vectors. We show that $\{\bar{a}, \bar{b}\}$ is a subtrace in $\boldsymbol{B}_r$ of type $\mathbf{k}$.

First notice that if $\bar{c}^1, \ldots, \bar{c}^{m+2} \in B_r$, then $f(\bar{c}^1, \ldots, \bar{c}^{m+2})$ always has a coordinate which is 0 (since $\bar{c}^1$ cannot be constant 1). Hence $\bar{a}, \bar{b}$ are not in the range of f. Next we show that for every unary polynomial p, if $p(\bar{a}) \in \{\bar{a}, \bar{b}\}$, then $p(\bar{b}) \in \{\bar{a}, \bar{b}\}$. (In other words, $\{\bar{a}, \bar{b}\}$ is a class of the congruence $\Theta(\bar{a}, \bar{b})$.) Indeed, the same argument as above yields that if such a p is not a constant or the identity map, then $p(x)=h(\bar{t}^1, \ldots, \bar{t}^m, p_1(x), p_2(x))$ such that for each $1 \leqslant j \leqslant r$ we have $(t^1_j, \ldots, t^m_j) \in \Sigma$. Moreover, $p_1(\bar{a}), p_2(\bar{a})$ must be in $\{a, b\}^r$, hence the definition of B_r implies that $p_1(\bar{a}), p_2(\bar{a}) \in \{\bar{a}, \bar{b}\}$. Thus we are done by induction. Therefore, by Lemma 2.7, $\{\bar{a}, \bar{b}\}$ is a subtrace. Finally choose $\bar{t}^1, \ldots, \bar{t}^m$ from S_r to satisfy that for each $1 \leqslant j \leqslant r$ we have $(t^1_j, \ldots, t^m_j) \in \Sigma$. This can be done easily, in many ways. Then the polynomials $h(\bar{t}^1, \ldots, \bar{t}^m, x, y)$ for the basic operations $h \in H$ of $\boldsymbol{C}$ show that the type of this subtrace cannot be $\mathbf{1}$ or $\mathbf{5}$, so it must be $\mathbf{k}$ by the results above. We have proved all assertions of Theorem 3.4, except (3) on the distribution of subdirectly irreducible algebras.

Suppose that $\boldsymbol{B}/\alpha$ is a subdirectly irreducible factor of $\boldsymbol{B}$ with type $\mathbf{k}$ monolith β/α. Then $\langle \alpha, \beta \rangle$ is a type $\mathbf{k}$ prime quotient of $\boldsymbol{B}$. Let $\{\bar{c}, \bar{d}\}$ be a corresponding subtrace. Choose this subtrace so that the set $K=\{j \mid 1 \leqslant j \leqslant r, c_j \neq d_j\}$ is as small as possible. Let $p(x, y)$ be a polynomial inducing a 1-snag on $\langle \bar{d}, \bar{c} \rangle$, so p and the corresponding vectors $\bar{t}^i$ have all the properties established above. We prove that the elements $\bar{t}^1, \ldots, \bar{t}^m, \bar{c}, \bar{d}, \bar{0}$ are pairwise incongruent modulo α, which yields that $|\boldsymbol{B}/\alpha| \geqslant m+3$ as desired.

Take a pair $(\bar{u}, \bar{v}) \in \beta$ which is not in α. We can define the set K' corresponding to this pair as above: $K' = \{j \mid 1 \leqslant j \leqslant r, u_j \neq v_j\}$. By 2.8 (4) of [4], there exists a unary polynomial g of $\boldsymbol{B}$ such that $\{g(\bar{u}), g(\bar{v})\}$ is an $\langle \alpha, \beta \rangle$-subtrace. If $u_j = v_j$, then this equality holds for the g-images, too, hence j is not in the set K'' corresponding to the subtrace $\{g(\bar{u}), g(\bar{v})\}$. Hence the set K is minimal for all pairs in $\beta \backslash \alpha$, not just for subtraces.

First suppose that $\bar{t}^i \alpha \bar{t}^j$ for some $i \neq j$. To simplify notation we may assume that $i = 1$ and $j = 2$. With $p(x, y) = h(\bar{t}^1, \bar{t}^2, \bar{t}^3, \ldots, \bar{t}^m, t^{m+1}(x, y), t^{m+2}(x, y))$ as given above, set $p'(x, y) = h(\bar{t}^1, \bar{t}^1, \bar{t}^3, \ldots, \bar{t}^m, t^{m+1}(x, y), \ t^{m+2}(x, y))$, and let $\bar{u} = p'(\bar{c}, \bar{c})$, $\bar{v} = p'(\bar{c}, \bar{d})$. Then $\bar{u} \alpha \bar{c}$ and $\bar{v} \alpha \bar{d}$, so by transitivity, $(\bar{u}, \bar{v}) \in \beta \backslash \alpha$. Let $K' = \{j \mid 1 \leqslant j \leqslant r, u_j \neq v_j\}$. Choose any $1 \leqslant j \leqslant r$, and consider the altered sequence $(t_j^1, t_j^1, t_j^3, \ldots, t_j^m)$. There are two cases. The first case is, when this altered sequence is still in Σ. Then $u_j = c_j$ and $v_j = d_j$. For such a j we therefore have that $j \in K'$ implies $j \in K$. In the second case, when this altered sequence is not in Σ, we have $u_j = v_j = 0$. Thus $K' \subseteq K$. This latter case occurs for every $j \in K$ with $t_j^1 = 2$, and here $c_j \neq d_j$. Thus K' is a proper subset of K, contradicting the minimality of K.

This contradiction proves that $(\bar{t}^i, \bar{t}^j) \notin \alpha$ if $i \neq j$. Now assume that $\bar{t}^1 \alpha \bar{w}$, where $\bar{w}$ is one of $\bar{c}$, $\bar{d}$, $\bar{0}$. Then replace $\bar{t}^1$ by $\bar{w}$ in the second coordinate of the definition of p'. The new p' yields $\bar{u} = \bar{v} = \bar{0}$, hence $\bar{c} \alpha \bar{d}$ by transitivity, which is a contradiction. Finally, to show that $\bar{c} \alpha \bar{0}$ and $\bar{d} \alpha \bar{0}$ are also impossible, use $h(\bar{t}^1, \ldots, \bar{t}^m, x, y)$ for the basic operations $h \in H$ of $\boldsymbol{C}$ to show that $\bar{c} \alpha \bar{0}$ and $\bar{d} \alpha \bar{0}$ imply each other, hence both imply $\bar{c} \alpha \bar{d}$ by transitivity, which is again a contradiction. Thus $\boldsymbol{B}/\alpha$ indeed has at least $m + 3$ elements.

Finally, to construct large subdirectly irreducible algebras, let α be the congruence on $\boldsymbol{B}_r$ with a single nontrivial block that contains all sequences having a 0 entry. Add the class $\{\bar{a}, \bar{b}\}$ to α to get the congruence β. Then β covers α, and the type of this quotient is $\mathbf{k}$. We show that $\boldsymbol{B}_r/\alpha$ is subdirectly irreducible.

Let ψ be a congruence of $\boldsymbol{B}$ strictly containing α. We need to show that $\beta \subseteq \psi$. First assume that $\bar{t}$ and $\bar{s}$ are different elements of S_r, which are ψ-congruent. We can assume, say, that $t_1 = 2$ and $s_1 = 1$. There are at least $m - 1$ occurrences of 1 in $\bar{t}$, so we can choose $\bar{t}^2, \ldots, \bar{t}^m$ from S_r such that for each $1 \leqslant j \leqslant r$ we have $(t_j^1, \ldots, t_j^m) \in \Sigma$, where $\bar{t}^1 = \bar{t}$. Consider the polynomials $p(x) = h(\bar{t}^1, \bar{t}^2, \bar{t}^3, \ldots, \bar{t}^m, x, x)$, and $p'(x) = h(\bar{s}, \bar{t}^2, \bar{t}^3, \ldots, \bar{t}^m, x, x)$ for some basic operation $h \in H$ of $\boldsymbol{C}$. Then $p(\bar{b}) \in \{\bar{a}, \bar{b}\}$ while the first coordinate of $p'(\bar{b})$ is 0, hence it is congruent to $\bar{0}$ modulo α. The same conclusion holds if $\bar{s}$ is any other element of $\boldsymbol{B}_r$. Finally, using the basic operations $h \in H$ of $\boldsymbol{C}$ we see that $\bar{a} \alpha \bar{0}$ and $\bar{b} \alpha \bar{0}$ imply each other, hence $\bar{a} \alpha \bar{b}$ by transitivity, showing that $\beta \subseteq \psi$. Thus $\boldsymbol{B}/\alpha$ is indeed subdirectly irreducible. In this proof we did not use that r is finite, so this method yields infinite subdirectly irreducibles, too. Thus the proof of Theorem 3.4 is complete. □

Theorem 3.5. *For each integer $m > 1$ there exists a four-element algebra $\boldsymbol{D}$ such that:*

(1) $\operatorname{typ}\{V(\boldsymbol{D})\} = \{\mathbf{1}, \mathbf{3}, \mathbf{5}\}$;

(2) $\mathbf{5} \in \operatorname{typ}\{S(\boldsymbol{D}^m)\}$ *but* $\operatorname{typ}\{S(\boldsymbol{D}^r)\} = \{\mathbf{1}, \mathbf{3}\}$ *for all* r, $1 \leqslant r < m$;

(3) *The variety generated by* $\boldsymbol{D}$ *contains arbitrarily large finite and arbitrarily large infinite subdirectly irreducible algebras of type* **5** *monolith, but the smallest such algebra has* $m+2$ *elements.*

Proof. Let $D=\{1,2,a,b\}$ and let Σ be as in the proof of Theorem 3.4. Define $f\colon D^{m+2}\to D$ as follows: for $\bar{d}\in D^m$ and $u,v\in D$

$$f(\bar{d},u,v)=\begin{cases} b & \text{if } \bar{d}\in\Sigma\setminus\{(2,1,1,\ldots,1)\},\quad \{u,v\}=\{a,b\};\\ a & \text{otherwise.}\end{cases}$$

We let $h\colon D^{m+2}\to D$ be such that $h(d_1,\ldots,d_{m+2})=a$ except $b=h(d_1,\ldots,d_m,b,b)$ for all $(d_1,\ldots,d_m)\in\Sigma$. So f induces on $\{a,b\}$ the binary operation $+$ given by $a+a=b+b=a$ and $a+b=b+a=b$, and $\langle a,b\rangle$ is a 2-snag by virtue of h. We show that $\boldsymbol{D}=\langle D,h,f\rangle$ satisfies the conditions. The main ideas of the proof of this theorem are the same as the ones in the previous proof. Therefore we only outline the argument, present the new tricks, and leave the details to the reader.

Consider the ordering given by $a<b$ on $\{a,b\}$, this induces a partial order on $\{a,b\}^r$, which we shall also denote by $<$. Take a subtrace $\{\bar{c}',\bar{d}'\}$ of a subalgebra $\boldsymbol{B}$ of $\boldsymbol{D}^r$, which has type different from **1**, say $\langle\bar{d}',\bar{c}'\rangle$ is a 1-snag. Then by arguments similar to those given at the start of the proof of Theorem 3.4 one may construct $\bar{c},\bar{d}\in\{a,b\}^r$ such that $\{\bar{c}',\bar{d}'\}$ and $\{\bar{c},\bar{d}\}$ are polynomially isomorphic (thus $\{\bar{c},\bar{d}\}$ is also a subtrace of $\boldsymbol{B}$ of the same type), and moreover, we have that $\bar{c}<\bar{d}$, and $h(\bar{t}^{\,1},\ldots,\bar{t}^{\,m},x,y)$ induces a 2-snag on $\{\bar{c},\bar{d}\}$ for appropriate elements $\bar{t}^{\,1},\ldots,\bar{t}^{\,m}$ of $B\cap\{1,2\}^r$.

As in the proof of Theorem 3.4, suppose that $\bar{t}^{\,1}$ satisfies that $t_j^1\neq 2$ for $j\in J$, where $J=\{j\mid 1\leqslant j\leqslant r,\ d_j\neq a\}$. Consider the polynomials $q(x,y)=f(\bar{t}^{\,1},\ldots,\bar{t}^{\,m},x,y)$ and $g(x,y)=q(q(x,\bar{c}),y)$. Then g yields an Abelian group addition on $\{\bar{c},\bar{d}\}$ with $\bar{c}$ being the zero element. Indeed, it is true for the coordinates belonging to J, since q induces $+$ for these, and the other coordinates of all relevant elements are a. Thus the type of $\{\bar{c},\bar{d}\}$ is **3** in this case.

Therefore, if the type of $\{\bar{c},\bar{d}\}$ is **5**, then, as in the proof of Theorem 3.4, we have $m\leqslant|J|\leqslant r$. The type of this subtrace cannot be **2**, since it has a 2-snag. To prove that it cannot be **4** either, it is easy to check that if p is a nonconstant unary polynomial of $\boldsymbol{B}$ with $p(\bar{c})\geqslant\bar{d}$, of least possible complexity, then $p(x)=f(\bar{t}^{\,1},\ldots,\bar{t}^{\,m},p_1(x),p_2(x))$ such that $\bar{t}^{\,1},\ldots,\bar{t}^{\,m}$ satisfy the condition of the previous paragraph, hence $\{\bar{c},\bar{d}\}$ has type **3**. Thus the type set of this variety is a subset of $\{\mathbf{1},\mathbf{3},\mathbf{5}\}$. The rest of the proof is similar to the proof of Theorem 3.4, and is left to the reader. $\square$

For any ordered pair of positive integers (p,q) there are, up to polynomial equivalence, only finitely many algebras of cardinality p in which all the basic operations have arity at most q. Therefore, there is an integer-valued function $t(p,q)$ such that if $\boldsymbol{A}$ is any algebra of size p in which every basic operation has arity at most q, then $\operatorname{typ}\{V(\boldsymbol{A})\}=\bigcup_{r\leqslant t(p,q)}\operatorname{typ}\{S(\boldsymbol{A}^r)\}$. For example, it is immediately verified that $t(p,1)=t(2,q)=1$ for all p and q. Indeed, if $q=1$, then our algebra is unary, and the variety it generates contains only types **1**. If $p=2$, then using the description of minimal algebras quoted in the Introduction it can be shown that for types **2**, **3** and **4** our two element algebra generates a congruence modular variety, and we are done by the theorem in [6] quoted at the beginning of this section, while for type **5** our algebra is polynomially equivalent to a semilattice, and the meet operation is given by a term. Although it is probably unlikely that an exact formula for $t(p, q)$ will be found, it would be of interest to have a good upper bound for this function.

A consequence of the next result is that for every finite algebra A, if $\mathbf{1} \in \operatorname{typ}\{V(A)\}$, then $\mathbf{1} \in \operatorname{typ}\{S(A^r)\}$ for a choice of $r \leqslant |A|^2$. A slightly different proof of Theorem 3.6 is found in [6].

Theorem 3.6. *Let $\mathcal{W}$ be a locally finite variety.*

(i) *If* $\mathbf{1}$ *is in* $\operatorname{typ}\{\mathcal{W}\}$, *then* $\mathbf{1}$ *is in* $\operatorname{typ}\{\boldsymbol{F}_{\mathcal{W}}(2)\}$.

(ii) *If* $\mathbf{2}$ *is in* $\operatorname{typ}\{\mathcal{W}\}$, *then either* $\mathbf{1}$ *or* $\mathbf{2}$ *is in* $\operatorname{typ}\{\boldsymbol{F}_{\mathcal{W}}(2)\}$.

Proof. Let $A \in \mathcal{W}$ be a finite algebra with $\mathbf{1}$ occurring in $\operatorname{typ}\{A\}$. Without loss of generality we may assume $0 \prec \beta$ in $\operatorname{Con} A$ and that $\operatorname{typ}(0, \beta) = \mathbf{1}$. By 7.2 of [4], there are no 1-snags in β. Let $(a, b) \in \beta - 0$ and let $\boldsymbol{C}$ be the subalgebra of A generated by $\{a, b\}$. We denote by β' the congruence $\beta|_C$ of $\boldsymbol{C}$. Since $(a, b) \in \beta'$, $\beta' > 0_C$. If $(c, d) \in \beta'$ is a 1-snag in $\boldsymbol{C}$, then (c, d) is also a 1-snag in A. So β' contains no 1-snags and 7.2 of [4] shows $\operatorname{typ}\{0, \beta'\} = \{\mathbf{1}\}$. We conclude $\mathbf{1} \in \operatorname{typ}\{\boldsymbol{F}_{\mathcal{W}}(2)\}$ since $\boldsymbol{C}$ is 2-generated.

If $\operatorname{typ}\{\mathcal{W}\}$ includes $\mathbf{2}$ then a similar argument gives that there are no 2-snags in β and no 2-snags in β'. Thus $\operatorname{typ}\{0, \beta'\} \subseteq \{\mathbf{1}, \mathbf{2}\}$ by 7.2 of [4]. $\square$

We conclude with an easy construction that shows that the result of Theorem 3.6 (i) cannot be extended to types other than **1**. (This result is related to Example 3 in Kearnes [5].)

Theorem 3.7. *For each* $i \in \{2, 3, 4, 5\}$ *and for each* $m > 1$ *there exists an algebra* A *such that* $\mathbf{i} \in \operatorname{typ}\{A\}$ *and for* $1 \leqslant n < m$, $\operatorname{typ}\{\boldsymbol{F}_{V(\mathbf{A})}(n)\} = \{\mathbf{1}\}$.

Proof. Let A be the set $\{0, 1, \ldots, m, a, b\}$. Define the operation $f_i : A^{m+2} \to A$ by setting $f_i(c_1, \ldots, c_m, c_{m+1}, c_{m+2}) = 0$ except if $c_j = j$ for all $j \leqslant m$ and $\{c_{m+1}, c_{m+2}\} \subseteq \{a, b\}$ in which case $f(1, 2, \ldots, m, x, y) \in (\operatorname{Pol}_2 A)|_{\{a, b\}}$ is given by

	a	b
a	a	b
b	b	a

$i = 2$

	a	b
a	b	a
b	a	a

$i = 3$

	a	b
a	a	b
b	b	b

$i = 4$

	a	b
a	a	a
b	a	b

$i = 5$

For $i \neq 4$ we let $A = \langle A, f_i \rangle$ and for $i = 4$ we let $A = \langle A, f_4, f_5 \rangle$. Note that the range of each f_i is $\{0, a, b\}$. If β is the equivalence relation on A for which $\{a, b\}$ is the only nonsingleton equivalence class, then β is easily seen to be a congruence of A and $\mathrm{typ}(0, \beta)=\mathbf{i}$. If $n<m$ and $p=f(x_i, \ldots, x_{i_{m+2}})$ is a term involving the variables $x_1, \ldots, x_n$, then p is identically equal to 0 since there exist $i_j=i_k$ for $1<j<k\leqslant m$. From this it follows that $\boldsymbol{F}_{V(\mathbf{A})}(n)$ is essentially unary for every $n<m$. □

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