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Stable properties of graphs

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Abstract

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For many properties P Bondy and Chvátal (1976) have found sufficient conditions such that if a graph $G+uv$ has property P then G itself has property P. In this paper we will give a generalization that will improve ten of these conditions.

1. Introduction

Our notation and terminology follows Berge [1] and Harary [7]. We denote the set of all graphs of order n by R_n. The distance between vertices u and v in the graph $G=(V(G), E(G))$ is denoted by $d_G(u, v)$. Let k be a positive integer. For each $u \in V(G)$ we denote by $N_G^k(u)$ and $M_G^k(u)$ the sets of all $v \in V(G)$ with $d_G(u, v)=k$ and $d_G(u, v) \leqslant k$, respectively.

The k-closure of G is the graph $C_k(G)$ obtained from G by recursively joining pairs of non-adjacent vertices whose degree-sum is at least k, until no such pair remains.

For many properties P, Bondy and Chvátal [2] have found sufficient conditions such that if a graph $G+uv$ has property P, then G itself has property P. In particular it is shown (by paraphrasing Ore's proof [10]) that if $G \in R_n$, $uv \notin E(G)$, $d_G(u)+d_G(v) \geqslant n$ and $G+uv$ is hamiltonian, then G is hamiltonian. Using this condition Bondy and Chvátal [2] have found the following sufficient condition for a graph to be hamiltonian: If the graph $C_n(G)$ is hamiltonian, then G is hamiltonian. In particular, if $n \geqslant 3$ and $C_n(G)=K_n$, then G is hamiltonian. It was noted in [2], that many generalizations of Dirac's condition [6] including those of Chvátal [4] and Las Vernas [9], guarantee that $C_n(G)=K_n$. It was shown in [5], that if $C_n(G)=K_n$ then $|E(G)| \geqslant \lceil (n+2)^2/8 \rceil$.

In this paper we will give a generalization that will improve the conditions of Bondy–Chvátal for ten properties considered in [2]. For example, we prove that if $G + uv$ is hamiltonian, $d_G(u, v) = 2$ and

$$d_G(u) + d_G(v) \geq |M_G^2(u)| + |N_G^3(u) \cap N_G^1(v)|$$

then G is hamiltonian. Using this condition, we define a new closure of the graph G, which has $C_n(G)$ as a spanning subgraph, and G is hamiltonian if and only if this new closure of G is hamiltonian. It is shown that for every $n \geq 6$ there is $G \in R_n$ such that $|E(G)| = 2n - 3$ and the new closure of G is a complete graph.

These results can be viewed as a step towards a unification of the various known results on the existence of hamiltonian cycles in undirected graphs.

We will use the methods of proof that were used in [2].

2. Stability and closures

Let P be a property defined on R_n and r be an integer.

Definition 1. The property P is (k, r)-stable, $k \geq 2$, if whenever $G + uv$ has property P, $d_G(u, v) = 2$ and

$$d_G(u) + d_G(v) \geq |M_G^k(u)| + |N_G^{k+1}(u) \cap N_G^1(v)| + r \tag{2.1}$$

then G itself has property P.

Remark 1. If $k \geq 3$ and $d_G(u, v) = 2$ then (2.1) is equivalent to

$$d_G(u) + d_G(v) \geq |M_G^k(u)| + r$$

because

$$N_G^{k+1}(u) \cap N_G^1(v) = \emptyset.$$

Remark 2. If $d_G(u, v) = 2$ then (2.1) is equivalent to

$$|N_G^1(u) \cap N_G^1(v)| \geq 1 + \sum_{j=2}^{k} |N_G^j(u) \setminus N_G^1(v)| + r$$

because

$$|M_G^k(u)| = 1 + \sum_{j=1}^{k} |N_G^j(u)|, \qquad d_G(u) = |N_G^1(u)|, \qquad d_G(v) = \sum_{j=1}^{3} |N_G^j(u) \cap N_G^1(v)|$$

and

$$N_G^j(u) \setminus N_G^1(v) = N_G^j(u), \qquad N_G^j(u) \cap N_G^1(v) = \emptyset \quad \text{for } j \geq 4.$$

From Definition 1 we have the following.

Proposition 1. *If property P is (k, r)-stable and* $m > k \geqslant 2$, $t > r$, *then*:
(a) *P is (m, r)-stable,*
(b) *P is (k, t)-stable.*

A property P is called $(n+r)$-stable [2] if whenever $G \in R_n$, $G + uv$ has property P and $d_G(u) + d_G(v) \geqslant n + r$, then G itself has property P.

Proposition 2. *If property P is (k, r)-stable,* $k \geqslant 2$ *and* $r \geqslant -1$, *then P is* $(n+r)$*-stable.*

Proof. Assume $G \in R_n$, $G + uv$ has property P and $d_G(u) + d_G(v) \geqslant n + r$. Clearly,

$$d_G(u, v) = 2 \quad \text{and} \quad d_G(u) + d_G(v) \geqslant |M_G^k(u)| + |N_G^{k+1}(u) \cap N_G^1(v)| + r.$$

Hence G has property P which completes the proof. □

In [2], the smallest integer $r(P)$ was found for many properties P such that P is $(n + r(P))$-stable.

In this paper we will find for ten of these properties P the smallest integer $k(P) \geqslant 2$ such that P is $(k(P), r(P))$-stable.

Definition 2. Let $G \in R_n$, $H \in R_n$ and let H be a supergraph of G. We shall say that H is a (k, r)-closure of G, $k \geqslant 2$, if

$$d_H(u) + d_H(v) < |M_H^k(u)| + |N_H^{k+1}(u) \cap N_H^1(v)| + r$$

for all $uv \notin E(H)$ with $d_H(u, v) = 2$ and there exists a sequence of graphs $H_1, \ldots, H_m$ such that $H_1 = G$, $H_m = H$ and for $1 \leqslant i \leqslant m-1$ $H_{i+1} = H_i + u_i v_i$, where $d_{H_i}(u_i, v_i) = 2$ and

$$d_{H_i}(u_i) + d_{H_i}(v_i) \geqslant |M_{H_i}^k(u_i)| + |N_{H_i}^{k+1}(u_i) \cap N_{H_i}^1(v_i)| + r.$$

A (k, r)-closure of a graph is certainly not unique. For example, the graph G in Fig. 1 has two (2, 0)-closures, namely $G + uv$ and $G + uw$.

It is not difficult to see that if $r \geqslant -1$ then $C_{n+r}(G)$ is a subgraph of each (k, r)-closure of G, $k \geqslant 2$.

From Definition 1 and 2 we have the following.

Proposition 3. *If P is (k, r)-stable,* $k \geqslant 2$ *and some (k, r)-closure of G has property P, then G itself has property P.*

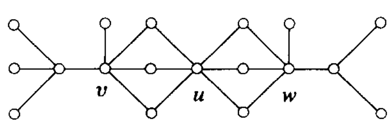

Fig. 1.

3. The hamiltonian property

Lemma 1. *Let $G \in R_n$, $n \geq 3$. If $u_1, u_2, \ldots, u_n$ is a hamiltonian path of G, $d_G(u_1, u_n) = 2$, and*

$$d_G(u_1) + d_G(u_n) \geq |M_G^2(u_1)| + |N_G^3(u_1) \cap N_G^1(u_n)| \tag{3.1}$$

then there is a m such that $2 \leq m \leq n-2$, $u_1 u_{m+1} \in E(G)$ and $u_n u_m \in E(G)$.

Proof. Let $N_G^1(u_1) = \{u_{i_1}, \ldots, u_{i_t}\}$. If $u_n u_{i_j - 1} \notin E(G)$ for every j, $1 \leq j \leq t$, then

$$|N_G^1(u_1) \cap N_G^1(u_n)| + |N_G^2(u_1) \cap N_G^1(u_n)| < |M_G^2(u_1)| - d_G(u_1).$$

But then

$$d_G(u_n) < |M_G^2(u_1)| + |N_G^3(u_1) \cap N_G^1(u_n)| - d_G(u_1)$$

because

$$d_G(u_n) = \sum_{j=1}^{3} |N_G^j(u_1) \cap N_G^1(u_n)|.$$

This contradicts (3.1) and completes the proof. □

Theorem 1. *The property of containing a hamiltonian cycle is* (2, 0)*-stable.*

Proof. Let $G \in R_n$, $n \geq 3$, $d_G(u, v) = 2$ and

$$d_G(u) + d_G(v) \geq |M_G^2(u)| + |N_G^3(u) \cap N_G^1(v)|.$$

Suppose that $G + uv$ is hamiltonian, but G is not. Then, G has a hamiltonian path $u_1, u_2, \ldots, u_n$ with $u_1 = u$, $u_n = v$. From Lemma 1, there is an integer m such that $2 \leq m \leq n-2$, $u_n u_m \in E(G)$ and $u_1 u_{m+1} \in E(G)$. But then G has the hamiltonian cycle $u_1 u_2 \cdots u_m u_n u_{n-1} \cdots u_{m+1} u_1$. This contradicts the hypothesis, and completes the proof. □

From Theorem 1 and Proposition 1 it follows that the property of containing a hamiltonian cycle is (3, 0)-stable. Hence, from Remark 1 we have the following.

Corollary 1. *Let $G \in R_n$, $n \geq 3$. If $d_G(u, v) = 2$, $d_G(u) + d_G(v) \geq |M_G^3(u)|$ and $G + uv$ is hamiltonian, then G is hamiltonian.*

Remark 3. If the (2, 0)-closure of G has the hamiltonian cycle C, then, by using Lemma 1, one can transform C into a hamiltonian cycle in G in exactly the same way that the hamiltonian cycle in $C_n(G)$ was transformed into a hamiltonian cycle in G (see [2]).

From Theorem 1 and Proposition 3 we obtain the following.

Corollary 2. *Let $G \in R_n$, $n \geq 3$. If K_n is the* $(2, 0)$*-closure of G, then G is hamiltonian.*

Theorem 2. *For every $n \geq 6$ there is $G \in R_n$ such that $|E(G)| = 2n - 3$ and K_n is the* $(2, 0)$*-closure of G.*

Proof. Let t be the integer part of the number $n/2$. Consider a sequence of graphs $G_1, \ldots, G_t$, such that $G_t = K_n$, $V(G_i) = \{u_1, u_2, \ldots, u_n\}$, $i = 1, \ldots, t$ and

$$E(G_{t-k+1}) = \{u_iu_j \mid 2k-1 \leq i < j \leq n\} \cup \{u_{2i-1}u_{2i},\ u_{2i-1}u_{2i+1},\ u_{2i}u_{2i+1},\ u_{2i}u_{2i+2} \mid i = 1, \ldots, k-1\}$$

for every k, $2 \leq k \leq t$. (For $n = 8$ the graphs G_1, G_2, G_3 are shown in Fig. 2.) Clearly

$$|E(G_1)| = 2n - 3 \quad \text{and} \quad |E(G_{t-k+2})| - |E(G_{t-k+1})| = 2n - 4k + 1,\ k = 2, \ldots, t.$$

We shall show that G_t is a $(2, 0)$-closure of G_1. For each k, $2 \leq k \leq t$, define $H_{k,0}, H_{k,1}, \ldots, H_{k,2n-4k+1}$ to be a sequence of graphs such that $H_{k,0} = G_{t-k+1}$, $H_{k,2n-4k+1} = G_{t-k+2}$ and

(1) if $k = t$, $n = 2t$ then $H_{k,1} = G_2 = G_1 + u_nu_{n-3}$,

(2) if $k < t$ or $n = 2t + 1$ then

$$H_{k,i+1} = \begin{cases} H_{k,i} + u_{n-i}u_{2k-2} & \text{for } i = 0, 1, \ldots, n-2k-1, \\ H_{k,i} + u_{2n-2k-i}u_{2k-3} & \text{for } i = n-2k, \ldots, 2n-4k. \end{cases}$$

It is not difficult to verify that if $2 \leq k \leq t$, $0 \leq i < 2n - 4k + 1$ and $H_{k,i+1} = H_{k,i} + u_pu_r$, then

$$d_{H_{k,i}}(u_p, u_r) = 2$$

and

$$d_{H_{k,i}}(u_p) + d_{H_{k,i}}(u_r) \geq |M^2_{H_{k,i}}(u_p)| + |N^3_{H_{k,i}}(u_p) \cap N^1_{H_{k,i}}(u_r)|.$$

Hence G_t is a $(2, 0)$-closure of G_1 and this completes the proof. □

4. Other properties

By C_s and P_s we mean a cycle and a path on s vertices, respectively.

Theorem 3. *Let n, s be positive integers with $4 \leq s \leq n$. Then the property of containing a C_s is* $(2, n - s)$*-stable.*

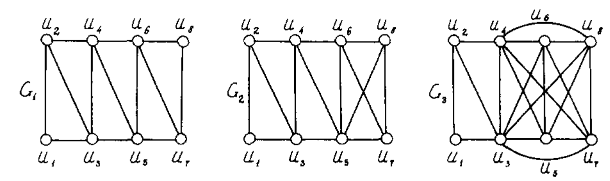

Fig. 2.

Proof. Let $G \in R_n$, $d_G(u, v) = 2$ and

$$d_G(u) + d_G(v) \geq |M_G^2(u)| + |N_G^3(u) \cap N_G^1(v)| + n - s. \tag{4.1}$$

From Remark 2 we have that (4.1) is equivalent to

$$|N_G^1(u) \cap N_G^1(v)| \geq 1 + |N_G^2(u) \setminus N_G^1(v)| + n - s. \tag{4.2}$$

If $G + uv$ contains a C_s but G does not, then G contains a path $u_1, u_2, \ldots, u_s$ with $u_1 = v$, $u_s = u$. Let H be the subgraph of G induced by $\{u_1, u_2, \ldots, u_s\}$. Then $H + uv$ is hamiltonian but H is not. Clearly, $v \in N_G^2(u) \setminus N_G^1(v)$ and

$$|N_G^1(u) \cap N_G^1(v)| \leq |N_H^1(u) \cap N_H^1(v)| + n - s. \tag{4.3}$$

From (4.2) and (4.3) we have $|N_H^1(u) \cap N_H^1(v)| \geq 1$, and so $d_H(u, v) = 2$. Now from Theorem 1 and Remark 2, it follows that

$$|N_H^1(u) \cap N_H^1(v)| < 1 + |N_H^2(u) \setminus N_H^1(v)|. \tag{4.4}$$

It's clear, that $|N_H^2(u) \setminus N_H^1(v)| \leq |N_G^2(u) \setminus N_G^1(v)|$. From (4.3) and (4.4) we can deduce that

$$\begin{aligned} |N_G^1(u) \cap N_G^1(v)| &\leq |N_H^1(u) \cap N_H^1(v)| + n - s \\ &\leq |N_H^2(u) \setminus N_H^1(v)| + n - s \leq |N_G^2(u) \setminus N_G^1(v)| + n - s. \end{aligned} \tag{4.5}$$

This contradicts (4.2) and completes the proof. □

Theorem 4. *Let n, s be positive integers such that s is even and $4 \leq s < n$. Then the property of containing a C_s is $(4, n - s - 1)$-stable.*

Proof. Let $G \in R_n$, $d_G(u, v) = 2$ and

$$d_G(u) + d_G(v) \geq |M_G^4(u)| + n - s - 1. \tag{4.6}$$

From Remark 2 we have that (4.6) is equivalent to

$$|N_G^1(u) \cap N_G^1(v)| \geq n - s + \sum_{j=2}^{4} |N_G^j(u) \setminus N_G^1(v)|. \tag{4.7}$$

If $G + uv$ contains a C_s but G does not, then G contains a path $u_1, u_2, \ldots, u_s$ with $u_1 = v$, $u_s = u$. Let H be the subgraph of G induced by $\{u_1, u_2, \ldots, u_s\}$. As in the proof of Theorem 3, we have (4.5). It's clear, that (4.5) and (4.7) imply

$$\begin{aligned} &|N_G^3(u) \setminus N_G^1(v)| = |N_G^4(u) \setminus N_G^1(v)| = 0, \\ &|N_H^1(u) \cap N_H^1(v)| = |N_H^2(u) \setminus N_H^1(v)| = |N_G^2(u) \setminus N_G^1(v)|, \end{aligned} \tag{4.8}$$

and

$$|N_G^1(u) \cap N_G^1(v)| = |N_H^1(u) \cap N_H^1(v)| + n - s.$$

Since $n > s$, u and v have a common neighbour w.

Clearly,

$$\{k \mid 2 \leq k \leq s - 2,\ u_n u_k \in E(G),\ u_1 u_{k+1} \in E(G)\} = \emptyset, \tag{4.9}$$

because in fact if $u_nu_k \in E(G)$ and $u_1u_{k+1} \in E(G)$ for some k, then $u_1u_2 \cdots u_ku_su_{s-1} \cdots u_{k+1}u_1$ is C_s in G.

In addition we have $u_1u_3 \notin E(G)$, for otherwise $u_1u_3u_4 \cdots u_swu_1$ is a C_s in G. Similarly, we have $u_su_{s-2} \notin E(G)$ for otherwise $u_1u_2 \cdots u_{s-2}u_swu_1$ is a C_s in G.

Let $N_H^1(u) \cap N_H^1(v) = \{u_{i_1}, \ldots, u_{i_t}\}$, $i_0 = 0$ and $i_1 < \cdots < i_t$ if $t \geqslant 2$. Then (4.9) and $u_1 \in N_H^2(u) \setminus N_H^1(v)$ imply that for j, $0 \leqslant j \leqslant t-1$, there exist r_j, such that $i_j < r_j < i_{j+1}$ and $u_{r_j} \in N_H^2(u) \setminus N_H^1(v)$. We can take $r_0 = 1$.

We will now show that $i_t = s-1$. Suppose $i_t < s-1$. Then (4.9) and $u_su_{s-2} \notin E(G)$ imply that there exists r_t such that $i_t < r_t \leqslant s-2$, $uu_{r_{t-1}} \in E(G)$, $uu_{r_t} \notin E(G)$ and $vu_{r_t} \notin E(G)$. But then $\{u_{r_i} \mid i = 0, 1, \ldots, t\} \subseteq N_H^2(u) \setminus N_H^1(v)$ and $|N_H^2(u) \setminus N_H^1(v)| \geqslant t+1$, which contradicts (4.8). Therefore $i_t = s-1$.

Next, note that if $2 \leqslant i \leqslant s-3$, then

$$u_iu_s \in E(G) \Rightarrow u_su_{i+1} \notin E(G). \tag{4.10}$$

Otherwise $u_1 \cdots u_iu_su_{i+1}u_{i+2} \cdots u_{s-1}u_1$ is a C_s in G.

We have that

$$d_H(u_3, u) \leqslant 4 \quad \text{and} \quad N_G^3(u) \setminus N_G^1(v) = N_G^4(u) \setminus N_G^1(v) = \emptyset.$$

Therefore $d_G(u_3, u) \leqslant 2$. If $d_G(u_3, u) = 1$, then from (4.9) and (4.10) we have $u_4 \in N_H^2(u) \setminus N_H^1(v)$. This implies $\{u_4, u_{r_0}, \ldots, u_{r_{t-1}}\} \subseteq N_H^2(u) \setminus N_H^1(v)$ and $|N_H^2(u) \setminus N_H^1(v)| \geqslant t+1$ which contradicts (4.8).

If $d_G(u_3, u) = 2$ and $i_1 \geqslant 4$ then $\{u_3, u_{r_0}, \ldots, u_{r_{t-1}}\} \subseteq N_H^2(u) \setminus N_H^1(v)$, which contradicts (4.8).

Let $d_G(u_3, u) = 2$ and $i_1 = 2$. Then $t \geqslant 2$ and $u_1u_{i_j-1} \notin E(G)$, $j = 1, \ldots, t$, because if $u_1u_{i_j-1} \in E(G)$ for some j, then $u_1u_{i_j} \cdots u_su_2u_3 \cdots u_{i_j-1}u_1$ is a C_s in G. It follows from (4.10) that $u_{i_j-1} \in N_H^2(u) \setminus N_H^1(v)$, $j = 1, \ldots, t$.

Also, $i_{j+1} - i_j = 2$ for every $j = 1, \ldots, t-1$, because if $i_{j+1} - i_j > 2$ for some j, then

$$\{u_{i_1-1}, \ldots, u_{i_t-1}, u_{1+i_j}\} \subseteq N_H^2(u) \setminus N_H^1(v) \quad \text{and} \quad |N_H^2(u) \setminus N_H^1(v)| \geqslant t+1,$$

which contradicts (4.8).

Therefore $s = 2t+1$, which contradicts the hypothesis, that s is even, and completes the proof. □

Fig. 3 (with $n = 10$, $s = 8$) and its obvious generalization show that the property of containing a C_s with $s = 2p < n$ is not $(3, n-s-1)$-stable for $s \geqslant 8$.

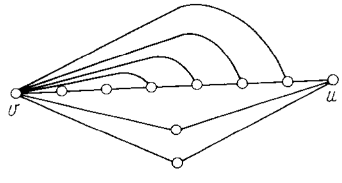

Fig. 3.

Theorem 5. *Let n, s be positive integers with $4 \leqslant s \leqslant n$. Then the property of containing a P_s is $(4, -1)$-stable.*

Proof. Let $G \in R_n$, $d_G(u, v) = 2$ and

$$d_G(u) + d_G(v) \geqslant |M_G^4(u)| - 1. \tag{4.11}$$

From Remark 2 we have that (4.11) is equivalent to

$$|N_G^1(u) \cap N_G^1(v)| \geqslant \sum_{j=2}^{4} |N_G^j(u) \setminus N_G^1(v)|. \tag{4.12}$$

Suppose $G + uv$ contains a P_s but G does not. Then $G + uv$ contains a path $u_1, u_2, \ldots, u_s$ with $u_m = u$, $u_{m+1} = v$ for some m, $1 \leqslant m \leqslant s-1$. Let $N_G^1(u) \cap N_G^1(v) = \{u_{i_1}, \ldots, u_{i_t}\}$, $i_0 = 1$, $i_{t+1} = s$, $i_0 < i_1 < \cdots < i_{t+1}$ and let $i_k < m < i_{k+1}$. Clearly,

$$\{j \mid 1 \leqslant j \leqslant s,\ u_m u_j \in E(G),\ u_{m+1} u_{j+1} \in E(G)\} = \emptyset$$

because if $u_m u_j \in E(G)$ and $u_{m+1} u_{j+1} \in E(G)$ for some j, then G contains a P_s where

$$P_s = \begin{cases} u_1 u_2 \cdots u_j u_m u_{m-1} \cdots u_{j+1} u_{m+1} \cdots u_s & \text{if } j < m, \\ u_1 u_2 \cdots u_m u_j u_{j-1} \cdots u_{m+1} u_{j+1} \cdots u_s & \text{if } j > m. \end{cases}$$

In addition we have $u_s u_m \notin E(G)$ and $u_1 u_{m+1} \notin E(G)$. Then for each j, $j \neq k$, $1 \leqslant j \leqslant t$, there is a u_{r_j} such that $i_j < r_j < i_{j+1}$, $uu_{r_j - 1} \in E(G)$, $uu_{r_j} \notin E(G)$ and $vu_{r_j} \notin E(G)$. Therefore $u_{r_j} \in N_G^2(u) \setminus N_G^1(v)$, $j \neq k$, $1 \leqslant j \leqslant t$, and

$$|N_G^1(u) \cap N_G^1(v)| \leqslant |N_G^2(u) \setminus N_G^1(v)|. \tag{4.13}$$

It follows from (4.12) and (4.13) that $N_G^3(u) \setminus N_G^1(v) = N_G^4(u) \setminus N_G^1(v) = \emptyset$ and

$$t = |N_G^1(u) \cap N_G^1(v)| = |N_G^2(u) \setminus N_G^1(v)|. \tag{4.14}$$

If $uu_1 \notin E(G)$ then $u_1 \in N_G^2(u) \setminus N_G^1(v)$. Then

$$\{u_{r_j} \mid j \neq k,\ 1 \leqslant j \leqslant k\} \cup \{u_1, v\} \subseteq N_G^2(u) \setminus N_G^1(v)$$

and $|N_G^2(u) \setminus N_G^1(v)| \geqslant t + 1$. This contradicts (4.14).

If $uu_1 \in E(G)$, then $i_1 > m$, for otherwise

$$u_{1+i_1} u_{2+i_1} \cdots u_m u_1 u_2 \cdots u_{i_1} u_{m+1} u_{m+2} \cdots u_s$$

is a P_s in G. Therefore

$$\{v, u_{r_1}, \ldots, u_{r_t}\} \subseteq N_G^2(u) \setminus N_G^1(v) \quad \text{and} \quad |N_G^2(u) \setminus N_G^1(v)| \geqslant t + 1.$$

This contradicts (4.14) and completes the proof. □

Fig. 4 (with $n = s = 7$) and its obvious generalization show that the property of containing a P_s is not $(3, -1)$-stable for $s \geqslant 7$.

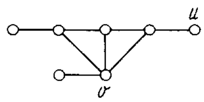

Fig. 4.

Theorem 6. *Let n, s be positive integers with* $4 \leq s \leq n$. *Then the property of containing a* P_s *is* (2, 0)*-stable.*

Proof. Let $G \in R_n$, $d_G(u, v) = 2$ and

$$d_G(u) + d_G(v) \geq |M_G^2(u)| + |N_G^3(u) \cap N_G^1(v)|. \tag{4.15}$$

From Remark 2 we have that (4.15) is equivalent to

$$|N_G^1(u) \cap N_G^1(v)| \geq 1 + |N_G^2(u) \setminus N_G^1(v)|. \tag{4.16}$$

Suppose $G + uv$ contains a P_s but G does not. Then $G + uv$ contains a path $u_1, u_2, \ldots, u_s$ with $u_m = u$, $u_{m+1} = v$ for some m, $1 \leq m \leq s - 1$. As in the proof of Theorem 5, we have $|N_G^1(u) \cap N_G^1(v)| \leq |N_G^2(u) \setminus N_G^1(v)|$. This contradicts (4.16) and completes the proof. □

Corollary 3. *Let n, s be positive integers with* $4 \leq s \leq n$. *Then the property of containing a* P_s *is* (3, 0)*-stable.*

Corollary 3 follows from Theorem 6 and Proposition 1. From Theorem 5, Corollary 3 and Remark 1 we have the following.

Corollary 4. *If* $d_G(u) + d_G(v) \geq \min\{|M_G^4(u)| - 1, |M_G^3(u)|\}$, $d_G(u, v) = 2$ *and* $G + uv$ *contains a* P_s, *then* G *contains a* P_s.

Theorem 7. *Let n, s be positive integers with* $s \leq n - 3$. *Then the property of being s-hamiltonian (see* [3]*) is* (2, *s*)*-stable.*

Proof. Let $G \in R_n$, $d_G(u, v) = 2$ and

$$d_G(u) + d_G(v) \geq |M_G^2(u)| + |N_G^3(u) \cap N_G^1(v)| + s. \tag{4.17}$$

From Remark 2 we have that (4.17) is equivalent to

$$|N_G^1(u) \cap N_G^1(v)| \geq 1 + |N_G^2(u) \setminus N_G^1(v)| + s. \tag{4.18}$$

Suppose that for some set W of at most s vertices of G, $(G + uv) - W$ is hamiltonian but $H = G - W$ is not. We have

$$|N_G^1(u) \cap N_G^1(v)| \leq |N_H^1(u) \cap N_H^1(v)| + s.$$

Together with (4.18) this implies that

$$|N_H^1(u) \cap N_H^1(v)| \geq 1 \quad \text{and} \quad d_H(u, v) = 2.$$

Then from Theorem 1 and Remark 2 we have

$$|N_H^1(u) \cap N_H^1(v)| < 1 + |N_H^2(u) \setminus N_H^1(v)|.$$

Hence

$$|N_G^1(u) \cap N_G^1(v)| \leqslant |N_H^1(u) \cap N_H^1(v)| + s \leqslant |N_H^2(u) \setminus N_H^1(v)| + s$$
$$\leqslant |N_G^2(u) \setminus N_G^1(v)| + s.$$

This contradicts (4.18) and completes the proof. □

The following Theorems 8–12 are obtained by using the same arguments as in [2].

Theorem 8. *Let n, s be positive integers with $s \leqslant n-3$. Then the property of being s-edge-hamiltonian (see [8]) is $(2, s)$-stable.*

Theorem 9. *Let n, s be positive integers with $s \leqslant n-4$. Then the property of being s-hamiltonian-connected (see [1]) is $(2, s+1)$-stable.*

Theorem 10. *Let n, s be positive integers with $s \leqslant n-2$. Then the property of containing $K_{2,s}$ is $(2, s-2)$-stable.*

Theorem 11. *Let n, s be positive integers with $s \leqslant n-2$. Then the property of being s-connected is $(2, s-2)$-stable.*

Theorem 12. *Let n, s be positive integers with $s \leqslant n-2$. Then the property of being s-edge-connected is $(2, s-2)$-stable.*

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